SOME NEW RESULTS IN RICCI CURVATURE

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The purpose of this note is to announce some new results (see [10]) concerning manifolds of positive Ricci curvature.

The motivation for this line of study is ultimately to understand the relationship between curvature and topology. The earliest result in this direction was the Gauss-Bonnet Theorem, which states that for a closed Riemannian 2-manifold, the Euler characteristic is proportional (by a factor of 2π) to the integral of the Gaussian curvature. As a consequence we have for example that any 2-manifold admitting a metric of everywhere positive curvature must have a positive Euler characteristic.

In dimensions greater than two, there are various competing notions of curvature: the sectional, the Ricci and the scalar. The scalar curvature, being the weakest, has proved the easiest to analyse, and much work has been carried out into understanding its topological implications.

It turns out that there are in fact no topological restrictions for negative scalar curvature in dimensions ≥ 3. In other words *any* closed manifold of dimension ≥ 3 can be equipped with a metric of negative scalar curvature - even a sphere!

The case of positive scalar curvature is more interesting. The topological implications are not fully known, but partial results include the following (see [9]):

Theorem. (Stolz) *Let M^n be a smooth, closed, simply-connected manifold with $n \geq 5$. If M is non-spin, then M admits a metric of positive scalar curvature. If M is spin, then M has a positive scalar curvature metric if and only if $\alpha(M) = 0$, where α is a*

certain (topologically defined) homomorphism of spin bordism into connective K-theory.

$$\alpha : \Omega_n^{spin} \longrightarrow ko_n.$$

The key to much of the progress with scalar curvature involves the concept of 'surgery', which we describe presently.

Given an embedding

$$\iota : D^n \times S^m \longrightarrow M,$$

where M is a manifold of dimension $n+m$, we form a new manifold $\hat{M}$ in the following way:

$$\hat{M} = (M \setminus \iota\,(D^n \overset{\circ}{\times} S^m) \amalg (S^{n-1} \times D^{m+1}))/ \sim$$

where $\circ$ denotes the interior and $\sim$ denotes identification of the boundaries via the map ι. (Note that $\partial(D^n \times S^m) = \partial(S^{n-1} \times D^{m+1}) = S^{n-1} \times S^n$.) This process is known as performing an m-surgery. If we wish to be more precise we speak of performing a surgery on the embedded sphere $\iota(* \times S^m) \subset M$. The number n is called the codimension of the surgery, and $\hat{M}$ is the result of the surgery.

The relevance of surgery to questions of curvature arises from the following theorem, which is due to Gromov and Lawson [2] and independently to Schoen and Yau [7].

Theorem. (Gromov, Lawson, Schoen, Yau) *Suppose M is a manifold of dimension ≥ 5 with a positive scalar curvature metric. Let $\hat{M}$ be the result of performing a surgery of codimension ≥ 3 on M. Then $\hat{M}$ has a positive scalar curvature metric.*

We turn our attention now to Ricci curvature. It has been established that there are no topological restrictions also for negative Ricci curvature in dimensions ≥ 3 (see [6]). Since a positive Ricci curvature metric is also a positive scalar curvature metric, any obstruction to admitting a positive scalar curvature metric is also an obstruction to admitting a positive Ricci curvature metric. The only known restriction to positive Ricci curvature which does not arise from positive scalar curvature is stated in the following theorem which is due to Myers:

Theorem. (Myers) *If M is compact and has a positive Ricci curvature metric then the fundamental group $\pi_1 M$ is finite.*

Although there are manifolds (such as $S^2 \times S^1$) which admit metrics of positive scalar curvature but not of positive Ricci curvature, none of the known examples are simply connected.

In the light of the progress made for positive scalar curvature, it is reasonable to look for surgery results in the realm of Ricci curvature. In [8], Sha and Yang prove such a result, though it only applies in very special circumstances.

Note that the normal bundle of the sphere S^m in the product $S^n \times S^m$ has a canonical trivialization, *ie* there is a canonical embedding

$$\iota : D^n \times S^m \longrightarrow \nu(* \times S^m) \subset S^n \times S^m.$$

Suppose that ι is actually an isometry, where the metric on $S^n \times S^m$ is a product of round metrics and the normal bundle fibres have constant radius. Sha and Yang show that provided $m+1$, $n \geq 2$, the result of performing any number of surgeries using such embeddings yields a manifold which admits a positive Ricci curvature metric.

Our first result is a surgery theorem with more flexibility than that of Sha and Yang. It can be shown that Sha and Yang's conclusion remains true if the embedding ι is replaced by an arbitrary one, provided $m+1 \geq n \geq 3$. More generally we have the following:

THEOREM A. *Suppose we have a manifold M of positive Ricci curvature together with an isometric embedding*

$$\iota : D^n_R(N) \times S^m(\rho) \longrightarrow M,$$

where $D^n_R(N)$ denotes a geodesic ball of radius R in the n-sphere with the round metric of radius N, and where $S^m(\rho)$ is the m-sphere with the round metric of radius ρ. Suppose further that $m+1 \geq n \geq 3$. We can twist ι to a non-isometric embedding by composing with a map

$$\tau : D^n \times S^m \longrightarrow D^n \times S^m$$

$$(x, y) \longmapsto (T(y)x, y)$$

where $T : S^m \to SO(n)$. Let $\{0\}$ denote the centre point of D^n. Performing surgery on $\iota(\{0\} \times S^n)$ using the map $\iota \circ \tau$, we again obtain a manifold of positive Ricci curvature provided the ratio $\frac{\rho}{N}$ is suitably small.

Using this result we can prove the following:

THEOREM B. *Homotopy spheres which bound parallelizable manifolds admit metrics of positive Ricci curvature.*

(Note that a homotopy sphere is a manifold homotopy equivalent to a sphere, and that a parallelizable manifold is a manifold with trivial tangent bundle.)

Many examples of Ricci positive manifolds are homotopy spheres. However, by a result of Hitchin, [4], this is not true for all homotopy spheres. Indeed some admit no metric of positive scalar curvature. One would like to find criteria for deciding whether a given homotopy sphere admits a Ricci positive metric or not.

We can divide the set of homotopy spheres into those which bound a parallelizable manifold and those which do not. It is reasonable to ask if this division mirrors the division by positive/negative Ricci curvature.

The diffeomorphism classes of homotopy spheres bounding parallelizable manifolds of dimension m form an abelian group under the connected sum operation. This group is denoted bP_m. It was shown by Kervaire and Milnor in [5] that $bP_{odd} = 0$, bP_{4k+2} is either 0 or $\mathbf{Z}_2$ (depending on k), and bP_{4k} is cyclic. In [3] Hernandez showed that a certain class of Brieskorn manifolds carry Ricci positive metrics. This class includes homotopy spheres representing the non-trivial element of those groups bP_{4k+2} which are isomorphic to $\mathbf{Z}_2$ (a case previously covered by Cheeger in [1]), as well as many elements in bP_{4k}. Until now, however, it was an open question whether in fact all such homotopy spheres admit Ricci positive metrics.

On the other hand, using these methods it can be shown that there is a homotopy sphere of dimension 8 which admits a Ricci positive metric, although it is not the boundary of a parallelizable manifold. Hence the converse to Theorem B is false.

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THE 36TH INTERNATIONAL MATHEMATICAL OLYMPIAD

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The 36th International Mathematical Olympiad took place in York University, Ontario, Canada, on 19th and 20th July 1995. 412 students participated from 73 countries and Ireland was represented by Robert Hayes (St Thomas Community College, Bray), Gavin Hurley (Coláiste an Spioraid Naoimh, Cork), Brian Jones (Gonzaga College, Dublin), Peter McNamara (East Glendalough School, Wicklow), Deirdre O'Brien (Mount Mercy College, Cork) and Gregory Wall (St Mary's Academy, Carlow). The team consisted of the top six performers in the Irish Mathematical Olympiad, which took place on 6th May 1995. The University of Limerick hosted a three-day intensive training session for the team from 5th to 7th July. The training involved staff from UL, UCC and UCD and some previous Irish Olympiad team members. The team leader was Gordon Lessells of UL and the deputy leader was Eugene Gath, also of UL. I accompanied Gordon as an "observer".

Gordon and I flew to Toronto on 12th July, whence we were taken to the University of Waterloo. The first job of the team leaders was to select six problems, from a short list of 28, which would form the two papers of the IMO exam. A local committee had chosen the short-listed questions from a total of about 200 problems which were submitted by the participating countries. The leaders and observers were free to work on the problems for 24 hours, without the distraction (?) of having the official solutions! I felt a bit disappointed because I only managed to solve four of the problems in that time, but felt a little less chastened on discovering that many of the team leaders had similar success! One