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ALGEBRAIC MINIMAL SURFACES IN R^4

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Abstract

There exists a natural correspondence between null curves in C^4 and 'free' curves on $\mathcal{O}(1) \oplus \mathcal{O}(1) \longrightarrow \mathsf{P}_1$; it underlies the existence of 'Weierstrass type formulae' for minimal surfaces in R^4. The construction determines correspondences for minimal surfaces in R^3, and constant mean curvature 1 surfaces in H^3; moreover it facilitates the study of symmetric minimal surfaces in R^4.

1. Introduction

Our purpose here is to describe a natural correspondence between null holomorphic curves in C^4, and 'free' holomorphic curves on the total space of the holomorphic vector bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ over P_1. Our main interest in the former derives from the fact that any minimal surface in R^4 may be described as the real part of such a curve. The correspondence is particularly useful in the study of algebraic minimal surfaces, that is, the real parts of null meromorphic curves in C^4.

The construction is closely related to the classical Klein correspondence between lines in P_3 and points of the quadric $Q_4 \subset \mathsf{P}_5$. In fact, compactifying C^4 to Q_4, and $\mathcal{O}(1) \oplus \mathcal{O}(1)$ to P_3, it may be understood in terms of classical osculation duality, cf. [10]. Here we work in the 'uncompactified picture'. We feel that this makes the differential geometry clearer, in particular the appearance of nullity, and moreover eases the discussion of the relationship with the other correspondences described below.

The correspondence underlies the Weierstrass-type formulae (10)–(13) below. These were found by Montcheuil [15], and studied at length by Eisenhart in [5] and [6]. The geometrical structure underlying the formulae was first exposed by Shaw [21]. This was framed in 'twistor terminology', in terms of the Klein construction. Unfortunately, this interesting paper has largely been

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overlooked. The results of §4 are essentially an amplification of the relevant part Shaw's paper. One should consult [13] for a higher dimensional analogue and [2] for connections with spinors and strings.

Lie discovered a duality between null curves in C^3 and 'free' curves on a singular quadric cone in P_3, see [3], [11], [17] and [18] for further details. In fact this may be understood, following Lie, in terms of classical osculation duality between curves in P_3 and P_3^*. This duality underlies the classical Weierstrass representation formulae, [23]. In §5 we see how this fits into the construction mentioned above and relate the two through the Euler sequence on P_1; moreover we see that the Weierstrass formulae may be derived from the formulae for null curves in C^4.

In [19], a duality between null curves in $\mathsf{P}SL(2,\mathsf{C})$ and 'free' curves on a non-singular quadric surface in P_3 is described. This is interesting mainly because Bryant [1] showed that the former project to H^3 to give all surfaces of constant mean curvature 1. This duality underlies an integrated version of the Bryant representation formula, cf. (18)–(21). In §6 we explain how to derive this duality from the correspondence for null curves in C^4. For other recent derivations, see [4], [7] and [8]. (For basic information about constant mean curvature 1 surfaces in H^3, in addition to [1], one should consult the seminal papers of Umehara and Yamada and their coauthors.)

In §7 we explain how to read various features of an algebraic minimal surface off its dual curve in $\mathcal{O}(1)\oplus\mathcal{O}(1)$; this means the total Gaussian curvature, end and branch point structures.

In §8 we show that the correspondence facilitates, through some elementary representation theory, the study of algebraic minimal surfaces in R^4 with symmetry.

2. Null Geometry of C^4

First we fix notation and review some basic concepts. Complexification of the Euclidean structure of R^4 gives a quadratic form on C^4 which determines the quadric hypersurface $Q_2 \subset \mathsf{P}_3$ of null directions:

$$Q_2 = \{\,[z] \in \mathsf{P}_3\,;\,(z,z) = z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\,\}.$$

The null vectors comprise the affine cone $C(Q_2) \subset \mathsf{C}^4$.

With respect to the coordinates given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} z_1 + iz_2 & z_3 + iz_4 \\ -z_3 + iz_4 & z_1 - iz_2 \end{pmatrix}, \tag{1}$$

Q_2 is described by $(ad - bc = 0)$. Recall that $Q_2 \cong \mathsf{P}_1 \times \mathsf{P}_1$: explicitly, consider the following Segré embedding $\Sigma : \mathsf{P}_1 \times \mathsf{P}_1 \longrightarrow Q_2$, where

$\Sigma([s_0, s_1], [t_0, t_1]) = [s_0 t_0, -s_1 t_0, s_0 t_1, -s_1 t_1]$. Let $\zeta_1 = s_0/s_1$, $\zeta_2 = t_0/t_1$; if neither a and b, nor a and c are both zero, then

$$\Sigma^{-1}([a, b, c, d]) = (-a/b, a/c) = (\zeta_1, \zeta_2), \tag{2}$$

equivalently, in (z_1, z_2, z_3, z_4)-coordinates:

$$\Sigma^{-1}([z_1, z_2, z_3, z_4]) = \left(-\frac{z_1 + iz_2}{z_3 + iz_4}, \frac{z_1 + iz_2}{-z_3 + iz_4}\right). \tag{3}$$

A two-dimensional subspace lying on $C(Q_2)$ is said to be *totally isotropic*; such a subspace projects to a line on Q_2. It is well-known that there are two disjoint families of lines on Q_2, each parameterised by a copy of P_1; two lines meet if and only if they are from different families, see [10] for further details.

This can be understood in terms of the Segré embedding. The two families of lines on Q_2 comprise curves of the form $\Sigma(\{\zeta_1\} \times \mathsf{P}_1)$, and $\Sigma(\mathsf{P}_1 \times \{\zeta_2\})$ respectively; the former are called the A-lines and the latter the B-lines. The corresponding families of isotropic subspaces of C^4 are accordingly referred to as A-planes and B-planes respectively.

A point $q \in Q_2$, lies at the intersection of an A-line with a B-line; these are thus uniquely determined. In fact, the union of these lines is the intersection of the tangent plane to Q_2 at q, with Q_2.

So, there is a P_1 of A-planes passing through the origin in C^4. Each such plane has two dimension's worth of affine translates. Consequently, the set of all affine A-planes in C^4 is parameterised by a rank 2 complex vector bundle over P_1. It is convenient for later calculations to describe this in the following way.

Let $\mathcal{O}(1) \longrightarrow \mathsf{P}_1$, denote the holomorphic line bundle of degree 1, and let $\pi : \mathcal{O}(1) \oplus \mathcal{O}(1) \longrightarrow \mathsf{P}_1$ be the projection map. With respect to an affine coordinate ζ on P_1, an element of $\mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$ takes the form $\sigma_{abcd}(\zeta) = (a + b\zeta, c + d\zeta)$, for some $(a, b, c, d) \in \mathsf{C}^4$, and hence $\mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \cong \mathsf{C}^4$.

REMARK 1. We use the same notation for the total space of a bundle and its sheaf of germs of local sections.

Consider the set of *null sections*:

$$\{\sigma \in \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \,;\, ad - bc = 0\},$$

it is easy to see that *the null sections are precisely the sections which vanish somewhere on* P_1.

Fix a point $\zeta_1 \in \mathsf{P}_1$ and consider the set Π_{ζ_1}, of global sections that vanish there. σ vanishes at ζ_1, means $a + b\zeta_1 = c + d\zeta_1 = 0$, so Π_{ζ_1} is the A-plane determined by $\zeta_1 = -a/b$, cf. (2). Fixing a point $p \in \mathcal{O}(1) \oplus \mathcal{O}(1)$, with $\pi(p) = \zeta_1$,

$$\Pi_p = \{\sigma \in \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \; ; \; \sigma(\zeta_1) = p\}$$

is just an affine translate of Π_{ζ_1}; thus $\pi^{-1}(\zeta_1)$ parameterises all the affine translates of Π_{ζ_1}. Letting Π denote the vector bundle on P_1 with total space $\cup_{\zeta \in \mathsf{P}_1} \Pi_\zeta$, this fact is displayed by the exact sequence

$$0 \longrightarrow \Pi \longrightarrow \mathsf{P}_1 \times \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \stackrel{\epsilon}{\longrightarrow} \mathcal{O}(1) \oplus \mathcal{O}(1) \longrightarrow 0,$$

where $\epsilon(\zeta, \sigma) = \sigma(\zeta)$.

In summary: a point of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ corresponds to an affine A-plane in $\mathsf{C}^4 \cong \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$. The image of the global section $\mathsf{P}_z = \sigma_z(\mathsf{P}_1)$, may be viewed as parameterising the set of affine A-planes in C^4 which pass through z.

Now fix the three dimensional subspace of C^4, given by $(z_4 = 0)$, or equivalently, $(b + c = 0)$.

Let $C(Q_1)$ be the affine cone in C^3 over $(z_1^2 + z_2^2 + z_3^2 = 0; z_4 = 0)$; the latter are equivalent to $(ad + b^2 = 0; b + c = 0)$. Clearly,

$$C(Q_2) \cap \mathsf{C}^3 = C(Q_1).$$

Observe that $\Sigma^{-1}(Q_1)$ is the diagonal $\Delta \subset \mathsf{P}_1 \times \mathsf{P}_1$. Each $q \in \Delta$, determines a pair Π_q^A, Π_q^B, an A-plane and B-plane respectively, such that $T_q Q_2 \cap Q_2 = [\Pi_q^A] \cup [\Pi_q^B]$. Observe that

$$q = \Pi_q^A \cap \Pi_q^B, \qquad \text{and} \qquad q^0 = \Pi_q^A + \Pi_q^B,$$

where $q^0 = \{z \in \mathsf{C}^4 \; ; \; (z, w) = 0, \text{ for all } w \in q\}$.

Recall that an affine plane in C^3 is said to be *null* if the restriction to it of the quadratic form $z_1^2 + z_2^2 + z_3^2$ is degenerate. An affine null line in C^3 lies on a unique affine null plane; for $q \in \Delta$, $q^0 \cap \mathsf{C}^3$, gives the unique null plane in C^3 that contains q.

Let $\mathcal{O}(2)$ denote the line bundle of degree 2 on P_1, and recall that $\mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(2)) \cong \mathsf{C}^3$. In appropriate coordinates the discriminant of $a + b\zeta + c\zeta^2$ becomes $z_1^2 + z_2^2 + z_3^2$; thus the non-zero *null sections* are precisely those possessing a *double root*. Points of $\mathcal{O}(2)$ determine affine null planes in C^3; simply fix a point and consider the set of all global sections passing through it. Thus $\mathcal{O}(2)$ parameterises the set of all affine null planes in C^3. Dually, the global section $\mathsf{P}_z = \sigma_z(\mathsf{P}_1)$, may be viewed as parameterising all the affine null planes through $z \in \mathsf{C}^3$. See [11] and [17] for further details. We now relate this to the correspondence for A-planes in C^4 described above.

Let Π^A and Π^B denote the obvious subbundles of $\underline{\mathsf{C}}^4 = Q_1 \times \mathsf{C}^4$ over Q_1. From elementary linear algebra there is the following exact sequence:

$$(4) \qquad 0 \longrightarrow (\Pi^A + \Pi^B)/\Pi^A \longrightarrow \underline{\mathsf{C}}^4/\Pi^A \xrightarrow{\Theta} \underline{\mathsf{C}}^4/(\Pi^A + \Pi^B) \longrightarrow 0$$

As we saw above, $\underline{\mathsf{C}}^4/\Pi^A$ parameterises the set of affine A-planes in C^4. Clearly, the last term parameterises the set of affine null planes in C^3: Θ is given by intersecting an affine A-plane with C^3, and taking the unique affine null plane in C^3 that contains the affine null line resulting from the intersection. The first term is isomorphic to the trivial bundle.

In fact, this is equivalent to the Euler sequence over P_1, cf. [10]. Let (ζ, η_1, η_2) be the usual coordinates on $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Fixing $\sigma_1, \sigma_2 \in \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1))$, determines a bundle map:

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{O}(2), \quad \text{where} \quad (\zeta, \eta_1, \eta_2) \longrightarrow (\zeta, \eta_1\sigma_1(\zeta) + \eta_2\sigma_2(\zeta)).$$

Setting $\sigma_1(\zeta) = 1, \sigma_2(\zeta) = -\zeta$, gives $\Theta(\zeta, \eta_1, \eta_2) = (\zeta, \eta_1 - \zeta\eta_2)$. It is easy to see that $\ker(\Theta)$ is the trivial line bundle, and thus (4) may be reformulated as:

$$(5) \qquad 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1) \xrightarrow{\Theta} \mathcal{O}(2) \longrightarrow 0.$$

The induced cohomology sequence gives:

$$(6) \qquad 0 \longrightarrow \mathsf{C} \longrightarrow \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \xrightarrow{\tilde{\Theta}} \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(2)) \longrightarrow 0,$$

where

$$\tilde{\Theta}(a + b\zeta,\ c + d\zeta) = a + (b - c)\zeta - d\zeta^2.$$

Observe that $\ker(\tilde{\Theta})$ is the line spanned by $(0, 0, 0, z_4)$, and $\mathsf{C}^3 = (z_4 = 0) \subset \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$ is mapped isomorphically to $\mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(2))$. The following is now clear:

PROPOSITION 2.1. (i) *The null sections in* $(z_4 = 0) \subset \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$ *map via* $\tilde{\Theta}$ *to the null (affine quadric) cone in* $\mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(2))$ *that comprises the zero section, together with the global sections with a double root somewhere on* P_1.

(ii) *For* $z \in (z_4 = 0) \subset \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$, *the* P_1 *of affine* A*-planes that pass through* z*, determines as above, the* P_1 *of affine null planes in* C^3 *that pass through* z.

3. Gauss Maps and 'Weierstrass Formulae'

Let X be a connected Riemann surface and suppose that $\omega : X \longrightarrow \mathsf{C}^4$ is a non-constant null holomorphic curve; i.e. $(\omega', \omega') = 0$, over X. The real part $\phi = (\omega + \bar{\omega})/2$ describes a branched minimal surface in R^4. In fact every such surface may be parameterised in this way; however in general, one may have to pass to the univeral cover of X to do so, because of real periods. See [12], [14] and [16] for further details and background information.

Let $G^+(2, \mathsf{R}^4)$ denote the Grassmann manifold of oriented two planes in R^4. If $\pi \in G^+(2, \mathsf{R}^4)$ has an oriented orthonormal basis $\{e_1, e_2\}$, then $[e_1 + ie_2] \in Q_2$; it is easy to check that this gives a diffeomorphism $G^+(2, \mathsf{R}^4) \cong Q_2$.

The *Gauss map* of ω, is given by $\gamma_\omega = [\omega'] : X \longrightarrow Q_2$. The preceeding observations show that it may be identified with γ_ϕ, the Euclidean Gauss map of ϕ, which is given by $\gamma_\phi(\xi) = d\phi(T_\xi X)$.

Following (3), write $G_\omega = \Sigma^{-1} \circ \gamma_\omega$; this gives the pair

$$G_\omega = (g^A_\omega, g^B_\omega) : X \longrightarrow \mathsf{P}_1 \times \mathsf{P}_1,$$

where

$$g^A_\omega = -\frac{\omega'_1 + i\omega'_2}{\omega'_3 + i\omega'_4}, \qquad g^B_\omega = \frac{\omega'_1 + i\omega'_2}{-\omega'_3 + i\omega'_4}. \tag{7}$$

REMARK 2. Note that g^A_ω and g^B_ω record the A-planes and B-planes, respectively, determined by the null directions described by the Gauss map γ_ω. If $\omega : X \longrightarrow (z_4 = 0)$, then $\gamma_\omega : X \longrightarrow Q_1$, and $g^A_\omega = g^B_\omega$, may be identified with the usual Gauss map $g_\omega : X \longrightarrow \mathsf{P}_1$, cf. [12].

If $\omega : X \longrightarrow \mathsf{C}^4$ is a non-constant null holomorphic curve such that, in (a, b, c, d)-coordinates, ω'_4 is not identically equal to $\pm i\omega'_3$, then there exists a holomorphic 1-form θ on X, such that with respect to (a, b, c, d)-coordinates:

$$\omega = \int (g^A_\omega g^B_\omega, -g^B_\omega, g^A_\omega, -1)\theta, \tag{8}$$

cf. (1); accordingly, in (z_1, z_2, z_3, z_4)-coordinates:

$$\omega = \frac{1}{2} \int (g^A_\omega g^B_\omega - 1, -i(1 + g^A_\omega g^B_\omega), -(g^A_\omega + g^B_\omega), i(g^B_\omega - g^A_\omega))\theta. \tag{9}$$

Conversely, given a holomorphic 1-form θ, such that none of the components in (8) have non-zero periods and moreover, at a pole of g^A_ω, g^B_ω or $g^A_\omega g^B_\omega$, θ has a zero of order at least equal to minus the pole, the above defines a null holomorphic curve $\omega : X \longrightarrow \mathsf{C}^4$, cf. [12].

REMARK 3. Recall that it follows from a theorem of Lawson that if g^A_ω or g^B_ω is constant, then there exists an orthogonal complex structure on R^4 with respect to which $\phi = \mathrm{Re}(\omega)$ is holomorphic, cf. [14].

Now we locally reparameterise the null curve by the first Gauss map variable $\zeta = \zeta_1 = g^A_\omega(\xi)$. Suppose that g^A_ω is non-constant and that the holomorphic functions $f_1(\zeta)$ and $f_2(\zeta)$ are such that

$$g^B_\omega \circ (g^A_\omega)^{-1}(\zeta) = \frac{f_1''}{f_2''}(\zeta) \qquad \text{and} \qquad \theta(\zeta) = -f_2''(\zeta)\, d\zeta.$$

Substituting into (8) gives:

$$\omega \circ (g^A_\omega)^{-1}(\zeta) = \int (-\zeta f_1''(\zeta), f_1''(\zeta), -\zeta f_2''(\zeta), f_2''(\zeta))\, d\zeta,$$

and hence

$$\omega \circ (g^A_\omega)^{-1}(\zeta) = (f_1(\zeta) - \zeta f_1'(\zeta), f_1'(\zeta), f_2(\zeta) - \zeta f_2'(\zeta), f_2'(\zeta)).$$

Converting to (z_1, z_2, z_3, z_4)-coordinates yields the formulae:

$$\omega_1 \circ (g^A_\omega)^{-1}(\zeta) = \frac{1}{2}(f_1(\zeta) - \zeta f_1'(\zeta) + f_2'(\zeta)) \tag{10}$$

$$\omega_2 \circ (g^A_\omega)^{-1}(\zeta) = \frac{i}{2}(-f_1(\zeta) + \zeta f_1'(\zeta) + f_2'(\zeta)) \tag{11}$$

$$\omega_3 \circ (g^A_\omega)^{-1}(\zeta) = \frac{1}{2}(f_1'(\zeta) + \zeta f_2'(\zeta) - f_2(\zeta)) \tag{12}$$

$$\omega_4 \circ (g^A_\omega)^{-1}(\zeta) = \frac{i}{2}(-f_1'(\zeta) + \zeta f_2'(\zeta) - f_2(\zeta)) \tag{13}$$

4. Duality for Null Curves in C^4

We view $\mathsf{C}^4 \cong \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$, and interpret nullity as in §2. Accordingly, to say that a non-constant holomorphic map $\omega : X \longrightarrow \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$ is null means

$$\frac{d\omega}{d\xi}(\xi)(\zeta) = \mathcal{O}[\zeta - g^A_\omega(\xi)].$$

For $\omega : X \longrightarrow \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$, non-constant and null, there exists a globally defined lift of g^A_ω; $\mathscr{A}_\omega : X \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1)$, given by

$$\mathscr{A}_\omega(\xi) = \omega(\xi)(g^A_\omega(\xi)).$$

We refer to this as the *Klein transform* of ω.

REMARK 4. $\mathscr{A}_\omega(\xi)$ gives the affine A-plane in C^4 that contains the null line $\gamma_\omega(\xi)$, and passes through $\omega(\xi)$.

LEMMA 4.1. *Suppose that $\omega : X \longrightarrow \mathrm{H}^0(\mathsf{P}_1, \mathscr{O}(1) \oplus \mathscr{O}(1))$ is null with g^A_ω non-constant. Then $\mathscr{A}_\omega$ determines ω.*

PROOF. Suppose that $(g^A_\omega)^{-1}$ exists on an open set $U \subset \mathsf{P}_1$ and write

$$\mathscr{A}_\omega \circ (g^A_\omega)^{-1}(\zeta) = (f_1(\zeta), f_2(\zeta))$$

over U. If $\omega(\xi)(\zeta) = (a(\xi) + b(\xi)\zeta,\ c(\xi) + d(\xi)\zeta)$, then

$$(f_1'(\zeta),\ f_2'(\zeta)) = (b \circ (g^A_\omega)^{-1}(\zeta),\ d \circ (g^A_\omega)^{-1}(\zeta)).$$

Hence, over U we have:

$$\begin{aligned} a \circ (g^A_\omega)^{-1}(\zeta) &= f_1(\zeta) - \zeta f_1'(\zeta), \qquad & b \circ (g^A_\omega)^{-1}(\zeta) &= f_1'(\zeta) \\ c \circ (g^A_\omega)^{-1}(\zeta) &= f_2(\zeta) - \zeta f_2'(\zeta), \qquad & d \circ (g^A_\omega)^{-1}(\zeta) &= f_2'(\zeta) \end{aligned}$$

Thus, by uniqueness of analytic continuation, $\mathscr{A}_\omega$ determines ω.

REMARK 5. Observe that this elucidates the geometric meaning of (10)–(13).

Let Spé$(\mathscr{O}(1) \oplus \mathscr{O}(1))$ denote the étalé space of the sheaf of germs of local holomorphic sections of $\mathscr{O}(1) \oplus \mathscr{O}(1)$, see [22]. There exists a canonical holomorphic map

$$\Omega : \mathrm{Sp\acute{e}}(\mathscr{O}(1) \oplus \mathscr{O}(1)) \longrightarrow \mathrm{H}^0(\mathsf{P}_1, \mathscr{O}(1) \oplus \mathscr{O}(1)),$$

that is given on stalks by:

$$\begin{aligned} &(\mathscr{O}(1) \oplus \mathscr{O}(1))_\zeta \\ &\qquad \longrightarrow (\mathscr{O}(1) \oplus \mathscr{O}(1))/(\mathscr{I}_\zeta^2 \otimes (\mathscr{O}(1) \oplus \mathscr{O}(1))) \cong \mathrm{H}^0(\mathsf{P}_1, \mathscr{O}(1) \oplus \mathscr{O}(1)), \end{aligned}$$

where $\mathscr{I}_\zeta$ denotes the ideal sheaf of germs of functions vanishing at ζ.

Let $\mathscr{G} \subset$ Spé$(\mathscr{O}(1) \oplus \mathscr{O}(1))$ denote the set of germs of global sections.

THEOREM 4.1. *The map $\Omega : \mathrm{Sp\acute{e}}(\mathscr{O}(1) \oplus \mathscr{O}(1)) \longrightarrow \mathrm{H}^0(\mathsf{P}_1, \mathscr{O}(1) \oplus \mathscr{O}(1))$ is null.*

Its Klein transform

$$\mathscr{A}_\Omega : \mathrm{Sp\acute{e}}(\mathscr{O}(1) \oplus \mathscr{O}(1)) \setminus \mathscr{G} \longrightarrow \mathrm{H}^0(\mathsf{P}_1, \mathscr{O}(1) \oplus \mathscr{O}(1)),$$

is given by evaluation, i.e. $\mathscr{A}_\Omega([\sigma]_\zeta) = \sigma(\zeta)$.

PROOF. By definition of Ω, for $[\sigma]_{\zeta_0} \in \text{Spé}(\mathscr{O}(1) \oplus \mathscr{O}(1))$, there exists a neighbourhood of ζ_0 on which

$$\sigma(\zeta) = \Omega([\sigma]_{\zeta_0})(\zeta) + \mathscr{O}[(\zeta - \zeta_0)^2].$$

Differentiating this equation in the local chart $[\sigma]_{\zeta_0} \longrightarrow \zeta_0$ on $\text{Spé}(\mathscr{O}(1) \oplus \mathscr{O}(1))$, yields

$$\frac{d\Omega}{d\zeta_0}([\sigma]_{\zeta_0})(\zeta) = \mathscr{O}[\zeta - \zeta_0].$$

Hence Ω is null, with $g^A_\Omega([\sigma]_{\zeta_0}) = \zeta_0$.

Observe that

$$\mathscr{A}_\Omega([\sigma]_{\zeta_0})(g^A_\Omega([\sigma]_{\zeta_0})) = \Omega([\sigma]_{\zeta_0})(\zeta_0) = \sigma(\zeta_0).$$

Suppose that $\omega : X \longrightarrow \mathrm{H}^0(\mathsf{P}_1, \mathscr{O}(1) \oplus \mathscr{O}(1))$ is null with g^A_ω non-constant. Let

$$\tilde{X} = \{\xi \in X \ ; \ \mathscr{A}_\omega(\xi) \text{ is transverse to } \pi^{-1}(g^A_\omega(\xi))\}.$$

Furthermore, let $\tilde{\mathscr{A}}_\omega : \tilde{X} \longrightarrow \text{Spé}(\mathscr{O}(1) \oplus \mathscr{O}(1))$, be the natural lift of $\mathscr{A}_\omega$ over $\tilde{X}$.

THEOREM 4.2. *If* $\omega : X \longrightarrow \mathrm{H}^0(\mathsf{P}_1, \mathscr{O}(1) \oplus \mathscr{O}(1))$ *is null with* g^A_ω *non-constant then* $\omega|_{\tilde{X}} = \Omega \circ \tilde{\mathscr{A}}_\omega$.

PROOF. First note $g^A_{\Omega\circ\tilde{\mathscr{A}}_\omega} = g^A_\Omega \circ \tilde{\mathscr{A}}_\omega$. Now,

$$\begin{aligned}\mathscr{A}_{\Omega\circ\tilde{\mathscr{A}}_\omega}(\xi) &= \Omega(\tilde{\mathscr{A}}_\omega(\xi))(g^A_{\Omega\circ\tilde{\mathscr{A}}_\omega}(\xi)) \\ &= \mathscr{A}_\Omega(\tilde{\mathscr{A}}_\omega(\xi)) \\ &= \mathscr{A}_\omega(\xi),\end{aligned}$$

and thus the result follows from Lemma 4.1.

5. Null Curves in C^3

There exists a canonical holomorphic map $\Omega_2 : \text{Spé}(\mathscr{O}(2)) \longrightarrow \mathrm{H}^0(\mathsf{P}_1, \mathscr{O}(2))$, that sends a germ to its 2-jet. It is easy to see that

$$\frac{d\Omega_2}{d\zeta_0}([\sigma]_{\zeta_0})(\zeta) = \mathscr{O}[(\zeta - \zeta_0)^2];$$

this means that Ω_2 is null, with Gauss map $g_{\Omega_2}(\zeta_0) = \zeta_0$.

For a null curve $\omega : X \longrightarrow \mathrm{H}^0(\mathsf{P}_1, \mathscr{O}(2))$, with g_ω non-constant, there exists a lift of g_ω, the *Gauss transform* $\Gamma_\omega : X \longrightarrow \mathscr{O}(2)$, given by $\Gamma_\omega(\xi) =$

$\omega(\xi)(g_\omega(\xi))$. (This gives the unique affine null plane in $\mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(2))$ through $\omega(\xi)$, that contains the null line $[\omega'(\xi)]$.) Moreover, $\omega|_{\tilde{X}} = \Omega_2 \circ \tilde{\Gamma}_\omega$, where

$$\tilde{X} = \{\xi \in X\ ;\ \Gamma_\omega(\xi) \text{ is transverse to } \pi_2^{-1}(g_\omega(\xi))\},$$

and $\tilde{\Gamma}_\omega;\ \tilde{X} \longrightarrow \mathrm{Sp\acute{e}}(\mathcal{O}(2))$, is the natural lift of Γ_ω over $\tilde{X}$.

This gives a correspondence between curves in $\mathcal{O}(2)$ and null curves in $\mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(2))$. Our purpose now is to relate this to the correspondence discussed in §4, via the Euler sequence, as described in §3.

First, consider $\Omega^{-1}(z_4 = 0)$: in particular note that

$$[(f_1,\ f_2)]_{\zeta_0} \in \Omega^{-1}(z_4 = 0) \iff [f_1']_{\zeta_0} = [\zeta f_2' - f_2]_{\zeta_0},$$

cf. [21]. Let $\Theta^e : \Omega^{-1}(z_4 = 0) \longrightarrow \mathrm{Sp\acute{e}}(\mathcal{O}(2))$ denote the map induced by Θ; cf. (5). The following is clear: for $\zeta_0' \in \mathsf{P}_1$,

LEMMA 5.1. $(\Theta^e)^{-1}([f]_{\zeta_0'}) = \left[\left(f - \frac{1}{2}\zeta_0 f', -\frac{1}{2}f'\right)\right]_{\zeta_0'}$.

Now suppose that $[(f_1,\ f_2)]_{\zeta_0'} \in \Omega^{-1}(z_4 = 0)$. On a neighbourhood U of ζ_0',

$$\Omega(\zeta_0)(\zeta) = (a(\zeta_0) + b(\zeta_0)\zeta,\ c(\zeta_0) + d(\zeta_0)\zeta),$$

where, writing $f = f_1 - \zeta_0 f_2$, from Lemma 5.1 and (10)–(13):

$$\begin{aligned} a(\zeta_0) &= f(\zeta_0) - \zeta_0 f'(\zeta_0) + \frac{1}{2}\zeta_0^2 f''(\zeta_0) \\ b(\zeta_0) &= \frac{1}{2}(f'(\zeta_0) - \zeta_0 f''(\zeta_0)) \\ c(\zeta_0) &= -b(\zeta_0) \\ d(\zeta_0) &= -\frac{1}{2}f''(\zeta_0) \end{aligned}$$

It follows that $\tilde{\Theta} \circ \Omega(\zeta_0) = A(\zeta_0) + B(\zeta_0)\zeta + C(\zeta_0)\zeta^2$, where

$$\begin{aligned} A(\zeta_0) &= f(\zeta_0) - \zeta_0 f'(\zeta_0) + \frac{1}{2}\zeta_0^2 f''(\zeta_0) \\ B(\zeta_0) &= f'(\zeta_0) - \zeta_0 f''(\zeta_0) \\ C(\zeta_0) &= \frac{1}{2}f''(\zeta_0) \end{aligned}$$

which, converting to $(z_1, z_2, z_3, 0)$-coordinates, give the classical Weierstrass formula for a null curve in C^3.

On the other hand, it is clear that $\Omega_2 \circ \Theta^e$ gives the same result, thus we have:

PROPOSITION 5.1. $\tilde{\Theta} \circ \Omega = \Omega_2 \circ \Theta^e$.

LEMMA 5.2. *If* $\psi_1, \psi_2 : X \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1)$, *are such that* $g^A_{\psi_1}$ *and* $g^A_{\psi_2}$ *are non-constant, and* $\Omega \circ \tilde{\psi}_1 = \Omega \circ \tilde{\psi}_2$, *then* $\psi_1 = \psi_2$.

PROOF. This follows immediately from

$$\psi_j \circ (g^A_{\Omega \circ \tilde{\psi}_j})^{-1}(\zeta_0) = \Omega(\tilde{\psi}_j \circ (g^A_{\Omega \circ \tilde{\psi}_j})^{-1}(\zeta_0))(\zeta_0).$$

In the next result we see that the Klein and Gauss transforms for a curve $\omega : X \longrightarrow (z_4 = 0)$, are related, via the Euler sequence, in the obvious way:

THEOREM 5.1. *For* $\omega : X \longrightarrow (z_4 = 0)$, *such that* g^A_ω *is non-constant,*

$$\Gamma_{\tilde{\Theta} \circ \omega} = \Theta \circ \mathscr{A}_\omega.$$

PROOF. First observe that $\tilde{\Theta} \circ \omega|_{\tilde{X}} = \tilde{\Theta} \circ \Omega \circ \tilde{\mathscr{A}}_\omega$. But, $\tilde{\Theta} \circ \omega|_{\tilde{X}} = \Omega_2 \circ \tilde{\Gamma}_{\tilde{\Theta} \circ \omega}$, and hence, from Proposition 5.1, $\Omega \circ \tilde{\mathscr{A}}_\omega = \Omega \circ (\Theta^e)^{-1} \circ \tilde{\Gamma}_{\tilde{\Theta} \circ \omega}$. The result now follows from Lemma 5.2.

6. Null Curves in $\mathsf{P}\,\mathrm{SL}(2, \mathsf{C})$

Translation of the Cartan-Killing form on $\mathsf{P}\,\mathrm{SL}(2, \mathsf{C})$ gives a holomorphic quadratic form Φ; the induced null cone in each tangent space endows $\mathsf{P}\,\mathrm{SL}(2, \mathsf{C})$ with a conformal structure. In [1], Bryant showed that holomorphic curves in $\mathsf{P}\,\mathrm{SL}(2, \mathsf{C})$ which are null with respect to this conformal structure project to H^3, hyperbolic space of curvature -1, to give surfaces of constant mean curvature 1.

In [19], it was shown that classical osculation duality between curves in P_3 and P_3^* induces a natural correspondence between null holomorphic curves in $\mathsf{P}\,\mathrm{SL}(2, \mathsf{C})$ and curves on the dual quadric $Q_2 \cong \mathsf{P}_1 \times \mathsf{P}_1$. This means that Bryant's representation can be integrated, at least locally, to yield 'free' Weierstrass type representation formulae for constant mean curvature 1 surfaces in H^3, in terms of a single holomorphic function $f(\zeta)$, cf. (18)–(21) below.

In this section we show that this correspondence and the resulting formulae can be derived from the correspondence described in §4.

The first point to note is that for a holomorphic map $\omega : X \longrightarrow \mathrm{SL}(2, \mathsf{C})$,

$$\omega^* \Phi = -4 \det(\omega')(d\xi)^2.$$

PROPOSITION 6.1. *The conformal structure on* $\mathrm{SL}(2, \mathsf{C})$ *determined by* Φ *is the same as that induced by the complexification of the Euclidean structure on* C^4.

Now, suppose that $\omega : X \longrightarrow \mathrm{SL}(2, \mathsf{C}) \subset \mathsf{C}^4$ is null, with g^A_ω non-constant. Locally reparameterising by $(g^A_\omega)^{-1}$, there exist holomorphic functions f_1 and f_2 such that

$$\omega \circ (g^A_\omega)^{-1}(\zeta) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} f_1 - \zeta f_1' & f_1' \\ f_2 - \zeta f_2' & f_2' \end{pmatrix}; \tag{14}$$

$ad - bc = 1$ means that $f_1 f_2' - f_1' f_2 = 1$ and hence

$$g^B_\omega \circ (g^A_\omega)^{-1} = \frac{f_1''}{f_2''} = \frac{f_1}{f_2} = -f \quad \text{(say)}. \tag{15}$$

Differentiating $f_1/f_2 = -f$ yields $f_2 = (f')^{-1/2}$, and thus:

$$f_1 = -f(f')^{-1/2} \tag{16}$$

$$f_2 = (f')^{-1/2} \tag{17}$$

Substitution into (14) gives:

$$a = -f(f')^{-1/2} + \zeta \left\{ (f')^{1/2} - \frac{1}{2} f (f')^{-3/2} f'' \right\} \tag{18}$$

$$b = -(f')^{1/2} + \frac{1}{2} f (f')^{-3/2} f'' \tag{19}$$

$$c = (f')^{-1/2} + \frac{1}{2} \zeta (f')^{-3/2} f'' \tag{20}$$

$$d = -\frac{1}{2} (f')^{-3/2} f'' \tag{21}$$

Now solving, as in [19],

$$\frac{a + b\zeta}{c + d\zeta} = -f, \qquad \frac{1}{(c + d\zeta)^2} = f', \qquad \frac{2d}{(c + d\zeta)^3} = -f'',$$

for a, b, c, d, gives essentially the same formulae.

REMARK 6. (i) In fact (18)–(21) differ slightly from the formulae in (3.1) of [19]; this is because there we solved $(\alpha\zeta + \beta)/(\gamma\zeta + \delta) = f$, etc.

(ii) Global versions may be derived from (22)–(25) below; cf. [19].

The geometrical relationship that underlies this derives from the holomorphic map

$$\Psi : \mathcal{O}(1) \oplus \mathcal{O}(1) \setminus \mathsf{P}_0 \longrightarrow \mathsf{P}_1 \times \mathsf{P}_1, \quad \text{where } \Psi(\zeta, \eta_1, \eta_2) = (\zeta, \eta_1/\eta_2),$$

and $\mathsf{P}_0 = (\eta_1 = \eta_2 = 0)$. The induced map on global sections, on restriction to $\mathrm{SL}(2, \mathsf{C}) \subset \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \setminus \{0\}$, gives the usual double covering

$$\tilde{\Psi} : \mathrm{SL}(2, \mathsf{C}) \longrightarrow \mathsf{P}\,\mathrm{SL}(2, \mathsf{C}).$$

Now, given $\omega : X \longrightarrow \mathrm{SL}(2, \mathsf{C})$, $\tilde{\Psi} \circ \omega : X \longrightarrow \mathsf{P}\,\mathrm{SL}(2, \mathsf{C})$, has a *Gauss transform* $\Gamma_{\tilde{\Psi}\circ\omega} : X \longrightarrow \mathsf{P}_1 \times \mathsf{P}_1$, cf. [19]. It is explained there that this map records the totally geodesic null hypersurfaces of $\mathsf{P}\,\mathrm{SL}(2, \mathsf{C})$ that osculate the curve and moreover that these are cut out by hyperplanes of P_3 which lie tangent to the quadric at infinity of $\mathsf{P}\,\mathrm{SL}(2, \mathsf{C})$, i.e. $(ad - bc = 0)$. It is clear from above that the A-planes determined by the lift of the curve into $\mathrm{SL}(2, \mathsf{C}) \subset \mathsf{C}^4$, give the same information, that is to say we have:

THEOREM 6.1. $\Gamma_{\tilde{\Psi}\circ\omega} = \Psi \circ \mathscr{A}_\omega$.

7. Metrical Aspects

Suppose that g, f_1, f_2 are non-constant meromorphic functions on a compact, connected Riemann surface Y; the following are global versions of (10)–(13):

$$\omega_1 = \frac{1}{2}\left\{ f_1 - g\frac{df_1}{dg} + \frac{df_2}{dg} \right\} \tag{22}$$

$$\omega_2 = \frac{i}{2}\left\{ -f_1 + g\frac{df_1}{dg} + \frac{df_2}{dg} \right\} \tag{23}$$

$$\omega_3 = \frac{1}{2}\left\{ \frac{df_1}{dg} + g\frac{df_2}{dg} - f_2 \right\} \tag{24}$$

$$\omega_4 = \frac{i}{2}\left\{ -\frac{df_1}{dg} + g\frac{df_2}{dg} - f_2 \right\} \tag{25}$$

$\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$, describes a null meromorphic curve $\omega : X \longrightarrow \mathsf{C}^4$, where X is Y punctured at the poles of the above. We derive various formulae from (22)–(25); similiar formulae can be derived from (10)–(13), which work locally on an arbitrary null curve with non-constant g^A.

If there exist $a, b, c, d \in \mathsf{C}$, such that $f_1 = a + bg$, and $f_2 = c + dg$, then the data describes a global section of $\mathcal{O}(1) \oplus \mathcal{O}(1)$, in which case osculation is degenerate and the null curve constant. *In the following we suppose that no such constants exist.*

First observe that if $\phi = \mathrm{Re}(\omega)$, then the metric induced on X by ϕ is:

$$ds_\phi^2 = \frac{1}{4}(1 + |g|^2)\left|\frac{dg}{d\xi}\right|^2 \left\{ \left|\frac{d^2 f_1}{dg^2}\right|^2 + \left|\frac{d^2 f_2}{dg^2}\right|^2 \right\} |d\xi|^2 \tag{26}$$

PROPOSITION 7.1. *For ϕ as above, the total Gaussian curvature of the induced metric is*

$$\int K\, dA_\phi = -2\pi \left\{ \deg(g) + \deg\left(\frac{d^2 f_1}{dg^2} / \frac{d^2 f_2}{dg^2} \right) \right\}$$

PROOF. It is well-known that

$$\int K\, dA_\phi = -2\pi \{ \deg(g_\omega^A) + \deg(g_\omega^B) \},$$

cf. [12], and the result follows immediately.

The next result, which follows easily from (26), characterises branch points in the induced metric in terms of the behaviour of g, f_1 and f_2:

PROPOSITION 7.2. *Suppose that at a point $\xi_0 \in X$, the local coordinate ξ, centred at ξ_0, is such that:*

$$g(\xi) = \xi^q; \quad f_1(\xi) = a_0 + a_p \xi^p + \cdots \quad \textit{and} \quad f_2(\xi) = b_0 + b_r \xi^r + \cdots$$

where p and r are positive integers.

(i) *If $q > 0$, then $ds_\phi^2(\xi_0) = 0$ if and only if $p \geq q + 2$ and $r \geq q + 2$.*

(ii) *If $q < 0$, then $ds_\phi^2(\xi_0) = 0$ if and only if $p \geq 2$ and $r \geq 2$.*

Next we characterise the 'ends' of ϕ, again this follows easily from (26):

PROPOSITION 7.3. *Suppose that at a point $\xi_0 \in X$, the local coordinate ξ, centred at ξ_0, is such that $g(\xi) = \xi^q$.*

(i) *If ξ_0 is a pole of f_1 or f_2, then it is an end of ϕ.*

(ii) *If $q > 0$, $f_1(\xi) = a_0 + a_p \xi^p + \cdots$ and $f_2(\xi) = b_0 + b_r \xi^r + \cdots$, where p, r are positive integers, then ξ_0 is an end of ϕ if and only if $p \leq q$ or $r \leq q$.*

8. Remarks on Symmetry

An invertible bundle map $A : \mathcal{O}(1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1)$, has the following form in local coordinates:

$$A(\zeta, \eta_1, \eta_2) = (\alpha(\zeta), a\eta_2 + b\eta_1, c\eta_2 + d\eta_1),$$

where $\alpha \in \mathrm{P}\,\mathrm{SL}(2, \mathbf{C})$, and $a, b, c, d \in \mathbf{C}$. Observe that A induces, via Ψ, $\hat{A} \in \mathrm{P}\,\mathrm{SL}(2, \mathbf{C}) \times \mathrm{P}\,\mathrm{SL}(2, \mathbf{C})$, where $\hat{A}(\zeta, \zeta_2) = (\alpha(\zeta), \beta(\zeta_2))$, with $\beta(\zeta_2) = (a + b\zeta_2)/(c + d\zeta_2)$. $\hat{A}$ determines A up to a scale factor $\lambda \in \mathbf{C}^*$, and thus

$$\mathcal{B} = \{\text{invertible bundle maps}\}/\sim$$

where $A \sim B$ means that they differ by a scale factor, is isomorphic to $\mathsf{P}\,\mathrm{SL}(2,\mathsf{C}) \times \mathsf{P}\,\mathrm{SL}(2,\mathsf{C})$. Any invertible bundle map acts linearly on $\mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$ and preserves the null cone of vanishing sections. This leads easily to the well-known isomorphism

$$\mathsf{P}\,\mathrm{SO}(4,\mathsf{C}) = \mathrm{SO}(4,\mathsf{C})/\{\pm I\} \cong \mathsf{P}\,\mathrm{SL}(2,\mathsf{C}) \times \mathsf{P}\,\mathrm{SL}(2,\mathsf{C}). \tag{27}$$

Cf. §18.2 in [9] for a related description.

Suppose that G is a subgroup of $\mathscr{B}$ and C is a G-invariant curve in $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Via (27), observe that G is isomorphic to a subgroup of the symmetry group of the null curve in C^4, generated by C.

REMARK 7. These observations facilitate the construction and study of minimal surfaces in R^4 with symmetry groups in $\mathrm{SO}(4,\mathsf{R})$; we leave this to be pursued by any interested reader.

Consider the real structure $\tau : \mathcal{O}(1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1)$, that is given in local coordinates by:

$$\tau(\zeta, \eta_1, \eta_2) = (-\bar{\zeta}^{-1},\ \bar{\eta}_2\bar{\zeta}^{-1},\ -\bar{\eta}_1\bar{\zeta}^{-1}). \tag{28}$$

Observe that τ induces the antipodal map $\alpha : \mathsf{P}_1 \longrightarrow \mathsf{P}_1$ on the zero section of $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Furthermore, τ induces the map

$$\tilde{\tau} : \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \longrightarrow \mathrm{H}^0(\mathsf{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1)),$$

given by: $\tilde{\tau}(\sigma_z) = \tau \circ \sigma_z \circ \alpha$.

It is easy to see that $\tilde{\tau}(\sigma_z) = \sigma_{\bar{z}}$, in the (z_1, z_2, z_3, z_4)-coordinates of (1), and hence the $\tilde{\tau}$-invariant sections correspond to the real slice, R^4, in those coordinates.

Now, if a curve C in $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is τ-invariant then the branched minimal immersion $\phi = \mathrm{Re}(\omega)$ associated to C, factors through C/τ.

REMARK 8. Similiar observations for the R^3 case were made in [3], see [20] for a family of elliptic examples.

EXAMPLE. The data $g(\zeta) = \zeta$, $f_1(\zeta) = \zeta^{-n}$, $f_2(\zeta) = \zeta^{n+1}$, where n is a positive even integer, substituted into (22)–(25), gives $\phi = \mathrm{Re}(\omega) : \mathsf{P}_1 \setminus \{0, \infty\} \longrightarrow \mathsf{R}^4$, of total curvature $-4(n+1)\pi$, which factor through $\mathsf{RP}_2 \setminus \{[0]\}$.

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