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Renormalisation group flow and geodesics in the $O(N)$ model for large N

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Abstract

A metric is introduced on the space of parameters (couplings) describing the large N limit of the $O(N)$ model in Euclidean space. The geometry associated with this metric is analysed in the particular case of the infinite volume limit in three dimensions and it is shown that the Ricci curvature diverges at the ultra-violet (Gaussian) fixed point but is finite and tends to constant negative curvature at the infra-red (Wilson–Fisher) fixed point. The renormalisation group flow is examined in terms of geodesics of the metric. The critical line of cross-over from the Wilson–Fisher fixed point to the Gaussian fixed point is shown to be a geodesic but all other renormalisation group trajectories, which are repulsed from the Gaussian fixed point in the ultra-violet, are not geodesics. The geodesic flow is interpreted in terms of a maximisation principle for the relative entropy. © 1998 Elsevier Science B.V.

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1. Introduction

In this paper the idea of a geometry on the space of field theories will be investigated, in particular within the context of an exactly soluble model – the $O(N)$ model in D-dimensions in the limit of large N. A geometry on the space of theories was used to great effect by Zamolodchikov in the case $D = 2$ [1], where a metric was defined on the space of couplings of the theory which led to deep insights into the nature of

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renormalisation flow and cross-over between fixed points (the c-theorem). This metric was essentially given by the two-point correlators of the primary fields of the theory, and has no unique generalisation to $D > 2$, where the concept of a primary field does not play such a central rôle. However, a related metric can be defined for $D > 2$, where the components are given by two-point correlators of the composite operators associated with the couplings [2]. The concept goes back much further in the statistical mechanics literature and can be traced to ideas of Fisher and Rao [3,4] - the Fisher information matrix and "relative entropy". In the context of ordinary statistics there is a book on the subject [5] and metrics on the space of thermodynamic states have been investigated in some detail by Ruppeiner and Weinhold [6,7]. A related metric in quantum mechanics has been proposed by Provost and Vallee [8]. Zamolodchikov appears to have been the first to use the idea in field theory, albeit only for two-dimensional field theories. Many attempts have been made to generalise Zamolodchikov's results to three and four dimensions [9], but these have mostly focused on attempts to prove a c-theorem in $D > 2$, rather than on the intrinsic *geometry* of the proposed metric.

The geometry itself is also of intrinsic importance. A change in the way the theory is parameterised would correspond to a general co-ordinate transformation and non-linear transformations are perfectly acceptable, provided all quantities are expressed in a manifestly general co-ordinate co-variant manner. For example the transformation from bare to renormalised parameters, which is in general non-linear, can be interpreted as a general co-ordinate transformation [10]. In particular any quantity which is a scalar under general co-ordinate transformations is automatically independent of the renormalisation scheme.

The geometry of the space of couplings was taken seriously in [2], where Gaussian models (free scalar field theories in a finite box) were investigated, and curvatures calculated. The concept of a connection on the infinite-dimensional space of theories, and its rôle in renormalisation group flow, was investigated in [11]. The metric studied in [2] was in a sense the infra-red limit of the Fourier transform of Zamolodchikov's metric (generalised to $D > 2$ where the concept of primary fields is not so well defined), and so does not contain as much information as that of Zamolodchikov, but it is still relevant to an analysis of the long distance behaviour of the theory.

The relation between the geometry and the renormalisation flow was investigated in [12], where it was observed that the renormalisation flow on the two-dimensional space parameterised by the mass and the expectation value $\langle\varphi\rangle$ of a scalar field is geodesic for free fields, provided $\langle\varphi\rangle = 0$ but not otherwise (except in $D = 2$, where all renormalisation trajectories are geodesic). This was however in the rather restricted cases of free field theories, where the renormalisation flow is just dictated by canonical dimensions, and the slightly less trivial case of the one-dimensional Ising model.

The purpose of this paper is to pursue these investigations for an interacting non-trivial model which is exactly soluble - the $O(N)$ model in the limit of $N \to \infty$. For $D = 3$, this model is non-trivial and has two fixed points - the Gaussian fixed point (free field theory) in the ultra-violet and the non-trivial Wilson-Fisher fixed point in the infra-red (which is equivalent to the spherical model [13]). The Ricci scalar diverges at the Gaussian fixed point but elsewhere the curvature is finite, tending to a negative constant in the infra-red. It is shown that, with the metric used here, the line of cross-over between the Gaussian and Wilson–Fisher fixed points is a geodesic and this is related to the concept of relative entropy in statistics.

In Section 2 the choice of metric that is used will be described, motivated by considerations of general co-ordinate invariance. Section 3 is devoted to the explicit determination of the metric and curvature for the $O(N)$ model in D-dimensions, for large N. This involves the inclusion of $1/N$ corrections, as the metric proves to be degenerate to lowest order. Section 4 specialises to the infinite volume limit in $D = 3$, where it is shown that the Ricci scalar, $R \rightarrow +\infty$ at the Gaussian fixed point, and $R \rightarrow -6\pi^2$ when any of the three parameters of the model (constant external source, the mass of the scalar field or the four-point coupling, λ) is large. In particular the infra-red fixed point corresponds to $\lambda \rightarrow \infty$. It is also shown that the line of cross-over, from the infra-red to the ultra-violet fixed point is a geodesic, and Section 5 is devoted to an interpretation of this result in terms of relative entropy. Section 6 contains a summary and conclusions.

There are two appendices, one containing some technical aspects of Legendre transforms, which are used in Section 3, and a second which gives the connection coefficients, also used in Section 3.

2. The metric

In this section a definition of a metric on the space of couplings will be given. The basic motivation follows that of Ref. [2]. Consider a field theory in D-dimensional Euclidean space with n couplings g^a, $a = 1, \ldots, n$, corresponding to operators $\hat{\Phi}_a(x)$ (in general composite). The definition of the reduced free energy (i.e. the free energy divided by the temperature) is

$$W(g) = -\ln Z(g)\,, \quad \text{where } Z(g) = \int \mathcal{D}\varphi\, e^{-S[\varphi]} \tag{1}$$

and $S[\varphi]$ is the action. This gives

$$1 = \int \mathcal{D}\varphi e^{-S[\varphi]+W} \Rightarrow dW = \langle dS \rangle, \tag{2}$$

where $dW = \partial_a W dg^a$ is a one-form and $dS = \partial_a S dg^a$ can be thought of as an operator valued one-form. In particular, if the action S is linear in the couplings, then

$$\partial_a S = \int d^D x\, \hat{\Phi}_a(x), \tag{3}$$

where $\hat{\Phi}_a(x)$ is the composite operator associated with the coupling g^a. Thus, if g^{a_0} are bare couplings, then $\hat{\Phi}_{a_0}(x)$ are bare operators. If one then defines renormalised couplings g^{a_R}, using some preferred scheme, then the renormalised operators are $\hat{\Phi}_{a_R}(x) = (Z^{-1})^{b_0}{}_{a_R}\hat{\Phi}_{b_0}(x)$, where the operator mixing matrix $(Z^{-1})^{b_0}{}_{a_R} = \partial g^{b_0}/\partial g^{a_R}$ is nothing other than a general co-ordinate transformation matrix for the co-variant vector with components $\hat{\Phi}_{a_R}(x)$. Thus the definition of $\hat{\Phi}_a(x)$ given in Eq. (3) is a general co-ordinate co-variant definition even when the action is non-linear in the couplings and is valid both for bare and renormalised couplings. Eqs. (2) and (3) are referred to as an "action principle" in [14].

The metric advocated by O'Connor and Stephens in [2] is determined by the infinitesimal line element on the n-dimensional space parameterised by g^a defined by

$$ds^2 = \langle (dS - dW) \otimes (dS - dW) \rangle. \tag{4}$$

In order to be able to pass to the infinite volume limit, it will be convenient to divide Eq. (4) by a factor $V = \int d^D x$, the volume of space, and use densities. Let

$$\tilde{\Phi}_a(x) = \hat{\Phi}_a(x) - \langle \hat{\Phi}_a(x) \rangle \tag{5}$$

and define

$$G_{ab} = \int d^D x \langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(0) \rangle. \tag{6}$$

This is the metric which will be investigated here. Obviously $G_{ab} = G_{ba}$ and under a general co-ordinate transformation $g^a \to g^{a'}(x)$

$$\partial_a S \to \partial_{a'} S = \frac{\partial g^b}{\partial g^{a'}} \partial_b S \tag{7}$$

so

$$G_{ab} \to G_{a'b'} = \frac{\partial g^c}{\partial g^{a'}} \frac{\partial g^d}{\partial g^{b'}} G_{cd} \tag{8}$$

has the correct transformation properties to be considered as a metric.

Of course if bare couplings are used then the $\hat{\Phi}_a(x)$ are divergent operators when the regulator is removed. One can either keep the regulator in place until the end of the calculation or transform to renormalised operators using a co-ordinate transformation - provided the formalism is manifestly co-variant it does not matter and the latter possibility allows a consistent analysis. However, the r.h.s. of Eq. (6) contains further divergences in general, either infra-red divergences due to the large x-behaviour or ultra-violet divergences due to the small x-behaviour. The usual procedure is to perform further subtractions, over and above any that may have already been used to obtain renormalised operators, so as to obtain a renormalised two-point function [15]. This will not be done here - rather G_{ab} will be defined using a regulator, connections and curvatures will be calculated first and only then will the regulator be removed. There is a good geometrical reason for this strategy. As explained above multiplicative renormalisation can be interpreted as a co-ordinate transformation and so does not change the geometry - the components of the metric look different but the geometry (in particular the Ricci scalar) is not changed. Subtracting extra terms which are non-linear in the couplings from (6) would however change the geometry and so would change the Ricci scalar. By avoiding such subtractions one can be confident that the resulting Ricci scalar is independent of the renormalisation scheme.

As noted in [2], for free field theories, the curvature remains finite even though the components of the metric diverge when the regulator is removed. For the large N limit of the $O(N)$ model in three dimensions, it will transpire that the curvature diverges at the Gaussian fixed point but not elsewhere.

Eq. (6) can be written in a manner more convenient for computations. Let

$$w = \frac{1}{V} W \tag{9}$$

be the reduced free energy density, so that $W = \int w \, d^D x$. Then Eq. (2) reads

$$\partial_a w = \frac{1}{V} \langle \partial_a S \rangle . \tag{10}$$

Differentiating a second time gives

$$\partial_a \partial_b w = \frac{1}{V} \left\{ \langle \partial_a \partial_b S \rangle - \langle \partial_a S \partial_b S \rangle + \langle \partial_a S \rangle \langle \partial_b S \rangle \right\} \tag{11}$$

or

$$G_{ab} = \int d^D x \, \langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(0) \rangle = \frac{1}{V} \langle \partial_a \partial_b S \rangle - \partial_a \partial_b w . \tag{12}$$

Despite appearances the right-hand side of (12) is co-variant under general co-ordinate transformations since, if $\partial_b S \to \partial_{b'} S = (\partial g^c / \partial g^{b'}) \partial_c S$ and $\partial_b w \to \partial_{b'} w = (\partial g^c / \partial g b') \partial_c w$ then

$$\begin{aligned} \partial_{a'} \partial_{b'} S &= \frac{\partial g^c}{\partial g^{a'}} \frac{\partial g^d}{\partial g^{b'}} \partial_c \partial_d S + \frac{\partial^2 g^c}{\partial g^{a'} \partial g^{b'}} \partial_c S, \\ \partial_{a'} \partial_{b'} w &= \frac{\partial g^c}{\partial g^{a'}} \frac{\partial g^d}{\partial g^{b'}} \partial_c \partial_d w + \frac{\partial^2 g^c}{\partial g^{a'} \partial g^{b'}} \partial_c w . \end{aligned} \tag{13}$$

So the inhomogeneous terms cancel when expectation values are taken, by virtue of Eq. (2). The analysis of this section has been general co-ordinate co-variant up to this point. Eqs. (3), (6) and (12) are valid even if renormalised couplings are used and the action is not linear in the couplings. If, however, one chooses parameters in which the action is linear (these would be the bare parameters of the theory), then Eq. (12) simplifies to

$$G_{ab} = -\partial_a \partial_b w . \tag{14}$$

The class of such co-ordinate systems is special, of course – only linear co-ordinate transformations are allowed. [2] Within this class, however, Eq. (14) says that the com-

[2] In some situations there is a natural complex structure on the space of parameters and a metric of the form (14) can be interpreted as a Kähler metric. The class of allowed co-ordinate transformations which preserve the form of (14) can then be extended to include any complex analytic transformation. An example is the Seiberg–Witten metric on the parameter space of $N = 2$ super symmetric Yang–Mills theory in four dimensions [16].

ponents of the metric can be determined if the partition function, and so w, is known as a function of the regularised bare parameters. For translationally invariant systems, this is equivalent to a knowledge of the effective potential, or the free energy. Another useful class of co-ordinates is that obtained by the non-linear co-ordinate transformations associated with the Legendre transformed variables, these can simplify the metric even further and will prove useful in the sequel – this class, and the resulting form of the metric in terms of the effective potential, is examined in detail in Appendix A.

When viewed in this light, some singularities in the metric can be given a more direct interpretation. For example, if one of the operators is φ^2 in a scalar field theory, the statistical physics interpretation of the co-efficient of $\frac{1}{2}\varphi^2$, $t = (T - T_c)/T_c$, is that it is the deviation from the critical temperature, and the second derivative of the free energy with respect to t is the specific heat, hence one expects some components of the metric (14) to diverge at critical points and in general one might expect the curvature to diverge there also. In fact for the $O(N)$ model at large N in three dimensions, the critical exponent for the specific heat, $\alpha = -1 + o(1/N)$, is negative at the infra-red (Wilson–Fisher) fixed point, so the specific heat is actually finite at $t = 0$ and it is only the third derivative of the free energy with respect to t that diverges. Calculation of the Ricci scalar however, reveals that it is finite all along the critical line between the infra-red and the ultra-violet fixed point, diverging only at the ultra-violet (Gaussian) fixed point. The non-analyticity of the free energy along the critical line is still reflected in the Ricci scalar however, in that it displays a discontinuity across the critical line.

3. The geometry of the $O(N)$ model

The model that will be investigated here is the $O(N)$ model in D Euclidean dimensions, in the limit of $N \to \infty$. This is an example of a non-trivial interacting field theory (for $D < 4$) which can be solved exactly. The model consists of a scalar field φ in the vector representation of $O(N)$, with components $\varphi^i, i = 1, \ldots, N$. The action is (really total energy since the space is Euclidean)

$$S = \int d^D x \left\{ \frac{1}{2}(\nabla\varphi)^2 + j \cdot \varphi + \frac{r}{2}\varphi^2 + \frac{u}{4!}(\varphi^2)^2 \right\}, \qquad (15)$$

where j is a constant external source and r and u are the bare mass and four-point coupling respectively. The aim of this section is to determine the metric, as defined in the previous section, and to investigate the resulting geometry in terms of the Levi–Civita connection and the curvature. The variables j, r and u are not particularly convenient for this purpose and it will prove expedient to transform to an alternative set, but first we outline the calculation of the partition function and the effective potential.

The partition function is

$$Z[j, r, u] = \int \mathcal{D}\varphi \, e^{-S}. \qquad (16)$$

The form of the scaling function for this model in the limit of large N was investigated in [17]. The analysis here will use the steepest descents method of [18], as implemented in [19]. First replace the φ^4 term with an effective field ψ

$$\begin{aligned}
&\exp\left[-\int d^D x \frac{u}{4!}(\varphi^2)^2\right] \\
&= \mathcal{N}\int \mathcal{D}\psi \exp\left\{\int d^D x \left[\frac{N}{2}\left(\psi - \sqrt{\frac{u}{12N}}\varphi^2\right)^2 - u\frac{(\varphi^2)^2}{4!}\right]\right\} \\
&= \mathcal{N}\int \mathcal{D}\psi \exp\left\{\int d^D x \left[\frac{N}{2}\psi^2 - \sqrt{\frac{uN}{12}}\psi\varphi^2\right]\right\}
\end{aligned} \tag{17}$$

where $\mathcal{N}$ is an irrelevant constant, independent of u. Now

$$\begin{aligned}
Z &= \mathcal{N}\int \mathcal{D}\psi\mathcal{D}\varphi \exp\left\{-\int d^D x \left[\frac{1}{2}(\nabla\varphi)^2 + \boldsymbol{j}.\boldsymbol{\varphi} + \frac{1}{2}M^2\varphi^2 - \frac{N}{2}\psi^2\right]\right\} \\
&= \mathcal{N}\int \mathcal{D}\psi\mathcal{D}\varphi \exp\left\{-\int d^D x \left[\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}M^2\left(\varphi + \frac{j}{M^2}\right)^2 - \frac{j^2}{2M^2} - \frac{N}{2}\psi^2\right]\right\}
\end{aligned} \tag{18}$$

where $M^2(\psi) = r + \sqrt{\frac{1}{3}Nu}\,\psi$ is an effective mass for the field φ. After a shift in φ, the φ integration is Gaussian, leading to an effective action for the field ψ

$$Z[j,r,u] = \mathcal{N}\int \mathcal{D}\psi\, e^{-S_{\mathrm{eff}}(\psi)}, \tag{19}$$

where

$$S_{\mathrm{eff}}(\psi) = -\int d^D x \left\{\frac{N}{2}\psi^2 + \frac{1}{2}\frac{j^2}{M^2(\psi)}\right\} + \frac{N}{2}\int d^D x \ln\{-\nabla^2 + M^2(\psi)\}. \tag{20}$$

Note that Z now depends only $j = |\boldsymbol{j}|$, as expected. So far the manipulations are exact, if formal. The method of steepest descents now allows one to evaluate the effective potential as a $1/N$ expansion. Expand $\psi(x)$ around a constant background,

$$\psi(x) = \psi_0 + \frac{1}{\sqrt{N}}\varepsilon(x), \tag{21}$$

where ψ_0 is chosen so that $\partial S_{\mathrm{eff}}/\partial\psi(x)|_{\psi(x)=\psi_0} = 0$. The function $\int d^D x \ln\{-\nabla^2 + M^2\}$ will appear so frequently in the following, that it will be convenient to give it a name. In momentum space

$$G(m^2) := \int \frac{d^D p}{(2\pi)^D}\ln(p^2 + m^2), \tag{22}$$

$$\dot{G}(m^2) := \frac{dG}{dm^2} = \int \frac{d^D p}{(2\pi)^D}\frac{1}{(p^2 + m^2)}, \tag{23}$$

$$\ddot{G}(m^2) := \frac{d^2 G}{(dm^2)^2} = -\int \frac{d^D p}{(2\pi)^D}\frac{1}{(p^2 + m^2)^2}, \tag{24}$$

where $m^2(\psi_0) = M^2|_{\psi=\psi_0} = r + \sqrt{\frac{1}{3}Nu}\,\psi_0$.

The function $G(m^2)$ is not uniquely specified here, it depends on boundary conditions and geometry, for example the integral could correspond to infinite Euclidean space or a D-dimensional torus each giving different eigenvalues for the Laplacian. If space is continuous, $G(m^2)$ must be rendered finite in some way e.g. by introducing a momentum cut-off. Alternatively, continuous space could be replaced by a finite lattice of points and the Laplacian becomes a matrix with $\int d^Dp/(2\pi)^D \to Tr$. In the following we shall work with a generic $G(m^2)$, it being understood that different geometries for D-dimensional space lead to different functions, $G(m^2)$.

The extremum condition now determines ψ_0 via

$$-N\psi_0 + \frac{1}{2}\frac{j^2}{m^4}\sqrt{\frac{Nu}{3}} + \frac{N}{2}\dot{G}(m^2)\sqrt{\frac{Nu}{3}} = 0, \tag{25}$$

where $\partial M^2(\psi(y))/\partial\psi(x) = \sqrt{\frac{1}{3}Nu}\,\delta(x-y)$ has been used. Eq. (25) determines ψ_0 as a function of j, r and u since $m^2(\psi_0) = r + \sqrt{\frac{1}{3}Nu}\,\psi_0$. Expanding S_{eff} in Eq. (20) around ψ_0, using (21) gives

$$S_{\text{eff}} = \frac{N}{2}V\left\{G(m^2) - \psi_0^2 - \frac{j^2}{Nm^2}\right\} + o(\varepsilon^2), \tag{26}$$

where again $V = \int d^Dx$ is the volume of space. Thus the reduced free energy density

$$w = -\frac{1}{V}\ln Z \tag{27}$$

is given (up to an irrelevant constant, independent of j, r and u) by

$$w = \frac{N}{2}\left\{G(m^2) - \frac{j^2}{Nm^2} - \psi_0^2\right\} + o(1) \tag{28}$$

with m^2 determined implicitly by Eq. (25). If we define $\lambda = \frac{1}{3}Nu$, $J = j/\sqrt{N}$ and $\tilde{w} = w/N$ then

$$\tilde{w} = \frac{1}{2}\{G(m^2) - J^2/m^2 - \psi_0^2\} + o\left(\frac{1}{N}\right) \tag{29}$$

with

$$\psi_0 = \frac{1}{2}\left\{\frac{J^2}{m^4} + \dot{G}(m^2)\right\}\sqrt{\lambda}, \tag{30}$$

from (25), and $m^2 = r + \sqrt{\lambda}\psi_0$. If the triple limit $N \to \infty, j \to \infty, u \to \infty$, such that λ and J are finite, is taken one finds

$$\tilde{w} = \frac{1}{2}\{G(m^2) - J^2/m^2 - \psi_0^2\} + o\left(\frac{1}{N}\right)$$

as the reduced free energy density for the model.

The effective potential, for constant J, is obtained from the Legendre transform with $\phi = \partial\tilde{w}/\partial J = -J/m^2$ so that $\phi = \langle\varphi\rangle/\sqrt{N}$,

$$\tilde{\Gamma}(\phi, r, \lambda) = \frac{1}{N}\Gamma(\phi, r, \lambda) = \tilde{w} - \phi J = \frac{1}{2}\{G(m^2) + m^2\phi^2 - \psi_0^2\} + o\left(\frac{1}{N}\right) \tag{31}$$

with

$$\psi_0 = \frac{1}{2}\{\phi^2 + \dot{G}(m^2)\}\sqrt{\lambda}. \tag{32}$$

Eliminating ψ_0 using $\psi_0 = (m^2 - r)/\sqrt{\lambda}$ and re-arranging gives

$$\tilde{\Gamma}(\phi, r, \lambda) = \frac{1}{2}\left\{G(m^2) - m^2\dot{G}(m^2) + \frac{m^4}{\lambda} - \frac{r^2}{\lambda}\right\} + o\left(\frac{1}{N}\right) \tag{33}$$

$$m^2 = r + \frac{\lambda}{2}\dot{G}(m^2) + \frac{1}{2}\lambda\phi^2 \tag{34}$$

which is the form of the effective action used in [19].

The metric will be difficult to calculate using the co-ordinates (ϕ, r, λ) because m is defined only implicitly through Eq. (34). It is easier to perform a second Legendre transform on the variable r to a new variable,

$$X := \frac{1}{2}\int d^D x\,\langle\varphi^2\rangle = \left.\frac{\partial\tilde{\Gamma}}{\partial r}\right|_{\phi,\lambda},$$

and define

$$\tilde{\Xi}(\phi, X, \lambda) = \tilde{\Gamma} - r\frac{\partial\tilde{\Gamma}}{\partial r}. \tag{35}$$

First note that (34) gives

$$\left.\frac{\partial m^2}{\partial r}\right|_{\phi,\lambda} = \frac{1}{1 - \frac{\lambda}{2}\ddot{G}} \tag{36}$$

thus $X = (m^2 - r)/\lambda$ or

$$X = \frac{1}{2}\{\dot{G}(m^2) + \phi^2\} \tag{37}$$

using (34). This gives

$$\tilde{\Xi}(\phi, X, \lambda) = \frac{1}{2}\{G(m^2) - m^2\dot{G}(m^2) + \lambda X^2\} \tag{38}$$

with $m^2(\phi, X)$ given implicitly by (37). Note that m^2 is independent of λ when expressed as a function of ϕ and X and it is this observation that simplifies the calculation of the metric and curvature when the (ϕ, X, λ) co-ordinate system is used.

Using (37) we find

$$\left.\frac{\partial m^2}{\partial X}\right|_{\phi,\lambda} = \frac{2}{\ddot{G}}, \qquad \left.\frac{\partial m^2}{\partial \phi}\right|_{X,\lambda} = -\frac{2\phi}{\ddot{G}}, \qquad \left.\frac{\partial m^2}{\partial \lambda}\right|_{\phi,\lambda} = 0. \tag{39}$$

The metric can now be determined using the results of Appendix A for the Hessian of a Legendre transform,

$$G_{ab} = N\begin{pmatrix} m^2 - 2\phi^2/\ddot{G} & 2\phi/\ddot{G} & 0 \\ 2\phi/\ddot{G} & \lambda - 2/\ddot{G} & 0 \\ 0 & 0 & 0 \end{pmatrix} + o(1), \tag{40}$$

i.e. the metric is degenerate at this order, since the Hessian of $\tilde{\Xi}$ has a zero-mode in the λ-direction. This degeneracy is lifted by including the $o(1/N)$ corrections to $\tilde{\Xi}$.

In order to determine the order one corrections to (40) one must calculate the order one corrections to the partition function. Consider therefore Eqs. (19) and (20). Including the order ε^2 terms from (21) gives

$$\begin{aligned} S_{\text{eff}} = {} & \frac{N}{2}V[G(m^2) - \psi_0^2 - J^2/m^2] \\ & + \frac{1}{4}\int d^D x \int d^D y\, \varepsilon(x)\left[-\lambda\frac{\delta(x-y)}{(-\nabla_x^2 + m^2)(-\nabla_y^2 + m^2)}\right. \\ & \left. -2\delta(x-y)\left(1 + \frac{\lambda J^2}{(m^2)^3}\right)\right]\varepsilon(y) + o(\varepsilon/\sqrt{N})^3. \end{aligned} \tag{41}$$

After a contour rotation [18], a Gaussian integral over ε gives, up to an irrelevant constant independent of J, r and λ,

$$w = \frac{N}{2}\left\{G(m^2) - J^2/m^2 - \psi_0^2\right\} + \frac{1}{2}\ln\det F + o\left(\frac{1}{N}\right), \tag{42}$$

where F is diagonal in momentum space,

$$F(p) = 1 + \frac{\lambda J^2}{(m^2)^3} + \frac{\lambda}{2}\int \frac{d^D q}{(2\pi)^D}\frac{1}{(p^2 + m^2)}\frac{1}{\{(p-q)^2 + m^2\}} \tag{43}$$

and

$$\ln\det F(p) = \int \frac{d^D p}{(2\pi)^D}\ln F(p) \equiv L(J, r, \lambda). \tag{44}$$

The comments made after Eqs. (22)–(24) also apply here, the function L depends on the geometry of D-dimensional space. It is shown in Appendix A that the Legendre transform of any differentiable function of the form

$$w(g) = w_0(g) + \frac{1}{2N}L(g) + o\left(\frac{1}{N^2}\right) \tag{45}$$

is

$$\tilde{\Gamma}(\phi) = \tilde{\Gamma}_0(\phi) + \frac{1}{2N}L(g(\phi)) + o\left(\frac{1}{N^2}\right), \tag{46}$$

where $\phi = dw/dg$ can be inverted to give $g(\phi) = g_0(\phi) + (1/N)g_1(\phi) + o(1/N^2)$ and $\tilde{\Gamma}_0(\phi) = w_0(g_0(\phi)) - g_0(\phi)\phi$. Thus the double Legendre transform of (42) is, using (37) and (38),

$$\tilde{\Xi}(\phi, X, \lambda) = \frac{1}{N}\Xi(\phi, X, \lambda) = \frac{1}{2}\left\{G(m^2) - m^2\dot{G}(m^2) + \lambda X^2\right\} + \frac{1}{2N}\int \frac{d^D p}{(2\pi)^D} \ln F(p) + o\left(\frac{1}{N^2}\right) \tag{47}$$

with $m^2(\phi, X)$ determined implicitly by

$$X = \frac{1}{2}\left\{\dot{G}(m^2) + \phi^2\right\} \tag{48}$$

and

$$F(p) = 1 + \frac{\lambda\phi^2}{m^2} + \frac{\lambda}{2}\int \frac{d^D q}{(2\pi)^D}\frac{1}{(q^2 + m^2)}\frac{1}{\{(p-q)^2 + (m^2)\}}. \tag{49}$$

The function $F(p)$ encodes the order one corrections to the metric. Expressing $L(p)$ in Eq. (44) as a function of (ϕ, X, λ) gives

$$G_{ab} = N\begin{pmatrix} m^2 - \frac{2\phi^2}{\ddot{G}} & \frac{2\phi}{\ddot{G}} & 0 \\ \frac{2\phi}{\ddot{G}} & \lambda - \frac{2}{\ddot{G}} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{\partial^2 L}{\partial\phi^2} & \frac{\partial^2 L}{\partial\phi\partial X} & 0 \\ \frac{\partial^2 L}{\partial\phi\partial X} & \frac{\partial^2 L}{\partial X^2} & 0 \\ 0 & 0 & -\frac{\partial^2 L}{\partial\lambda^2} \end{pmatrix} + o\left(\frac{1}{N}\right). \tag{50}$$

Note that G_{ab} is positive definite since $\ddot{G} < 0$ and $L'' = \partial^2 L/\partial\lambda^2 < 0$.

The connection coefficients can now be evaluated in a straightforward but tedious manner (remembering that the matrix is curl free in the chosen co-ordinate system $G_{ab,c} = G_{ac,b}$, etc.) and they are enumerated in Appendix B.

The components of the Ricci tensor are

$$R^X{}_X = R^\lambda{}_\lambda = -\frac{1}{2L''}\frac{(m^2\ddot{G} - 2\phi^2)}{\{\lambda(m^2\ddot{G} - 2\phi^2) - 2m^2\}}\left[\frac{L'''}{L''} + \frac{(m^2\ddot{G} - 2\phi^2)}{\{\lambda(m^2\ddot{G} - 2\phi^2) - 2m^2\}}\right] + o\left(\frac{1}{N}\right) \tag{51}$$

and $R^a{}_b = o(1/N)$ otherwise. As a reminder, a dot denotes $\partial/\partial m^2$ while a prime denotes $\partial/\partial\lambda$.

The geometry is essentially such that all of the sectional curvature is in the X–λ planes for constant ϕ. The sectional curvature in the X–ϕ and λ–ϕ planes is of order $1/N$, (see, e.g., Ref. [20] p. 46). Thus, to this order, all of the geometry is encapsulated in the Ricci scalar, which is twice the Gaussian curvature of the surfaces of constant ϕ,

$$\mathcal{R} = -\frac{1}{L''}\frac{(m^2\ddot{G} - 2\phi^2)}{\{\lambda(m^2\ddot{G} - 2\phi^2) - 2m^2\}}\left[\frac{L'''}{L''} + \frac{(m^2\ddot{G} - 2\phi^2)}{\{\lambda(m^2\ddot{G} - 2\phi^2) - 2m^2\}}\right] + o\left(\frac{1}{N}\right). \tag{52}$$

with $\ddot{G}$ and L given in Eqs. (24) and (44) and m^2 (ϕ, X) (the effective mass) defined implicitly in Eq. (37).

Having obtained $\mathcal{R}$ we can of course use whatever co-ordinate system we wish, and it is convenient now to change from (ϕ, X, λ) to (ϕ, m^2, λ). It would have been very tedious to have used (ϕ, m^2, λ) from the start because they are related to (ϕ, X, λ) non-linearly and so neither Eq. (14), nor its analogous form for Legendre transformed variables, is valid in the (ϕ, m^2, λ) system. The actual form of $\mathcal{R}(\phi, m^2, \lambda)$ now depends on

$$\ddot{G}(m^2) = -\int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2 + m^2)^2} \tag{53}$$

and

$$L = \int \frac{d^D p}{(2\pi)^D} \ln\left[1 + \frac{\lambda\phi^2}{m^2} + \frac{\lambda}{2}\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+m^2)} \frac{1}{((p-q)^2 + m^2)}\right]. \tag{54}$$

The geometry of D-dimensional space has not yet been specified – it could be infinite Euclidean space or a D-dimensional torus are even a lattice with a finite set of points, in which case ∇^2 is a matrix and $\int d^D p/(2\pi)^D \to$ Trace.

In the next section we shall examine the geometry for infinite three-dimensional Euclidean space, using a cut-off to regularise the integrals – this case is of special interest, because the model is known to exhibit two fixed points – one is the Gaussian fixed point leading to free field theory in the UV direction and the other is the Wilson–Fisher fixed point in the IR direction, which is equivalent to the spherical model in the $N \to \infty$ limit [13].

4. The $O(N)$ model in three dimensions

The geometry on the space of couplings described by Eq. (52) will now be examined for the case of $D = 3$, flat, Euclidean space. The integral in (53) is finite for $D = 3$ and gives

$$\ddot{G}(m^2) = -\frac{1}{2\pi^2}\int_0^\Lambda \frac{p^2 dp}{(p^2+m^2)^2} = -\frac{1}{4\pi^2}\left\{\frac{1}{m}\tan^{-1}\left(\frac{\Lambda}{m}\right) - \frac{\Lambda}{\Lambda^2 + m^2}\right\}. \tag{55}$$

For simplicity the asymptotic form

$$\lim_{\Lambda\to\infty} \ddot{G}(m^2) = -\frac{1}{8\pi m} \tag{56}$$

will be used below, but it should be borne in mind that the final expression is only valid for $m/\Lambda \ll 1$. The q-integral in Eq. (54) is also finite and gives

$$L = \frac{1}{2\pi^2}\int_0^\Lambda p^2 dp \ln\left\{1 + \frac{\lambda\phi^2}{m^2} + \frac{\lambda}{8\pi p}\tan^{-1}\left(\frac{p}{2m}\right)\right\}, \tag{57}$$

where a cut-off, Λ, has been introduced because the p-integral diverges. It is particularly easy to take derivatives of L with respect to λ, the nth derivative being

$$L^{(n)} = \frac{(-1)^{n-1}(n-1)!}{2\pi^2}\int_0^{\Lambda} p^2 dp \left[\frac{\frac{\phi^2 p}{m^2} + \frac{1}{8\pi}\tan^{-1}\left(\frac{p}{2m}\right)}{p + \frac{\lambda\phi^2 p}{m^2} + \frac{\lambda}{8\pi}\tan^{-1}\left(\frac{p}{2m}\right)}\right]^n, \tag{58}$$

(note that $\partial/\partial\lambda|_{\phi,X} = \partial/\partial\lambda|_{\phi,m^2}$ since $\partial m^2/\partial\lambda|_{\phi,X} = 0$ from Eq. (39)).

Using (56) in (52) gives the Ricci scalar as

$$\mathcal{R} = -\frac{(\frac{m}{16\pi}+\phi^2)}{L^{(2)}\left\{\lambda(\frac{m}{16\pi}+\phi^2)+m^2\right\}}\left[\frac{L^{(3)}}{L^{(2)}} + \frac{(\frac{m}{16\pi}+\phi^2)}{\left\{\lambda(\frac{m}{16\pi}+\phi^2)+m^2\right\}}\right] + o(1/N), \tag{59}$$

where $L^{(2)}$ and $L^{(3)}$ are defined in (58). If the limit $\Lambda \to \infty$ is taken, $L^{(n)} \sim \Lambda^3 \Rightarrow \mathcal{R} \sim 1/\Lambda^3 \to 0$, but this is really throwing away important geometric information. Better is to observe that $\mathcal{R}$ has dimensions of mass^{-3}, ϕ^2 and λ both have dimensions of mass in 3D so, following Zinn-Justin [18], define dimensionless parameters

$$\bar\lambda = \frac{\lambda}{16\pi\Lambda}, \quad \bar\phi^2 = \frac{16\pi\phi^2}{\Lambda}, \quad \bar m = \frac{m}{\Lambda} \quad \text{and} \quad \bar{\mathcal{R}} = \Lambda^3\mathcal{R}. \tag{60}$$

The last equation here is equivalent to a conformal rescaling by Λ^{-3} which renders the metric dimensionless. Now define

$$\bar L^{(n)} := (16\pi)^n\frac{L^{(n)}}{\Lambda^{3-n}} = \frac{(-1)^{n-1}(n-1)!(16\pi)^n}{2\pi^2} \times \int_0^1 z^2 dz \left[\frac{\bar\phi z + 2\bar m^2\tan^{-1}(z/2\bar m)}{\bar m^2 z + \bar\lambda\bar\phi z + 2\bar\lambda\bar m^2\tan^{-1}(z/2\bar m)}\right]^n, \tag{61}$$

in terms of which the rescaled Ricci scalar is

$$\bar{\mathcal{R}} = -\frac{1}{\bar L^{(2)}}\frac{(\bar m+\bar\phi^2)}{\{\bar m^2+\bar\lambda(\bar m+\bar\phi^2)\}}\left[\frac{\bar L^{(3)}}{\bar L^{(2)}} + \frac{(\bar m+\bar\phi^2)}{\{\bar m^2+\bar\lambda(\bar m+\bar\phi^2)\}}\right] + o(1/N), \tag{62}$$

which is finite even when $\Lambda \to \infty$, provided $\bar\phi$, $\bar m$ and $\bar\lambda$ are kept finite and are not all zero. The Ricci scalar is shown in Figs. 1–4, where it is graphed as a function of $\bar m$ and $\bar\phi^2$ for four values of $\bar\lambda$, $\bar\lambda = 0.1$, $\bar\lambda = 0.2$, $\bar\lambda = 0.3$ and $\bar\lambda = 5.0$. In order to produce these graphs, the integrals in (61) were performed numerically.

The Ricci scalar is infinite at $\bar\lambda = \bar m = \bar\phi^2 = 0$, corresponding to the Gaussian fixed point, but is finite elsewhere. If either of the two variables, $\bar\phi$, or $\bar\lambda$ becomes large, then $\bar{\mathcal{R}}$ tends to a negative constant.

$$\bar{\mathcal{R}}|_{\bar\lambda\to\infty} = \bar{\mathcal{R}}|_{\bar\phi\to\infty} = -6\pi^2 + o\left(\frac{1}{N}\right). \tag{63}$$

The limit for large $\bar m$ can be obtained from (52), (53) and (54) directly, by taking this limit before performing the integrals, avoiding the constraint $\bar m \ll 1$. One finds again

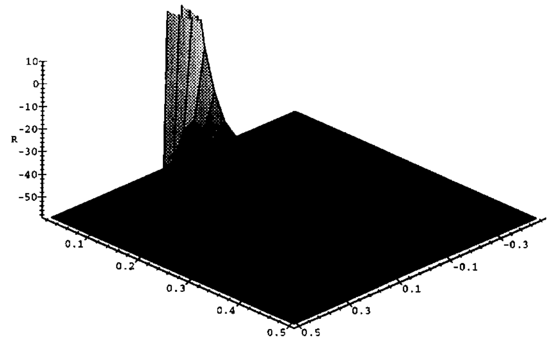

Fig. 1. Ricci scalar for $\bar{\lambda} = 0.1$.

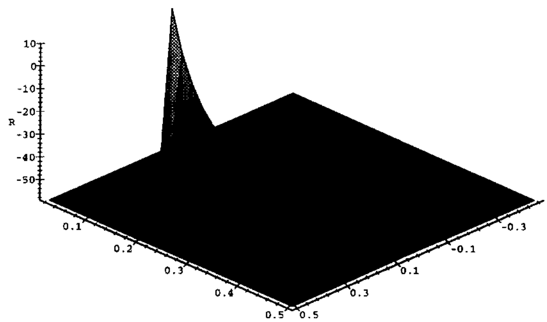

Fig. 2. Ricci scalar for $\bar{\lambda} = 0.2$.

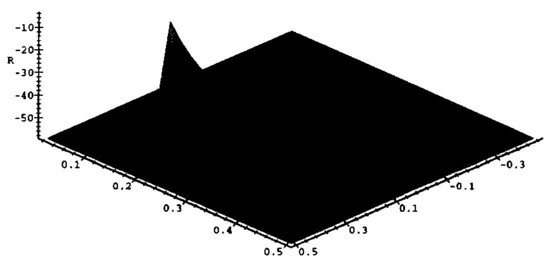

Fig. 3. Ricci scalar for $\bar{\lambda} = 0.3$.

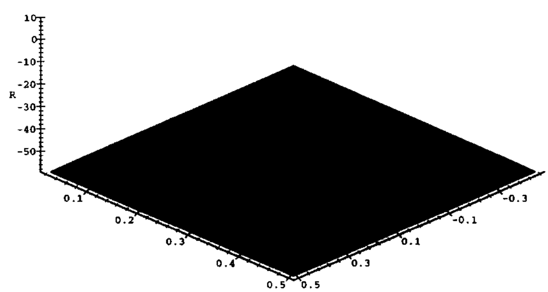

Fig. 4. Ricci scalar for $\bar{\lambda} = 5.0$.

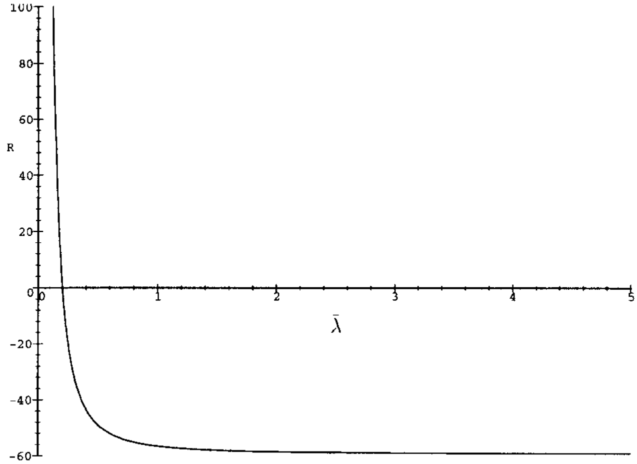

Fig. 5. Ricci scalar along the critical line $\bar{\phi} = 0, \bar{m} = 0$.

$$\bar{\mathcal{R}}|_{\bar{m}\to\infty} = -6\pi^2 + o\left(\frac{1}{N}\right), \tag{64}$$

which is the same value as one gets by näively putting $\bar{m} \to \infty$ in Eq. (62).

Of particular interest is the curvature along the critical line. Setting $\bar{\phi} = 0$ first and then letting $\bar{m} \to 0$ gives

$$\begin{aligned} \bar{\mathcal{R}}|_{\bar{\phi}=0,\bar{m}\to 0} &= \left(\frac{2}{\bar{\lambda}^2}\right) \frac{\left\{1 + 6\pi\bar{\lambda} + 4(\pi\bar{\lambda})^2 - 4\pi\bar{\lambda}(1+\pi\bar{\lambda})^2 \ln\left(1 + \frac{1}{\pi\bar{\lambda}}\right)\right\}}{\left\{1 + 2\pi\bar{\lambda} - 2\pi\bar{\lambda}(1+\pi\bar{\lambda}) \ln\left(1+\frac{1}{\pi\bar{\lambda}}\right)\right\}^2} \\ &\longrightarrow \begin{cases} -6\pi^2 & \bar{\lambda} \to \infty \\ 2/\bar{\lambda}^2 & \bar{\lambda} \to 0 \end{cases} \end{aligned} \tag{65}$$

which is shown in Fig. 5. There is actually a discontinuity in $\bar{\mathcal{R}}$ across this line, if we keep $\bar{m} = 0$, which is the critical line in the $\bar{\phi} - \bar{\lambda}$ plane. For $\bar{\phi} \neq 0$ and $\bar{m} = 0$, $\bar{\mathcal{R}} = 0$ $\forall\ \bar{\lambda}$, but for $\bar{\phi} = 0$ and $\bar{m} \to 0$, $\bar{\mathcal{R}} \neq 0$, (except at one value of $\bar{\lambda} \approx 0.2$). This behaviour is shown in Fig. 6, where $\bar{\mathcal{R}}$ is graphed as a function of $\bar{\phi}$ for $\bar{m} = 0$ and a generic value of $\bar{\lambda}$. The discontinuity in $\bar{\mathcal{R}}$ is due to the non-analyticity of the specific heat at the critical point. As explained in the introduction, the metric is defined in terms of second derivatives of the free energy and the Riemann tensor involves third derivatives of the reduced free energy with respect to any one parameter (e.g. temperature), hence one naively expects the Ricci scalar to diverge at a critical point. This does not happen here, except at the Gaussian fixed point $\bar{\lambda} = 0$, because the specific heat exponent $\alpha = -1 + o(1/N)$ is negative for the $O(N)$ model in three dimensions – the Ricci scalar is finite, but discontinuous, i.e. its derivative diverges at the critical line.

Let us examine the critical line, $\bar{m} = 0$, more closely for a fixed value of $\bar{\lambda}$. In three dimensions $\dot{G}(m^2)$ diverges, so introducing a cut-off, Eq. (23) yields

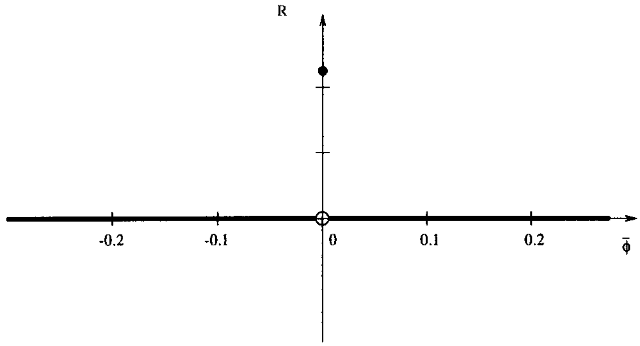

Fig. 6. Ricci scalar for $\bar{m} = 0$ and a fixed value of $\bar{\lambda}$.

$$\dot{G}(m^2) = -\frac{m}{4\pi} - \frac{\Lambda}{2\pi^2} \tag{66}$$

so $m = -4\pi\{\dot{G}(m^2) - \dot{G}(0)\}$. Eq. (34) can now be solved to give $m(r, \phi, \lambda)$, or in terms of dimensionless quantities

$$\bar{m} = \sqrt{\bar{\lambda}^2 + \frac{\bar{\lambda}\bar{\phi}^2}{2} + t} - \bar{\lambda}, \tag{67}$$

where $t = (r + \lambda\dot{G}(0)/2)/\Lambda^2$ is the reduced temperature. The addition of $\lambda\dot{G}(0)/2$ to the bare parameter, r, is the usual mass shift. The critical temperature $t = 0$, gives $\bar{m} = 0$ for $\bar{\phi} = 0$ (vanishing external field) but for $t < 0$, $\bar{m} = 0$ for $\bar{\phi} = 4|t|/\bar{\lambda}$, which is the critical line. Along a line specified by $\bar{\phi} \neq 0$, $\bar{m} = 0$ and a fixed value of $\bar{\lambda}$, the specific heat is finite [17] – the line along which it diverges lies in the unstable region and is known as the pseudo-spinodal line, this only coincides with the critical line, $\bar{m} = 0$, for $\bar{\phi} = 0$.

Finally, let us consider renormalisation group flow. Following Zinn-Justin [18], define β-functions for the three parameters $\bar{\phi}, t, \bar{\lambda}$ by

$$\beta^{\bar{\phi}} = \Lambda\frac{d\bar{\phi}}{d\bar{\Lambda}} = -\frac{1}{2}\bar{\phi}, \qquad \beta^{t} = \Lambda\frac{dt}{d\bar{\Lambda}} = -2t, \qquad \beta^{\bar{\lambda}} = \Lambda\frac{d\bar{\lambda}}{d\lambda} = -\bar{\lambda}. \tag{68}$$

These are simply the canonical dimensions since $(\bar{\phi}, t, \bar{\lambda})$ are bare parameters, which are finite for finite cut-off Λ. In terms of the variables ϕ, X and λ, let $\bar{X} = X/\Lambda$ be dimensionless and then

$$\beta^{\bar{\phi}} = -\frac{1}{2}\bar{\phi}, \qquad \beta^{\bar{X}} = -\bar{X}, \qquad \beta^{\bar{\lambda}} = -\bar{\lambda}. \tag{69}$$

These represent a vector flow on the space of parameters and we shall now investigate the dynamics of this vector flow, in particular we can ask: how is this flow related to geodesics of the metric (50)?

For any curve $\bar{\phi}(\Lambda), \bar{X}(\Lambda), \bar{\lambda}(\Lambda)$ parameterised by Λ, the geodesic equation is

$$\frac{d^2x^\mu}{d\Lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\Lambda}\frac{dx^\sigma}{d\Lambda} = -cx^\mu, \tag{70}$$

where $c(x)$ is a function which allows for the possibility that Λ might not be an affine parameter [21]. Using the connection coefficients in Appendix C, one finds that the condition that both the order N and the order one contributions satisfy Eq. (70), with $\bar{\phi}, \bar{X}$ and $\bar{\lambda}$ all order one, is very restrictive and the only solution is $\bar{X} = \bar{\phi} = 0$, though $\bar{\lambda}$ can be non-zero, provided c is a function of $\bar{\lambda}$ alone, given by

$$c(\bar{\lambda}) = 1 + \frac{\bar{\lambda}\bar{L}^{(3)}}{2\bar{L}^{(2)}}. \tag{71}$$

This leads to the somewhat surprising result that the line of crossover from the Wilson–Fisher fixed point to the Gaussian fixed point is a geodesic, but none of the other RG trajectories is. The physical significance of this result will be examined in the next section.

5. Relative entropy

In this section a physical interpretation of the geodesic flow, unveiled in the previous section, is given. The metric used in the previous analysis is related to the concept of relative entropy in statistical mechanics (for a consideration of relative entropy in field theory, see Ref. [22]). For a discrete probability distribution $p_i(g)$, $i = 1, \dots, r$, depending on some set of parameters g^a, $a = 1, \dots, n$, the relative entropy of g^a relative to $g^{a'}$ is defined to be [23]

$$\mathcal{S}_R(g, g') = -\sum_{i=1}^{r} p_i(g) \ln \{p_i(g)/p_i(g')\}. \tag{72}$$

For a continuous probability distribution, the discrete sum becomes a functional integral with

$$p_i(g) \to \frac{e^{-S[\varphi, g]}}{Z(g)} \tag{73}$$

so

$$\mathcal{S}_R(g, g') = \langle S(g) \rangle_g - \langle S(g') \rangle_g + W(g') - W(g), \tag{74}$$

where $W(g) = -\ln Z(g)$, and all expectation values use the measure appropriate to g as in Eq. (73), not g'.

Dividing by the volume of D-dimensional space, so as to work with specific quantities, one defines the relative entropy per unit volume to be

$$s_R(g, g') = -\frac{1}{V}\{\langle S(g') \rangle_g - \langle S(g) \rangle_g\} + w(g') - w(g). \tag{75}$$

If $g^{a'} = g^a + \delta g^a$, with δg^a small, we have

$$S(g') = S(g) + \delta g^a \partial_a S(g) + \frac{1}{2}\delta g^a \delta g^b \partial_a \partial_b S(g) + \dots,$$
$$w(g') = w(g) + \delta g^a \partial_a w(g) + \frac{1}{2}\delta g^a \delta g^b \partial_a \partial_b w(g) + \dots \tag{76}$$

and the terms linear in δg cancel in (75), since $\partial_a w = \frac{1}{V}\langle \partial_a S\rangle$, giving

$$-s_R(g, g+\delta g) = \frac{1}{2}\left\{\frac{1}{V}\langle \partial_a \partial_b S\rangle - \partial_a \partial_b w\right\}\delta g^a \delta g^b + o(\delta g)^3. \tag{77}$$

Thus the metric defined in (12) is completely equivalent to the infinitesimal relative entropy, and the distance between two points g_A and g_B along a curve $g(\tau)$, parameterised by τ, is given by

$$d = \int_{\tau_A}^{\tau_B} \sqrt{-2s_R}\, dt = \int_{\tau_A}^{\tau_B} \sqrt{G_{ab}\dot{g}^a \dot{g}^b} d\tau, \tag{78}$$

where $\dot{g}^a = dg^a/d\tau$. Note that $S_R(g, g') \neq S_R(g', g)$ for finite $g' - g$ so S_R itself cannot be interpreted as a distance function.

The conclusions of the previous section can now be rephrased by saying that, in the large N limit of the $O(N)$ model in three dimensions, the line of crossover between the Wilson–Fisher fixed point and the Gaussian fixed point is a line of extremal relative entropy. At least for the segment of the line along which $\bar{\mathcal{R}} < 0$, i.e.

$$\bar{\lambda}_0 \approx 0.2 < \bar{\lambda} < \infty, \tag{79}$$

where $\bar{\lambda}_0$ is the value of $\bar{\lambda}$ at which $\bar{\mathcal{R}} = 0$ (Fig. 5), one can be confident that the relative entropy is maximised, since there can be no conjugate points for $\bar{\mathcal{R}} < 0$ [24].

6. Conclusions

The concept of a geometry on the space of couplings, and its relation to the vector flow of the renormalisation group equation, has been investigated in the particular case of the $O(N)$ model in the limit of $N \to \infty$. The space of couplings in this case is three dimensional and can be parameterised by the vacuum expectation of the field, a mass and the φ^4 coupling. The metric adopted,

$$G_{ab} = \int d^D x\, \langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(0)\rangle, \tag{80}$$

is the matrix given by taking the zero momentum limit of two-point correlators of the composite operators associated with the couplings, which ought to capture the infra-red behaviour of the theory, but would not be expected to give useful information in the ultra-violet. This is borne out when the curvature is calculated and in $D = 3$, in the infinite volume limit, the Ricci scalar is found to diverge at the Gaussian (ultra-violet) fixed point, but is finite (and negative) at the Wilson–Fisher (infra-red) fixed point. This statement is independent of the co-ordinate system used – it does not matter whether one uses bare or renormalised couplings. This is not true of the components of the metric – divergences in the metric could be due to either genuine singularities of the geometry or could be merely co-ordinate artifacts due, for example, to a parameterisation using bare couplings.

In particular, the Ricci scalar is a smooth monotonically increasing function along the RG trajectory between the Wilson–Fisher and the Gaussian fixed points, although it is not differentiable in one of the directions transverse to the line of cross-over – reflecting the fundamental non-analyticity of the free energy at the critical line.

It was also noted in Section 4 that this cross-over line is a geodesic in the geometry described here but none of the other RG trajectories, which miss the Gaussian fixed point, is a geodesic. This property is equivalent to the statement that the relative entropy is maximised along this curve. The geodesic nature of some renormalisation group trajectories in simpler models was noted in [12], and would seem to hint at a possible variational formalism for the RG, but this requires further study.

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Appendix A. Legendre transforms

Consider a differentiable function $w(g)$ of n variables $g^a, a = 1, \dots, n$, and the corresponding Hessian

$$G_{ab} = -\frac{\partial^2 w}{\partial g^a \partial g^b} \equiv -w_{ab}. \qquad \text{(A.1)}$$

If one changes variables to $(g^{a'}) = (\phi_1, g^2, \dots, g^n)$, where

$$\phi_1(g) = \frac{\partial w}{\partial g^1} \qquad \text{(A.2)}$$

is the Legendre transform variable of g^1, then the corresponding co-ordinate transformation matrix is

$$\frac{\partial g^{a'}}{\partial g^{b}} = \begin{pmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \tag{A.3}$$

and the inverse matrix is

$$\frac{\partial g^{b}}{\partial g^{a'}} = \begin{pmatrix} \frac{1}{w_{11}} & -\frac{w_{12}}{w_{11}} & \dots & -\frac{w_{1n}}{w_{11}} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}. \tag{A.4}$$

Treating G_{ab} as a tensor (which requires endowing the original co-ordinate system, g^a, with a very special status as explained in the introduction) one finds

$$G_{a'b'} = \left[\left(\frac{\partial g}{\partial g'}\right)^T\right]_{a'}^{\ b} G_{ab} \left(\frac{\partial g}{\partial g'}\right)^{b}_{\ b'} = \begin{pmatrix} \tilde{\Gamma}_{11} & 0 & \dots & 0 \\ 0 & -\tilde{\Gamma}_{22} & \dots & -\tilde{\Gamma}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & -\tilde{\Gamma}_{2n} & \dots & -\tilde{\Gamma}_{nn} \end{pmatrix}, \tag{A.5}$$

where $\tilde{\Gamma}(\phi_1, g^2, \dots, g^n) = \{w(g) - \phi_1 g^1\}|_{\phi_1 = \partial w/\partial g^1}$ is the Legendre transform of w,

$$\tilde{\Gamma}_{11} = \frac{\partial^2 \tilde{\Gamma}}{\partial(\phi_1)^2} = -\frac{1}{w_{11}} \quad \text{and} \quad \tilde{\Gamma}_{\bar{a}\bar{b}} = \frac{\partial^2 \tilde{\Gamma}}{\partial g^{\bar{a}} \partial g^{\bar{b}}}$$

with $\bar{a}, \bar{b} = 2, \dots n$.

In deriving this result it is important to remember that, for $\bar{a} = 2, \dots, n$

$$\left.\frac{\partial \tilde{\Gamma}}{\partial g^{\bar{a}}}\right|_{\phi_1} = \left.\frac{\partial w}{\partial g^{\bar{a}}}\right|_{g^1} + \frac{\partial w}{\partial g^1} \cdot \left.\frac{\partial g^1}{\partial g^{\bar{a}}}\right|_{\phi_1} - \phi_1 \left.\frac{\partial g^1}{\partial g^{\bar{a}}}\right|_{\phi_1} = \left.\frac{\partial w}{\partial g^{\bar{a}}}\right|_{g^1}, \tag{A.6}$$

where $g^1(\phi_1, g^2, \dots, g^n)$ is determined by inverting the function $\phi_1(g^1, \dots, g^n) = \partial w(g)/\partial g^1$. Similarly

$$\left.\frac{\partial^2 \tilde{\Gamma}}{\partial g^{\bar{a}} \partial g^{\bar{b}}}\right|_{\phi_1} = \left.\frac{\partial^2 w}{\partial g^{\bar{a}} \partial g^{\bar{b}}}\right|_{g^1} + \left.\frac{\partial^2 w}{\partial g^{\bar{a}} \partial g^1} \frac{\partial g^1}{\partial g^{\bar{b}}}\right|_{\phi_1} = w_{\bar{a}\bar{b}} - \frac{w_{1\bar{a}} w_{1\bar{b}}}{w_{11}}, \tag{A.7}$$

since $\partial g^1/\partial g^{\bar{b}} = -w_{1\bar{b}}/w_{11}$ from (A.4).

Alternatively, since the complete Legendre transform, $\Psi(\phi)$, where $\phi_1 = \partial w/\partial g^1$, $\phi_2 = \partial \tilde{\Gamma}/\partial g^2|_{\phi_1} = \partial w/\partial g^2|_{g^1}$, etc., has the property that the matrix

$$\frac{\partial^2 \Psi(\phi)}{\partial \phi_a \partial \phi_b} \equiv \Psi^{ab} \tag{A.8}$$

is the inverse of $\partial^2 w/\partial g^a \partial g^b$, one has

$$ds^2 = G_{ab} dg^a dg^b = \Psi^{ab} d\phi_a d\phi_b. \tag{A.9}$$

The same argument can now be applied to the Legendre transform of $\Psi(\phi)$ in one variable, $\tilde{\Xi}(\phi_1,\dots,\phi_{n-1},g^n)=\Psi(\phi)-\phi_n g^n$, to deduce that, in the co-ordinate system $\phi^{\hat{a}}=(\phi_1,\dots,\phi_{n-1},g^n)$,

$$G_{\hat{a}\hat{b}}=\begin{pmatrix} -\tilde{\Xi}_{11} & \dots & -\tilde{\Xi}_{1,n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\tilde{\Xi}_{1,n-1} & \dots & -\tilde{\Xi}_{n-1,n-1} & 0 \\ 0 & \dots & 0 & \tilde{\Xi}_{nn} \end{pmatrix} \tag{A.10}$$

and this is the form of the metric used in the text, with $n=3$ and $\phi^{\hat{1}}=\phi$, $\phi^{\hat{2}}=X$, $g^3=\phi^{\hat{3}}=\lambda$.

Finally, a proof will be given of Eq. (46) in the text. For simplicity we consider a function $w(g)$ of only one argument, but the results apply equally well to a function of more than one variable. Let

$$w(g)=w_0(g)+\frac{1}{2N}L(g)+o\left(\frac{1}{N^2}\right), \tag{A.11}$$

where w_0 and L are independent functions of g, and N is some large parameter. The Legendre transform variable is

$$\phi(g)=\frac{\partial w_0}{\partial g}+\frac{1}{2N}\frac{\partial L}{\partial g}+o\left(\frac{1}{N^2}\right)=\phi_0(g)+\frac{1}{N}\phi_1(g)+o\left(\frac{1}{N^2}\right) \tag{A.12}$$

and

$$\tilde{\Gamma}(\phi)=w_0(g)+\frac{1}{2N}L(g)-\phi\cdot g+o\left(\frac{1}{N^2}\right). \tag{A.13}$$

Inverting Eq. (A.12), one obtains

$$g(\phi)=g_0(\phi)+\frac{1}{N}g_1(\phi)+o\left(\frac{1}{N^2}\right), \tag{A.14}$$

where $g_0(\phi)$ and $g_1(\phi)$ are functions of ϕ. Therefore, Taylor expanding $w_0(g)$ and $L(g)$,

$$\begin{aligned} w_0(g)&=w_0(g_0)+\frac{1}{N}g_1\cdot\left.\frac{\partial w_0}{\partial g}\right|_{g_0}+o\left(\frac{1}{N^2}\right),\\ L(g)&=L(g_0)+o\left(\frac{1}{N}\right) \end{aligned} \tag{A.15}$$

leads to

$$\begin{aligned} \tilde{\Gamma}(\phi)&=w_0(g_0)+\frac{1}{N}g_1\left.\frac{\partial w_0}{\partial g}\right|_{g_0}+\frac{1}{2N}L(g_0)-\left(g_0(\phi)+\frac{1}{N}g_1(\phi)\right)\phi+o\left(\frac{1}{N^2}\right)\\ &=w_0(g_0(\phi))+\frac{1}{2N}L(g_0(\phi))-\phi\cdot g_0(\phi)+o\left(\frac{1}{N^2}\right) \end{aligned} \tag{A.16}$$

since

$$\phi = \left.\frac{\partial w_0}{\partial g}\right|_{g_0} + o\left(\frac{1}{N}\right).$$

Hence

$$\begin{aligned}\tilde{\Gamma}(\phi) &= \tilde{\Gamma}_0(\phi) + \frac{1}{2N}L(g_0(\phi)) + o\left(\frac{1}{N^2}\right)\\ &= \tilde{\Gamma}_0(\phi) + \frac{1}{2N}L(g(\phi)) + o\left(\frac{1}{N^2}\right),\end{aligned} \tag{A.17}$$

where

$$\tilde{\Gamma}_0(\phi) = w_0(g_0) - \phi \cdot g_0,$$

which is Eq. (46).

Appendix B. Connection coefficients for the $O(N)$ model

The metric is given in Eq. (50) in the (ϕ, X, λ) co-ordinate system,

$$G_{ab} = \begin{pmatrix} \gamma_{ij} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} L_{\phi\phi} & L_{\phi X} & 0 \\ L_{\phi X} & L_{XX} & 0 \\ 0 & 0 & -L_{\lambda\lambda} \end{pmatrix} + o\left(\frac{1}{N}\right),$$

where γ_{ij} is the 2 x 2 matrix

$$\gamma_{ij} = N\begin{pmatrix} m^2 - \frac{2\phi^2}{\ddot{G}} & \frac{2\phi}{\ddot{G}} \\ \frac{2\phi}{\ddot{G}} & \lambda - \frac{2}{\ddot{G}} \end{pmatrix}$$

and $L_{\phi\phi} = \partial^2 L/\partial\phi^2$, etc. The functions $\ddot{G}$ and L are given in Eqs. (53) and (54) and the effective mass $m^2(\phi, X)$ is defined implicitly in Eq. (37). The inverse metric is

$$G^{ab} = \begin{pmatrix} \gamma^{ij} & 0 \\ 0 & -\frac{2}{L_{\lambda\lambda}} + o(1/N) \end{pmatrix}$$

with

$$\gamma^{ij} = \frac{1}{N\det\gamma}\begin{pmatrix} \lambda - \frac{2}{\ddot{G}} & -\frac{2\phi}{\ddot{G}} \\ -\frac{2\phi}{\ddot{G}} & m^2 - \frac{2\phi^2}{\ddot{G}} \end{pmatrix} + o\left(\frac{1}{N^2}\right).$$

The evaluation of the connection coefficients is simplified by the observation that, in the co-ordinate system used here, the metric is curl free,

$$G_{ab,c} = G_{cb,a} = G_{ca,b} := G_{abc},$$

thus

$$\Gamma^a_{bc} = \frac{1}{2}G^{ad}G_{dbc}.$$

Explicitly one finds

$$
\begin{aligned}
\Gamma^X_{XX} &= \frac{2m^2\dddot{G}}{\det\gamma(\ddot{G})^3} + o\left(\frac{1}{N}\right), \\
\Gamma^X_{X\phi} &= -\frac{2\phi}{\det\gamma(\ddot{G})^3}(m^2\dddot{G} + \ddot{G}) + o\left(\frac{1}{N}\right), \\
\Gamma^X_{\phi\phi} &= -\frac{1}{\det\gamma(\ddot{G})^3}\left\{m^2(\ddot{G})^2 + \phi^2(2m^2\dddot{G} + 4\ddot{G})\right\} + o\left(\frac{1}{N}\right), \\
\Gamma^\phi_{XX} &= -\frac{2\lambda\phi\dddot{G}}{\det\gamma(\ddot{G})^3} + o\left(\frac{1}{N}\right), \\
\Gamma^\phi_{X\phi} &= \frac{1}{\det\gamma(\ddot{G})^3}\left\{\lambda(\ddot{G})^2 - 2\ddot{G} + 2\lambda\phi^2\dddot{G}\right\} + o\left(\frac{1}{N}\right), \\
\Gamma^\phi_{\phi\phi} &= \frac{\phi}{\det\gamma(\ddot{G})^3}\left\{4\ddot{G} - 3\lambda(\ddot{G})^2 - 2\lambda\phi^2\dddot{G}\right\} + o\left(\frac{1}{N}\right), \\
\Gamma^X_{X\lambda} &= \frac{1}{2\det\gamma}\left(m^2 - \frac{2\phi^2}{\ddot{G}}\right) + o\left(\frac{1}{N}\right), \\
\Gamma^\phi_{X\lambda} &= -\frac{\phi}{\det\gamma\,\ddot{G}} + o\left(\frac{1}{N}\right), \\
\Gamma^X_{\phi\lambda} &= o\left(\frac{1}{N}\right), \qquad \Gamma^\phi_{\phi\lambda} = o\left(\frac{1}{N}\right), \\
\Gamma^\lambda_{XX} &= \frac{N}{L_{\lambda\lambda}} + o(1), \qquad \Gamma^\lambda_{X\phi} = \frac{L_{X\phi\lambda}}{2L_{\lambda\lambda}} + o\left(\frac{1}{N}\right),, \\
\Gamma^\lambda_{\phi\phi} &= \frac{L_{\phi\phi\lambda}}{2L_{\lambda\lambda}} + o\left(\frac{1}{N}\right), \qquad \Gamma^\lambda_{X\lambda} = \frac{L_{X\lambda\lambda}}{2L_{\lambda\lambda}} + o\left(\frac{1}{N}\right), \\
\Gamma^\lambda_{\phi\lambda} &= \frac{1}{2}\frac{L_{\phi\lambda\lambda}}{L_{\lambda\lambda}} + o\left(\frac{1}{N}\right), \qquad \Gamma^X_{\lambda\lambda} = o\left(\frac{1}{N}\right), \\
\Gamma^\phi_{\lambda\lambda} &= o\left(\frac{1}{N}\right), \qquad \Gamma^\lambda_{\lambda\lambda} = \frac{L_{\lambda\lambda\lambda}}{2L_{\lambda\lambda}} + o\left(\frac{1}{N}\right),
\end{aligned}
$$

where $L_{\lambda\lambda} = \partial^2 L/\partial\lambda\partial\lambda$, etc. Using these expressions, the components of the Ricci tensor in Eq. (51) can be verified.

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