# NONCOMMUTATIVE ANTICOMMUTATIVE RINGS

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An associative ring R is said to be *anticommutative* if $xy + yx = 0$ for all $x,y \in R$. If R has characteristic 2, then the concepts of commutativity and anticommutativity coincide, but $Z_3$, with the usual addition and trivial multiplication, shows that an anticommutative ring need not have characteristic 2.

If a ring R satisfies $x^2 = 0$ for all $x \in R$ then clearly R is anticommutative, but not conversely. However, if R is anticommutative it is easy to verify that R satisfies each of the following identities.

(i) $2x^2 = 0$ (ii) $(xy - yx)^2 = 0$ (iii) $x^2y - yx^2 = 0$.

Frequently, when looking at commutativity theorems for rings, one requires counterexamples to show that certain conditions are not sufficient for commutativity. For example, if $(xy)^2 = x^2y^2$ for all $x,y \in R$ and either of the following conditions holds then R is commutative:

(a) R has unity; (b) R has no non-zero nilpotent elements.

To show that some such additional condition is necessary, it is enough to produce a non-commutative ring in which $x^2 = 0$ for all $x \in R$. In this note, for finite rings, we pose the question, *"what is the order of a smallest noncommutative anti-commutative ring?"* and show that the answer is 27. Since this number is odd, we see that it is also the answer to the question, *"what is the order of a smallest noncommutative ring satisfying the identity $x^2 = 0$?"*.

First of all we produce a ring of order 27 with the desired properties. Let $\Lambda = (a_{ij})$ be the ring of those 4x4 matrices with entries in the field $Z_3$, such that $a_{ij} = 0$ if $j \leq$ $a_{23} = 0$, $a_{24} = a_{13}$ and $a_{34} = -a_{12}$. Then it is easily checked that R is a noncommutative anticommutative ring of order 27. In more abstract terms, R can be expressed as follows: if $C_n$ is the cyclic group of order n and $\oplus$ denotes the direct sum of groups, then $(R,+) \cong C_3 \oplus C_3 \oplus C_3 = \langle a\rangle \oplus \langle b\rangle \oplus \langle c\rangle$, where $a^2 = b^2 = c^2 = ac = bc = ca = cb = 0$, $ab = -ba = c$ determines the multiplicative operation in R.

We proceed to show that no ring of order less than 27 can be both noncommutative and anticommutative, so let R be a ring with these properties. Since every finite ring is the direct sum of rings of prime-power order and since a direct sum of rings is anticommutative if and only if each of its direct summands is anticommutative, we may confine our attention to rings of prime-power order. If $(R,+)$ is cyclic, then R is commutative - this eliminates rings of prime order and if $|R| = p^2$ for some prime p, we may assume $(R,+) \cong C_p \oplus C_p$. Clearly, we may also eliminate rings of characteristic 2. Thus we need only consider the following values of $|R|$ with corresponding structures for $(R,+)$:

(i) $|R| = 8$, $(R,+) \cong C_2 \oplus C_4$;

(ii) $|R| = 9$, $(R,+) \cong C_3 \oplus C_3$;

(iii) $|R| = 16$, $(R,+) \cong C_2 \oplus C_8$, $C_2 \oplus C_2 \oplus C_4$, $C_4 \oplus C_4$;

(iv) $|R| = 25$, $(R,+) \cong C_5 \oplus C_5$.

We can eliminate 9 and 25 using the following result.

<u>LEMMA</u>. *If p is an odd prime, then $C_p \oplus C_p$ cannot be the additive group of a noncommutative anticommutative ring.*

<u>PROOF</u>. Let R be a counterexample and let $(R,+) = \langle a\rangle \oplus \langle b\rangle$. Since R is anticommutative, $x.x + x.x = 0$ for all $x \in R$, so $x^2 = 0$, since $|R|$ is odd. If $ab = 0$ then $ab + ba = 0 \Rightarrow ba = 0 = ab$, so R is commutative, a contradiction. Suppose that $ab = ra + sb$ where $r,s \in Z_p$. Then $a^2b = ra^2 + sab$, so $sab = 0$, and so $s = 0$. Finally $ab = ra$, so $ab^2 = rab = 0$ which gives $ab = 0$, a contradiction.

Next, we suppose that $(R,+) \cong C_2 \oplus C_4$ or $C_2 \oplus C_8$ and $R = \langle a\rangle \oplus \langle b\rangle$, where b has order 4 or 8. In either case, $2ab = (2a)b = 0$, so $2ab = ab + ba$, and R is commutative. The case $(R,+) \cong C_2 \oplus C_2 \oplus C_4$ is dismissed in a similar manner. We are left with the possibility that $(R,+) \cong C_4 \oplus C_4$. Suppose that $(R,+) = \langle a\rangle \oplus \langle b\rangle$ where $4a = 4b = 0$. Consider first the case where $a^2 = b^2 = 0$. Then we get a contradiction, as in the proof of the lemma. Thus we may assume that one generator (a say) satisfies $a^2 \neq 0$. Since $2a^2 = 0$, $a^2 \in \{2a, 2b, 2a+2b\}$, the set of elements of order 2 in R. Suppose first that $a^2 = 2a$ and let $ab = ra + sb$, where $r,s \in Z_4$. Then $2ab = a^2b = a(ab) = ra^2 + sab = 2ra + sab$. This gives $(sr)a + s(s-2)b = 0$. Hence s is even and if $s \neq 0$, r is even. This implies that ab has order 2 and so R is commutative, a contradiction. Thus $s = 0$ and $ab = ra$, $r = \pm 1$. Then $ab^2 = (ab)b = rab = r^2a = a$, so $2ab^2 = a(2b^2) = 0 = 2a$, a contradiction.

Finally, we may suppose that $a^2 = 2b$, since if $a^2 = 2a+2b$ we may replace b in the basis by a+b. If $ab = ra + sb$, we get $(rs)a + (2r+s^2)b = 0$. Thus s is even and if $s \neq 0$, r is even also, so ab has order 2, a contradiction. Hence $s = 0$, so $2r + s^2 = 0$, r is even, $2ab = 0$ and we are finished.

Let S be the ring of order 32 where $(S,+) = \langle a\rangle \oplus \langle b\rangle$ $C_8 \oplus C_4$, with $a^2 = 4a$, $b^2 = 2b$, $ab = -ba = 2a$. Then S is a noncommutative anticommutative ring of order a power of 2. By our previous analysis, S is a smallest such 2-ring and in addition, S is a smallest such ring of even order. Finally, we observe that S is a smallest ring of the desired type such that $(S,+)$ is a 2-generator group.

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