Information Diffusion on Social Networks

A dissertation submitted for the degree of
Master of Science

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Declaration

I herewith declare that I have produced this manuscript without the prohibited assistance of third parties and without making use of aids other than those specified.

The thesis work was conducted from January 2011 to August 2012 under the supervision of Dr. Oliver Mason in the Hamilton Institute, National University of Ireland Maynooth. This work was supported by the Irish Higher Educational Authority(HEA) PRTLI Network Mathematics Grant.

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Special thanks to all the people I've met in the Hamilton Department who have made my time here so enjoyable, in particular to Buket Benek Gursey for her friendship and for many interesting conversations.

Finally I would like to thank my friends for the support and entertainment that they have given me over the past two years. I am a better person for having known them and am very grateful to have done so. I thank my family for doing their best to ensure that I had a good education.

The world will never starve for want of wonders; but only for want of wonder.

G. K. Chesterton

Abstract

In this thesis we model the diffusion of information on social networks. A game played on a specific type of graph generator, the iterated local transitivity model, is examined. We study how the dynamics of the game change as the graph grows, and the relationship between properties of the game on a graph initially and properties of the game later in the graph's development. We show that, given certain conditions, for the iterated local transitivity model it is possible to predict the existence of a Nash equilibrium at any point in the graph's growth. We give sufficient conditions for the existence of Nash Equilibria on star graphs, cliques and trees. We give some results on potential games on the iterated local transitivity model.

Chapter 2 provides an introduction to graph properties, and describes various early graph models. Chapter 3 describes some models for online social networks, and introduces the iterated local transitivity model which we use later in the thesis. In Chapter 4 various models for games played on networks are examined. We study a model for competitive information diffusion on star graphs, cliques and trees, and we provide conditions for the existence of Nash Equilibria on these. This model for competitive information diffusion is studied in detail for the iterated local transitivity model in Chapter 5. We discuss potential games in Chapter 6 and their existence on the iterated local transitivity model. We conclude with some suggestions on how to extend and develop upon the work done in this thesis.

CHAPTER 1

Introduction

Recent advances in communications and the emergence of social networking sites such as Facebook and Twitter [29] have greatly increased the power of individual agents to disseminate information. This provides strong motivation for analysing the mechanisms of information propagation and the role played by the individual in the process. We approach this problem by modelling the structure of the network of individuals as a graph and using game theoretic concepts to capture the process of information diffusion. Thus the two central mathematical themes of the thesis are graph theory and game theory.

Graphs have been the subject of study for many years [44]. The study of graph theory began with Euler in 1735. He wrote a paper that is regarded as the first in the history of graph theory [23]. There had been a long standing problem in the city of Königsberg, of finding a walk through the city, consisting of four landmasses and seven bridges, which would cross each bridge exactly once. A map of the area is given below, with the river in blue, the bridges in green and the landmasses numbered. Euler rewrote the problem in abstract terms, representing the bridges as edges and the landmasses as vertices, as shown below. The structure shown below is called a graph or network, with vertices in blue that are connected by edges.

Graphs have since been used in many areas, in modelling relations in physical, biological and social systems, such as connections between people, the internet, food webs, metabolic networks, neural networks and many more.

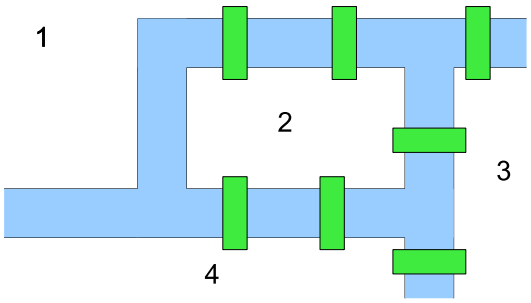


Figure 1.1: Map of Königsberg

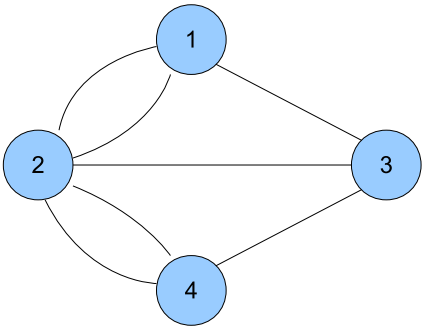


Figure 1.2: Graph representation of Königsberg

Game theory is the study of strategic decision making. It is useful because many human activities can be viewed as games on a network in which a person's utility is determined by the behaviour of others that are in some sense nearby. Complex behaviour can emerge as local changes in a network may have global effects. Game theory was first used to model how human populations behaved. It is now employed in many fields such as economics, political science, computer science and biology. A classic example of a game is the prisoner's dilemma. There are two players in this game, each of which can choose either to cooperate or to defect. Both players gain if they cooperate, but if only one cooperates then the one that defects will gain more. If both defect both gain very little. A diagram of the payoffs each player receives is given below.

Sociologists realised in the 1930s the importance of patterns of connections between people in understanding our society. They discovered many of the

	Player 2 cooperates	Player 2 defects
Player 1 cooperates	Both receive a payoff of 10	Player 1 receives no payoff, Player 2 receives a payoff of 20
Player 1 defects	Player 2 receives no payoff, Player 1 receives a payoff of 20	Both receive a payoff of 5

Figure 1.3: Prisoner's dilemma

small scale properties of graphs that are familiar today. However at this time it was difficult to collect much data as typical studies would involve the circulation of questionnaires, which is labour intensive.

Understanding social networks is important in epidemiology, behavioural sciences, marketing and many other areas. Social networks are self organizing and complex. In the last twenty years online social networks have become part of the fabric of our lives. Websites such as Facebook, Twitter, MySpace, LinkedIn and Bebo have received a huge number of page views. They are used daily by millions of people and the data generated by these networks have made it possible to study social networks on a large scale. Datasets involving millions of vertices, rather than fewer than a hundred, are now analysed. This required new techniques for analyzing the data, as there are too many vertices for information to be obtained in an ad hoc manner from a picture of the graph. Different properties of the graph become important, as individual vertices are no longer relevant. It has also become necessary to create graph generators, that can replicate properties of real world graphs. This makes it possible to run simulations and can provide insight into the processes that cause various graph properties to emerge. Ideally a graph generator is simple, fast, realistic, and requires few input parameters.

In this thesis a game played on a specific type of graph generator is examined. The game models information dissemination. We study how the dynamics of the game changes as the graph grows, and the relationship between properties of the game on the graph initially and properties of the game after the graph

has grown. We look for the existence of Nash Equilibria on the initial graph and then on the later graph. For the graph generator we chose, given certain conditions, it is possible to predict the existence of a Nash equilibrium at any point of the graph's growth, once the initial graph has been examined. A Nash Equilibrium is a solution concept of a game in which no player can increase its payoff by unilaterally changing its strategy, and each player is aware of the other players strategies. We give some results on potential games on this graph model.

Chapter 2 provides an introduction to graph properties, and describes various early graph models. Chapter 3 describes some models for online social networks, and introduces the iterated local transitivity model which we use later in the thesis. In Chapter 4 various models for games played on networks are examined. We study a model for competitive information diffusion on star graphs, cliques and trees, and we provide conditions for the existence of Nash Equilibria on these. These results have been submitted to Information Processing Letters. This model for competitive information diffusion is studied in detail for the iterated local transitivity model in Chapter 5. The results from this chapter have been submitted to Discrete Applied Mathematics. We discuss potential games in Chapter 6 and their existence on the iterated local transitivity model. We conclude with some suggestions on how to extend and develop upon the work done in this thesis.

CHAPTER 2

Background

In this chapter we provide some important basic definitions and concepts used in graph theory. As the networks studied became larger, it became impossible to examine their graphs using the naked eye, thus it became necessary to develop different methods of analyzing them. Numerous different measures have been developed over the years to characterize the structure and behaviour of networks, some of which we look at here. We examine three early models for generating graphs, each of which captures different properties of real world graphs.

2.1 Definitions

Some basic terms used in graph theory are defined below. Many definitions in graph theory have unfortunately not yet been standardized, so it is important to be aware of possible differences in meaning between sources. For the most part the definitions given in Newman's review paper [44] are followed here.

An *undirected graph* is a graph consisting of a set of vertices $V(G)$ and a set of undirected edges $E(G)$ of the form $\{v, w\}$, for $v, w \in V(G)$. There may be more than one edge between any two vertices, and there may be an edge that connects a vertex to itself. To simplify notation we write vw (or wv) to denote the edge $\{v, w\}$ [44]. The graph G in Figure 2.1 is undirected.

A *directed graph* is a graph consisting of a set of vertices $V(G)$ and a set

of directed edges $E(G)$ of the form (v, w), where (v, w) is an ordered pair and $v, w \in V(G)$ [44].

A *loop* is an edge that connects a vertex to itself.

A *simple graph* is an undirected graph that has no loops and no more than one edge between any two vertices. Graph G in Figure 2.1 is simple.

The *degree* of a vertex v, $deg_G(v)$, is the number of edges incident to the vertex, with loops counted twice [44]. The degree of vertex 10 in the graph G below is four.

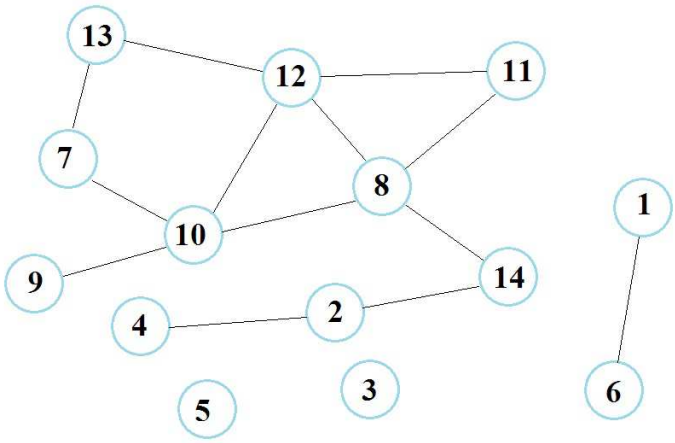


Figure 2.1: An undirected graph G

A *walk* is sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2 ... e_n, v_n$ such that for $1 \leq i \leq n$, the edge e_i links vertices v_{i-1} and v_i. The *length* of a walk is the number of edges in the path, counting multiple edges multiple times. The length of the walk $13, 12, 11, 8, 10$ is four.

A *path* is a sequence of vertices and distinct edges such that from each of its vertices there is an edge to the next vertex in the sequence, and no vertices other than the first and last may be repeated. The walk $13, 12, 11, 8, 10$ on the graph above is an example of a path. A *cycle* is a closed path, a sequence of distinct vertices and edges that starts and terminates at the same vertex [37]. $10, 8, 12$ is an example of a cycle in the graph above.

A *geodesic path* is a shortest path from one vertex to another [44]. The geodesic path between $12, 14$ is $12, 8, 14$. The *diameter* of a graph is the length of the longest geodesic path between two vertices [44]. The diameter of the graph G in Figure 2.1 is not defined as the graph is not connected.

For a set X, we denote the cardinality of X by $|X|$.

The *degree distribution* of a graph, $P(k)$, describes the fraction of vertices in the graph of degree k for $k = 0, 1, 2..$ [4].

$$P(k) = \frac{|v : \{deg_v(G) = k\}|}{|V|}$$

where $|V|$ is the number of vertices in the graph.

A *connected component* of an undirected graph is a subgraph in which any two vertices in the subgraph are connected to each other by a path [37]. Vertices $(1, 6)$ form a connected component in G. The *giant component* of a graph is the connected component with the greatest number of vertices [40] in the graph. The giant component in G consists of the vertices $(2, 4, 7, 8, 9, 10, 11, 12, 13, 14)$.

A *connected triple* of vertices refers to a vertex with edges to an unordered pair of vertices.

A *triangle* refers to three vertices a, b, c such that the edges ab, bc and ac exist.

We loosely define *hubs* to be vertices with high degree [37].

The *distance* $d(u, v)$ between u, v in $V(G)$ is the length of a shortest path between u and v. For a set $S \subseteq V(G)$ and $u \in V(G)$, we define $d(u, S) = \min\{d(u, v) : v \in S\}$.

2.2 Graph Properties

It is not evident which properties of a graph are the most important. We choose which properties to examine based on the problem or application at hand. Properties that have attracted the most attention recently are explained below. In this section we describe some features that have been used to qualitatively describe a variety of social, biological, and technological networks.

2.2.1 Clustering

Clustering, or transitivity is a property most commonly associated with social networks. The analogy here is that there is a high probability that a friend of your friend is also your friend. This implies that there are a high number of triangles in the graph relative to the number in an Erdos Renyi random graph (defined later) with the same number of vertices and edges. The clustering coefficient of the graph is given by

$$C = 3 \times \frac{N_{\triangle}}{N_3}$$

where $N_{\triangle}$ is the number of triangles in the graph, and N_3 is the number of connected triples of vertices. The factor three accounts for the fact that each triangle is involved in three connected triples, one centred on each vertex. Graph G_1 in the figure below has clustering coefficient $C = 3 \times \frac{2}{8} = \frac{3}{4}$. An

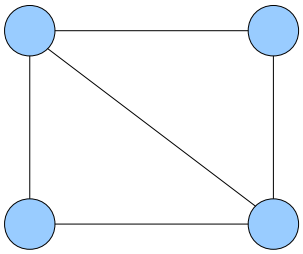


Figure 2.2: Graph G_1

alternative and also widely used expression for clustering is the local clustering coefficient C_i of vertex i [52]. This quantifies how close the neighbours of i are to being a complete graph (a clique). If k_i is the degree of vertex i, l_i the number of edges between neighbours of i, then:

$$C_i = (\frac{N_{\triangle} \; connected \; to \; vertex \; i}{N_3 \; centred \; on \; vertex \; i}) = \frac{2l_i}{k_i(k_i - 1)}$$

Using this an alternative definition of the network clustering coefficient is

$$\tilde{C} = \frac{1}{N}\sum_i C_i$$

This may result in different values for C and $\tilde{C}$ since vertices of higher degree could be involved in a larger number of triangles than vertices of smaller degree.

2.2.2 Assortativity

Assortative mixing occurs when vertices of the same type in the network tend to be connected. Disassortative mixing occurs when vertices of different type tend to be connected. A typical example of this in social networks is mixing by race [44], and more generally we tend to associate with people who are similar to us in some way. In a graph theoretic context, a particularly important example of assortative mixing is degree correlation, the tendency of vertices of high degree to associate with other vertices of high degree, or vertices of low degree to associate with other vertices of low degree. This occurs mainly in social networks, whereas disassortative mixing is characteristic of biological and technical networks, for example the internet [43].

One of the most appealing ways of measuring assortativity is to calculate Pearson's correlation coefficient r [43] between vertices at the end of each edge, and average over the edges.

$$r = \frac{\frac{1}{N}\sum_i j_i k_i - [\frac{1}{N}\sum_i \frac{1}{2}(j_i + k_i)]^2}{\frac{1}{N}\sum_i \frac{1}{2}(j_i^2 + k_i^2) - [\frac{1}{N}\sum_i \frac{1}{2}(j_i + k_i)]^2}$$

where j_i, k_i are the degrees of the vertices at the ends of the ith edge with $i = 1, .., N$. It is generally positive for assortative mixing, negative for disassortative mixing, and zero if there is no correlation between vertex degrees.

2.2.3 Community Structure

Communities are particularly relevant to social networks, and may be defined loosely in various ways. A community is a subgraph of a graph whose vertices are tightly connected [11], ie there is a higher density of edges within communities than between them. The ability to identify communities provides insight into how the network function and topology affect each other. The strongest definition of a community requires that all pairs of individuals in

the community are connected. In an undirected graph a *clique* is a subset of the graph's vertices such that every pair of vertices is connected by an edge. There are numerous algorithms for finding communities in graphs [47], [53], [25], many relying on predetermining the number of communities.

The Girvan-Newman algorithm [28] is an example of an algorithm used for finding communities in graphs. They define the edge betweenness of a given edge as the number of shortest paths between pairs of vertices that run through the edge. Communities are constructed here by removing edges with highest betweenness from the graph to reveal the community structure. These edges will generally lie between communities. By examining the graph at different stages in the algorithm, the communities can be identified. The algorithm is as follows:

1. Calculate the betweenness for all edges in the network.
2. Remove the edge with the highest betweenness.
3. Recalculate the betweenness for all edges affected by the removal.
4. Repeat from step 2 until no edges remain.

The betweenness centrality must be recalculated at each step, which involves a lot of computation. This algorithm works well but is too slow to use on very large networks.

2.2.4 Temporal Evolution Laws

Many real world graphs are dynamically evolving over time, such as the world wide web, the internet and social networks. If vertices are added to a graph over time then new properties of the graph may emerge. Two key temporal evolution laws are the densification power law and the fact that the diameter of real world graphs is generally found to decrease over time.

The number of edges $E(t)$ and vertices $N(t)$ of many real world graphs obey the *densification power law*,

$$E(t) \propto N(t)^a$$

a is typically between one and two. Thus real graphs become denser as they grow. This is in contrast to some graph models such as the preferential attachment model, where the average degree remains constant over time [9].

Let $g(d)$ denote the fraction of connected vertex pairs whose shortest connecting path has length at most d. The effective diameter of the network is defined to be the value of d at which $g(d)$ achieves the value 0.9. The effective diameter of many real world networks has been observed to shrink or stabilize as the graph grows with time [36].

2.2.5 Betweenness Centrality

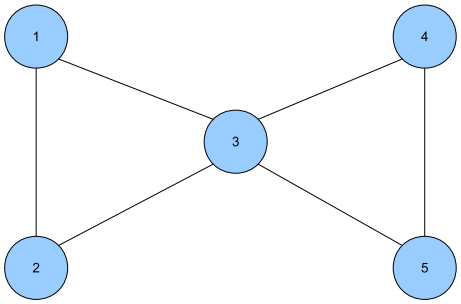
Betweenness centrality is a measure of a vertex's importance in a graph, and gives insight into the graph's structure. It is a more useful measure of the load placed on a vertex and the vertex's importance than the vertex's degree. The latter is only a local effect while the former is more global to the network. However betweenness centrality is unfortunately costly to compute.

The betweenness B_j of a vertex j is obtained by counting the number of geodesic paths going through it [26].

$$B_j = \sum_{ik} \frac{\sigma(i,j,k)}{\sigma(i,k)}$$

where $\sigma(i,j,k)$ is the number of shortest paths between i,k passing through j, and $\sigma(i,k)$ the total number of shortest paths between i,k. Betweenness scales with the number of pairs of vertices, so the betweenness of a vertex in an undirected graph can be normalised by dividing by $\frac{(N-1)(N-2)}{2}$, where N is the number of vertices in the component that the vertex is a part of. The betweenness centrality of vertex 3 in the graph below is 4, while the normalised betweenness centrality is 4/6.

This measure assumes that all relevant interactions follow a shortest path, which may not be the case [37]. This can be generalized by taking into account that shortest paths aren't the only important paths [8]. Information for example will spread through many paths, not only the shortest paths. An alternative notion of betweenness can be developed for such scenarios. In this case the betweenness of a vertex is the probability of it being visited by a given search algorithm. The betweenness of a vertex here is different to that calculated using only shortest paths.



2.2.6 Resilience

Resilience is the increase in the geodesic path length of a graph due to a random removal of a percentage of the graph's vertices. Different types of networks can display very different levels of resilience. Scale free networks (further discussed later in the chapter) are quite resilient, but if removal is targeted at high degree vertices they can quickly become disconnected. For example the increase in distance between vertices in the internet is almost entirely unaffected by random vertex removal. This is because many vertices have very low degree. However it is vulnerable to deliberate attack on its highest-degree vertices [5].

Networks such as the internet and power grids are also susceptible to cascading failure [44]. If a network is carrying a flow, vertices individually experience a load, and normally this load doesn't exceed the capacity of that vertex. If flows on the network change, or vertices are added or removed, dynamic adjustment of flows on individual vertices occurs automatically, keeping all vertices loaded below capacity. Cascading failures occur when a heavily loaded vertex is lost. The redistribution of flow may cause other vertices to exceed their capacity causing them also to fail. Hence the number of failed or stressed vertices increases, propagating throughout the network. The entire network may be affected. This can happen across power grids, when a small outage can spread across a large area [31].

2.3 Early Graph Models

The random graph model of Erdos and Renyi [22] is considered the most basic graph model in complex networks. However, it fails to describe many important properties of real world networks, so it was necessary to create new models. The small world model proposed by Watts and Strogatz [52] uses a simple idea to create graphs with a high clustering coefficient, and with a small diameter. Barabasi and Albert [9] showed that the degree distribution of many real systems follows a power law. They proposed a scale free model to explain this.

2.3.1 The Erdos Renyi Random Graph

The first probabilistic generative model was the Erdos Renyi random graph model, where each pair of vertices has an identical, independent probability p of being joined by an edge. k is the degree of a vertex and $\langle k \rangle$ is the mean degree of the vertices in the graph. The study of this basic model has led to a rich mathematical theory. In many ways it fails to match the properties of real world graphs [11], but it is still very important.

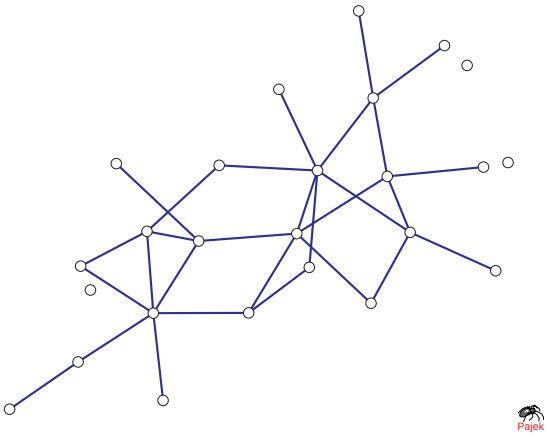


Figure 2.3: An Erdos Renyi Graph.

This model has a small expected geodesic path length, of order $ln\ N$, where N is the number of vertices in the graph. It does not demonstrate local clustering [11]. The expected clustering coefficient of an Erdos Renyi graph is $C =< k > /N$, since there are $pk(k-1)/2$ edges between the neighbours of a

vertex with degree k, out of a maximum possible number of $k(k-1)/2$ [11]. Hence, ER random graphs have a vanishing C for fixed $< k >$ in the limit of large system size. As $N \to \infty$, $< k >$ diverges if p is fixed, instead $p = \frac{< k >}{N-1}$ is used.

The probability of a random vertex having degree k is binomial,

$$P(k) = \binom{N-1}{k}(p^k)(1-p)^{N-1-k}$$

For a large number of vertices N and the average degree $< k >$ fixed, the degree distribution is approximately Poisson rather than a power law distribution.

$$P(k) \cong \frac{< k >^k \, e^{-<k>}}{k!}$$

There are some hubs in the Erdos Renyi model, but real world graphs have many more.

2.3.2 The Watts and Strogatz Small-World Model

Many real world networks display the small world property, there is a short path from any vertex to any other vertex in the graph. For example, in social networks where an edge represents friendship, there is a short path length from any person to any other person in the world, giving rise to the idea of 'six degrees of separation' [38]. Many different kinds of networks display small world properties, including road maps, electric power grids, metabolite processing networks, networks of brain neurons, and social networks [52]. In numerous real world networks there is also a higher level of clustering than would be found in a random graph. The Watts and Strogatz model tries to account for the small world property and clustering in as simple a way as possible. However the model produces graphs that are homogenous in degree so it doesn't have hubs or a scale free degree distribution.

To construct the Watts-Strogatz graph, take a regular ring lattice, a graph with N vertices each connected to 2κ neighbours, κ on each side, with $N \gg \kappa \gg log(N) \gg 1$. Each edge is rewired with probability β. Rewiring is done by replacing edge (N_i, N_j) with (N_i, N_k), where N_i, N_j, N_k are vertices in the graph, and k is chosen with uniform probability from all possible values that avoid loops and multiple edges. This introduces $\beta\kappa N$ rewired edges. When $\beta = 0$ we have the original lattice, with a lot of short cycles but a large

geodesic path length, and when $\beta \to 1$ the network becomes a random graph with a short geodesic path but short cycles. If β is between 0 and 1 both a short geodesic path and a high number of short cycles are present. In the figure below, $N = 15, \kappa = 2, \beta = 0.3$.

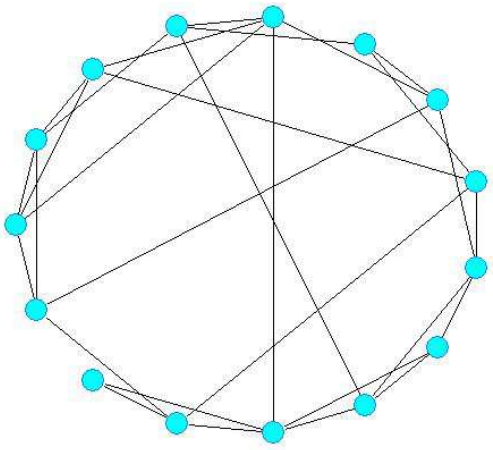


Figure 2.4: A Watts Strogatz Graph.

2.3.3 Scale Free Networks of Barabasi and Albert

There are many examples of real world networks where the structural changes are ruled by the dynamical evolution of the system, such as in a social network when new individuals join over time. Graph models were developed which attempted to reproduce the growth processes taking place in real networks. The Barabasi Albert model of network growth was one of the first which allowed the generated network to grow through the addition of vertices and edges at each time step. Many of their results are heuristic but have been made rigorous by Bollobas and Riordan [12]. The degree distribution of the network is scale free, mirroring many real world networks, such as the world wide web, citation networks and networks of Hollywood actors. The degree distribution $P(k)$, where k is the degree of a vertex, follows a power law

$$P(k) \sim k^{-\gamma}$$

Thus the model accounts for the formation of hubs in a network.

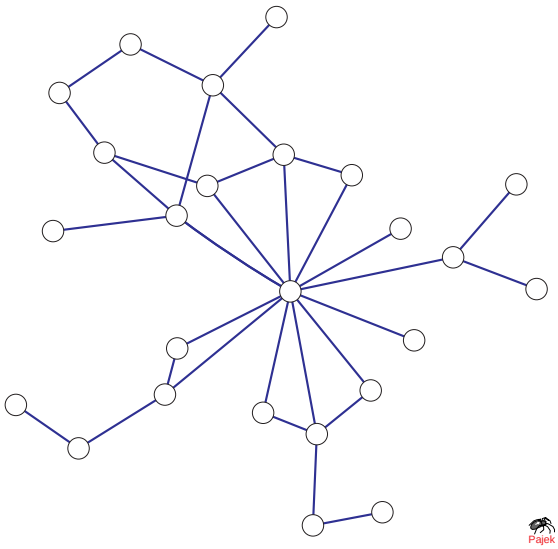


Figure 2.5: A Barabasi Albert Graph.

The graph in this model is generated using preferential attachment rules. This means that vertices of higher degree are more likely to receive new edges than vertices of lower degree. Unfortunately though, the preferential attachment model doesn't allow links to be added between old vertices.

To construct this network start with m_0 vertices, each with degree of at least one, and for each new vertex add $m \leq m_0$ edges. The probability of a vertex i receiving an edge from new vertex j is proportional to the degree of i.

$$P(j \to i) = \frac{k_i}{\sum_x k_x}$$

This network model has an interesting property, it isn't very vulnerable (in the sense that the geodesic path length doesn't increase much) to a random removal of a percentage of its vertices, but if the removal of vertices is targeted at the vertices of highest degree the network can become disconnected easily [5].

This model fails to account for the fact that in many real world networks some vertices that have been added at a late stage can become hubs with very high degree. The original preferential attachment model rules this possibility out. To remedy this problem, a parameter called fitness was introduced in [10], where vertices with a higher level of fitness are more likely to become hubs. Each new vertex i is given a fitness η_i, where η_i is chosen from some distribution $\rho(\eta)$. The probability Π of a vertex i receiving an edge from a new vertex j depends on the degree k_i and on the fitness η_i of i.

$$\Pi(j \to i) = \frac{\eta_i k_i}{\sum_x k_x \eta_x}$$

This allows late arriving vertices with high levels of fitness to become hubs.

2.4 Hierarchical structure

Many different kinds of networks exhibit hierarchical structure [48], [16], [30], in which vertices divide into groups and into groups of groups and so on. The existence of hierarchy can explain and be used to reproduce many important properties of networks, such as right-skewed degree distributions, high clustering coefficients and short geodesic path lengths. A knowledge of hierarchical structure can be used to predict missing links in partly known networks. It is useful to be able to do this as our knowledge of many networks is substantially incomplete and being able to predict links can greatly reduce the amount of data that must be collected before analysis can be done [21], [49].

Conventionally, hierarchical structure is represented by a tree, or dendrogram, in which closely related pairs of vertices have lowest common ancestors that are lower in the tree than those of more distantly related pairs. The probability of a connection between two vertices depends on their degree of relatedness [15]. Each internal vertex r of the dendrogram is given a probability p_r and each pair of vertices for which r is the lowest common ancestor is connected with probability p_r.

The lowest common ancestor of two vertices is defined as the lowest vertex in the hierarchical random graph that has both the vertices as descendants. The lowest common ancestor of vertices five and six in the graph below is vertex one. Communities at different levels of organization are disjoint. If

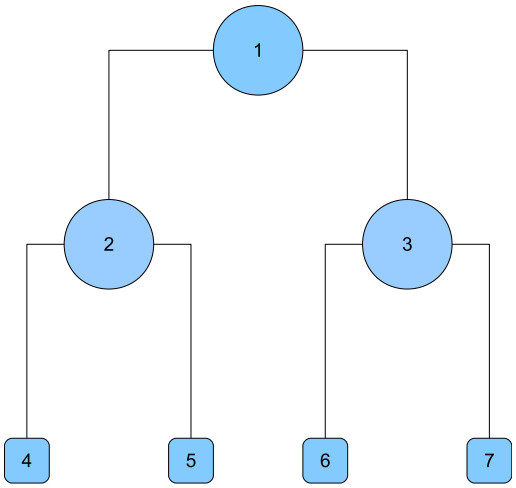


Figure 2.6: A Hierarchical Random Graph.

p_r decreases as we move further up the tree then the network generated will demonstrate assortative mixing, and vice versa. If p_r varies randomly throughout the tree then the network can capture both assortative and disassortative structure.

The trees created using real data can be used to generate new networks with the same hierarchical structure. These resampled networks have similar degree distributions, clustering coefficients, and distributions of shortest path lengths between pairs of vertices, despite the fact that none of these properties is explicitly represented in the hierarchical random graph. This set of new trees

can be used to create a consensus tree, which captures the topological features that appear consistently across all or a large fraction of the trees and typically is a better summary of the networks structure than any individual tree.

If this method for analyzing graphs is compared to other models for predicting missing links then it does better than most models for all networks other than assortative networks. For instance an algorithm based upon the shortest path will work better for assortative networks. For a predator prey network a shortest path algorithm predicts connections between predators that don't exist in reality. The hierarchical method thus makes accurate predictions for a wider range of network structures than previous methods, as it is capable of expressing both assortative and disassortative structure in the same model.

CHAPTER 3

Online Social Network Models

3.1 Introduction

Online Social Networks (OSNs) have existed since the internet was first created, for example a graph formed by people emailing each other forms an OSN. In recent times however, OSNs are used by many more people than ever before. The emergence of websites such as Flickr, Facebook, Youtube and Twitter etc have made it easier to study and analyze these networks, such as the work done in [29], [3], [39], in which the authors examined OSNs such as Facebook, Cyworld, Orkut, Youtube and Livejournal, and their properties, on a large scale. Various features that these networks have in common have been discovered, some of which are common to many different types of networks and some of which are unique to social networks. It has become possible to create models of these networks which reflect some of the observed properties, for example in [17], [32], [46].

It is useful to be able to model networks as we can discover which initial assumptions give rise to specific network properties and we can predict future network growth. It allows us to predict the flow of information and other resources through networks, and enables simulation on networks of arbitrary size. Many models have been proposed in recent years to create representations of OSNs that reflect specific properties of real OSNs. A model is chosen in a given case based upon which properties it is important to replicate. For instance, if one wishes to study local properties of networks then one would

choose a model which is particularly good at replicating local properties of real world networks.

Here we examine various proposed models, each of which goes about constructing a graph model for an OSN in a different way. We look at a community based graph model, a model created using Kronecker multiplication, and then graphs models that were recently suggested in a paper by Leskovic et al [35]. Finally we consider the Iterated Local Transitivity graph model of Bonato et al, which we will use in later chapters.

3.2 A Community based Model

3.2.1 Introduction

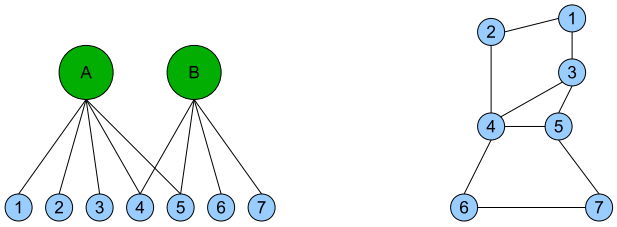
Social networks are divided up into communities or groups in the real world, based on distinctions such as locality, class, race, age etc. This is the basis of the model proposed in [45]. It captures the properties of clustering and positive assortative mixing well. Positive assortative mixing is a feature of social networks but is uncommon in other kinds of networks [43].

3.2.2 Method

The model is as follows:

- there are N individuals and M groups
- an individual can belong to more than one group
- if two individuals are in one group then there is a probability p that there is an edge between them
- r_m is the probability that an individual belongs to m groups
- s_n is the probability that a group contains n individuals

In the figure below there are two groups, A, B, and seven individuals. The left hand figure is a bipartite graph where an edge between an individual and a group represents membership. The actual graph generated using p with edges drawn in is shown on the right hand side.
Intuitively, individuals that are members of bigger groups have higher degree



than individuals that are members of smaller groups, and are likely to be linked to other individuals of high degree. This results in assortative mixing.

3.3 Kronecker Multiplication Model [34]

3.3.1 Introduction

The Kronecker product is a concept of matrix analysis [51]. It is an operation on two matrices of any size and results in a block matrix. The Kronecker product of an m by n matrix A and a p by q matrix B is

$$A \otimes B = \begin{pmatrix} a_{11}B & .. & a_{1n}B \\ .. & .. & .. \\ a_{m1}B & .. & a_{mn}B \end{pmatrix}$$

For example

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix}$$

The Kronecker product $G \otimes H$ of the adjacency matrices of two graphs G and H is a matrix such that edge $(x_{ij}, x_{kl}) \in V(G \otimes H)$ if and only if $(x_i, x_k) \in V(G)$ and $(x_j, x_l) \in V(H)$ where x_{ij} and x_{kl} are vertices in $V(G \otimes H)$, and x_i, x_j, x_k, x_l are the corresponding vertices in $V(G)$ and $V(H)$.

It can be used to generate adjacency matrices for undirected graphs that possess a number of important properties [34], listed below. Graphs generated in this way display many of the static graph properties of real world social networks, and some of the temporal evolution laws. In particular they:

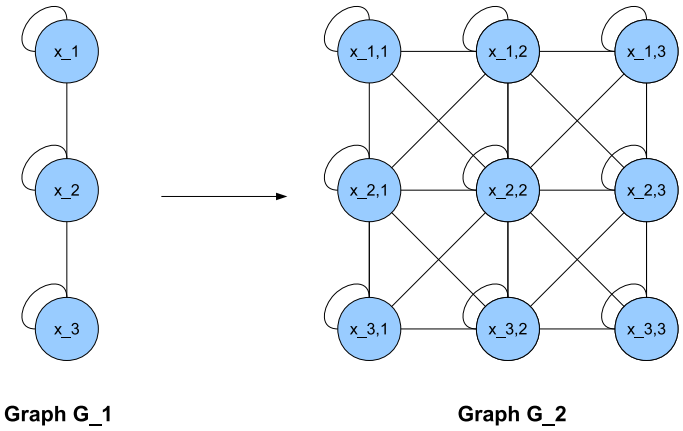
- have heavy tails for the degree distribution

- have heavy tails for the eigenvalues and eigenvectors
- have small diameters
- obey the densification power law

No other graph generator had succeeded in simultaneously capturing all these properties. This graph generator also leads to tractable analysis and rigorous proofs.

3.3.2 Method

We start with an initiator graph G_1, where N_1 is the initial number of vertices in the graph and E_1 is the initial number of edges. Self similar graphs are generated recursively, using multiple iterations of the Kronecker product. In all the graphs analysed here it is assumed that there exists a self loop on every vertex. This ensures that if G_1 is a connected graph then G_t is connected for all t.

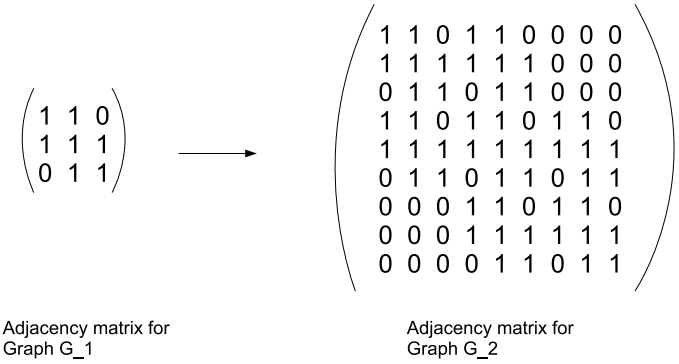


The k^{th} Kronecker power of G_1 is defined as

$$G_1^{[k]} = G_k = G_1 \otimes G_1 .. \otimes G_1 = G_{k-1} \otimes G_1$$

Thus a graph G_k has $N_k = N_1^k$ vertices, and $E_k = E_1^k$ edges. For example in the graph above, $G_2 = G_1 \otimes G_1$, $E_1 = 7$, $N_1 = 3$, $E_2 = 49$ and $N_2 = 9$

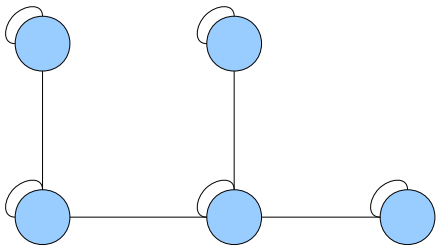
The adjacency matrices of the above graphs are provided below.



3.3.3 Properties of Kronecker Graphs

Kronecker graphs have multinomial degree distributions

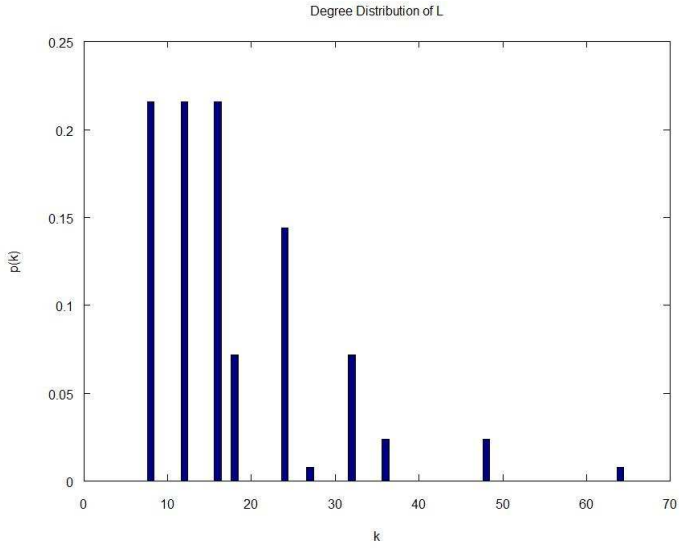
Suppose a graph G_1 has the degree sequence $d_1, d_2..d_{N_1}$. Taking $G_1 \otimes G_1$ a vertex of degree d is expanded into N_1 vertices of degrees $d \times d_1, ..d \times d_{N_1}$. So the degree of a vertex in the graph G_k is of the form $d_{i_1} \times d_{i_2} \times ...d_{i_k}$, where $i_1, ..i_k \in (1, ..N_1)$. There is one vertex of each ordered combination, giving a multinomial distribution on the degrees of G_k. Consider a graph L. The



degree distribution of $L \otimes L \otimes L = L_3$ is given below.

The Kronecker graph G_k has a multinomial distribution for its eigenvalues, and the components of each eigenvector of the Kronecker graph G_k follow a multinomial distribution.

If G_1 has eigenvalues $\lambda_1, \lambda_2, ..\lambda_{N_1}$, by the properties of Kronecker multiplication [51] the eigenvalues of G_k are the k^{th} Kronecker power of the vector $(\lambda_1, \lambda_2, ..\lambda_{N_1})$. The eigenvalue distribution is a multinomial. Similarly the components of each eigenvector of the Kronecker graph G_k follow a multinomial



distribution.

Kronecker graphs follow the Densification Power Law (DPL) with densification exponent $a = log(E_1)/log(N_1)$**.**
G_k has $N_k = N_1^k$ vertices and $E_k = E_1^k$ edges, so $E_k = N_k^a$ where $a = log(E_1)/log(N_1)$. a is independent of k.

If G_1 **has diameter** d **and a self-loop on every vertex, then for every** k**, the graph** G_k **also has diameter** d**, so the diameter is constant.**

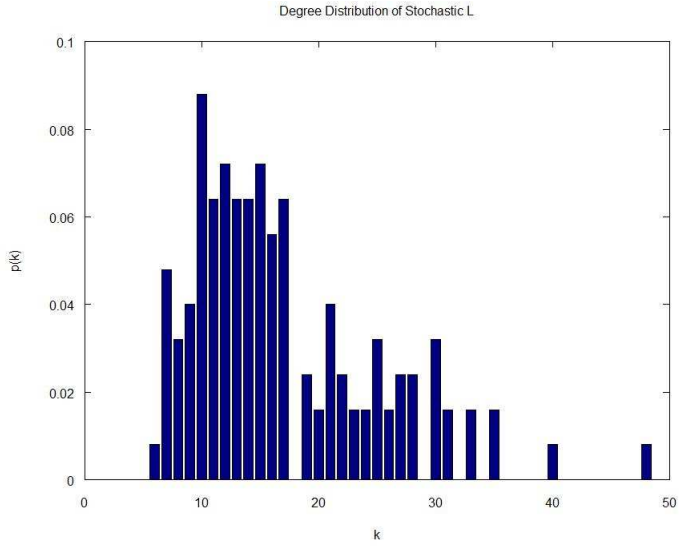
The q-effective diameter d_e is defined as the minimum d_e such that, for at least some fraction or quantile q of the pairs of vertices in the graph that are part of the same component, the path length is at most d_e. It is a more robust quantity than the diameter.
If G_1 **has diameter** d **and a self-loop on every vertex, then for every q, the q-effective diameter of** G_k **converges to** d **(from below) as** k **increases.**

3.3.4 Stochastic Kronecker Graphs

The nature of Kronecker powering produces staircase effects in the degrees and spectral quantities, as can be seen in the degree distribution of the graph L_3 above, because individual values appear multiple times. Introducing a stochastic element produces smoother degree and eigenvalue distributions,

which better reflect real world graphs. This can be done as follows. P_1 is an $N_1 \times N_1$ matrix such that p_{ij} is the probability that edge (i, j) is present. The k^{th} Kronecker power $P_1^{[k]} = P_k$ is computed, and then for each entry p_{uv} of P_k an edge between vertices u and v is included with probability $p_{u,v}$.

Two parameters α, β are used. P_1 is created by replacing $0, 1$ in the adjacency matrix of G_1 with β and $\alpha, (\beta \leq \alpha)$ respectively. The resulting probability matrices maintain with some random noise the self-similar structure of the Kronecker graphs. The degree distribution of a stochastic version of L_3, with $\alpha = 0.9,\ \beta = 0.1$ is given below. It is evidentally a lot smoother than the previous degree distribution.



For small values of α and β, stochastic Kronecker graphs have many small disconnected components; for large values they have a giant component with small diameter. In between, they exhibit behavior suggestive of a phase transition. For a carefully chosen set of (α, β), the diameter is large, and a giant component just starts emerging [34].

3.4 Graph Evolution Models of Leskovec et al

A further class of models, introduced in [36], are motivated by a desire to find some local model of behaviour which naturally leads to the macroscopic densification power laws and shrinking effective diameter. The graphs here

are directed, in contrast to the Kronecker graph model. The densification power law is obeyed in the Community Guided Attachment (CGA) models, while the Forest Fire model also exhibits shrinking diameter and heavy tailed in and out degree distributions.

3.4.1 Community Guided Attachment (CGA) model

It is desirable to create a model in which the densification exponent arises indirectly from the construction of the graph model, rather than directly defining the exponent. This allows insight to be gained into the origin of the densification exponent.

Power laws often appear in combination with self-similar datasets [1], so this model's approach involves two steps based on self similarity. The first step is that close (in terms of distance, relation or type etc) vertices are more likely to be linked than vertices that are not close. The second is that there is a numerical measure of the difficulty in forming links across communities, quantified by the difficulty constant c which is explained below.

A tree Γ of height H is constructed of constant fanout b. The fanout b is the number of branches that each of the tree leaves produce at each step. A tree leaf is a vertex of degree one. The root vertex is the vertex with no incoming links, and the height is the length of the longest downward path to a leaf from the root vertex. The value of the fanout of the tree below is two and the height H is three. The vertices V in the graph G are the leaves of the tree, where

$$n = |V| = b^H$$

and are green in the Figure 3.1. Let $h(v, w)$ define the standard tree distance between two leaves v, w. This is the height of their closest common ancestor. The function $f(h)$ gives the probability of there being an edge between v and w, where h is the height of their closest common ancestor. It is desirable for f to be scale free, ie $f(h)/f(h-1)$ is constant, so define $f(h) = f(0)c^{-h}$. $f(0)$ is set to one, yielding

$$f(h) = c^{-h}$$

where $c \geq 1$. So as c increases, cross community links become harder to form. This model obeys the densification power law, and has exponent $a = 2 - log_b(c)$, when $c < b$.

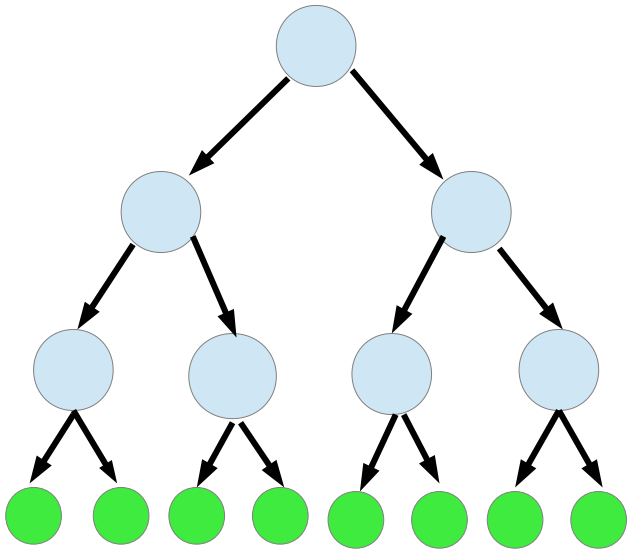


Figure 3.1:

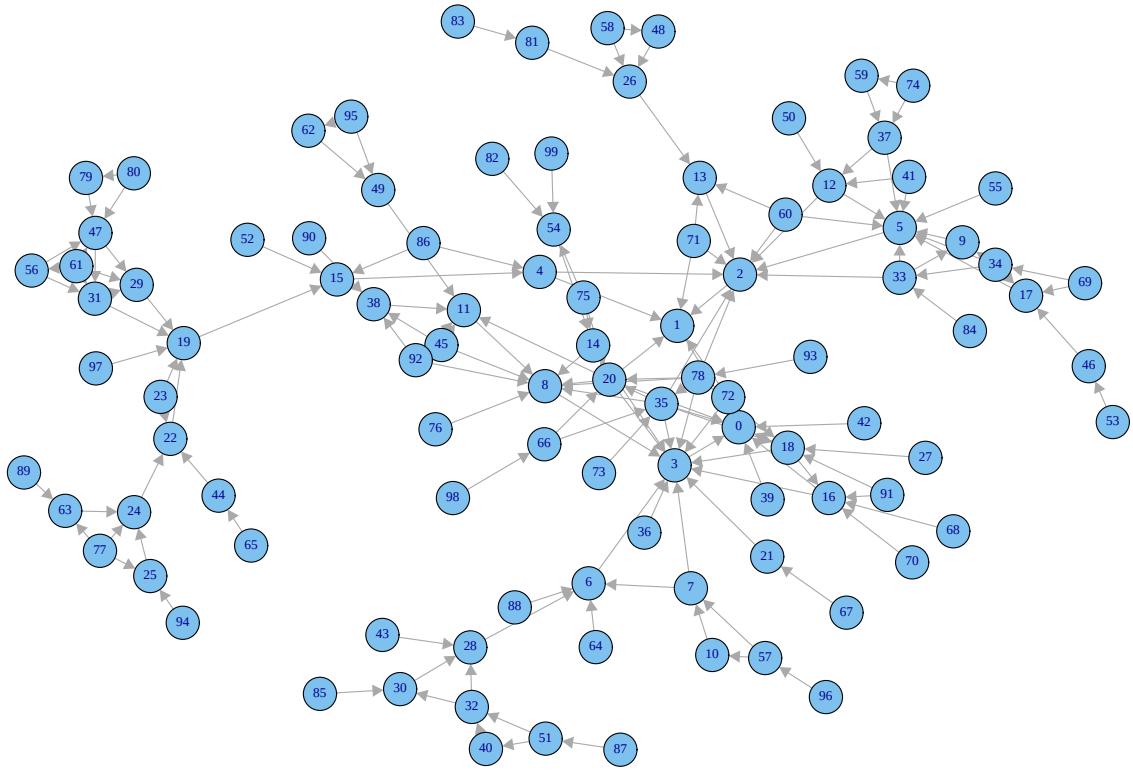
3.4.2 Dynamic Community Guided Attachment model

It is advantageous to create models which are time evolving, as they generally capture more properties of real networks than static graph models, since most real world networks change over time. Let $d(v, w)$ be the geodesic distance between v and w in Γ, considering Γ as an undirected graph, that is the length of the path from v up to the least common ancestor of v and w, plus the length of the path from this least common ancestor down to w. Vertices are added to the tree one at a time. A new vertex v creates an outlink to vertex w with probability $c^{-d(v,w)/2}$. The vertices of the graph model are now not only the leaves of the tree Γ but also the internal vertices of Γ. In timestep t the tree grows from height $t-1$ to height t, adding b new leaves to each current leaf. The tree distance between vertices v and w of Γ is defined as the length of a path between them in Γ. Similarly to the previous model this model obeys the densification power law and has exponent $a = 2 - log_b(c)$, when $c < b$. For $c < b^2$ it also yields a heavy tailed degree distribution, unlike the basic model. This model, unlike the previous one, produces a graph with a heavy tailed distributions of in-degrees.

3.4.3 Forest Fire Model

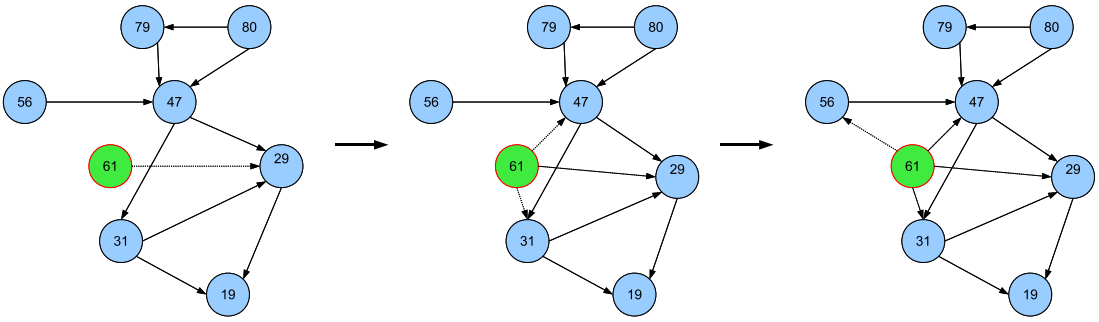
The preceding models do not capture the properties of shrinking effective diameters and heavy tailed out degree distributions that have been observed in real graphs. The forest fire model succeeds in doing so, as well as having heavy tailed in-degrees, a community structure, and obeying the densification power law. In this model, vertices arrive individually over time.

Given an initial graph G, a new vertex v initially links to a vertex w, chosen uniformly at random from the existing graph, and then begins creating (burning) new outlinks. Random numbers x, y are generated, which are geometrically distributed with means $p/(1-p)$ and $rp/(1-rp)$ respectively. p is the forward burning probability, and r the backward burning ratio. v selects x outlinks and y inlinks of w. $w_1, w_2...w_{x+y}$ denotes the other ends of the selected links. v forms outlinks to $w_1, w_2...w_{x+y}$, and then applies the previous step recursively to each of $w_1, w_2...w_{x+y}$. Vertices cannot be visited twice.



The graph G above was created using the forest fire model, with parameters $p = r = 0.25$. An important feature of this model is that some vertices burn a large number of edges and so form a large number of outlinks. Dense or sparse

matrices can be generated depending on the forward and backward burning parameters. In the subgraph below of graph G, vertex 61 initially links to vertex 29. It links to vertices 47 and 31 at the next step and to 56 at the final step.



3.5 Iterated Local Transitivity (ILT) model

3.5.1 Introduction

We now examine the OSN model which we shall study in detail in later chapters. A model [13] that we use to generate graphs is examined below. Bonato et al developed a deterministic model for online social networks based on transitivity and local knowledge in social interactions. The ILT provably satisfies a number of both global and local properties, namely:

- the densification power law
- decreasing average distance
- higher clustering than in random graphs with same average degree
- constant diameter

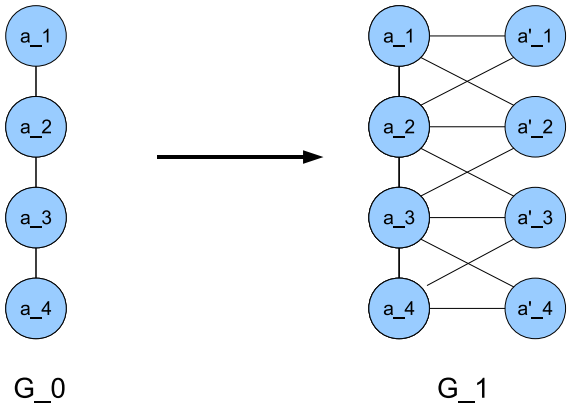
Given an initial graph as a starting point, vertices are repeatedly added over time which clone each existing vertex, so that the new vertices form an independent set. (An *independent set* is a set of vertices in a graph, no two of which are connected.) At each time step, and for every vertex v, a new vertex appears, and an edge is added between it and every neighbour of v. The model incorporates local knowledge only in its growth, since each vertex joins only to neighbours of an existing vertex.

3.5.2 Method

The ILT model generates simple, finite, undirected graphs $(G_t : t \geq 0)$. The only parameter is the fixed, finite, connected graph G_0. To form G_{t+1}, for each vertex $v \in V(G_t)$ add a clone v', which is joined to v and all the neighbours of v at time t. The set of new vertices at time $t+1$ form an independent set of cardinality $|V(G_t)|$. Let $deg_t(v)$ be the degree of a vertex at time t, the number of vertices of G_t be n_t, and the number of edges be e_t. Then $n_t = 2^t n_0$, and

$$deg_{t+1}(v) = 2deg_t(v) + 1$$
$$deg_{t+1}(v') = deg_t(v) + 1$$

The graph below shows the evolution of a graph G_0. Vertices $a_1, a_2, a_3, a_4 \in V(G_0)$ and $a'_1, a'_2, a'_3, a'_4 \in V(G_1) \backslash V(G_0)$. a'_1 is the clone of a_1 and so on.



3.5.3 Main Results

The key properties of the ILT model were considered in [13]. We recall some of the most relevant of these here. The ILT model displays properties such as higher clustering than in random graphs with the same average degree, and smaller spectral gaps for both their normalized Laplacian and adjacency matrices than in random graphs. The volume of G_t is defined as

$$vol(G_t) = \sum_{x \in V(G_t)} deg_t(x) = 2e_t$$

Theorem 3.5.1. *For $t > 0$ the average degree of G_t is*

$$\frac{e_t}{n_t} = \left(\frac{3}{2}\right)^t \left(\frac{vol(G_0)}{n_0} + 2\right) - 2$$

This shows that the ILT model obeys the densification power law $e_t \propto n_t^a$, with exponent $a = \frac{log3}{log2}$. This is a reasonable figure for the exponent in comparison with real world graphs [36].

The Wiener index $W(G_t)$ of G_t is defined as follows:

$$W(G_t) = \frac{1}{2} \sum_{x,y \in V(G_t)} d(x,y)$$

The average distance of G_t is:

$$L(G_t) = \frac{W(G_t)}{\binom{n_t}{2}}$$

The ultimate average distance of G_0 is:

$$UL(G_0) = \lim_{t \to \infty} L(G_t)$$

assuming the limit exists. This is a new graph parameter, introduced in [13].

The following lemma is important when considering the diameter of a graph and the distance between vertices as the graph evolves. The distance in G_{t+1} between some $v \in G_t$ and $w' \in G_{t+1}$, a clone of some $w \in G_t$, is the same as the distance between v and w in G_t. The distance between v and w is unchanged.

Lemma 3.5.1. *Let x, y be vertices in G_t with $t > 0$. Then*

$$d_{t+1}(x', y) = d_{t+1}(x, y') = d_{t+1}(x, y) = d_t(x, y)$$

and

$$d_{t+1}(x', y') = \begin{cases} d_t(x,y) \; if \; xy \notin E(G_t) \\ d_t(x,y) + 1 = 2 \; if \; xy \in E(G_t) \end{cases}$$

Theorem 3.5.2. *(Average Distance)*
For $t > 0$,

$$W(G_t) = 4^t(W(G_0) + (e_0 + n_0)(1 - (3/4)^t))$$

For $t > 0$,

$$L(G_t) = \frac{4^t(W(G_0) + (e_0 + n_0)(1 - (3/4)^t))}{4^t n_0^2 - 2^t n_0}$$

For all graphs G_0,

$$UL(G_0) = \frac{W(G_0) + e_0 + n_0}{n_0^2}$$

Further $UL(G_0) \leq L(G_0)$ *iff* $W(G_0) \geq (n_0 - 1)(e_0 + n_0)$.

The average distance of G_t is bounded above by $diam(G_0) - 1$, and the condition above for $UL(G_0) \leq L(G_0)$ holds for large cycles and paths, thus for many initial graphs G_0 the average distance decreases at each step, a property which has been observed in OSNs and other complex networks.

If $\tilde{C}(G_t) = \frac{1}{N}\sum_i C_i$ is the network clustering coefficient of graph G_t, then the following theorem holds.

Theorem 3.5.3.

$$\tilde{C}(G_t) = O\left(\left(\frac{7}{8}\right)^t t^2\right)$$

$\tilde{C}(G_t)$ tends to zero much faster in a random graph than for the ILT model. This is important since in many real world networks and in particular in social networks, vertices tend to create tightly knit groups characterised by a relatively high density of ties.

The normalized Laplacian of a graph [10], relates to important graph properties. Let A denote the adjacency matrix of a graph G such that no vertices in the graph are isolated, and let D denote the diagonal adjacency matrix of G. Then the normalized Laplacian of G is

$$L = I - D^{-1/2}AD^{-1/2}$$

Let

$$0 = \lambda_0 \leq \lambda_1 ... \leq \lambda_{n-1} \leq 2$$

denote the eigenvalues of L. The spectral gap of the normalized Laplacian is

$$\lambda = max\{|\lambda_1 - 1|, |\lambda_{n-1} - 1|\}$$

Theorem 3.5.4. *For* $t \geq 1,\ \lambda(G_t) > \frac{1}{2}$

This contrasts markedly with the expansion properties of random graphs and the preferential attachment model. Social networks organize into communities, and as a result, their adjacency matrices possess bad expansion properties realized by small gaps between their first and second eigenvalues [17].

The spectral properties of the adjacency matrices of the ILT model are characterised in the following result. Let $\rho_0(t) \geq |\rho_1(t)| \geq ..$ denote the eigenvalues of the adjacency matrix of G_t.

Theorem 3.5.5. *The eigenvalues of the adjacency matrix of G_{t+1} are:*

$$\frac{\rho \pm \sqrt{(\rho^2 + 4(\rho+1)^2)}}{2}$$

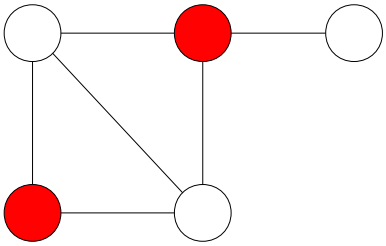
for every ρ which is an eigenvalue of G_t.

3.5.4 Stochastic Version of the ILT model

When new individuals join a real world social network, they may already be friends. This point is taken into account in the stochastic version of the ILT model, ILT(p). New vertices added to the graph G_t are linked to each other with independent probability $p(k)$, where k is the number of vertices added at time $t+1$. This model retains many properties of the original ILT model. The average distance may only decrease, and the clustering coefficient may only increase. ILT(p) generates graphs following a densification power law with exponent $log(3+\delta)/log\ 2$ where $0 \leq \delta \leq 1$, depending on the value of p.

3.5.5 Domination number

An interesting property of the ILT model is that the domination number of the graphs G_t for $t > 0$ remain the same as for the graph G_0. A dominating set for a graph $G = (V, E)$ is a subset D of V : every vertex not in D is joined to at least one member of D by some edge. The *domination number* is the number of vertices in a smallest dominating set. In the graph below an example of a dominating set is the set of the red vertices.



CHAPTER 4

Games on Networks

4.1 Introduction

The study of games is important in economics, computer science, the social sciences, and in studying both human and animal behaviour. It is used in economics and business to model interacting agents. This can be of particular interest in areas such as viral marketing and advertising, which are related to areas such as epidemic spreading [27], [20]. The recent large increase in interest in the study of social networks makes the study of games on these networks of more importance than ever. Much of human behaviour can be viewed as a game in which a person's utility is determined by the behaviour of those who are in some sense close to them, such as friends, associates, or trading partners [2], [14].

We examine various models for games on networks below. In each game there is a set of players (or agents) who are trying to maximize their individual utilities (or payoffs). The utility of a player is defined in a different way for each game. For some games there is a small set of players who choose a starting vertex on a graph in such a way as to maximize the number of other vertices which they capture, which is in this case their utility. In other games every vertex in the graph is a player that has to choose between two options, to adopt or not to adopt an idea (or innovation/rumour). The utility of a player here is calculated based on the decisions the player and each of its neighbours makes.

A player's strategy refers to the decision that a player makes in a game, for example for some games it is which vertex to initially select. The best response of a player is the decision the player can make which maximizes its utility, given the other players' decisions. A strategy combination is a set of strategies for each player which fully specifies the decisions each player will make given the other players' decisions. It gives the best response of each player to the other players' decisions. We will refer to a strategy profile here as the set of decisions which the players have made at a given point in the game.

A central theme in the study of games is the existence of Nash equilibria. A pure Nash Equilibrium occurs if no player can increase its utility by unilaterally changing its decision. From here on pure Nash Equilibria will be referred to as Nash Equilibria. There may be multiple Nash equilibria for a single game, and some games may have no Nash equilibria. A Nash equilibrium does not necessarily mean maximising the sum of the utilities for all the players involved; in many cases, all the players might improve their utilities if they were allowed to cooperate.

We will use the stag hunt game to illustrate the above concepts. This is a coordination game with two Nash Equilibria. There are two players in the game, and each player has two choices, hunt a stag or hunt a hare. In order to receive a payoff for hunting a stag a player must have the other player's cooperation, but the payoff if both players hunt a stag is substantially greater than for a hare. The two Nash Equilibria occur when both players choose to hunt a stag or both players choose to hunt a hare. The utility matrix for this situation is given below. Thus for player one, the strategy combination is as

	Stag	Hare
Stag	4,4	0,1
Hare	1,0	1,1

follows. If player two plays hare then player one will play hare, and if player two plays stag, player one will do likewise.

4.2 Diffusion and cascading behaviour in random networks

4.2.1 Introduction

Our work is motivated by the problem of information diffusion on social networks. We now discuss a number of game theoretic models of this phenomenon. A simple model of diffusion on networks is presented by Lelarge [33]. All the models in this section are from this paper. The results here are restricted to deterministic infinite graphs. In this case every vertex in the graph is a player. The player's behaviour here is a result of strategic choice. The players play a co-ordination game with binary choice. The players decide which state to be in based upon their neighbours' decisions. For example the different states could represent subscription to differing mobile phone networks, in which there are benefits for users subscribing to the same network. This model has many applications, particularly in word of mouth advertising.

4.2.2 The model

Each vertex chooses to either be in state A or state B. If there is an edge between vertices i, j then:

- if i and j are in state A then both vertices receive a payoff of q
- if i and j are in state B then both vertices receive a payoff of $1 - q$
- if i and j are in different states then they both receive no payoff

The total payoff that each vertex receives is the sum of the payoffs it receives from each neighbour. Let N_i^A, N_i^B be the total number of neighbours of i that choose A and B respectively, and d_i be the degree of i.

- The total payoff that i receives if i is in state A is $q(d_i - N_i^B)$
- The total payoff that i receives if i is in state B is $(1 - q)N_i^B$

So if $N_i^B > qd_i$ the payoff for choosing state B is higher, thus vertex i will choose state B, otherwise it will choose state A. Lelarge examines the case of diffusion in the large population limit when the underlying graph is a random network $G(n, d)$ with n vertices and where d is a given degree sequence. Clearly the cases where all vertices play either A or B are equilibria of the network.

4.2.3 Permanent adoption model

In this model [33] all vertices start in state A, and then a small number are forced to adopt state B. Vertices that adopt state B at the start are forced to play it forever. All vertices other than this initial seed now apply a best response update. This is called the permanent adoption model. It is useful in describing a situation in which an idea/product/opinion is spread through a network starting from an initially small group of people.

For a graph $G = (V, E)$ and a parameter q, the largest connected component of the subgraph in which all vertices have degree less than q^{-1} is considered. The vertices in this component are called pivotal players: if only one pivotal player switches from A to B then the whole set of pivotal players will eventually switch to B in the permanent adoption model. For a player $v \in V$, we denote by $C(v, q)$ the final number of players playing B in the permanent adoption model with parameter q, when the initial state consists of only v playing B, all other players playing A. Informally, $C(v, q)$ is the size of the cascade induced when only player v starts in state B initially.

If B can spread to the whole population from a finite set of initial adopters then a contagion is said to occur. There is a contagion threshold such that contagion occurs if and only if q is less than the contagion threshold [42]. This threshold is always at most $1/2$. The lower q is the easier the diffusion spreads.

If q is sufficiently low, there are two thresholds for the average degree λ, $1 < \lambda_i(q) < \lambda_s(q)$ such that a global cascade for a fixed q is only possible for $\lambda \in (\lambda_i(q); \lambda_s(q))$. For example for Erdos Renyi graphs

- If λ is close to 1 the diffusion doesn't branch much and progresses along paths, until it is stopped by a high degree vertex.
- If $\lambda > \lambda_s(q)$ there are too many high degree vertices, which stop the diffusion process. These vertices are locally stable, in that they have

many neighbours, so a high proportion of these neighbours need to have adopted B in order for them to adopt it.

Connectivity plays an ambiguous role, it allows for spread of A and B but diffusion is limited by high degree vertices which are very stable.

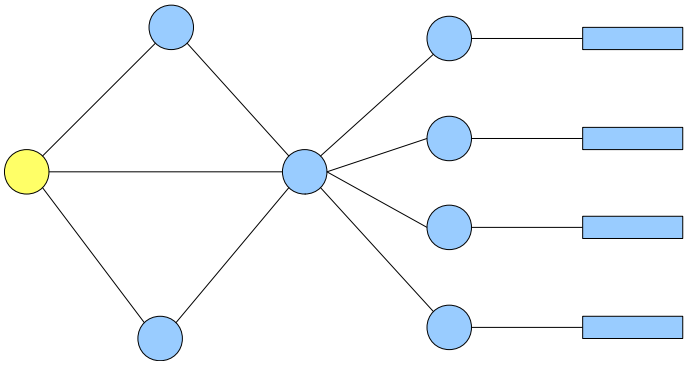
This also explains why social networks can display a great stability in the presence of continual small shocks that are as large as the shocks that ultimately generate a global cascade. Cascades are an example of the robust yet fragile nature of many complex systems

4.2.4 Nonmonotonic Model

Another possible model [33], the nonmonotonic model, is one in which the initial vertices that are forced to play B also apply the best response update after time $t = 0$. In this model, if the dynamic converges, the final state will be an equilibrium of the game. An equilibrium of the game is a fixed point of the best response dynamics. For example, the states in which all players play A or all players play B are trivial equilibria of the game. In contrast the permanent adoption model does not necessarily lead to an equilibrium of the game as the initial vertices forced to play B don't apply the best response update.

The yellow player in the graph below, where the rectangles represent a long line of players, induces a global cascade in the permanent adoption model but not in the nonmonotonic model, for $q = 1/3$. In the nonmonotonic model the graph will oscillate from the state where only the yellow player plays B and the state where only the yellow players' neighbours play B. If a player induces a global cascade for the nonmonotonic model, it will also induce a global cascade in the permanent adoption model, but as illustrated by the example, the converse is not true in general.

Thus a pivotal player doesn't necessarily induce a global cascade in the nonmonotonic model. If there exists a finite set of initial adopters causing a complete cascade for the permanent adoption model, it is also possible to find another set of initial adopters leading to a complete cascade for the nonmonotonic model [42]. Hence the contagion threshold is the same for both models. If two pivotal players that are neighbours are switched from A to B then the



whole set of pivotal players will eventually switch to B in the nonmonotonic model. So the contagion threshold is the same in both models, and both models will have exactly the same dynamics if started with the set of pivotal players playing B and all other players playing A.

4.3 Voronoi Games

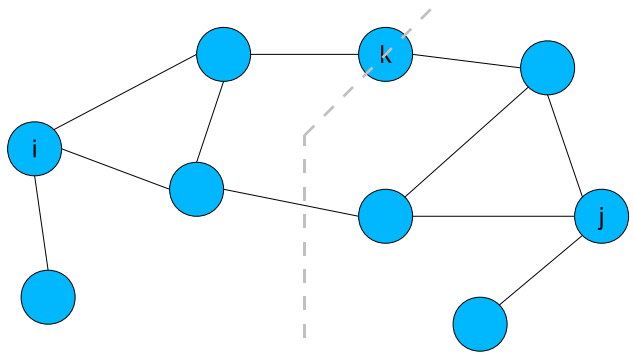
The Voronoi game is quite similar to the competitive game of Alon et al [7] which is a major focus of our later discussion. There are κ players, each of which selects a vertex on the graph. A strategy profile is the set of vertices selected by each player. The Voronoi game is represented as the pair $\langle G, \kappa \rangle$, where $[1, \kappa] = \{1, .., \kappa\}$ is the set of players, and $\kappa \geq 2$. The utility of a player i in this case measures the size of its Voronoi cell. The Voronoi cell of player i if strategy profile $\mathbf{s}$ is used is the set:

$$Vor_i(s) = \{v \in V(G) | d(s_i, v) \leq d(s_{i'}, v) \; for \; each \; player \; i' \in [\kappa]\}$$

If a vertex k is at equal distance to j closest players then it is called a boundary vertex. Each of the closest players receives a payoff of $\frac{1}{j}$ from k. Clearly the total payoff of all the players in the graph remains constant, $|V(G)|$.

In the graph below, if the strategy profile $\mathbf{x} = (i, j)$ is used then $U_1(\mathbf{x}) = U_2(\mathbf{x}) = 4.5$. The dashed line represents the boundary of the two Voronoi cells, and k is a boundary vertex. In a Nash Equilibrium no player can unilaterally increase its utility by changing to another vertex. In Voronoi games two players are permitted to choose the same initial vertex. The multiplicity of a vertex v using strategy profile $\mathbf{s}$ is:

$$\mu_v(\mathbf{s}) = |\{i' \in [\kappa] | v \in Vor_{i'}(\mathbf{s})\}|$$



The utility of player i using strategy profile $\mathbf{s}$ is:

$$U_i(\mathbf{s}) = \sum_{v \in Vor_i(\mathbf{s})} \frac{1}{\mu_v(\mathbf{s})}$$

4.3.1 Nash Equilibria for Voronoi Games

It is difficult to establish whether or not Nash Equilibria exist on a given graph, as in general every possible strategy profile must be checked to see if any player can benefit from deviating from it. Even if the locations of the players are given, enumeration of the Voronoi cells is a combinatorial bottleneck. In light of this Feldmann et al [24] attempted to find graphs in which the existence of Nash Equilibria was dependent on the properties of the graph and explicit calculation of the utilities of each strategy profile was unnecessary.

4.3.2 Relevant definitions

The pair of vertices u, v are said to be antipodal if $d(u, v) = \text{diam}(G)$. For some $V' \subseteq V(G)$

$$\Omega(V') = \{u \in V(G) | d(u, v) \textit{ is the same for all vertices } v \in V'\}$$

Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there is a bijection $\varphi : V \to V'$ such that for each pair of vertices $u, v \in V, \{u, v\} \in E$ if and only if $\{\varphi(u), \varphi(v)\} \in E'$; so φ preserves both edges and non-edges. The bijection φ is called an isomorphism from G to G'. An automorphism of G is an isomorphism from G to itself.

The graph G is vertex transitive if for each pair of vertices $u, v \in V$ there exists an automorphism ϕ of G such that $\phi(u) = v$. The graph G is generously vertex

transitive if for each pair of vertices $u, v \in V$ there exists an automorphism ϕ of G such that $\phi(u) = v$ and $\phi(v) = u$. Informally if G is generously vertex transitive each pair of vertices can be swapped. Any complete graph is generously vertex transitive.

A graph G is friendly if G is generously vertex transitive, and if for any pair of antipodal vertices $\alpha, \beta \in V(G)$, and for any arbitrary vertex $\gamma \in V(G)$, γ is on a shortest path between α and β.

4.3.3 Results of Feldmann et al

For the case where the number of players $\kappa = 2$ if a graph is generously vertex transitive then every strategy profile is a Nash Equilibrium. If G is vertex transitive with $\kappa = 2$ and there exist vertices α, β such that $U_1(\langle \alpha, \beta \rangle) \neq U_2(\langle \alpha, \beta \rangle)$ then the Voronoi game has no Nash Equilibrium.

Given a game on a friendly graph G with $\kappa = 4$ then for any arbitrary antipodal pair α, β the profile $\mathbf{s} = (\alpha, \alpha, \beta, \beta)$ is a Nash Equilibrium. If two players are not allowed to occupy the same vertex then for a bipartite friendly graph G with $\kappa = 4$ the game has a Nash Equilibrium at $\mathbf{s} = (\alpha, \beta, \delta, \gamma)$ where α, β and γ, δ are two distinct antipodal pairs. For the case of $\kappa = 3$ Feldmann et al were unable to bypass explicit enumeration of the Voronoi cells.

The class of graphs for which Feldmann et al's results hold is very restrictive, and illustrates the considerable difficulty of finding Nash Equilibria and characterising Nash Equilibria for games on networks.

4.4 Games on Networks [18]

4.4.1 Introduction

Diffusion of innovation can be examined from a game theoretic perspective on strategic decisions made by individuals in a social context. For example a mobile phone user's choice of whether to adopt an unlimited text messaging plan or to adopt a pay per message plan is influenced by their friends' decisions. Choosing a particular strategy in this case becomes more attractive as more friends choose it. This is an example of a symmetric two player base game on a social network. The general question of whether or not properties, such as the existence of pure Nash Equilibria, of the base game carry over to the networked

game has been examined in [18]. Multiplayer games on social networks directly correspond to the model used here, and have proven to be models of many natural scenarios involving strategic interaction.

4.4.2 The model

The game M played between two players is a symmetric co-ordination game, that is one in which a player only receives a positive payoff if they make the same selection as the other player. Both players receive identical payoffs. The players share a strategy set $S = \{s_1, .., s_k\}$. Both players receive payoffs of v_i if both choose $s_i,\ i \in \{1, .., k\}$, and both receive a payoff of zero if they choose differing strategies. This game is referred to as the k-strategy co-ordination game with payoffs $\{v_1, .., v_k\}$. The strategies are ordered so that $v_1 \leq v_2 .. \leq v_k$.

Every vertex in a graph G is a player. Each player chooses a strategy from S and simultaneously plays that strategy against each of its neighbours in G. $\mathbf{s} = (s_1, .., s_n)$ is a strategy profile of the choice each player has made. Player i receives as its payoff $p_i(\mathbf{s})$ the sum of the payoffs from playing the game M with each neighbour. The resulting game is denoted $M \oplus G$. The set of players is V, the set of vertices in G. For example if G is a two vertex graph with one edge $M \oplus G$ is isomorphic to M.

A strategy profile $\mathbf{s}$ is a Nash Equilibrium if no player can unilaterally deviate from $\mathbf{s}$ to improve their payoff. The social welfare of $\mathbf{s}$ is $\sum p_i(\mathbf{s})$. The social optimum OPT is the strategy profile $\mathbf{s}$ that maximizes $\sum p_i(\mathbf{s})$. The 'worst' N.E. WNE is the Nash Equilibrium that minimizes $\sum p_i(\mathbf{s})$. The 'best' Nash Equilibrium BNE is the Nash Equilibrium that maximizes $\sum p_i(\mathbf{s})$. The price of anarchy POA is OPT/WNE. The price of stability POS is OPT/BNE. $\Gamma_G(u)$ is the set of neighbours of u.

4.4.3 Best Response Dynamics (BRD)

Best Response Dynamics (BRD) is an algorithm that produces a sequence of strategy profiles by repeatedly allowing a player i to update their strategy to a strategy in S_i which maximizes $p_i(\mathbf{s})$, holding all other strategies constant. If BRD terminates then it terminates at a Nash Equilibrium, however it may not terminate even if a Nash Equilibrium exists. BRD is said to always converge if this process terminates regardless of the initial strategy profile s.

4.4.4 Observations made by Davis et al

There are few positive results regarding the existence of Nash Equilibria and BRD always converging, essentially nothing can be inferred in either direction about the existence of Nash equilibria in M and the existence of pure Nash equilibria in $M \oplus G$. The limited exception arises in the case of a bipartite graph B.

Let B be an arbitrary bipartite graph.

- If a base game M has a Nash Equilibrium, then the game $M \oplus B$ has a Nash Equilibrium.
- If a base game M has no Nash Equilibria, then there may or may not exist a Nash Equilibrium on $M \oplus B$.
- If BRD always converges on M then it may or may not always converge on $M \oplus B$.
- If BRD always converges on $M \oplus B$ then it always converges on M.

Let G be an arbitrary non-bipartite graph.

- If a base game M has a Nash Equilibrium, then there may or may not exist a Nash Equilibrium on $M \oplus G$.
- If a base game M has no Nash Equilibria, then there may or may not exist a Nash Equilibrium on $M \oplus G$.
- If BRD always converges on M then it may or may not always converge on $M \oplus G$.
- If BRD always converges on $M \oplus G$ then it may or may not always converge on M.

The proof that if a base game M has a Nash Equilibrium, then the game $M \oplus B$ has a Nash Equilibrium, described in [18], is given below. It is clear that the properties of the bipartite graph allow for there to be a direct relationship between the games on M and on $M \oplus G$.

Lemma 4.4.1. *For every bipartite graph G and for every base game M that has a pure Nash equilibrium, the game $M \oplus G$ has a pure Nash equilibrium too.*

Proof. Let M be an arbitrary base game that has a Nash equilibrium $\mathbf{s} = (a, b)$. Let $G = \langle L \cup R, E \rangle$ be an arbitrary bipartite graph. Consider the strategy profile $\mathbf{p}$ in which $p(u) = a$ for $u \in L$ and $p(u) = b$ for $u \in R$. It is claimed that $\mathbf{p}$ is a Nash equilibrium in $M \oplus G$. Let

$p_u(a) = u$'s payoff for playing a in M when the other player plays according to $\mathbf{s}$

$\hat{p}_u(a) = u$'s payoff for playing a in $M \oplus G$ when the other player plays according to $\mathbf{p}$

Then we have $\hat{p}_u(a) = d(u)p_u(a)$ where $d(u)$ denotes the degree of vertex u in G. No player wants to deviate from $\mathbf{s}$ in M, so no vertex wants to deviate from $\mathbf{p}$ in $M \oplus G$. Thus $\mathbf{p}$ is a Nash equilibrium in $M \oplus G$. □

4.5 Competitive Information Diffusion

4.5.1 Diffusion

A simple model of competitive information diffusion on graphs was introduced in [7]. This model considers the diffusion process as a competitive game taking place on a graph that captures the underlying social structure. The model considers players who wish to spread their idea through the network. It has applications in areas such as viral marketing. The players are initially assigned vertices which they "colour" at the first timestep in the diffusion process. At each following timestep, uncoloured vertices adjacent to vertices that are already coloured are coloured according to the following rules. If two or more vertices of different colour neighbour an uncoloured vertex, then in the next timestep this vertex is coloured grey. This colour grey does not propagate through the network. If a vertex is neighboured by vertices coloured grey and one other colour then the uncoloured vertex takes this colour. If an uncoloured vertex is adjacent to vertices of only one colour, then the uncoloured vertex takes this colour. All other uncoloured vertices remain uncoloured. This represents the spread of an idea through a social network. The diffusion process ends when no further vertices can be coloured. The utility of a player is the number of vertices that it has coloured at the end of the diffusion process.

In the graph below the diffusion process is shown. There are two players 1 and 2, that colour the vertices green and blue respectively. The diffusion process terminates at timestep $t = 2$.

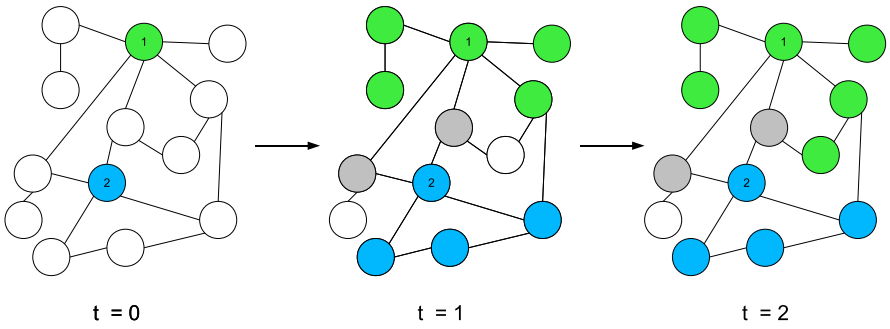


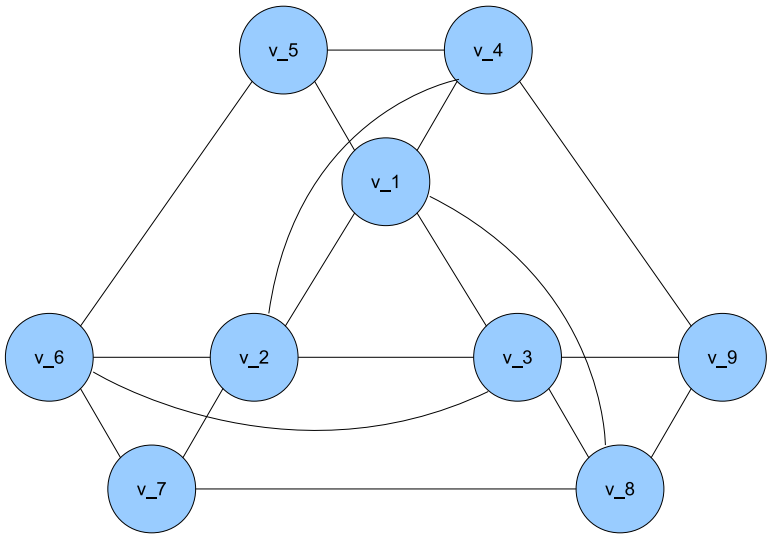
Figure 4.1: The diffusion process

Alon et al give a sufficient condition for a graph of diameter two to have a Nash Equilibrium [6]. Let $D(G)$ be the diameter of a graph G.

Theorem 4.5.1. *Let G be a graph and let the set of players $\{1, .., n\}$ such that $D(G') \leq 2$ for every G' that is obtained from G by removing $n-1$ vertices along with their neighbours. Then the game admits a Nash equilibrium, which can be found in polynomial time.*

They initially claimed that every graph of diameter two would admit a Nash Equilibrium. If the grey vertices were allowed to propagate then this would indeed hold, but Takehara et al [50] demonstrated that this was incorrect given that they do not. Although a graph may have diameter two, this does not imply that the diffusion process will terminate by time $t = 1$. Takehara et al provide the following example. If the strategy profile $\mathbf{x} = (v_1, v_9)$ is used on the graph below then vertices v_6 and v_7 are not coloured until time $t = 2$. This graph has no Nash Equilibrium for two players.

For any $x_1 \in V(G)$ and $x_2 \in \{v_4, .., v_9\}$ there exists some $v \in \{v_1, v_2, v_3\}$ such that $U_2(x_1, x_2) < U_2(x_1, v)$. So starting on an 'inner' vertex will always give a vertex a higher utility than starting on an 'outer' vertex. It is the case that: $U_2(v_1, v_2) < U_2(v_1, v_3)$, $U_2(v_2, v_3) < U_2(v_2, v_1)$ and $U_2(v_3, v_1) < U_2(v_3, v_2)$, ie once both players are starting on one of $\{v_1, v_2, v_3\}$, it is always beneficial for one of the players to change their starting vertex.



Alon et al also show that a graph of diameter three may not admit a Nash Equilibrium [7], and note that this result may be extended to graphs of any diameter greater than three, or for any number of agents greater than two.

4.6 Some Results

4.6.1 Star Graphs and Cliques

Certain classes of graphs are guaranteed to have a Nash Equilibrium for the diffusion game described above for any number of players. For the remainder of the chapter let the set of players be $\{1, .., n\}$ on a graph G, and $V(G) = \{v_1, .., v_N\}$ be the set of vertices in the graph. The game played on these graphs is described in Alon's paper [7] and in the previous section.
A star graph is a tree on N vertices such that one vertex has degree $N - 1$ and all other vertices have degree 1.

Lemma 4.6.1. *A star graph S will have a Nash Equilibrium for any number of players N_A.*

Proof. Consider a strategy profile $\mathbf{x} = (x_1, .., x_n)$, such that a player 1 starts on the internal vertex. There exists an edge between player 1 and every other $v \in V(S)$. Thus the diffusion process will terminate at $t = 1$. The utility of

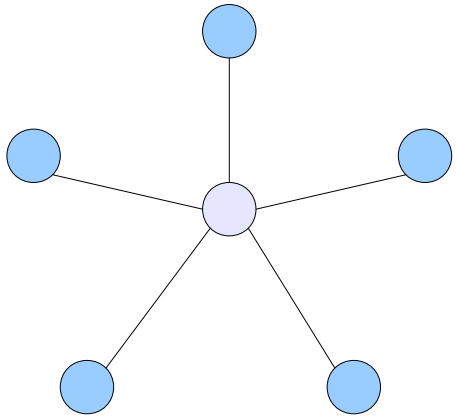


Figure 4.2: The star graph S_5, with blue leaves and a grey internal vertex

player 1 is $U_1(\mathbf{x}) = N - (n-1)$, and cannot be improved, since $n-1$ of the vertices are coloured by other the players. The utility $U_j(\mathbf{x}) = 1$ for $j \neq 1$ since every player $p_j, j \neq 1$ is on a leaf of the graph, and only has an edge to player 1. The players $p_j, j \neq 1$ cannot improve their utility since they can only move to another leaf of the graph. Thus $\mathbf{x}$ is a Nash Equilibrium. □

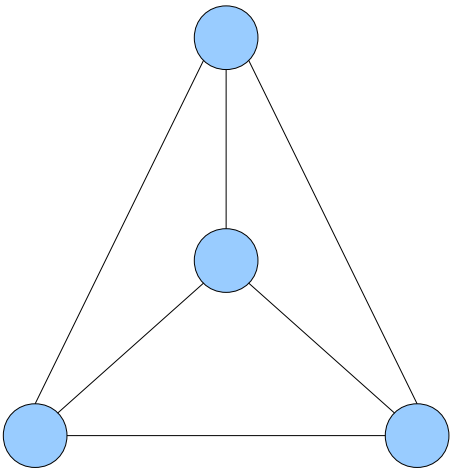


Figure 4.3: The clique K_4

Lemma 4.6.2. *Any strategy profile on a clique C is a Nash Equilibrium for any number of players n.*

Proof. There exists an edge between every pair of vertices $v, w \in C$. Consider a strategy profile $\mathbf{x}$. If the number of players is greater than one, then every vertex that is coloured white at time $t = 0$ is linked to more than one player, thus is coloured grey at time $t = 1$. Therefore the utility of every player is one, regardless of which vertex is initially chosen. Every strategy profile is a Nash Equilibrium. If there is only one player, then regardless of which vertex the player initially starts on this player will colour every vertex on the graph at time $t = 1$, so again every strategy profile is a Nash Equilibrium. □

We later present some results concerning the existence of N.E. on trees. We first note the following fact.

Example 4.6.1. *Let T be a tree. A game played on T may not have a Nash Equilibrium, as shown below.*

Consider a game on the tree below with seven vertices and three players. Let

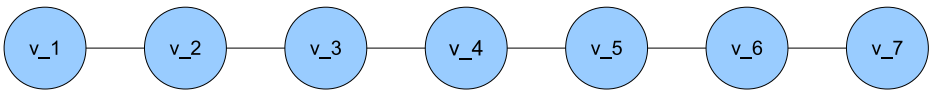


Figure 4.4: The tree T

us assume without loss of generality that strategy profile $\mathbf{x} = (v_i, v_j, v_k)$ where $i < j < k$. We will show that this profile cannot maximize the utility of every player simultaneously, and thus cannot exhibit a Nash Equilibrium.

If $v_i v_j \notin E(T)$ then $U_1(\mathbf{x})$ is improved by player 1 instead choosing vertex v_{j-1}.

If $v_j v_k \notin E(T)$ $U_3(\mathbf{x})$ is improved similarly to the previous case by player 3 choosing vertex v_{j+1}.

If $v_i v_j \in E(T)$ and $v_j v_k \in E(T)$ then $U_2(\mathbf{x}) = 1$, and it is always possible

for player 2 to improve their utility by moving to either v_{i-1} or v_{k+1}

No strategy profile can satisfy the three players simultaneously, so there exists no Nash Equilibrium on this graph.

4.6.2 Cliques

Lemma 4.6.3. *Take a graph G containing a clique C. If more than one player initially selects vertices in C, then at the end of the diffusion process each player will have coloured one vertex in C.*

Proof. Every $v \in C$ that is not initially selected by a player is neighboured by more than one player, so is coloured grey at $t = 1$. Thus every player only colours itself out of all the vertices in the clique. □

For the following result, we introduce a graph parameter, $CD(G)$, which is the minimal number of cliques into which the vertices of G can be partitioned. Formally, $CD(G)$ is the minimal k such that there exist vertex disjoint cliques $C_1, \ldots, C_k$ in G with $V(G) = V(C_1) \cup \cdots V(C_k)$. $CD(G) = 2$ in the graph G in Figure 4.5 below, where the vertices in one clique are coloured blue, and in the other clique yellow.

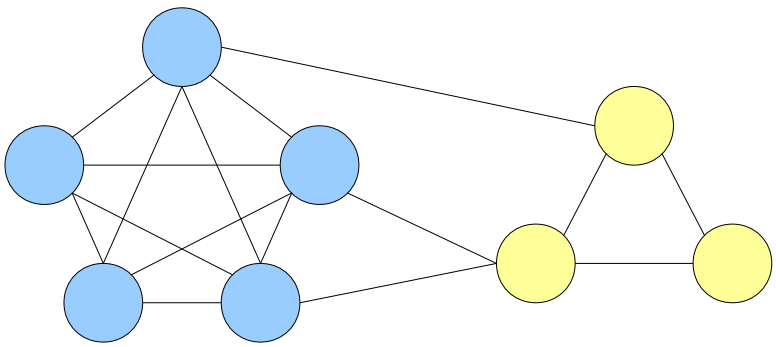


Figure 4.5: The graph G

Proposition 4.6.1. *Let G be a connected graph and consider the diffusion process $\boldsymbol{D}$ on G. If $CD(G) = 2$, then there exists a Nash equilibrium for $\boldsymbol{D}$ on G for any $n \geq 2$.*

Proof. Label the two cliques C_1 and C_2. There exists at least one edge, say vw between C_1 and C_2 since the graph is connected. Let player 1 select $v \in V(C_1)$ and player 2 select $w \in V(C_2)$, so $\mathbf{x} = (v, w)$, and let the number of edges between any $c \in V(C_1)$ and w be γ. Consider the case where there are two agents.

There are $\gamma - 1$ grey vertices in C_1 at the end of the diffusion process, since there are γ edges between C_1 and w, and there is an edge between v and w. Then the utility of player 1 is

$$U_1(\mathbf{x}) = |C_1| - \gamma + 1$$

Suppose player 1 instead selects some element of $c \in C_2$, $\mathbf{x}' = (c, w)$ From Lemma 4.6.3 player 1 can only colour itself in C_2. Thus

$$max\ U_1(\mathbf{x}') = 1 + |C_1| - \gamma$$

Suppose player 1 instead colours some other element of C_1. If there is no edge between this vertex and w then player 1 colours one fewer vertex, as an extra vertex in C_1 is coloured grey. If there exists an edge between this vertex and w then it will colour the same vertices as v. Thus it is not beneficial for player 1 to select a vertex other than v. The same reasoning applies to player 2. So $\mathbf{x} = (v, w)$ is a Nash Equilibrium for two players.

Now suppose there are n players on the graph, and the strategy profile is now $x_n = (v, w, v_1, .., v_{n-2})$. By Lemma 4.6.3 any player other than 1 or 2 will have a utility of one, since every vertex in the graph is either neighboured by player 1 or player 2, so it cannot benefit any of these players to move.

Players 1 cannot benefit by moving to any $c \in V(C_2)$, as demonstrated above. By moving to any other vertex in C_1 player 1 can at best equal the utility it has by selecting v. Thus $\mathbf{x}$ is a Nash Equilibrium on G.

□

4.6.3 Trees

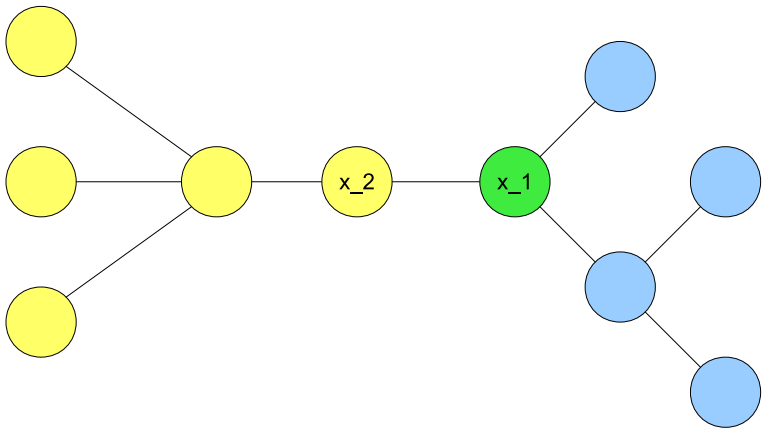
Lemma 4.6.4. *Let there be a game on a tree T with two players. If player 1 initially selects a vertex v, the vertex w selected by player 2 which maximizes player 2's utility is a neighbour of v.*

Proof. Suppose player one selects vertex v, and the degree of v is l. T is a tree, so every edge is a bridge. Then the deletion of v will divide the graph into l connected components $C_1, .., C_l$. Let vertex η_i be the neighbour of vertex v that is an element of C_i.

Player 2 cannot colour vertices in more than one of these components, since the path between any pair of components must include v. Thus the maximum number of vertices that player 2 can colour is $max\ |V(C_a)|$ for $a \in 1, .., l$.

Suppose player 2 initially selects $\eta_y \in C_y$, a neighbour of v. For any $v_k \in C_y$ there exists a path $v, \eta_y, .., v_k$. This path is unique since T is a tree. So player 1 cannot colour any $v_k \in C_y$, thus the utility of player 2 is $|C_y|$. So if player 2 selects $\eta_y \in C_y$ it will colour all $v_k \in C_y$. Thus the maximum number of vertices player 2 will colour starting from a neighbour of x_1 is $max\ |V(C_a)|$ for $a \in \{1, .., l\}$. This is the maximum number of vertices that it is possible for player 2 to colour, so player 2 cannot improve its payoff by moving to some $v_j \notin N_v$. □

For example it can be seen that the tree below is split into three components if x_1 is removed, and that selecting a neighbour of x_1 gives the highest utility for player two.



Lemma 4.6.5. *Let there be a game on a tree T with two players, and let $\boldsymbol{x} = (v, w)$ be a strategy profile such that v, w are neighbours. If player 2 moves to some $\eta \in N_v \backslash \{w\}$, with $\boldsymbol{x'} = (v, \eta)$ then $U_2(\boldsymbol{x'}) < U_1(\boldsymbol{x})$*

Proof. Suppose the two graph components formed when edge $\{v, w\}$ is removed are C_v, C_w. Player 1 colours all the vertices in C_v, and player 2 colours all the vertices in C_w.

By Lemma 4.6.4 the vertex which maximizes player 2's utility is a neighbour of v. Suppose $\mathbf{x}' = (v, \eta)$ where $\eta \in C_v, \eta \in N_v \backslash \{w\}$. Player 2 in this case cannot colour any elements of C_w, since the path to any $c \in C_w$ must involve v. Player 2 colours some subset of $C_v \backslash v$, so

$$max\ U_2(\mathbf{x}') = |V(C_v)| - 1 < U_1(x)$$

□

Lemma 4.6.6. *Let there be a game on a tree T with two players. If a strategy profile $\boldsymbol{x}$ is such that x_1, x_2 are neighbours, there are no grey vertices in the graph once the diffusion process is completed, and $U_1(\boldsymbol{x}) + U_2(\boldsymbol{x}) = |V(T)|$.*

Proof. x_1, x_2 are at distance one, and have no common neighbour, since T is a tree. If the edge between x_1 and x_2 is removed, the graph is split into two connected components, C_1 and C_2, since every edge is a bridge. There is a unique path between any two vertices, so player 2 cannot colour any vertex in C_1, as any path to a vertex in C_1 must involve x_1. Likewise player 1 can colour no vertices in C_2. Player 1 then colours every vertex in C_1 and player 2 colours every vertex in C_2. Every vertex in the graph is coloured by player 1 or player 2 at the end of the diffusion process, and no vertex is coloured grey. Thus $U_1(\mathbf{x}) + U_2(\mathbf{x}) = |V(T)|$. □

Proposition 4.6.2. *Let $\boldsymbol{x} = (v, w)$ be a strategy profile on a tree T such that v, w are neighbours, where $|U_1(\boldsymbol{x}) - U_2(\boldsymbol{x})|$ is minimized, given given that the two elements of the strategy profile are neighbours. Then $\boldsymbol{x}$ is a Nash Equilibrium. So a game on any tree has a Nash Equilibrium for two players.*

Proof. Let $U_1(\mathbf{x}) \geq U_2(\mathbf{x})$. Suppose player 1 can increase its utility by moving. By the previous lemma the vertex with best payoff is a neighbour of player 2. But the maximum utility of a neighbour of player 2 other than v, is less than $U_2(\mathbf{x})$, by Lemma 4.6.5. We know that $U_1(\mathbf{x}) \geq U_2(\mathbf{x})$, thus moving cannot benefit player 1.

Now suppose that it benefits player 2 to move, $U_2(\mathbf{x}') > U_2(\mathbf{x})$, then, as every vertex in the graph is coloured at the end of the diffusion process and there are no grey vertices, $U_1(\mathbf{x}') < U_1(\mathbf{x})$.

Case 1: $U_2(\mathbf{x}') \geq U_1(\mathbf{x})$
$U_2(\mathbf{x}')$ can colour at most $U_1(\mathbf{x}) - 1$ vertices by Lemma 4.6.5, so the above case cannot occur.

Case 2a: $U_2(\mathbf{x}') < U_1(\mathbf{x})$, and $U_1(\mathbf{x}') > U_2(\mathbf{x}')$
Now, we know that $U_1(\mathbf{x}) > U_1(\mathbf{x}')$, and $U_2(\mathbf{x}') > U_2(\mathbf{x})$. So $U_1(\mathbf{x}) + U_2(\mathbf{x}') > U_1(\mathbf{x}') + U_2(\mathbf{x})$, ie

$$U_1(\mathbf{x}) - U_2(\mathbf{x}) > U_1(\mathbf{x}') - U_2(\mathbf{x}')$$

which contradicts our initial assumption, that $|U_1(\mathbf{x}) - U_2(\mathbf{x})|$ is minimized.

Case 2b: $U_2(\mathbf{x}') < U_1(\mathbf{x})$, and $U_1(\mathbf{x}') \leq U_2(\mathbf{x}')$
The maximum value of $U_2(\mathbf{x}')$ is $U_1(\mathbf{x}) - 1$ from Lemma 4.6.5, and the minimum value of $U_1(\mathbf{x}')$ is $U_2(\mathbf{x}) + 1$, since player 1 will now colour all the vertices which player 2 formerly coloured, as well as itself. Then the maximum difference between the two

$$max|U_2(\mathbf{x}') - U_1(\mathbf{x}')| = U_1(\mathbf{x}) - U_2(\mathbf{x}) - 2 < U_1(\mathbf{x}) - U_2(\mathbf{x})$$

which contradicts our initial assumption, that $|U_1(\mathbf{x}) - U_2(\mathbf{x})|$ is minimized.

Then $\mathbf{x}$ is a Nash Equilibrium. □

In the tree T below there is one Nash Equilibrium, $x = (v, w)$ as shown, where v has coloured the yellow vertices and w the blue. $|U_1(\mathbf{x}) - U_2(\mathbf{x})| = |7 - 5| = 2$ is clearly minimized for v, w neighbours. Every Nash Equilibrium on a tree is optimal.

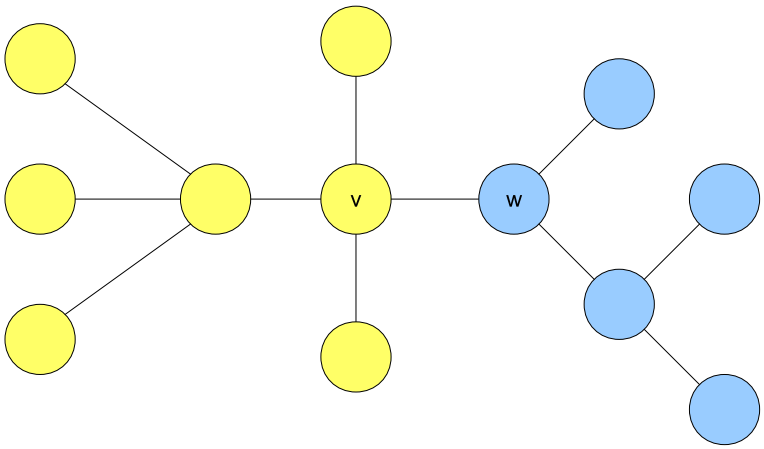


Figure 4.6: The tree T

CHAPTER 5

Diffusion on the ILT model

5.1 Introduction and Notation

In this chapter we study the information diffusion model of Alon et al on the Iterated Local Transitivity (ILT) model, both of which were separately discussed in earlier chapters. In particular, we show that for two competing players an independent Nash Equilibrium on the initial graph remains a Nash Equilibrium for all subsequent times. We also describe an example showing that this conclusion does not hold for general Nash Equilibria in the ILT process. The graphs we consider are finite and undirected.

5.1.1 The Diffusion Process D

We now provide formal notation for the diffusion process [7] discussed earlier. Consider a graph G with vertex set $V(G)$, $|V(G)| = N$ and a set of players indexed as $[1, n] = \{1, \ldots, n\}$. At time 0, each player $i \in [1, n]$ selects a seed vertex, x_i, in $V(G)$, which is labelled (or coloured) i. We shall only consider strategy profiles in which all of the x_i are distinct. If the set $\{x_1, \ldots, x_n\}$ is an independent set, we say that the strategy profile $\mathbf{x}$ is independent. All other vertices at time 0 are labelled 0 (corresponding to white vertices in [7]). In addition to the labels $0, 1, \ldots, n$ we also use the label -1 to denote grey vertices. In keeping with the original model of [7], grey vertices do not propagate. For $v \in V(G)$ and $t \geq 0$, we use $l^t(v)$ to denote the label of v at time t.

The labelling map l^0 is given by

$$l^0(v) = \begin{cases} i & \text{if } v = x_i \text{ where } 1 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

At each subsequent time $t \geq 1$, we define l^t as follows. If $l^{t-1}(v) \neq 0$, then $l^t(v) = l^{t-1}(v)$ (so only vertices labelled 0 can change their label). For vertices v with $l^{t-1}(v) = 0$, let $\mathcal{L}(v) = \{l^{t-1}(w) : w \in N_v\}$ denote the set of labels of the neighbours of v. Then:

- if $\mathcal{L}(v) \cap [1, n] = \{i\}$, then $l^t(v) = i$;
- if $|\mathcal{L}(v) \cap [1, n]| \geq 2$ then $l^t(v) = -1$;
- $l^t(v) = l^{t-1}(v)$ otherwise.

For $i \in [1, n]$, we denote by $L_i(t)$ those vertices labelled i at exactly time t. Formally $L_i(t) = \{v \in V(G) : l^{t-1}(v) = 0, l^t(v) = i\}$. The process *terminates* at some time t if $L_i(t+1)$ is empty for all $i \in [1, n]$. Thus no new vertices are labelled $i \in \{1, \dots n\}$ in the time step $t \to t+1$. Note however, that it is possible for vertices to be labelled -1 in the step $t \to t+1$. As we are only interested in vertices ultimately labelled $i \in [1, n]$, this is unimportant. Clearly, the process must terminate before time $t = N - n$.

5.1.2 Iterated Local Transitivity (ILT) Graphs

In [13], the *Iterated Local Transitivity (ILT)* model for online social networks was introduced. We have outlined this model in Chapter 3.

5.2 Preliminary Results

We are interested in the following question for the ILT model discussed in Section 5.1. When does the existence of a Nash Equilibrium in the initial graph G_0 imply the existence of a Nash Equilibrium in G_t for all $t \geq 0$?

We adopt the following notation. For a connected graph G, $\hat{G}$ denotes the graph obtained through applying one step of the ILT process to G. If a strategy profile $\mathbf{x} = (x_1, \dots, x_n)$ consists entirely of vertices from $V(G)$, we say that it is a strategy profile in G. For $1 \leq i \leq n$, $t \geq 0$, $\hat{L}_i(t)$ denotes the set of vertices

in $\hat{G}$ labelled i at *exactly* time t. We also use $W(t)$ ($\hat{W}(t)$) to denote the *white* vertices in G ($\hat{G}$) at time t. Formally, $W(t) = \{v \in V(G) : l^t(v) = 0\}$ ($\hat{W}(t) = \{v \in V(\hat{G}) : l^t(v) = 0\}$).

For a set $U \subseteq V(G)$ ($U \subseteq V(\hat{G})$), N_U ($\hat{N}_U$) denotes the neighbours of U in G ($\hat{G}$).

Definition 5.2.1. *Let a connected graph G, a set of players $[1, n] = \{1, \ldots, n\}$ and a strategy profile $\mathbf{x}$ be given. If the diffusion process $\boldsymbol{D}$ terminates at time T on G, the utility $U_i(\mathbf{x})$ of player $i \in [1, n]$ is given by*

$$U_i(\mathbf{x}) = |\{v \in V(G) : l_T(v) = i\}|$$

Informally, the utility of player i is the total number of vertices in G labelled i when the process terminates. We use $\hat{U}_i$ to denote utilities in $\hat{G}$.

Given a strategy profile $\mathbf{x}$, a vertex $v \notin \{x_1, \ldots, x_n\}$ and $i \in [1, n]$, we denote by $\mathbf{x}_{-i}(v)$ the profile given by

$$\mathbf{x}_{-i}(v) = (x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots x_n).$$

We next recall the definition of Nash Equilibrium.

Definition 5.2.2. *Let a connected graph G and a set of players $[1, n] = \{1, \ldots, n\}$ be given. A strategy profile $\mathbf{x}$ is a Nash Equilibrium for the process $\boldsymbol{D}$ on G if $U_i(\mathbf{x}) \geq U_i(\mathbf{x}_{-i}(v))$ for all $i \in [1, n]$ and all $v \in V(G) \backslash \{x_1, \ldots, x_n\}$.*

Note the following simple facts, which follow immediately from the definition of the process $\mathbf{D}$.

Lemma 5.2.1. *A vertex v is in $L_i(t+1)$ if and only if: (i) $v \in W(t)$; (ii) $v \in N_{L_i(t)}$; (iii) $v \notin N_{L_j(t)}$ for all $j \in [1, n] - i$.*

Lemma 5.2.2. *A vertex v is in $W(t+1)$ if and only if: (i)$v \in W(t)$; (ii) $v \notin N_{L_j(t)}$ for all $j \in [1, n]$.*

Proposition 5.2.1. *Let $(x_1, \ldots, x_n)$ be a strategy profile in a connected graph G. Then for $1 \leq i \leq n$*

$$\hat{L}_i(1) = \begin{cases} L_i(1) \cup (L_i(1))' \cup \{x_i'\} & \text{if } \mathbf{x} \text{ is independent} \\ L_i(1) \cup (L_i(1))' & \text{otherwise} \end{cases}.$$

$$\hat{W}(1) = W(1) \cup W(1)'.$$

Proof. Note that $\hat{W}(0) = W(0) \cup (W(0))' \cup \{x_1', \ldots, x_n'\}$ and that $\hat{N}_{x_j} = N(x_j) \cup (N_{x_j})' \cup \{x_j'\}$ for $1 \leq j \leq n$. As $x_i' \in \hat{N}_{x_j}$ if and only if $x_i \in N_{x_j}$, the result follows readily from Lemma 5.2.1. □

The following proposition clarifies the relationship between $\hat{L}_i(t)$ and $L_i(t)$ for $t = 2, 3, \ldots$.

Proposition 5.2.2. *Let $(x_1, \ldots, x_n)$ be a strategy profile in a connected graph G. Then for $1 \leq i \leq n$, $t \geq 2$,*

$$\hat{L}_i(t) = L_i(t) \cup (L_i(t))'.$$

$$\hat{W}(t) = W(t) \cup W(t)'.$$

Proof. We prove the result by induction on t. First, we use Proposition 5.2.1 to establish the result for the case $t = 2$. Lemma 5.2.1 implies that a vertex $w \in V(\hat{G})$ is in $\hat{L}_i(2)$ if and only if: $w \in \hat{W}(1)$; $w \in \hat{N}_{\hat{L}_i(1)}$; $w \notin \hat{N}_{\hat{L}_j(1)}$ for $j \in [1, n] - i$. Using Proposition 5.2.1, we can see that $\hat{N}_{x_i} \cap \hat{W}(1)$ is empty (this follows from Lemma 5.2.2 as $\hat{N}_{x_i'}$ is given by $\{x_i\} \cup N_{x_i}$) . Furthermore, for $1 \leq k \leq n$

$$\hat{N}_{L_k(1) \cup L_k(1)'} = \hat{N}_{L_k(1)} = N_{L_k(1)} \cup N'_{L_k(1)}.$$

It follows from Lemma 5.2.1 that $\hat{L}_i(2) = L_i(2) \cup L_i(2)'$. The conclusion $\hat{W}(2) = W(2) \cup W(2)'$ follows from Proposition 5.2.1 and Lemma 5.2.2.

Now assume that the result is true for some $t \geq 2$. Lemma 5.2.1 implies that $w \in \hat{L}_i(t+1)$ if and only if: $w \in \hat{W}(t)$; $w \in \hat{N}_{\hat{L}_i(t)}$; $w \notin \hat{N}_{\hat{L}_j(t)}$ for $j \in [1, n] - i$. Using the induction hypothesis we see that for $k \in [1, n]$

$$\hat{N}_{\hat{L}_k(t)} = \hat{N}_{L_k(t) \cup L_k(t)'} = \hat{N}_{L_k(t)} = N_{L_k(t)} \cup N'_{L_k(t)}.$$

As in the previous paragraph, it follows that

$$\hat{L}_i(t+1) = L_i(t+1) \cup (L_i(t+1))'.$$

Moreover, combining the induction hypothesis with Lemma 5.2.2 yields $\hat{W}(t+1) = W(t+1) \cup W(t+1)'$. □

5.3 Nash Equilibria and the ILT Model

In [7], the existence of Nash equilibria for the diffusion process was investigated on graphs of low diameter. We shall provide conditions under which such equilibria are guaranteed to exist for G_t in the ILT model for all t.

The following lemma shows that in the 2-player case, neither player can improve their utility by unilaterally changing from x_i to its clone x_i'.

Lemma 5.3.1. *Let (x_1, x_2) be a strategy profile in G. Then*

$$\hat{U}_1(x_1', x_2) \leq \hat{U}_1(x_1, x_2).$$

Proof. We use $\bar{l}^t$ to denote the labelling map for the profile (x_1', x_2), and l^t for the labelling map for (x_1, x_2). It is clear that $\bar{l}^1(v) = 1$ implies $l^1(v) = 1$, and that $l^1(v) = 2$ implies $\bar{l}^1(v) = 2$. Suppose that $\hat{U}_1(x_1', x_2) > \hat{U}_1(x_1, x_2)$. Then there must exist some $t > 1$ and v such that $\bar{l}^t(v) = 1$, $l^t(v) \neq 1$. Let t_0 be the minimal $t > 1$ for which this occurs. It is immediate that $l^{t_0-1}(v) \neq 1$. If $t_0 - 1 = 1$, then this implies that $\bar{l}^{t_0-1}(v) \neq 1$. If $t_0 - 1 > 1$, then as t_0 is minimal, it also follows that $\bar{l}^{t_0-1}(v) \neq 1$. We can thus conclude that $\bar{l}^{t_0-1}(v) = 0$.

As $\bar{l}^{t_0}(v) = 1$, there is some $w_1 \in \hat{N}_v$ with $\bar{l}^{t_0-1}(w_1) = 1$ and there exists no $w \in N_v$ with $\bar{l}^{t_0-1}(w) = 2$. We know that $l^{t_0}(v) \neq 1$ and $l^{t_0-1}(w_1) = 1$. It follows from this that there must exist some $w_2 \in \hat{N}_v$ with $l^{t_0-1}(w_2) = 2$. Moreover, we know that $\bar{l}^{t_0-1}(w_2) \neq 2$. This implies that $t_0 - 1 > 1$. Thus if we define t_1 to be the minimum $t > 1$ for which there exists u with $\bar{l}^t(u) \neq 2$, $l^t(u) = 2$, we can see that $1 < t_1 < t_0$. A similar argument to that used above will show that there must exist some w_3 in $\hat{N}_u$ such that $l^{t_1-1}(w_3) \neq 1$, $\bar{l}^{t_1-1}(w_3) = 1$. As $1 < t_1 < t_0$ (and this cannot happen for $t = 1$ so that $t_1 - 1 > 1$), this contradicts the minimality of t_0. This shows that $\hat{U}_1(x_1', x_2) \leq \hat{U}_1(x_1, x_2)$ as claimed. □

The example in Figure 5.1 below shows that the previous result need not hold for 3 or more players. If $\mathbf{x} = (v_1, v_2, v_3)$ and $\mathbf{x}_1 = (v_1, v_2', v_3)$, then $\hat{U}_2(x) < \hat{U}_2(x')$.

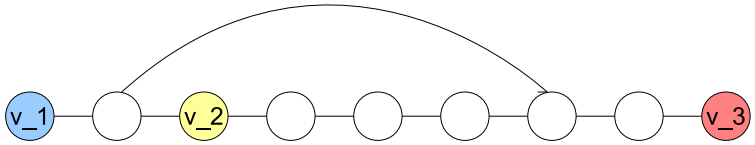


Figure 5.1: Utility increases by choosing clone

The next lemma, which follows from Proposition 5.2.2, shows how the utility $U_i(\mathbf{x})$ of an player on G relates to its utility $\hat{U}_i(\mathbf{x})$ on $\hat{G}$ for a strategy profile $\mathbf{x}$ in G.

Lemma 5.3.2. *Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a strategy profile in G. Then*

$$\hat{U}_i(\mathbf{x}) = \begin{cases} 2U_i(\mathbf{x}) - 1 & \text{if } x_i \text{ is neighboured by some } x_j \\ 2U_i(\mathbf{x}) & \text{if } x_i \text{ is not neighboured by some } x_j \end{cases}$$

Proof. This result follows immediately from Proposition 5.2.1 and Proposition 5.2.2. □

For the remainder of this section, we will use the above result to investigate the relationship between Nash equilibria on G and $\hat{G}$.

Proposition 5.3.1. *Let $\mathbf{x} = (x_1, .., x_n)$ be a strategy profile in G. If $\mathbf{x}$ is a Nash Equilibrium for $\boldsymbol{D}$ in $\hat{G}$ then $\mathbf{x}$ is also a Nash Equilibrium for $\boldsymbol{D}$ in G.*

Proof. For any strategy profile $\mathbf{y} = (y_1, \ldots, y_n)$ in G, $1 \leq i \leq n$, let $f_i(\mathbf{y}) \in \{0, 1\}$ be the indicator function of whether y_i neighbours some y_j, $j \neq i$. Let $\mathbf{x}_0$ be a strategy profile $\mathbf{x}_{-i}(v)$ for some $v \in V(G) \backslash \{x_1, \ldots, x_n\}$. As $\mathbf{x}$ is a Nash Equilibrium in $\hat{G}$, we have

$$2U_i(\mathbf{x}_0) - f_i(\mathbf{x}_0) = \hat{U}_i(\mathbf{x}_0) \leq \hat{U}_i(\mathbf{x}) = 2U_i(\mathbf{x}) - f_i(\mathbf{x}).$$

Rearranging the above inequalities, we see that

$$U_i(\mathbf{x}_0) \leq U_i(\mathbf{x}) + \frac{f_i(\mathbf{x}_0) - f_i(\mathbf{x})}{2} \leq U_i(\mathbf{x}) + \frac{1}{2}.$$

As both $U_i(\mathbf{x}_0)$ and $U_i(\mathbf{x})$ are integers, it follows that $U_i(\mathbf{x}_0) \leq U_i(\mathbf{x})$. As this is true for all $i \in \{1, \ldots, n\}$, $v \in V(G)$, it follows that $\mathbf{x}$ is a Nash Equilibrium in G as claimed. □

The previous proposition shows that in order for a strategy profile $\mathbf{x}$ selected from the vertices of G to be a Nash Equilibrium in $\hat{G}$, it is necessary for $\mathbf{x}$ to be a Nash Equilibrium in G. It is tempting to conjecture that the converse will be true: that any Nash Equilibrium on G will also be a Nash Equilibrium on $\hat{G}$. The example in Figure 5.2 below shows that this is not true, even for the simple case of 2 players. Moreover, for this example, the process $\mathbf{D}$ has no Nash Equilibrium on $\hat{G}$.

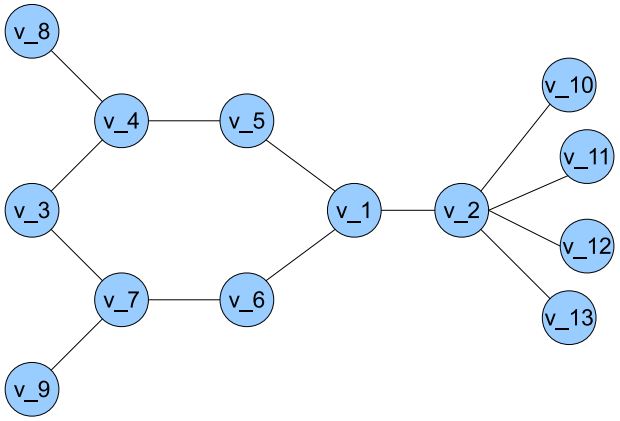


Figure 5.2: Nash Equilibrium on G, no Nash Equilibrium on $\hat{G}$

Discussion of Example in Figure 5.2

We claim that $\mathbf{x} = (v_1, v_2)$ is a Nash Equilibrium. First note that $U_1(\mathbf{x}) = 8$, $U_2(\mathbf{x}) = 5$. If we remove the edge v_1v_2 then G splits into two connected components. Let C_i denote the component containing v_i for $i = 1, 2$. Consider a strategy profile (v, v_2) with $v \neq v_1$, $v \in C_1$. For such a profile, it is clear that v_1 will either be labelled 2 or -1. In either case, $U_1(v, v_2) \leq 7$. On the other hand $U_1(v, v_2) = 1$ for any $v \in C_2$. Thus player 1 cannot improve their utility by unilaterally changing strategy. Now consider player 2. Again $U_2(v_1, v) = 1$ for any $v \in C_2$ with $v \neq v_2$. So consider $v \in C_1$. By symmetry, it is enough to consider $v \in \{v_3, v_4, v_5, v_8\}$. For $v \in \{v_4, v_5, v_8\}$, it is easy to see that vertices $\{v_6, v_7, v_9\}$ cannot be labelled 2 so in each of these cases $U_2(v_1, v) \leq 4$. Finally, $U_2(v_1, v_3) = 5$. Thus, player 2 cannot unilaterally improve their utility and (v_1, v_2) is a Nash Equilibrium as claimed.

Moreover, (v_1, v_2) is the only Nash Equilibrium in G. Consider any strategy

profile (x_1, x_2). If both x_1 and x_2 are in C_2, then one player has utility at most 1 and this player can increase their utility by changing to v_1. On the other hand, if both x_1 and x_2 are in C_1, then (as v_1 will either be labelled i for some $i \in \{1, 2\}$ or -1), one player will have utility at most $\frac{|C_1|}{2} = 4$, while their utility would be at least 5 if they change to v_2. Therefore, any Nash Equilibrium (x_1, x_2) must have one vertex in each component. Without loss of generality, suppose (x_1, x_2) is a Nash Equilibrium with $x_1 \in C_1$, $x_2 \in C_2$. We claim that $x_1 = v_1, x_2 = v_2$. Suppose this is not the case. First assume $x_1 \neq v_1$. If v_1 is labelled 1 (when the process terminates), then player 2 will improve their utility by changing x_2 to v_1. On the other hand, if v_1 is not labelled 1, then player 1 will increase their utility by changing to v_1. This shows that $x_1 = v_1$. A similar argument shows that $x_2 = v_2$ so (v_1, v_2) is the only Nash Equilibrium in G as claimed.

The Nash Equilibrium (v_1, v_2) is not independent so it follows from Lemma 5.3.2 that $\hat{U}_2(v_1, v_2) = 9$. However, for this example, we have seen that $U_2(v_1, v_3) = 5$ and as $v_1v_3 \notin E(G)$, it follows that $\hat{U}_2(v_1, v_2) = 10$. Thus (v_1, v_2) is not a Nash Equilibrium in $\hat{G}$. Using very similar arguments to those employed above, we can show that any Nash Equilibrium (x_1, x_2) in $\hat{G}$ must have $x_1 \in \{v_1, v_1'\}$, $x_2 \in \{v_2, v_2'\}$ or $x_2 \in \{v_1, v_1'\}$, $x_1 \in \{v_2, v_2'\}$. Without loss of generality, consider the former case. For both (v_1, v_2'), $(v_1'v_2')$, the utility of player 2 is 9; in both cases, the utility of player 2 increases to 10 if they change their seed vertex to v_3. The only remaining possibility for a Nash Equilibrium is (v_1', v_2). In this case, player 2 has utility 10. However, if player 2 changes their seed vertex to v_1, we find $U_2(v_1', v_1) = 14$ so this is not a Nash Equilibrium either. Thus we see that $\hat{G}$ has no Nash Equilibrium, while G does.

It is instructive to highlight some key factors involved in the construction of the above example. First, G has a *unique* (up to permutation) Nash Equilibrium (v_1, v_2) which is not independent, and there exists a vertex $v_3 \notin \{v_1, v_2\}$ with $U_2(v_1, v_3) = U_2(v_1, v_2)$. These conditions imply that (v_1, v_2) is not a Nash Equilibrium in $\hat{G}$. The structural properties of G ensuring a unique Nash Equilibrium, when inherited by $\hat{G}$ rule out the possibility of a Nash Equilibrium existing in $\hat{G}$. Before finishing our discussion here, note that it is possible for $\hat{G}$ to have a Nash Equilibrium even if there exists no independent Nash Equilibrium on G. To see this, consider a simple line graph G with 4

vertices $V(G) = \{v_1, \ldots, v_4\}$, $E(G) = \{v_1v_2, v_2v_3, v_3v_4\}$ and two players. The only Nash Equilibrium are (v_2, v_3) or (v_3, v_2). However is is easy to see that these are again Nash Equilibria on $\hat{G}$.

We now show that an independent Nash Equilibrium for $\mathbf{D}$ on G will also be a Nash Equilibrium for $\mathbf{D}$ on $\hat{G}$.

Proposition 5.3.2. *If $\mathbf{x} = (x_1, x_2)$ is an independent Nash Equilibrium for the diffusion process $\boldsymbol{D}$ in G then $\mathbf{x}$ is a Nash Equilibrium in $\hat{G}$.*

Proof. As $\mathbf{x}$ is a Nash Equilibrium in G, we know that for any $i \in \{1, 2\}$, $v \in V(G)\backslash\{x_1, x_2\}$,

$$U_i(\mathbf{x}_{-i}(v)) \leq U_i(\mathbf{x}). \tag{5.2}$$

As $x_1x_2 \notin E(G)$, we can conclude from Lemma 5.3.2 that

$$\hat{U}_i(\mathbf{x}) = 2U_i(\mathbf{x}) \tag{5.3}$$

Furthermore, it follows from Lemma 5.3.2 that $\hat{U}_i(\mathbf{x}_{-i}(v)) \leq 2U_i(\mathbf{x}_{-i}(v))$ so we can immediately conclude from (5.2) and (5.3) that

$$\hat{U}_i(\mathbf{x}_{-i}(v)) \leq \hat{U}_i(\mathbf{x}). \tag{5.4}$$

To complete the proof, note that for $w = v'$, where $v \in V(G)$, we know from Lemma 5.3.1 that

$$\hat{U}_i(\mathbf{x}_{-i}(w)) \leq \hat{U}_i(\mathbf{x}_{-i}(v)).$$

Combining this with the previous arguments, we see that for any $v \in V(\hat{G})\backslash\{x_1, x_2\}$,

$$U_i(\mathbf{x}_{-i}(v)) \leq \hat{U}_i(\mathbf{x}).$$

This completes the proof. □

The above result immediately yields the following conclusion concerning the ILT model.

Corollary 5.3.1. *Consider the ILT graph model with a connected initial graph G_0. Consider the diffusion process $\boldsymbol{D}$ corresponding to a set of players $\{1, 2\}$. If there exists an independent Nash Equilibrium $\mathbf{x} = (x_1, x_2)$ for $\boldsymbol{D}$ on G_0, then there exists a Nash Equilibrium for $\boldsymbol{D}$ (with players $\{1, 2\}$) on G_t for all $t \geq 0$.*

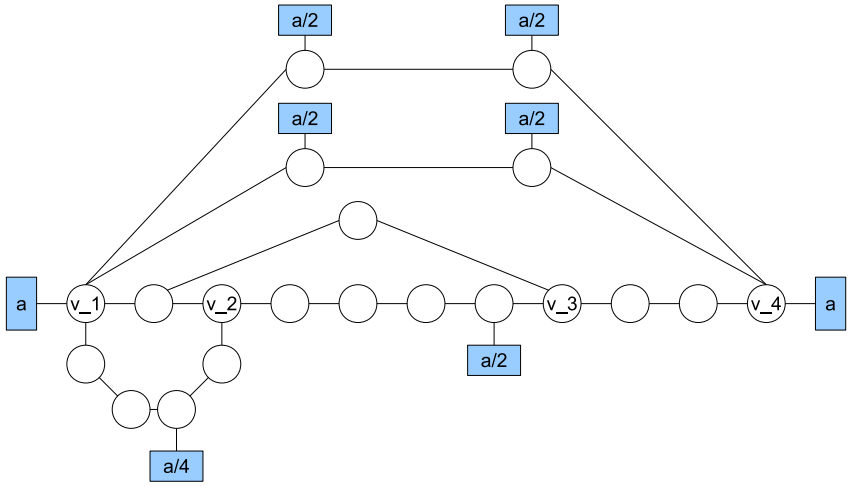


Figure 5.3: Independent Nash Equilibrium on G that is not a Nash Equilibrium on $\hat{G}$

As a final point, we note that as in the case of Lemma 5.3.1, the situation becomes more complicated when we consider three or more players. In Figure 5.3, the rectangles represent a large collection of leaves (vertices of degree 1). Assume that a represents 20 leaves, and that there are 3 players. There is a Nash Equilibrium at $\mathbf{x} = (v_1, v_3, v_4)$ in G. However, it can be verified that this is not a Nash Equilibrium in $\hat{G}$. In fact, $\hat{U}_2(v_1, v_2', v_4) > \hat{U}_2(v_1, v_3, v_4)$, and $\mathbf{x}_2 = (v_1, v_2', v_4)$ is a Nash Equilibrium.

CHAPTER 6

Potential Games

6.1 Introduction

In this chapter, we discuss a topic that is related to the work of previous chapters. We consider the existence of potential functions for the competitive information diffusion model on the ILT graph. Potential functions are an important concept in game theory. A game is said to be a potential game if the incentive of all players to change their strategy can be expressed using a single global function called the potential function. The potential function can be used to analyze the equilibrium properties of games, as the payoffs of all players are mapped into one function, and the set of pure Nash equilibria can be found by locating the local optima of the potential function.

6.2 Definitions

Let $\mathbf{x}_1,\ \mathbf{x}_2$ be two strategy profiles such that $\mathbf{x}_1 = (a_1, .., a_n)$ and $\mathbf{x}_2 = (a_1, .., a_{s-1}, v, a_{s+1}, ..)$ where $v \neq a_m$ for $m \in \{1, .., n\}$. We introduce some definitions from Monderer and Shapley's seminal paper [41]. A finite strategic game Γ is an *exact potential game* if there exists a potential function such that for every player $i \in \{1, .., n\}$ and for all pairs $\mathbf{x}_1,\ \mathbf{x}_2$

$$U_i(\mathbf{x}_1) - U_i(\mathbf{x}_2) = \phi(\mathbf{x}_1) - \phi(\mathbf{x}_2)$$

Γ is an *ordinal potential game* if there exists a potential function such that for every player $i \in \{1, .., n\}$ and for all $\mathbf{x}_1,\ \mathbf{x}_2$

$$U_i(\mathbf{x}_1) - U_i(\mathbf{x}_2) > 0 \Leftrightarrow \phi(\mathbf{x}_1) - \phi(\mathbf{x}_2) > 0$$

Γ is a *general ordinal potential game* if there exists a potential function such that for every player $i \in \{1, .., n\}$ and for all $\mathbf{x}_1,\ \mathbf{x}_2$

$$U_i(\mathbf{x}_1) - U_i(\mathbf{x}_2) > 0 \Rightarrow \phi(\mathbf{x}_1) - \phi(\mathbf{x}_2) > 0$$

A path on the set of strategy profiles is a sequence $\gamma = (\mathbf{x}_1, \mathbf{x}_2, ..)$ such that for every $k \geq 1$ $\mathbf{x}_k, \mathbf{x}_{k-1}$ differ in exactly one element. A path $(\mathbf{x}_1, \mathbf{x}_2, ..)$ is an *improvement path* if for all $k \geq 2\,; U_i(\mathbf{x}_k) > U_i(\mathbf{x}_{k-1})$ where $\mathbf{x}_k$ and $\mathbf{x}_{k-1}$ differ in the ith element. A game has the *finite improvement property* (FIP) if every improvement path is finite.

A useful result from [41] is that a game Γ has the FIP if and only if Γ has a generalized ordinal potential.

6.3 Potential Games on the ILT model

Consider again the game theoretic model of information diffusion on the ILT model. Following on from our work in the previous chapter, we consider the following general question. If there exists a potential function for the game on a graph G, does this imply that there exists one on $\hat{G}$. First let's examine a simple example. Consider the graph G in Figure 6.1 below. We now establish the existence of a potential function ϕ on the graph G below.

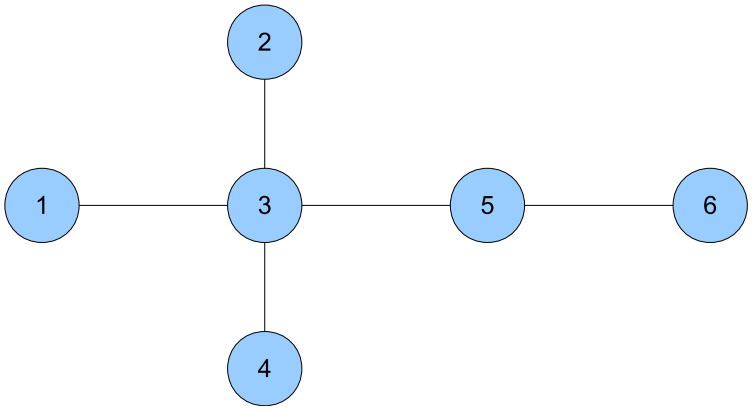


Figure 6.1: Graph G

Let

$$M_{x_i} = \{v : d(x_i, v) = 1, v \notin \mathbf{x}, d(x_j, v) \neq 1 \; \forall \; i \neq j\}$$

and let the potential function be $\phi(\mathbf{x}) = |\bigcup_{i=1}^{3} M_{x_i}|$

If $\mathbf{x}_1$ and $\mathbf{x}_2$ are strategy profiles for three players differing only in element i, then

$$U_i(\mathbf{x}_1) > U_i(\mathbf{x}_2) \Rightarrow \phi(\mathbf{x}_1) > \phi(\mathbf{x}_2)$$

here. The maximum value of ϕ is $\phi_{max}(\mathbf{x}) = 3$ which occurs only at the Nash Equilibria, which exist on the strategy profiles $(3, 5, 6), (2, 3, 5)$ and $(1, 3, 5)$.

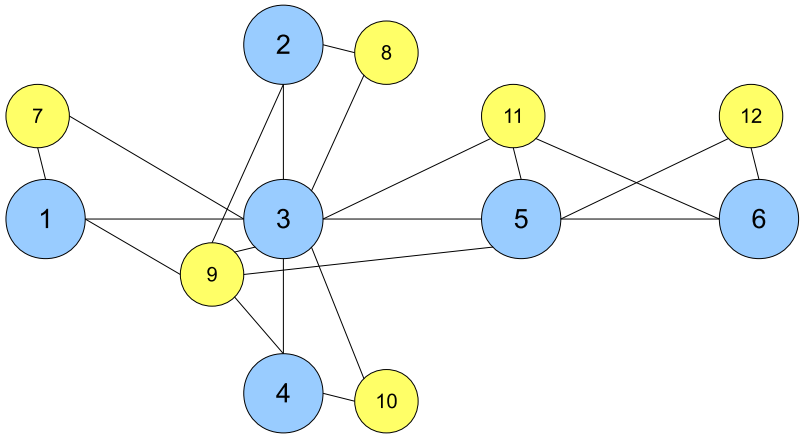


Figure 6.2: Graph $\hat{G}$

As before $\hat{G}$ denotes the graph G after applying one step of the ILT process. We wish to see if there is a simple relation between the potential function ϕ on G and $\hat{\phi}$ on $\hat{G}$. We try a simple candidate potential function, which unfortunately fails to work.

Let

$$\hat{\phi}(\mathbf{x}) = \phi(\Pi(x))$$

where $\Pi(\mathbf{x}) = \Pi(x_1, ., x_n) = (\pi(x_1), .., \pi(x_n))$, and for all $v \in V(\hat{G})$ $\pi(v)$ maps v to a vertex in $V(G)$

$$\pi(v) = \begin{cases} w & \text{if } v = w' \text{ for some } w \in V(G) \\ v & \text{if } v \in V(G) \end{cases}$$

If $\hat{\phi}$ is a generalized ordinal potential for a game on $\hat{G}$ then

$$U_i(\mathbf{x}_1) - U_i(\mathbf{x}_2) > 0 \Rightarrow \phi(\mathbf{x}_1) - \phi(\mathbf{x}_2) > 0$$

for every player $i \in \{1, .., n\}$ and for all $\mathbf{x}_1,\ \mathbf{x}_2$ such that $\mathbf{x}_1$ and $\mathbf{x}_2$ are strategy profiles differing only in element i.

However, if we choose strategy profiles $\mathbf{x}_1 = (5, 10, 12)$ and $\mathbf{x}_2 = (8, 10, 12)$, then $\hat{U}_1(\mathbf{x}_1) = 7$, $\hat{U}_1(\mathbf{x}_2) = 2$, and $\hat{\phi}(\mathbf{x}_1) = 0$, $\hat{\phi}(\mathbf{x}_2) = 0$, thus

$$\hat{U}_1(\mathbf{x}_1) > \hat{U}_1(\mathbf{x}_2) \text{ but } \hat{\phi}(\mathbf{x}_1) = \hat{\phi}(\mathbf{x}_2)$$

So $\hat{\phi}$ is not a general ordinal potential on $\hat{G}$.

In general it is difficult to relate potential functions on G to potential functions on $\hat{G}$. However, the following simple fact does hold.

Lemma 6.3.1. *If there exists a generalized ordinal potential on $\hat{G}$ then there exists a generalized ordinal potential on G.*

Proof. Let $\hat{\phi}$ be a generalized ordinal potential function on $\hat{G}$. We try to construct a generalized ordinal potential ϕ on G. We define $\phi(v_1, .., v_n) = \hat{\phi}(v_1, .., v_n)$ where $v_i \in V(G)$ for all i. Now for $\mathbf{x}$ such that $x_i \in V(G)$ for all i,

$$\hat{U}_i(\mathbf{x}) = \begin{cases} 2U_i(\mathbf{x}) - 1 & \text{if } x_i \text{ is neighboured by some } x_j \\ 2U_i(\mathbf{x}) & \text{if } x_i \text{ is not neighboured by some } x_j \end{cases}$$

so for all $\mathbf{x}_2$ such that $\mathbf{x}_2$ is a strategy profile on G and $\mathbf{x}$ differs from $\mathbf{x}_2$ only in element i,

$$U_i(\mathbf{x}_2) > U_i(\mathbf{x}) \Rightarrow \hat{U}_i(\mathbf{x}_2) > \hat{U}_i(\mathbf{x})$$

We know that

$$\hat{U}_i(\mathbf{x}_2) > \hat{U}_i(\mathbf{x}) \Rightarrow \hat{\phi}(\mathbf{x}_2) > \hat{\phi}(\mathbf{x})$$

thus

$$U_i(\mathbf{x}_2) > U_i(\mathbf{x}) \Rightarrow \phi(\mathbf{x}_2) > \phi(\mathbf{x})$$

So if there exists a generalized ordinal potential on $\hat{G}$ then there exists a generalized ordinal potential on G.

□

This is equivalent to the next lemma.

Lemma 6.3.2. *If $\hat{G}$ has the FIP then G has the FIP, and thus has a generalized ordinal potential.*

Proof. Let $\mathbf{x}_1,\ \mathbf{x}_2$ be strategy profiles where all the elements of $\mathbf{x}_1$ and $\mathbf{x}_2$ are elements of G, and the profiles only differ in element i. Suppose that in $\hat{G}$ $\hat{U}_i(\mathbf{x}_1) > \hat{U}_i(\mathbf{x}_2)$. Then in G it is the case that $U_i(\mathbf{x}_1) \geq U_i(\mathbf{x}_2)$, since

$$\hat{U}_i(\mathbf{x}) = \begin{cases} 2U_i(\mathbf{x}) - 1 & \text{if } x_i \text{ is neighboured by some } x_j \\ 2U_i(\mathbf{x}) & \text{if } x_i \text{ is not neighboured by some } x_j \end{cases}$$

Thus if an improvement path terminates in $\hat{G}$ clearly it will terminate in G. If $\hat{G}$ has the FIP then every improvement path in $\hat{G}$ will terminate. Since in this case every improvement path in G is finite, G has the FIP. □

We define a loop here as a sequence of strategy profiles $\mathbf{x}_1, \mathbf{x}_2, ..\mathbf{x}_z$ such that x_i, x_{i+1} differ in exactly one element for all $i \in 1, .., z-1$.

$$U_k(\mathbf{x}_i) < U_k(\mathbf{x}_j)$$

Lemma 6.3.3. *A game has a loop if and only if it does not have a general ordinal potential.*

Proof. If a loop exists, a potential game cannot exist since for some sequence of uniquely deviating strategy profiles, with $\mathbf{x}_1 = \mathbf{x}_z$

$$\phi(\mathbf{x}_z) > \phi(\mathbf{x}_{z-1}) > .. > \phi(\mathbf{x}_1)$$

but $\mathbf{x}_1 = \mathbf{x}_z$, so the graph has no general ordinal potential game.

If there are no loops in any improvement path in the game Γ, then every improvement path must be finite if the number of vertices is finite, since the path cannot then be longer than the number of strategy profiles on the graph. Any maximal finite improvement path ends in a Nash Equilibrium, so the graph must have a Nash Equilibrium. Also since a graph has the FIP if and only if it has a generalized ordinal potential then Γ must be a general ordinal potential game. □

If a game on a graph is path independent then there is at most one improvement path between any two strategy profiles. Path independence implies that no loop exists, but not vice versa, as there could be two improvement paths of different lengths with the same initial and final vertices.

Lemma 6.3.4. *Every improvement path in G will be an improvement path in $\hat{G}$.*

Proof. Let K be the set of all improvement paths in G, and K' be the set of these paths in $\hat{G}$. For every path $(\mathbf{x}_1, \mathbf{x}_2, ..) \in K$, $U_i(\mathbf{x}_k) > U_i(\mathbf{x}_{k-1})$ for $k > 1$ where $\mathbf{x}_k$ and $\mathbf{x}_{k-1}$ differ in the ith element. Then, since

$$(*) \quad \hat{U}_i(\mathbf{x}) = \begin{cases} 2U_i(\mathbf{x}) - 1 & \text{if } x_i \text{ is neighboured by some } x_j \\ 2U_i(\mathbf{x}) & \text{if } x_i \text{ is not neighboured by some } x_j \end{cases}$$

we have that

$$\hat{U}_i(\mathbf{x}_k) > \hat{U}_i(\mathbf{x}_{k-1}) \; for \; k > 1$$

□

The following lemma contains an example of a graph G in which every improvement path is finite, but has an improvement path in $\hat{G}$ which is not finite. Thus there may be new improvement paths in $\hat{G}$ which are not finite.

Lemma 6.3.5. *A game with a N.E. in G and $\hat{G}$, and which has an exact potential game in G does not necessarily have a generalized potential game in $\hat{G}$. This also shows that a game with a N.E. in G and $\hat{G}$, and which has a generalized ordinal potential game in G does not necessarily have a generalized ordinal potential game in $\hat{G}$.*

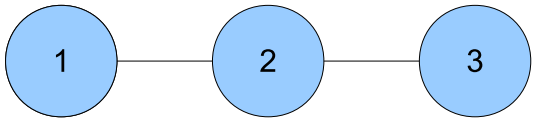


Figure 6.3: Graph G

Proof. To show that the graph G in Figure 6.3 has an exact potential game, we must show that there exists a potential function ϕ such that for every player $i \in \{1, .., n\}$ and for all pairs of strategy profiles $\mathbf{x}_1,\ \mathbf{x}_2$ which only differ in the ith element

$$U_i(\mathbf{x}_1) - U_i(\mathbf{x}_2) = \phi(\mathbf{x}_1) - \phi(\mathbf{x}_2)$$

If we use the potential function

$$\phi = |\bigcup_{i=1}^{3} N_{x_i}|$$

where:

$$N_{x_i} = \{v : d(x_i, v) = 1\}$$

we can show that the game has an exact potential game on G by checking every possible deviation from a starting strategy profile.

$$\mathbf{x}_1 = \{1, 2\} \to \{1, 3\} = \mathbf{x}_2,\ U_i(\mathbf{x}_1) - U_i(\mathbf{x}_2) = 2 - 1,\ \phi(\mathbf{x}_1) - \phi(\mathbf{x}_2) = 3 - 2$$
$$\mathbf{x}_1 = \{1, 2\} \to \{3, 2\} = \mathbf{x}_2,\ U_i(\mathbf{x}_1) - U_i(\mathbf{x}_2) = 1 - 1,\ \phi(\mathbf{x}_1) - \phi(\mathbf{x}_2) = 3 - 3$$
$$\mathbf{x}_1 = \{1, 3\} \to \{2, 3\} = \mathbf{x}_2,\ U_i(\mathbf{x}_1) - U_i(\mathbf{x}_2) = 1 - 2,\ \phi(\mathbf{x}_1) - \phi(\mathbf{x}_2) = 2 - 3$$

We construct a directed graph where the vertices of the graph are strategy profiles. A directed edge is drawn from between two vertices if and only if unilaterally changing the vertex to which an agent is assigned in the strategy profile will result in an increase in utility for this agent.

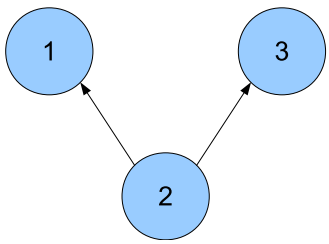


Figure 6.4: Graph of all strategy profiles for G

The numbered vertices in Figure 2 below represent the strategy profiles as follows:

$$1 = (1, 2), 2 = (1, 3), 3 = (2, 3)$$

So as we can see from Figure 2 all the improvement paths are finite, and terminate at the Nash Equilibria $(1, 2)$ and $(2, 3)$. Therefore the game on G has the finite improvement property and has a generalized ordinal potential game.

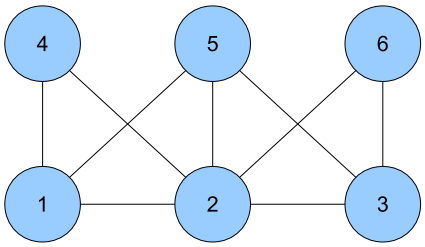


Figure 6.5: $\hat{G}$

The game on $\hat{G}$ has no potential game, since we can construct an infinite improvement path out of the following sequence of strategy profiles:

$$\{1, 6\} \to \{3, 6\} \to \{3, 4\} \to \{1, 4\} \to \{1, 6\}$$

The fact that $\hat{G}$ has no potential game can also be seen from the graph of strategy profiles for $\hat{G}$ below. The Nash Equilibrium in G remains a Nash Equilibrium in $\hat{G}$. This is then an example of a graph with a Nash equilibrium but no potential game. The numbered vertices in Figure 4 represent the strategy profiles as follows:

$1 = (1, 2), 2 = (1, 3), 3 = (1, 4), 4 = (1, 5), 5 = (1, 6), 6 = (2, 3)7 = (2, 4), 8 = (2, 5)$

$9 = (2, 6), 10 = (3, 4), 11 = (3, 5), 12 = (3, 6), 13 = (4, 5), 14 = (4, 6), 15 = (5, 6)$

□

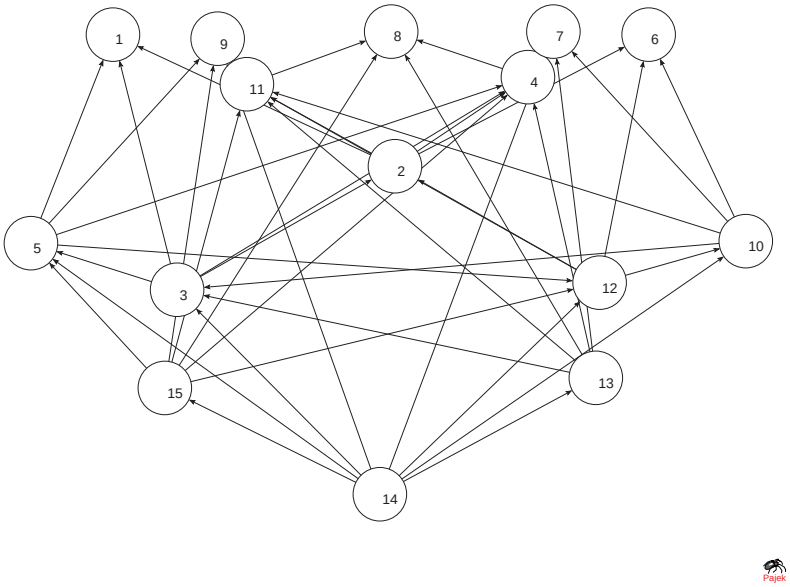


Figure 6.6: Graph of all strategy profiles for $\hat{G}$

CHAPTER 7

Conclusions

Motivated by the importance of information dissemination on modern online social networks, we have studied a competitive diffusion process on the iterated local transitivity model.

Chapters 2 and 3 provided some background on graph models, properties of graphs, and online social network models. Chapter 3 introduced the iterated local transitivity model which we used later in the thesis. In chapter 4 various models for games played on networks were examined. We studied a model for competitive information diffusion on star graphs, cliques and trees, and we provided conditions for the existence of Nash Equilibria on these. We have shown that the model of competitive information diffusion studied in [7], [50] always admits a Nash Equilibrium on a tree when the number of agents is two. While trees are a restrictive class of graphs, they can serve as an idealisation of hierarchical structures that arise in many social networks. Identifying other structures which guarantee the existence of a Nash Equilibrium and characterising these when they exist remains a challenging question for future research.

The model for competitive information diffusion was studied in detail for the iterated local transitivity model in chapter 5. Specifically, we have studied the model [13] for two competing agents and shown that an independent Nash Equilibrium on the initial graph G_0 will remain a N.E. for all $t > 0$. We have also described a counterexample to illustrate that this will not hold for non-independent Nash Equilibria. We showed that if two competing agents select an independent equilibrium strategy at some time, then neither of them

can benefit from unilaterally changing their strategy at any subsequent time in the evolution of the underlying network. This suggests that, in this case the alterations in the underlying social network do not effect the dynamics of the game. It would be interesting to test the extent to which this conclusion holds for real social networks, for which time-course data can be obtained. We discussed potential games in chapter 6 and their existence on the iterated local transitivity model.

The work here can be viewed as a starting point for a significant programme on the timely topic of information diffusion on social networks. There are several possible ways to extend our results. These include investigating alternative models of information diffusion and rumour spread [19], on the ILT model in the spirit in which epidemic models have been analysed on graphical models recently [27], [20]. It would also be interesting to investigate the possibility of obtaining similar results for the diffusion process on the stochastic ILT models described in [13] or on other models of online social networks such as Kronecker graph models [34].

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