# Global structure and finite-size effects in the $f(\alpha)$ of diffusion-limited aggregates

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The multifractal spectrum $f(\alpha)$ characterizing the scaling properties of the growth probability on the boundary of radial diffusion-limited aggregates is known to exhibit strong finite size effects. We demonstrate that there exists a correlation between these finite size effects and those present in measurements of the angular width of the fjords which lie between the principal cluster arms. We subsequently conclude that it is the evolution in the global structure of the clusters which is responsible for the slow convergence in $f(\alpha)$ and discuss how this global structure induces a phase transition in $f(\alpha)$.

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## I. INTRODUCTION

Diffusion-limited aggregation (DLA) is a simple stochastic growth model which spontaneously gives rise to complex fractal structure [1]. In the standard off-lattice radial version of the model, an immobile cluster is grown on a plane about some fixed initial seed by the adherence of diffusing particles which are sequentially released from random points lying on a circle surrounding the cluster. The clusters produced are fractal with a dimension $D \approx 1.71$ [2] and possess a global structure or geometry which may be summarily described as consisting of a finite number of principal arms which radiate outward from the central seed point (see Fig. 3).

At finite cluster sizes, this structure is, however, not invariant with growth. Instead, the number of these principal cluster arms is found to slowly increase, while the angular width of the main fjords between these arms exhibits a corresponding decrease [3,4]. This behavior is, in turn, primarily responsible for other reported phenomena such as the anomalous scaling of the width of the growth zone, the related multiscaling description of DLA, and the increase in the filling ratio with growth [5–8].

In this paper, we shall argue that the changes in the global structure of radial DLA clusters is also responsible for another anomalous phenomenon, namely, the "negative-moment problem" relating to the scaling behavior of the growth probability measure [9].

This measure has, in fact, attracted much research attention since the inception of the model, primarily because of its complex scaling properties. Due to the scale invariance of both the branch structure and the Laplacian field which determines the probability of growth occurring along a cluster boundary, its distribution is found to be nontrivially self-similar. It is thus a multifractal measure, and one of the earliest examples of its type to be discovered occurring naturally in a physical system [10].

The characterization of the self-similar properties of such a measure involves the evaluation of its multifractal spectrum $f(\alpha)$ which is typically a convex nonzero function over some interval $[\alpha_{min}, \alpha_{max}]$ [11]. However, estimates of $f(\alpha)$ obtained from the growth probability of radial DLA clusters were found to exhibit significant finite size effects at large values of $\alpha$ [12–15]. This phenomenon is, in turn, intimately related to the apparent nonpower law scaling of the minimal growth probability with cluster growth [12,15–18], the anomaly which lies at the heart of the "negative-moment problem."

Though much effort has been spent determining the asymptotic form of $f(\alpha)$ and the precise dependence of the minimal growth probability on the cluster mass $M$ [12,15–20], there has been little investigation into the actual origins of the finite size effects in $f(\alpha)$. The purpose of this paper is to primarily address this issue.

In Sec. II, we begin by first evaluating finite size estimates of $f(\alpha)$ obtained from off-lattice radial DLA clusters and argue that it is the global structure of the clusters which determines the form of these estimates at large $\alpha$. We then present numerical data which demonstrates that the finite size effects present at such large $\alpha$ are simply a reflection of the aforementioned changes which occur in the global structure with growth.

In Sec. III, we also discuss the dichotomy that exists in the structure of radial DLA clusters. That is, their structure may be decomposed into two essentially independent entities, namely, the $n$-fold global structure of the principal cluster arms, and the fractal structure of the branches lying within these arms. We discuss how this dichotomy subsequently generates two quite disparate subsets in the growth probability measure and argue that the union of two such sets presents a mechanism for the appearance of a phase transition in the multifractal spectra.

## II. FINITE SIZE EFFECTS IN $f(\alpha)$

### A. Calculation of $f(\alpha)$

We began our study by first growing 100 radial off-lattice DLA clusters up to a maximum size of 60 000 particles. Taking these clusters at different stages in their growth, the growth probability along their boundary was calculated by first forcing each off-lattice cluster onto a square lattice before subsequently solving Laplace's equation $\nabla^2\phi=0$ with the appropriate boundary conditions ($\phi=0$ on all cluster perimeter sites and $\phi=1$ on a circular boundary surrounding the cluster) [21]. A standard relaxation method was used to solve this boundary value problem and the growth probability estimated by calculating the gradient of the Laplacian field $\phi$ at the cluster perimeter sites.

 

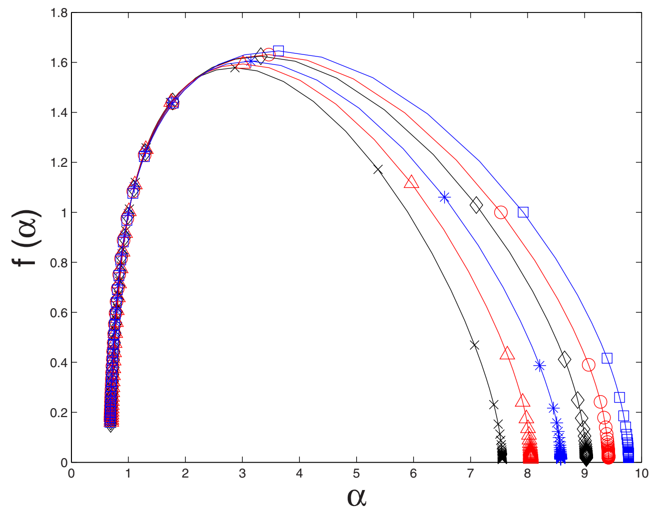


FIG. 1. (Color online) Estimates of $f(\alpha)$ obtained from 100 DLA clusters when the cluster mass $M=1875$ (×), 3750 (△), 7500 (*), 15 000 (◇), 30 000 (○), and 60 000 (□) particles.

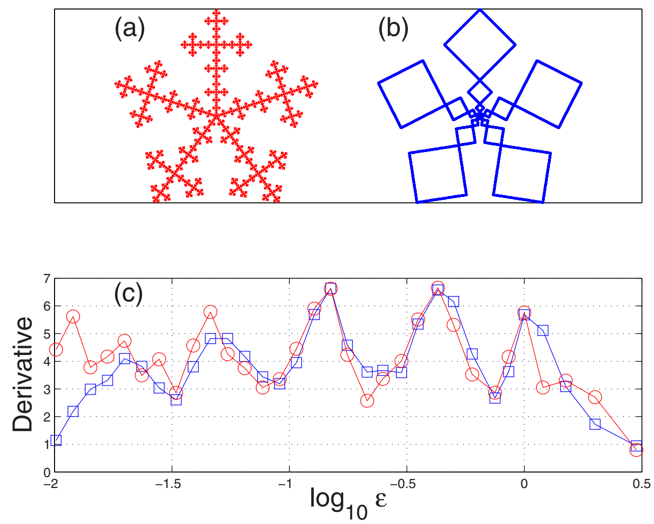


FIG. 2. (Color online) Derivative of $\frac{1}{(q-1)}\log \chi_q$ vs $\log \epsilon$ plots for $q=-9$ obtained from the harmonic measure on the boundary of our multiarmed fractal object (○) and on its envelope (□).

Subsequently, the scaling properties of this measure, commonly referred to as the harmonic measure, were investigated by implementing the method of moments on each individual cluster [22,23]. Briefly, this method involves the coarse graining of a cluster with radius of gyration $R_g$, by laying down upon it a uniform grid of boxes of side $l$. The total growth probability $\mu_i(l)$ contained inside the $i$th box is then calculated before computing, for some real value $q$, the quantity $\chi_q(l)=\Sigma_i \mu_i^q(l)$ over all boxes having nonzero measure. This process is repeated for different values of $l$ to obtain a doubly logarithmic plot of $\chi_q(l)$ which, if the measure is indeed self-similar, should take on a linear form with slope $\tau(q)$ [11].

Taking values of $q$ from −9 to +9 in steps of 0.06, as done in all subsequent multifractal work reported in this paper, such plots were averaged over the ensemble of 100 clusters to obtain the necessary quenched $\frac{1}{(q-1)}\langle \log \chi_q \rangle$ vs $\log \epsilon$ plots ($\epsilon = l/R_g$).

Identifying the linear regions in these plots, their slopes yield an estimate of $D(q)=\tau(q)/(q-1)$, the generalized spectrum of dimensions [22], from which the associated $f(\alpha)$ function may be obtained via the Legendre transformation [11]

$$\alpha(q) = \frac{d}{dq}\tau(q), \quad f(\alpha(q)) = q\alpha(q) - \tau(q). \tag{1}$$

The resultant estimates of $f(\alpha)$ are shown in Fig. 1. As previously reported in the literature, these estimates fail to converge to a single well-defined $f(\alpha)$ and we instead observe a characteristic divergence in its tail at large $\alpha$ as the clusters increase in size. The precise cause of these finite size effects is the issue which we shall now address.

### B. Global structure and the minimal measure

At large $\alpha$, the $f(\alpha)$ function characterizes the scaling properties of the minimal elements in the set $\{\mu_i\}$. In fact, for the smallest such element $\mu_{min}$, we expect that $\mu_{min} \sim \epsilon^{\alpha_{max}}$ if the distribution of the measure is statistically self-similar. However, it has long been known that for radial DLA, $\mu_{min}$ decreases faster than a power law with the cluster radius $R_g$, and that the right-hand side of $f(\alpha)$ diverges as a consequence [12,15–18]. If we are to find an explanation for the finite size effects in $f(\alpha)$, we must therefore endeavor to understand the reason for the anomalous scaling of $\mu_{min}$.

Let us thus begin by stating that in a DLA cluster, we expect the coarse-grained minimal measure $\mu_{min}$ to be generally found in the so-called frozen zone about the central seed point, where the screening to any incoming particle is a maximum. Critically for what is to follow, it transpires that the scaling of $\mu_{min}$ is, as a consequence, determined by the coarse wedgelike structure of the principal cluster arms, i.e., by the global structure of the cluster.

As a graphic demonstration of this point, the multiarmed fractal of Fig. 2(a) was constructed as a crude model of a DLA cluster, the principal arms of which possess a simple self-similar structure identical to that of the Vicsek snowflake fractal (see Fig. 8) [24]. Calculating the harmonic measure on its boundary, the method of moments was again implemented and the necessary $\frac{1}{(q-1)}\log \chi_q$ vs $\log \epsilon$ plots obtained. Note that at large negative $q$, it is $\mu_{min}$ which begins to dominate the scaling behavior of $\chi_q(\epsilon)=\Sigma_i \mu_i^q$. The derivative of the log-log plot for $q=-9$ is thus shown in Fig. 2. Also shown, however, is the derivative of the log-log plot, again for $q=-9$, obtained by coarse graining the harmonic measure existing on the envelope of the multiarmed fractal [see Fig. 2(b)]. We observe that the derivatives of the two log-log plots are essentially identical.

This thus demonstrates that the scaling of $\mu_{min}$ is determined by the global structure and geometry of the object and not by the fine detail of its internal self-similar structure. In this paper, we claim that this is also the situation with the harmonic measure existing on the boundary of DLA clusters.

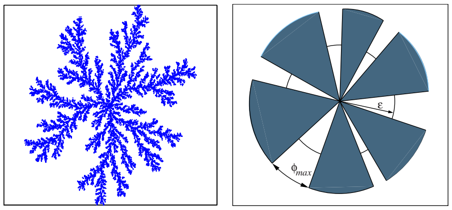


FIG. 3. (Color online) On the left is shown a DLA cluster. Also shown is an approximation of its global structure. As $\epsilon \to 0$, the harmonic measure contained within a circle of radius $\epsilon$ is expected to scale as $\mu(\epsilon) \sim \epsilon^{\pi/\phi_{max}}$.

### C. Correlation between $\alpha_{max}$ and $\phi_{max}$

In Fig. 3, we crudely model the global structure of a DLA cluster about its seed point by a series of wedges of varying angular widths. We shall show later in more detail that the harmonic measure $\mu_{tip}$ existing within a distance $\epsilon$ from the tip of a wedge of exterior angle $\phi$ scales as $\mu_{tip} \sim \epsilon^{\pi/\phi}$ [20,25]. Assuming the minimal measure to exist about the seed point, as a first approximation, we may thus expect that in a DLA cluster

$$\mu_{min}(\epsilon) \underset{\epsilon \to 0}{\sim} \epsilon^{\pi/\phi_{max}}, \tag{2}$$

where $\phi_{max}$ is the maximum angle existing between any two adjacent principal cluster arms. As $\mu_{min} \sim \epsilon^{\alpha_{max}}$, it follows that

$$\alpha_{max} \sim 1/\phi_{max}. \tag{3}$$

We thus predict a correlation between $\phi_{max}$, a parameter relating to the global structure, and $\alpha_{max}$, the scaling exponent of the coarse-grained minimal measure $\mu_{min}$.

With growth, however, it is known that the angle $\phi_{max}$ decreases very slowly [3] as the DLA clusters become more compact, and the distribution of the mass more homogeneous [4,8].

As such, our simple analysis suggests that this decrease in $\phi_{max}$ may be responsible for the increase in $\alpha_{max}$ observed in Fig. 1. [Incidentally, it also suggests that $\alpha_{max}$ is intimately related to the apparent exponent characterizing the power-law tail in $P(r,M)$, the radial growth probability distribution [26].]

To test this hypothesis, from Fig. 1 we thus obtained estimates of $\alpha_{max}$ at cluster masses of $M=1875$, 3750, 7500, 15 000, 30 000, and 60 000 particles. To obtain values of $\phi_{max}$ from individual clusters, we followed a scheme originally used when studying the distribution of gap sizes obtained from circular crosscuts of DLA clusters [3]. There, a circular crosscut is defined as being simply the intersection of a cluster with a circle of radius $0.75R_g$ cenetered on the seed point. The angular width of the largest gap in such a crosscut should therefore yield an estimate of $\phi_{max}$. Values of $\phi_{max}$ were thus measured from our set of 100 DLA clusters and the ensemble averages $\langle \phi_{max} \rangle$ obtained at each of the cluster masses quoted above.

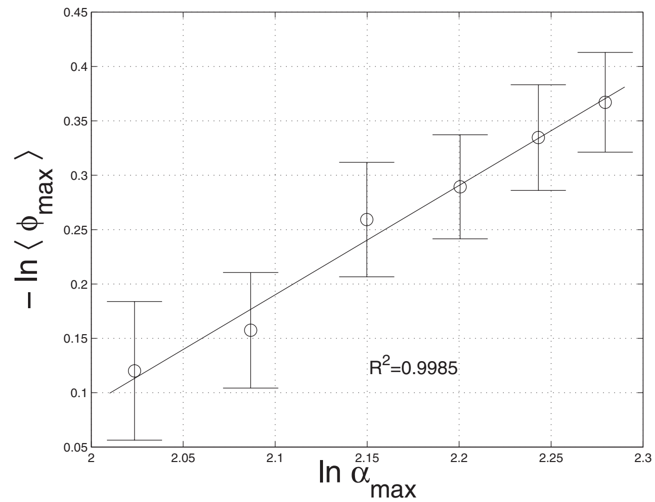


FIG. 4. Data obtained from 100 clusters at six different cluster sizes. A linear fit of the form $y=mx+c$ yields the values $m=1.01\pm0.25$ and $c=-1.92\pm0.55$. Error bars and errors quoted denote 95% confidence limits.

In Fig. 4, we show the resultant plot of $\log 1/\langle \phi_{max} \rangle$ against $\log \alpha_{max}$ which clearly illustrates a linear correlation between these two quantities. We conclude therefore that the finite size effects observed in $f(\alpha)$ are linked to the slow large-scale structural changes which occur in DLA with growth.

Furthermore, as it has previously been shown that $\langle \phi_{max} \rangle$ approaches a finite nonzero value with an order of magnitude estimate of $\approx 0.45$ rad as $M \to \infty$ [27], our results suggest that the asymptotic value of $\alpha_{max}$ is also finite, a conclusion which is supported by other arguments and numerical results in the literature [5,19,20,28]. In fact, we find that extrapolating the fit in Fig. 4 to $\langle \phi_{max} \rangle = 0.45$ yields the asymptotic estimate $\alpha_{max} \approx 15$.

## III. PHASE TRANSITION IN $f(\alpha)$

The presence of a phase transition in the multifractal spectra of the growth probability of DLA has long been a subject of much debate. The existence of such a transition in Fig. 1 is betrayed by a characteristic bunching of the data points at large $\alpha$, though, using alternative methods, it also manifests itself as linearity in the tail of $f(\alpha)$ [12,19,20].

In this section, we discuss the possibility that this phase transition is simply the result of the union of two disparate subsets in our measure, each having entirely different scaling properties. As a simple illustration of this mechanism, let us first examine the scaling properties of the harmonic measure lying on the boundary of a wedge of exterior angle $\phi$.

### A. Harmonic measure on a wedge

To obtain the harmonic measure along the boundary of a wedge such as that shown in Fig. 5 requires solving the Laplace equation subject to constant Dirichlet boundary conditions. This may easily be done analytically via the method

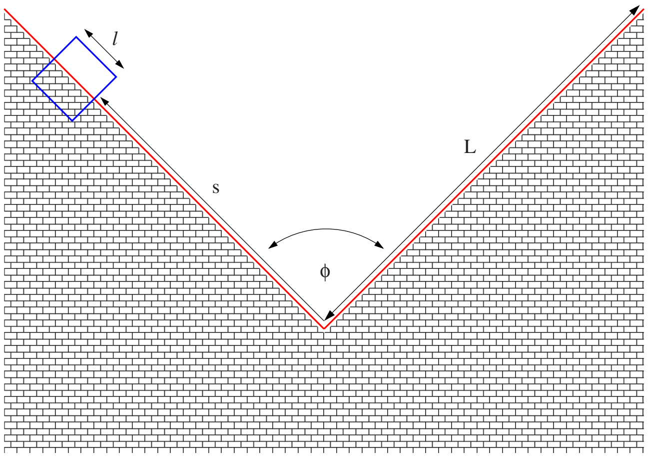


FIG. 5. (Color online) Wedge of exterior angle $\phi$. Harmonic measure within the box of side $l$ is given by Eq. (6).

of conformal maps if one can obtain the complex function which maps the unit circle in the complex plane onto the wedge boundary.

One can verify that the function

$$\Gamma(w) = \left( i\frac{w+1}{w-1} \right)^{1/\eta} \tag{4}$$

satisfies this criterion when we have $\eta = \pi/\phi$ [20,25]. Using Eq. (4) one can subsequently calculate the harmonic measure density on the wedge boundary at a distance $s$ from its tip (see Fig. 5) to be

$$E(s) \sim s^{\eta-1}. \tag{5}$$

Restricting our attention to the wedge section $0 \leqslant s \leqslant L$, we therefore have that the normalized measure $\mu_s(\epsilon)$ contained within a box of side $l$ located at a distance $s$ from the tip is given by

$$\mu_s(\epsilon) = \int_s^{s+l} E(x)dx = \left(\frac{s}{L} + \epsilon\right)^\eta - \left(\frac{s}{L}\right)^\eta, \tag{6}$$

where again $\epsilon = l/L$.

Laying down a grid of boxes, each of side $l$, along our wedge boundary, we may use this expression to calculate $\chi_q(\epsilon) = \Sigma_i \mu_i^q$, the moments of the coarse-grained measure distribution, which can be shown to behave as

$$\chi_q(\epsilon) \sim A(q)\epsilon^{q-1} - \epsilon^{\eta q}. \tag{7}$$

Critically, there are two terms in Eq. (7) above. Defining $q_c = -1/(\eta-1)$, for $q > q_c$, the second term becomes negligible as $\epsilon \to 0$ so that the first term dominates, while the converse is true for $q < q_c$. We thus have a phase transition occurring in the scaling properties of $\chi_q(\epsilon)$ at the critical value $q_c$.

As $\chi_q(\epsilon) \sim \epsilon^{\tau(q)}$, from Eq. (7) it is easily shown that

$$\tau(q) \sim \begin{cases} \eta q, & q \leqslant q_c, \\ q-1, & q > q_c, \end{cases} \tag{8}$$

and we see that this phase transition betrays its existence by a characteristic kink in the $\tau(q)$ function at $q_c$ (see Fig. 6). Subsequently taking the Legendre transform

$$f(\alpha) = \inf_q \{q\alpha - \tau(q)\} \tag{9}$$

of Eq. (8) yields the linear $f(\alpha)$ of Fig. 6.

The origins of this phase transition may in fact be gleaned quite easily from a cursory inspection of Eq. (6). At the wedge tip ($s=0$) we see that $\mu_s(\epsilon) \sim \epsilon^\eta$, whereas at any other point along the boundary we obtain $\mu_s(\epsilon) \sim \epsilon$ asymptotically as $\epsilon \to 0$. Our measure thus exhibits usual power-law scaling $\mu \sim \epsilon^\alpha$, of which, critically, there are two distinct types, each being characterized by the Hölder exponent values $\alpha = \eta$ and 1. For $q > q_c$, it is the measure characterized by the latter type of singularity which dominates $\chi_q(\epsilon)$ as $\epsilon \to 0$. As we decrease $q$ past $q_c$, however, $\chi_q(\epsilon)$ suddenly becomes dominated by the measure ($\alpha = \eta$) at the wedge tip and a phase transition results.

From this simple heuristic example, we thus see that the coexistence of two quite distinct subsets of the harmonic measure, each with their own unique scaling properties, can give rise to a phase transition in the multifractal spectra. We shall now argue that a similar phenomenon is responsible for the phase transition in the $f(\alpha)$ of radial DLA.

### B. Dichotomy in DLA

Though rarely stated explicitly, there is an essential dichotomy in the structure of radial DLA clusters. That is, they possess both the self-similar *fractal branch structure* present within the confines of the principal cluster arms, and a *global structure* as prescribed by the radial $n$-fold symmetry of these arms. It is important to note that these two entities are effectively independent of each other, the former being a product of the diffusion process inherent in the model, while the latter is determined by the specific boundary conditions used when growing the clusters (For example, choosing instead to release particles from a line far above a linear substrate, rather than from a circle surrounding a seed point, produces clusters which lack the $n$-fold symmetry of radial DLA [29].)

As we shall now demonstrate, a Laplacian field interacting with an object possessing such dichotomy in its structure may give rise to two subsets of the harmonic measure with wholly different scaling properties, and a phase transition may occur in its multifractal spectra as a result.

As an illustration of this idea, let us again use the multiarmed fractal of Fig. 2(a), an object which possesses $n$-fold symmetry and internal fractal structure like radial DLA. In Fig. 7 we show the $f(\alpha)$ spectra calculated for the harmonic measure existing on the boundary of both the multiarmed fractal and the simple homogeneous snowflake fractal of Fig. 8. Let us denote these spectra by $f_m(\alpha)$ and $f_s(\alpha)$, respectively.

We find that to the left of their maxima, the $f(\alpha)$ spectra are virtually indistinguishable. This must obviously be a con-

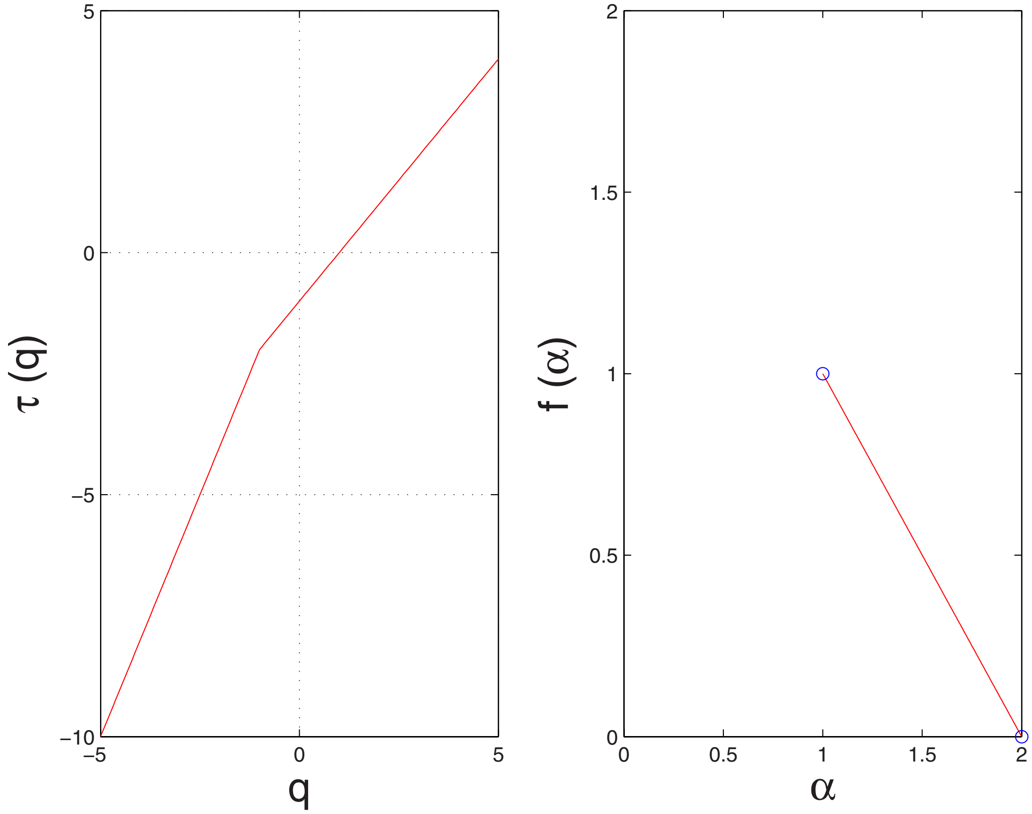


FIG. 6. (Color online) The $\tau(q)$ and $f(\alpha)$ functions characterizing the scaling properties of the harmonic measure on the boundary of a wedge of exterior angle $\phi=\pi/2$.

sequence of the identical internal fractal structure which both objects share. However, at larger $\alpha$ values, where $\mu_{min}$ begins to dominate the scaling properties of $\chi_q(\epsilon)$, the spectra diverge. This is not altogether surprising, as we have already established that the scaling of $\mu_{min}$ is determined by the coarse global structure, which is different in both objects.

Of most immediate interest, however, is the apparent presence of a phase transition in $f_m(\alpha)$. This is evidenced by the bunching of the data points at large $\alpha$, a feature which is absent in $f_s(\alpha)$. This bunching indicates that when $q$ is decreased past some finite critical value $q_c$, the sum of the moments $\chi_q(\epsilon)=\Sigma_i\mu_i^q$ suddenly becomes dominated by the minimal measure $\mu_{min}\sim\epsilon^{\alpha_{max}}$, so that $\alpha(q)$ essentially becomes independent of $q$ and equal to $\alpha_{max}$ as a consequence.

As with the $f(\alpha)$ characterizing the harmonic measure on a simple wedge, this phase transition is triggered by the union of two disparate subsets of the measure. In the case of the wedge, we had a one-dimensional subset on which the measure $\mu$ contained inside a circle of radius $\epsilon$ scaled simply as $\mu\sim\epsilon$, and a second subset consisting of a single point, the wedge tip, where $\mu\sim\epsilon^{\eta}$.

Analogously, for our multiarmed fractal, we have a multifractal subset whose complicated scaling properties are characterized by $f_s(\alpha)$, and a second subset, again consisting of a single point at the center of the object, about which $\mu\sim\epsilon^{\alpha_{max}}$. The scaling properties of these two subsets are different because they each have contrasting origins. Whereas those of the multifractal subset are determined by the internal fractal structure of the principal arms, it is instead the global structure which governs the scaling about the central point, i.e., it is the global structure which determines the value of $\alpha_{max}$.

It is thus the dichotomy present in the structure of the multiarmed fractal which is responsible for the creation of these two disparate subsets and the phase transition which subsequently arises. Note, for example, the absence of any phase transition in $f_s(\alpha)$, as no such dichotomy is found in the simple snowflake fractal of Fig. 8 where the global and internal fractal structure are one and the same.

In the light of this knowledge, we present the following conjecture as the primary result of this section: It states that it is the dichotomy in the structure of radial DLA which triggers the phase transition in the multifractal spectra of its growth probability.

It is, in fact, interesting to note that the estimates of $f(\alpha)$ obtained from DLA clusters in Fig. 1 are very similar in form to $f_m(\alpha)$, particularly the bunching of the data points about $f(\alpha)=0$ at $\alpha_{max}$. Note that, though not shown, we also obtained estimates of $f(\alpha)$ by implementing the histogram method [23] on both DLA clusters and the multiarmed fractal of Fig. 2(a), and again found they shared similar features, most notably the appearance of a linear tail at large $\alpha$.

## IV. SUMMARY AND CONCLUDING REMARKS

In this paper, we have addressed two separate but related issues, namely, the origins of both the phase transition and

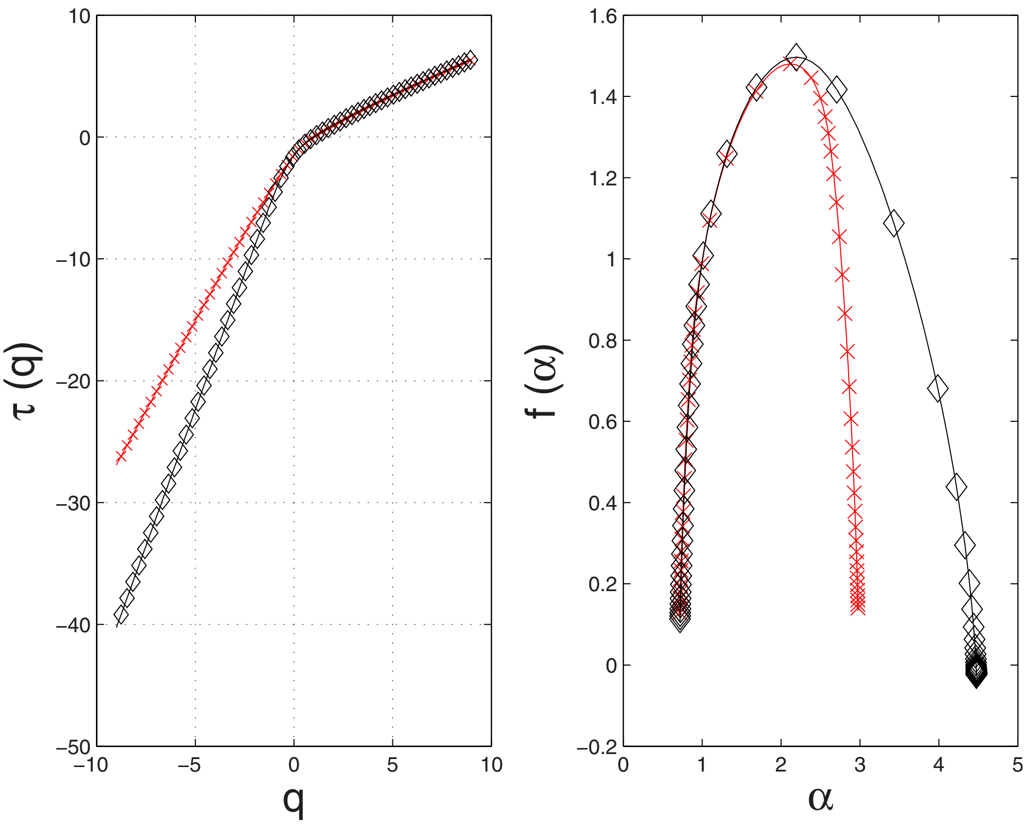


FIG. 7. (Color online) The $\tau(q)$ and $f(\alpha)$ functions evaluated from the harmonic measure on the boundary of the snowflake fractal ($\times$) of Fig. 8 and the multiarmed object ($\diamond$) of Fig. 2(a).

the finite size effects which are present in the $f(\alpha)$ of the growth probability of radial DLA. We have found that it is the coarse global structure (or geometry) of the clusters which plays the key role in both phenomena. That this is so, is, essentially, a consequence of the control which this global structure exercises over the scaling of the coarse-grained minimal measure $\mu_{min}$, which, it transpires, determines the form of $f(\alpha)$ at large $\alpha$.

The idea that the global structure of an object can have some effect on the $f(\alpha)$ characterizing the scaling of the harmonic measure existing along its boundary is not new [30,31]. In the DLA work of Turkevich and Scher [30], for example, an estimate of $\alpha_{min}$ was obtained through consideration of the global geometry of the clusters. Likewise, we have shown here that the value of $\alpha_{max}$ is similarly determined by this geometry.

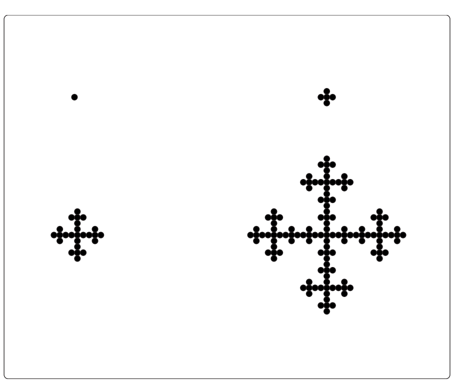

FIG. 8. First four stages in the construction of the snowflake fractal. The structure above is found within the arms of the object in Fig. 2(a).

However, as a result of slow changes which occur in the global structure of the clusters with growth, specifically the decrease in $\langle \phi_{max} \rangle$, the ensemble average of the maximum angle existing between adjacent principal cluster arms, this value of $\alpha_{max}$ increases and we thus observe, at large $\alpha$, the divergence in the $f(\alpha)$ that is characteristic of radial DLA. This lies contrary to much of the early work on DLA which typically attributed the divergence in the $f(\alpha)$ to non-power law scaling of $\mu_{min}$ originating, not from the slow evolution in the global structure of clusters as we claim, but rather from either some underlying stochastic multiplicative process [13,32,33] or from the formation of some specific fjord structure [16,34].

Our work suggests that this divergence does not continue indefinitely, however, and that asymptotically, $\alpha_{max}$ attains a finite limiting value and $\mu_{min}$ scales as a simple power law. The divergence present in $f(\alpha)$ estimates of radial DLA may thus be viewed as benign finite size effects and not as evidence of a phase transition, as was done in most early studies.

Nevertheless, other evidence exists which supports the case that a phase transition occurs in the multifractal spectra. There is, for example, the sudden domination of $\chi_q(\epsilon)$ by $\mu_{min}$ at some critical value $q_c$, the corresponding divergence in the second order derivative of the free energy $\tau(q)$ at this same value, and the linearity exhibited in the tail of $f(\alpha)$ [12,19,20].

As such linearity is characteristic of the harmonic measure lying on the boundary of a wedge, its presence in the $f(\alpha)$ of radial DLA may be seen as simply a consequence of the wedgelike global form of the principal cluster arms about the seed point, this argument being, in effect, akin to the geometric interpretation of the phase transition originally put forward in [20].

In more general terms though, our work has led us to the conclusion that the phase transition arises because there exists a dichotomy in the structure of radial DLA clusters. That is, they possess both *nonfractal global structure*, as prescribed by the radial configuration of the principal cluster arms, and the internal *self-similar fractal structure* of the branches within these arms. Consequently, the scaling behavior of the measure about, for example, the seed point, being determined by the global structure, is thus of a fundamentally different nature to the scaling found within the arms. We thus have two quite disparate subsets in our measure, a situation which is known to lead to the manifestation of a phase transition in its multifractal spectra [35].

As a final point, it should be noted that in this paper we have only examined clusters grown in a radial geometry. Using different boundary conditions, however, one may produce clusters whose global structure differs markedly from the $n$-fold symmetry of radial DLA. Nevertheless, one might still expect the global structure, whatever its form, to be responsible for determining the scaling properties of $\mu_{min}$ (which exists somewhere within the frozen zone of the clusters), so that it introduces scaling in the harmonic measure which is alien to that induced by the internal fractal branch structure.

This has, in fact, already been shown to be the case with cylindrical DLA [29,36], where the global structure introduces an exponential decay in the growth probability which is at variance with the multifractal properties of the growth probability found within the growth zone [37]. As predicted in this original work, a phase transition in $f(\alpha)$ must thus inevitably result. In conclusion, it thus seems likely that phase transitions in the $f(\alpha)$ of the growth probability may be a robust characteristic of 2D DLA in general, though the precise nature of such transitions may be dependent on the boundary conditions.


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