Psychological Aspects of Market Crashes

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Abstract

This paper analyzes the sensitivity of market crashes to investors' psychology in a standard general equilibrium framework. Contrary to the traditional view that market crashes are driven by large drops in aggregate endowments, we argue from a theoretical standpoint that individual anticipations of such drops are a necessary condition for crashes to occur, and that the magnitude of such crashes are positively correlated with the level of individual anticipations of drops.

1 Introduction

Sudden crashes are common features of financial markets. For instance, the 1994 Peso crisis in Mexico saw lending rates rising by four hundred percent over four months. Psychological factors are believed to play an important role in such situations, although the actual mechanisms linking such factors and market volatility are not yet fully explored.

This paper analyzes the sensitivity of market crashes to investors' psychology in a standard general equilibrium framework. Contrary to the traditional view that market crashes are driven by large drops in aggregate endowments, we argue from a theoretical standpoint that individual anticipations of such drops are a necessary condition for crashes to occur, and that the magnitude of such crashes are positively correlated with the level of individual anticipations. Anticipations of changes in fundamentals, driven by psychological factors as described later or also simply by rational expectations, are shown here to be a key explanatory factor of market volatility.

In a General Equilibrium economy with incomplete markets where no trader is constrained in borrowing in equilibrium, we make explicit a link between future albeit uncertain endowments drops and their anticipations sustaining any crash level. We quantify this relationship and illustrate the above findings through numerical simulations in the framework of Mehra and Prescott (1985). In particular, we show that when anticipations are not high enough then a crash may not occur as a result of a drop in endowments. We thus establish that changes in fundamentals alone may not trigger a crash.

The basic insight of our results is that, when anticipating a future albeit uncertain drop in aggregate endowments, traders take immediate financial positions to hedge against this event. The hedging can only be achieved by purchasing assets paying off positive dividends in this event, thus current purchasing prices are high and in turn returns are low at the time dividends are paid. This intuition also shows why agents must not be constrained in borrowing, since otherwise the demand for assets would be bounded above and prices could not be high enough to generate significant crashes. The importance of limiting borrowing possibilities in similar situations can be found in Hong and Stein (2003), although in this last reference short-sale constraints prevent bearish investors from initially participating in markets and revealing their information through prices.

Psychological factors affecting market volatility in our analysis are both at individual and social levels. At individual level, those factors can be for instance herding, market rumors, fear of contagion or panic (or possibly all those issues together, see Shiller, 2000). We do not sort which one seems most likely, but rather we argue that every psychological factor leading to an individual anticipation contributes to both a crash occurrence and its magnitude. Our analysis emphasizes that anticipations of changes in fundamentals are a key element to a crash, regardless of their formations and predictive accuracy. At social level, it must also be true that bearish sentiments must be shared by most traders (rumors or panic reached the whole market for instance); that is, market exuberance leading to a significant increase in market volatility (as in Shiller, 2000) must emerge from a behavioral correlation across market participants about the occurrence of a change in fundamentals, regardless of whether such anticipations are justified.

In more details, we analyze a standard General Equilibrium economy under uncertainty, where infinitely-lived agents with heterogenous beliefs trade infinitively-live assets, or Lucas' trees as in Lucas (1978), in sequential markets that need not be complete. To rule out the possibility of rolling over individual debts, we assume that agents cannot borrow more than the net present value of their future endowments (wealth constraint).

We define an $\varepsilon-$crash (for $\varepsilon > 0$) as an event where the return of every traded asset paying off positive dividends in this event is below ε. Provided that agents are not wealth-constrained in equilibrium (this occurs with complete markets for instance), we find that if there is a high enough lower bound on the probability that every agent assigns to a low enough upper bound on a next-period drop in aggregate endowment, then a market crash occurs with positive probability next period. The magnitude of the crash (the "ε") depends directly on the bounds found above. It is easy to derive from the proof of this result that if agents are constrained in borrowing in equilibrium, then low crashes cannot occur because equilibrium assets' demands are bounded above, and thus asset prices cannot reach high levels.

We also carry out numerical simulations in the well-known framework of Mehra and Prescott (1985) to make explicit the direct relationship between the above bounds sustaining an arbitrary magnitude of crash. In this setting, we show that for a given level of drop, the higher the level of anticipation the higher the magnitude of the crash. Highest crash magnitudes are associated with the highest levels of anticipations, and high anticipations significantly intensify the crash magnitudes regardless of the drop level.

Our results rely on the Inada conditions to obtain, although those conditions alone cannot lead to a crash unless traders largely agree upon a variation in fundamentals. The intuition is that the marginal disutility of a low consumption level on a particular history, typical of Inada conditions, can be compensated in terms of ex-ante utility by a low probability assigned to this history by every agent. Thus in this situation, a low contingent consumption need not be largely hedged against and a crash may not occur.

Our findings are consistent with the empirical findings above, although our theoretical explanation differs from that in Lee (1998). Indeed, Lee justifies crashes by information flows varying with private information, and crashes occur as an informational cascade when enough signals of bad times are released by traders. In contrast, we argue that crashes are driven by financial decisions motivated by the anticipation of future albeit uncertain variations in market fundamentals. This behavior requires a group coordination (or large agreement) about the actual state of the economy, although it goes beyond the idea of private information as in Lee. The coordination that we require for crashes to occur can stem from rational expectations, erratic beliefs generated by psychological factors (see Allen *et al.* (2005) or Shiller, 2000) even if it incorporates as well the situation raised in Lee (1998) and Ho and Stein (2003).

2 The model

In this section, a formal description of the model is given. Time is discrete and continues forever. In every period $t \in N_+$, a state is drawn by nature from a set $S = \{1, ..., L\}$, where L is strictly greater than 1. Before defining how nature draws the states, we first need to introduce some notations.

Denote by S^t ($t \in N \cup \{\infty\}$) the $t-$Cartesian product of S. For every history $s_t \in S^t$ ($t \in N$), a cylinder with base on s_t is defined to be the set $C(s_t) = \{s \in S^\infty | \ s = (s_t, ...)\}$ of all infinite histories whose t initial elements coincide with s_t. Define the set Γ_t ($t \in N$) to be the $\sigma-$algebra which consists of all finite unions of cylinders with base on S^t.[1] The sequence $(\Gamma_t)_{t \in N}$ generates a filtration, and define Γ to be the $\sigma-$algebra generated by $\cup_{t \in N} \Gamma_t$. Given an arbitrary probability measure Q on (S^∞, Γ), we define $dQ_0 \equiv 1$ and dQ_t to be the Γ_t-measurable function defined for every $s_t \in S^t$ ($t \in N_+$) as

$$dQ_t(s) = Q(C(s_t)) \text{ where } s = (s_t, ...).$$

Given data up to and at period $t-1$ ($t \in N$), the probability according to Q of a state of nature at period t, denoted by Q_t, is

$$Q_t(s) = \frac{dQ_t(s)}{dQ_{t-1}(s)} \text{ for every } s \in S^\infty,$$

with the convention that if $dQ_{t-1}(s)$=0 then $Q_t(s)$ is defined arbitrarily.

In every period and for every finite history, nature draws a state of nature according to an arbitrary probability distribution P on (S^∞, Γ). To simplify

[1]The set Γ_0 is defined to be the trivial $\sigma-$algebra, and $\Gamma_{-1} = \Gamma_0$.

the analysis, we assume that $P_{s_t} > 0$ for every history s_t.

To conclude this section, we define the operators E^Q to be the expectation operator associated with Q. Finally, we say that a finite history $s_{t+p} \in S^{t+p}$ *follows* a finite history $s_t \in S^t$ $(t, p \in N)$, denoted by $s_{t+p} \hookrightarrow s_t$, if there exists $s \in S^p$ such that $s_{t+p} = (s_t, s)$.

2.1 The agents

In this section, economic agents are described. There is a finite number $I \geq 1$ of infinitely-lived agents behaving competitively.

There is a single consumption good available in every period t $(t \in N_+)$. Denote by $c^i_{s_t}$ the consumption of an agent i $(i = 1, ..., I)$ in history $s_t \in S^t$ $(t \in N_+)$. In every period t $(t \in N_+)$ and in every history $s_t \in S^t$, every agent i $(i = 1, ..., I)$ is endowed with $w^i_{s_t} > 0$ units of consumption goods.

In every period $t \in N$, and after the realization of the history $s_t \in S^t$, the agents trade $L \geq 1$ infinitely-lived assets, or Lucas' trees as in Lucas (1978). Every tree j $(j = 1, ..., L)$ yields a dividend $d^j_{s_t} > 0$ of units of consumption good in history s_t. Let d_{s_t} denote the vector $(d^1_{s_t}, ..., d^L_{s_t})$ for every s_t. The supply of every tree is assumed to be 1 in every history.

The aggregate endowment w_{s_t}, in every history s_t $(s_t \in S^t$ and $t \in N_+)$, is given by

$$w_{s_t} = \sum_i w^i_{s_t} + \sum_j d^j_{s_t}.$$

The price in history s_t of one share of the tree j $(1 \leq j \leq L)$ is denoted by $q^j_{s_t}$, for every $s_t \in S^t$ and $t \in N_+$. Let q_{s_t} denote the vector $(q^1_{s_t}, ..., q^L_{s_t})$

for every history s_t.

A *portfolio* θ^i for every agent i is a vector $(\theta_{s_t})_{s_t \in S^t, t \in N_+}$ of shares held of the J trees in every history s_t, where $\theta_{s_t} = (\theta^j_{s_t})_j$ is the vector of holdings in history s_t and $\theta^j_{s_t}$ is the holding of j in this same history s_t. Every agent i has an initial portfolio θ^i_0 at date 0.

Every agent i does not have any information about P, the true probability measure from nature draws the states; however agent i has a subjective belief about nature represented by a probability measure P^i on (S^∞, Γ). We assume that $dP^i_t(s) > 0$ for every infinite history s and every period t, to avoid problems of existence as pointed in Araujo and Sandroni (1999).

Every agent derives some utility in any history from consuming the only consumption good present in the economy. We assume that agent i ranks all the possible future consumption sequences $c = (c_{s_t})_{s_t \in S^t, t \in N_+}$ according to the utility function

$$U^i(c) = E^{P^i}\left(\sum_{t \in N_+} (\beta_i)^t u_i(c_t)\right), \tag{1}$$

where $\beta_i \in (0,1)$ is the intertemporal discount factor, u_i is a strictly increasing, strictly concave and continuously differentiable function. We assume that u^i satisfies the Inada condition, namely $(u_i)'(c) \mapsto \infty$ as $c \mapsto 0$.

The budget constraint faced in every history s_t by agent i is

$$\begin{aligned} c_{s_t} + \sum_j q^j_{s_t}\theta^j_{s_t} &\leq w^i_{s_t} + \sum_j d^j_{s_t}\theta^j_{s_{t-1}} + \sum_j q^j_{s_t}\theta^j_{s_{t-1}} && (2) \\ c_{s_t} &\geq 0, && (3) \end{aligned}$$

where $s_t \hookrightarrow s_{t-1}$. The left-hand side of (2) represents the purchase of consumption good at price normalized to 1 plus the purchase of new shares of trees at current prices, and the right-hand side is the endowment plus the dividends payments from previous holdings plus the proceeds from selling the current holdings of trees at current prices.

Given the constraints faced by the traders, we also need to rule out the possibility of rolling over any debt through excessive future borrowing, also known as Ponzi's scheme. Consider any vector of prices that is *arbitrage-free.* As argued in Hernandez and Santos (1996), when a vector of prices q is arbitrage-free there exists a sequence of positive numbers $\{\pi_{s_t}\}_{s_t \in S^t, t \in N_+}$ with $\pi_{s_0} = 1$ such that

$$\pi_{s_t} q^j_{s_t} = \sum_{s \hookrightarrow s_t} \pi_s d^j_s,$$

for every j $(j = 1, ..., J)$ and s_t $(s_t \in S^t$ and $t \in N_+)$. We now assume that every agent cannot borrow more than the present value of her current endowment at such prices. Formally, we assume that for any vector of prices q that is arbitrage-free, every portfolio strategy satisfies the *wealth constraints*

$$q_{s_t}\theta_{s_t} \geq -\frac{1}{\pi_{s_t}} \sum_{s_\tau \in C(s_t)} \pi_{s_\tau} w^i_{s_\tau} \text{ for every } s_t. \tag{4}$$

This constraint naturally rules out Ponzi's scheme, and it is chosen arbitrarily among many others. Hernandez and Santos (1996) gives six other constraints ruling out Ponzi's schemes and shows that they are all equivalent when markets are complete. Markets are not assumed to be complete though.

For every i, we define the budget set $B^i(q)$ faced by agent i at prices q as follows. If q is arbitrage-free, the budget set $B^i(q)$ is the set of sequences (c, θ) that satisfy conditions (2)-(4) above. If now the vector of prices has an arbitrage opportunity, define $B^i(q)$ as the set of sequences (c, θ) that satisfy conditions (2)-(3) only.

Definition 1 *An equilibrium is a sequence* $(\bar{c}^i, \bar{\theta}^i)_i$ *and a system of prices* $\bar{q}$ *such that*

1. *taking prices* $\bar{q}$ *as given, for every* i *the vector* $(\bar{c}^i, \bar{\theta}^i)$ *is solution to the program consisting of maximizing (1) subject to* $(c, \theta) \in B^i(\bar{q})$*, and*

2. *for every history* s_t *we have that* $\sum_i \bar{c}^i_{s_t} = w_{s_t}$ *and* $\sum_i \bar{\theta}^i_{s_t} = 1$.

The above definition requires that, taking prices as given, every agent sequentially chooses consumption plans and portfolio holdings so as to maximize her expected utility, and markets for consumption good and trees all clear in every history. It is also straightforward to see that the equilibrium prices are arbitrage-free. Indeed, if otherwise then every agent will have an infinite demand for at least one tree in at least one history, and Condition 2 in the above definition will always be violated. By a similar reasoning, it is easy to check that equilibrium prices must be strictly positive.

2.2 Market crashes

We next describe the notion of *market crashes* occurring in financial markets. This notion focuses on arbitrarily low returns on traded trees. For every system of asset prices q, define first the *return* of tree j ($j = 1, ..., J$) in history s_{t+1}, when purchased in history s_t, as

$$R^j_{s_{t+1}} = \frac{q^j_{s_{t+1}} + d^j_{s_{t+1}}}{q^j_{s_t}} \tag{5}$$

if $q^j_{s_t} > 0$, and arbitrarily otherwise. With this notion, we can describe our notion of market crash.

Definition 2 *For every $\varepsilon > 0$, an ε-crash occurs in history s_t if $R^j_{s_t} < \varepsilon$ for every asset j such that $d^j_{s_t} > 0$.*

A market crash in a given history is thus defined as an arbitrarily low return on every asset paying off strictly positive dividend in this history. In the remainder of the paper, we are primarily interested in finding conditions leading to arbitrarily low market crashes. In particular, we analyze how individual anticipations of variations in market fundamentals can generate crashes as described above.

3 Beliefs and market crashes

In this section, we study how market crashes are linked to anticipations of variations in aggregate endowment, and what level of social coordination about the anticipations is needed for a crash to occur.. Our result gives a set of sufficient conditions on beliefs and aggregate endowments leading to arbitrarily low market crashes.

Proposition 3 *Consider any equilibrium such that Constraint (4) does not bind for every agent. Fix also* $\varepsilon > 0$ *and consider any history* s_t*. There exist positive constants* $\gamma < 1$ *and* $\delta < 1$ *such that if* $\frac{w_{s_t}}{w_{s_{t-1}}} \leq \gamma$ *for some predecessor* s_{t-1} *of* s_t*, and if* $P^i_{s_t} > \delta$ *for every* i*, then an* $\varepsilon-$*crash occurs in history* s_t*.*

Proposition 3 states that, for any given crash magnitude, one can find regions of parameters on endowment drop and drop anticipation sustaining this crash. For this result to occur, agents must not be wealth-constraint in equilibrium. The intuition of this result is given in the Introduction. A natural case where the wealth constraints do not bind in equilibrium for every agent is when markets are complete, as a straightforward consequence of Theorem 3.3 in Hernandez and Santos (1996).

Proposition 3 implicitly states that an endowment drop alone may not sustain a crash. For a crash to happen, two other conditions must be met. First, there must be high enough individual anticipations about an endowment drop next period; second, this sentiment must be shared by every agent in the economy. It will appear clearly in the next section that, when those conditions on anticipations are not met, an endowment drop alone may not sustain a crash.

Quantifying the relationship between the parameters γ and δ sustaining an arbitrary magnitude of crash is a central question of our analysis. Making this link explicit in our general setting would lead to a very complex and cumbersome technical analysis. Instead, we give next section the main quantitative features of this relationship in the well-known framework of Mehra and Prescott (1985) through numerical simulations. The choice of this framework is motivated by tractability reasons, and also for its large impact in terms of macroeconomic analysis.

4 Numerical simulations

We now carry out some numerical simulations to find regions for the parameters described in Proposition 3 sustaining arbitrary levels of crashes. We narrow down our model to that in Mehra and Prescott (1985), with the difference that we do not assume any condition on the endowment process and we allow for arbitrary beliefs. Our first simulation gives a region for the parameters δ and γ sustaining a given crash magnitude for various levels of risk-aversion. The second simulation shows that, for a given level of endowment drop this time, the higher the anticipation the higher the crash magnitude. The third simulation is a 3D-representation of crash magnitudes as a function of both drops and anticipations, illustrating the intuitions given in the Introduction.

We now assume, following Mehra and Prescott (1985), that in every period two states only can occur. We also assume that there is one agent only within the economy (a representative agent) forming subjective belief about economic uncertainty. Even if strong in appearance, this last assumption has already been largely justified in terms of macroeconomic analysis. Proposition 3 still remains relevant in this setting, the only conceptual loss is that it rules out the need for social coordination about the anticipation of the crash (this can be regarded as implicitly assumed). The representative agent has a utility function of the form

$$U(c) = E^P \left(\sum_{t \in N_+} \beta^t u(c_t) \right),$$

where P is an arbitrary belief process, where $\beta \in (0,1)$ is a constant, and where the function u is defined as

$$u(x) = \frac{x^{1-\alpha} - 1}{1-\alpha},$$

for some $\alpha > 0$ (this parameter is the coefficient of risk-aversion of the agent). We show in Appendix B that the asset structure is irrelevant to carry out our simulations, provided that the agent is not constrained in borrowing in equilibrium.

Fix now any history s_{t-1}, let $\bar{s}_t \hookrightarrow s_{t-1}$ be the history following s_{t-1} where the crash is expected and let $s_t \hookrightarrow s_{t-1}$ be the other history following s_{t-1}. In Appendix B, we show that

$$R^j_{\bar{s}_t} \leq \frac{1}{\beta} \cdot \frac{1}{P_{\bar{s}_t}} \cdot \left(\frac{w_{\bar{s}_t}}{w_{s_{t-1}}} \right)^\alpha \tag{6}$$

for every security j as before, and regardless of the asset structure provided that the agent is not constraint in borrowing in equilibrium. In particular, Inequality (6) shows that the upper-bound on equilibrium returns depends only the parameters γ, δ, α and β. The following numerical simulations are generated directly from this last inequality.

From now on, we fix $\beta = 0.9$ since this parameter does not a critical role in our analysis. Our first simulation provides a parameters region sustaining a .85-crash, which corresponds to a drop of 15% in price of all assets traded (assuming no dividend is paid).

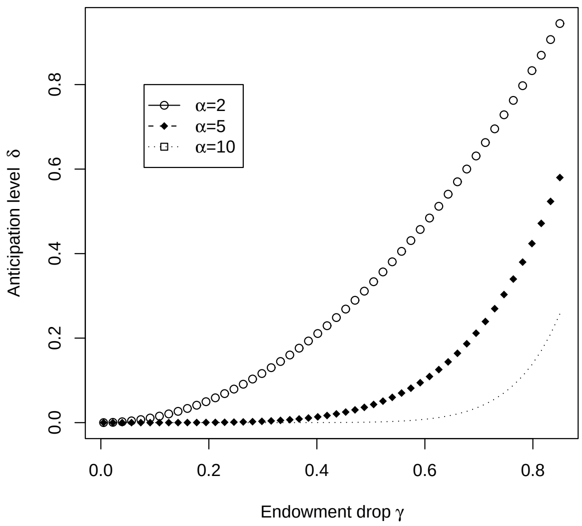

Figure 1: Parameters region sustaining a .85-crash (15% price drop)

Figure 1 simultaneously displays such regions for various level of risk-aversion. For every curve, any point of parameters above the curve sustains the .85-crash. For instance, for an agent with a level of risk-aversion of 5, any 20% drop in endowment next period that is anticipated with probability of at least .5 in the current period will trigger a .85-crash next if the drop actually occurs. Figure 1 also shows that, for those last values, any anticipation level below .5 may not trigger the crash, as is explained in the Introduction. This last point implies that the crash occurs independently of the true probability of a drop next period, showing that the anticipation (together with the drop of course) has driven the crash.

The next figure gives us a way to visualize the effect of drop anticipations on the magnitude of a crash, given a particular drop of endowment next period. We fix a 20% drop in the following simulation.

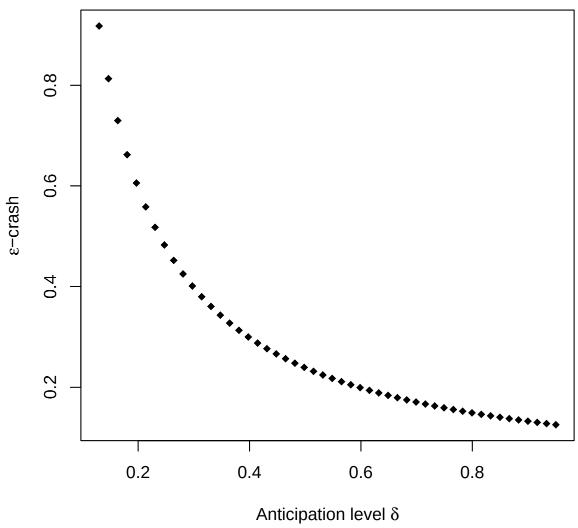

Figure 2: Crash magnitude as a function of the anticipation δ ($\alpha = 10$)

Figure 2 provides the direct link between the magnitude of the crash and the anticipation of the drop. Its main implication is that, for a fixed drop of endowment, the higher the anticipation the higher the crash magnitude. The intuition of this point is also given in the Introduction.

Figure 3 below maps crash magnitudes as a function of both the anticipation levels and endowments drops. Regions of relatively low endowment drops

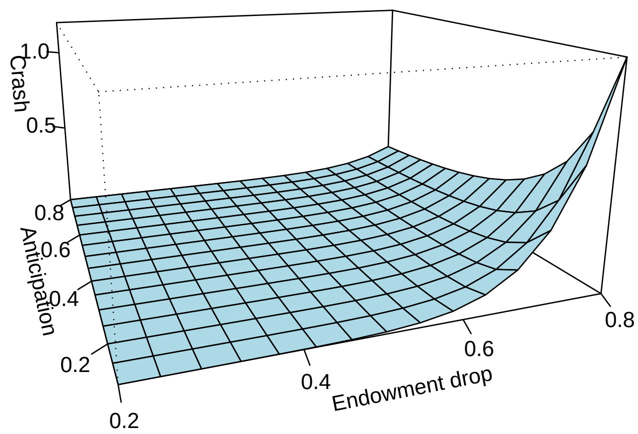

Figure 3: ϵ-crash as a function of the anticipation and drop ($\alpha = 10$)

and low anticipations leads to moderate crashes. Regions of high anticipations of drops trigger the highest levels of crash, and such high anticipations significantly intensify the crash magnitudes regardless of the drop level. Provided that anticipations are high enough, severe drops in endowment lead to severe crashes (as is commonly believed), but our point is to show that anticipations do intensify this phenomena. That is, psychological factors as described here turn crashes from bad to significantly worse.

5 Conclusion

Our main result states that, as long as every agent is not constrained in borrowing, a future albeit uncertain drop in endowments that is currently anticipated with high enough probability by every agent will trigger a crash if the drop actually occurs. Moreover, our simulations show that the crash magnitude is positively correlated with the commonly agreed anticipation level. Section 4 shows that those conditions are tight; that is, an endowment drop may not trigger a crash if the anticipation level is not high enough. The basic insight is that, when expecting future low endowments, agents will increase their demand for securities to hedge against this event. This, in turn, will raise the purchasing price of those securities and therefore will lower their returns. In particular, to arbitrarily increase their holdings and thus to induce such crashes, agents must not be constrained in their borrowing capabilities.

The psychological factors that we put forth in our study are two-fold. First, any factor leading an individual to believe in the occurrence of a drop is relevant because such beliefs will act as self-fulfilling prophecies. Those beliefs can stem from instance from herding, market rumors, fear of contagion or panic (or possibly all those issues together). We do not sort which one seems most likely, but rather we point out that they are all relevant because they lead to the same phenomena: a crash anticipation. Second, it must be true that all the agents in the economy agree on the anticipation (herding or rumors have reached the whole market for instance). This second point must occur so that anticipations can have a significant effect on prices formation. This view is somewhat consistent with that in Lee (1998) where crashes ar driven by successive releases of public information on the actual state of the economy.

Finally, the fact that with incomplete markets agents must not be wealth constrained for arbitrary level of crashes to occur suggests a natural policy recommendation. Indeed, limiting agents' borrowing abilities when bad times are largely anticipated appears as a natural way to reduce the magnitude of market crashes.

A Proof of Proposition 3

We next prove our result. The strategy of our proof goes as follows. We first find an equilibrium relationship between beliefs, aggregate endowments and equilibrium returns. We then derive our result by simply using the Inada conditions to generate an arbitrarily high marginal utility as endowments drop, forcing equilibrium returns to drop as well.

Consider the program of any agent i, consisting of maximizing (1) subject to $(c, \theta) \in B^i(\bar{q})$ and taking as given any arbitrage-free and strictly positive asset prices. Since we assume that Constraint (4) does not bind, and since we know by the Inada conditions that Constraint (3) does not bind as well, the Lagrangian to this program rewrites as

$$\mathcal{L} = \sum_{s_t} dP^i_{s_t} \beta_i^t u_i(c_{s_t}) + \sum_{s_t} \mu_{s_t} \left[w^i_{s_t} + \sum_j d^j_{s_{t-1}} \theta^j_{s_t} - c_{s_t} + \sum_j q^j_{s_t} (\theta^j_{s_t} - \theta^j_{s_{t-1}}) \right],$$

where for every history s_t the real number $\mu_{s_t} > 0$ is the Lagrange multiplier associated with the Constraint (2). Taking the first-order conditions with respect to every variable yields the following relationships for every history s_{t-1} and asset j

$$dP^i_{s_{t-1}} \cdot \beta_i^{t-1} \cdot u_i'(c_{s_{t-1}}) \quad = \quad \mu_{s_{t-1}} \text{ and} \tag{7}$$

$$\sum_{s_t \hookrightarrow s_{t-1}} \mu_{s_t} \cdot [d^j_{s_t} + q^j_{s_t}] \quad = \quad \mu_{s_{t-1}} \cdot q^j_{s_{t-1}}, \tag{8}$$

Rearranging terms gives

$$\sum_{s_t \hookrightarrow s_{t-1}} dP^i_{s_t} \cdot \beta_i^t \cdot u_i'(c_{s_t}) \cdot [d^j_{s_t} + q^j_{s_t}] = dP^i_{s_{t-1}} \cdot \beta_i^{t-1} \cdot u_i'(c_{s_{t-1}}) \cdot q^j_{s_{t-1}}, \tag{9}$$

and by (5) and some simplifications we obtain the desired relationship

$$\sum_{s_t \hookrightarrow s_{t-1}} P^i_{s_t} \cdot \beta_i \cdot u_i'(c_{s_t}) \cdot R^j_{s_t} = u_i'(c_{s_{t-1}}). \tag{10}$$

With the above relationship, we can prove our result. Fix $\varepsilon > 0$ and a history $\bar{s}_t$. It is easy to see that, for every $\delta > 0$ such that $P^i_{\bar{s}_t} > \delta$ for every i, there exists an agent, denoted by $\delta(i)$, such that for the history $\bar{s}_{t-1}$ such that $\bar{s}_t \hookrightarrow \bar{s}_{t-1}$ we have that $c^{\delta(i)}_{\bar{s}_{t-1}} \geq \frac{w_{\bar{s}_{t-1}}}{I}$ in equilibrium.

Since u_i satisfies the Inada conditions for every i, this last remark implies that the expression $u'_{\delta(i)}(c^{\delta(i)}_{\bar{s}_{t-1}})$ is bounded away from $+\infty$ for every $\delta > 0$.

Also, since $c^{\delta(i)}_{\bar{s}_t} \leq w_{\bar{s}_t}$ and by the Inada conditions, a low enough value of aggregate endowment $w_{\bar{s}_t}$ in history $\bar{s}_t$ will increase the left-hand side of (10) above to an arbitrary high level. Thus, as δ converges to 1 and $w_{\bar{s}_t}$ converges to 0, for (10) to hold for agent $\delta(i)$ it must be true that $R^j_{\bar{s}_t}$ converges to 0 for every j such that $d^j_{\bar{s}_t} > 0$. Thus, it is straightforward to find the two constants δ and γ as described in Proposition 3 satisfying $R^j_{\bar{s}_t} < \varepsilon$ for every j as above.

We have thus shown that, for the constants δ and γ found above, an ε-crash occurs in history $\bar{s}_t$. The proof is now complete.

B An inequality for numerical simulations

We now derive an upper-bound on equilibrium returns that depends on the parameters described in Proposition 3, under the assumptions of Section 4. This uniform upper-bound readily allows for the numerical simulations given in this last section.

Fix any history s_{t-1}, and let $\bar{s}_t \hookrightarrow s_{t-1}$ and $s_t \hookrightarrow s_{t-1}$ be defined as in Section 4. Consider any security j such that Equation (10) holds for those histories. Given the shape of our utility function, and since the consumption of the representative agent must be the aggregate endowment in every history, Equation (10) rewrites as

$$P_{\bar{s}_t}\left(\frac{1}{w_{\bar{s}_t}}\right)^{\alpha} R^j_{\bar{s}_t} + (1 - P_{\bar{s}_t})\left(\frac{1}{w_{s_t}}\right)^{\alpha} R^j_{s_t} = \frac{1}{\beta}\left(\frac{1}{w_{s_{t-1}}}\right)^{\alpha}, \tag{11}$$

for every security j as described above. Rearranging terms gives

$$R^j_{\bar{s}_t} = \frac{1}{P_{\bar{s}_t}}(w_{\bar{s}_t})^{\alpha}\left[\frac{1}{\beta}\left(\frac{1}{w_{s_{t-1}}}\right)^{\alpha} - (1 - P_{\bar{s}_t})\left(\frac{1}{w_{s_t}}\right)^{\alpha} R^j_{s_t}\right]. \tag{12}$$

Moreover, since in equilibrium it must be true that $R^j_{s_t} > 0$, we obtain the following inequality

$$R^j_{\bar{s}_t} \leq \frac{1}{\beta} \cdot \frac{1}{P_{\bar{s}_t}} \cdot \left(\frac{w_{\bar{s}_t}}{w_{s_{t-1}}} \right)^{\alpha}$$

for every security j as above. The right-hand side of Inequality (B) depends on the parameters described in Proposition 3, together with the intertemporal discount factor β and the coefficient of risk-aversion α. This last inequality directly yields the numerical simulations presented in Section 4.

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