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Some Results on Vertex-Minimal Triangulations of Manifolds

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Contents

Abstract

This thesis presents some results on vertex-minimal (simplicial) triangulations of manifolds. We are interested in triangulations that have nice geometric and combinatorial properties.

In the first chapter, we list some definitions used throughout the thesis.

In the second chapter, we give an elementary construction of the Witt design on 22 points, and a combinatorial description of the only known vertex-minimal triangulation of real projective 4-dimensional space. We show that the 16-vertex complex we describe triangulates $\mathbb{R}P^4$ by constructing a 4-dimensional combinatorial sphere which can be easily seen to be a double cover of our complex.

In the third chapter, we give two geometric constructions of the 16-vertex $\mathbb{R}P^4$.

In the fourth chapter, we give a purely combinatorial description of a 15-vertex triangulation of an 8-manifold that has the same cohomology as the quaternionic projective plane $\mathbb{H}P^2$, and is conjectured to be homeomorphic to $\mathbb{H}P^2$.

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Chapter 1

Basic Definitions and Background

1.1 Definitions of frequently used terms

We give some definitions which are used throughout this thesis.

Definition 1. An *abstract simplicial complex* is a finite collection $\mathcal{S}$ of finite sets such that any subset of an element of $\mathcal{S}$ is also an element of $\mathcal{S}$.

Elements of an abstract simplicial complex are called simplices. The dimension of a simplex S is defined to be $|S|-1$. The dimension of an abstract simplicial complex is defined as the largest dimension of a simplex in S

The *vertex set* of an abstract simplicial complex $\mathcal{S}$ is the union of all its elements.

A *geometric realization* $|\mathcal{S}|$ of an abstract simplicial complex $\mathcal{S}$ is constructed by taking a map σ from its vertex set V to the vertices of the standard $(|V|-1)$-simplex Δ in $\mathbb{R}^{|V|-1}$ and mapping each simplex S to the subsimplex of Δ spanned by the vertices of $\sigma(S)$.

The *closure* $Cl(\mathcal{T})$ of a subset $\mathcal{T} \subset \mathcal{S}$ is the smallest subcomplex of $\mathcal{S}$ containing $\mathcal{T}$. So

$$Cl(\mathcal{T}) = \{T' \subset T | T \in \mathcal{T}\}.$$

The *star* $St(\mathcal{T})$ of a subset $\mathcal{T} \subset \mathcal{S}$ is the set of all simplices of $\mathcal{S}$ containing any element of $\mathcal{T}$.

Definition 2. The *link* $lk(S)$ of a simplex $S \in \mathcal{S}$ is defined as

$$lk(S) = Cl(St(\{S\})) \setminus St(Cl(\{S\})).$$

The link of a simplex $S \in \mathcal{S}$ is the simplicial complex $\mathcal{T}$ containing all simplices T of $\mathcal{S}$ such that $S \cap T = \phi$ for any $T \in \mathcal{T}$, but $T \cup S \in \mathcal{S}$ for all $T \in \mathcal{T}$.

The *join* $S * S'$of two disjoint simplices $S, S' \in \mathcal{S}$ is the simplex $S \sqcup S'$.The join $\mathcal{T} * \mathcal{T}$ of two subsets $\mathcal{T}, \mathcal{T}' \subset S$ where any element of $\mathcal{T}$ is disjoint from any element of $\mathcal{T}'$ is the set of simplices

$$\{T \sqcup T' | T \in \mathcal{T}, T' \sqcup \mathcal{T}'\}.$$

Definition 3. The suspension of a simplicial complex $\mathcal{S}$ at the two points x, y, where $\{x\}, \{y\} \notin \mathcal{S}$ is the complex $\{\{x\}, \{y\}\} * \mathcal{S}$, denoted $S\mathcal{S}$.

Geometrically, this is the join of $\mathcal{S}$ with the 0-sphere, and in the case of the n-sphere S^n gives $SS^n = S^{n+1}$.

Definition 4. The *f-vector* of a d-dimensional simplicial complex $\mathcal{S}$ is the vector $[f_0, f_1, \ldots, f_d] \in \mathbb{N}^{d+1}$, where f_i is the number of i-dimensional faces of $\mathcal{S}$.

We say that a simplicial complex is *pure* if all its maximal (with respect to inclusion) simplices have the same dimension. We call a maximal simplex of a complex $\mathcal{S}$ a *facet* of $\mathcal{S}$. All complexes we deal with in this thesis are pure, and we sometimes describe a complex by means of listing its facets, whereby the complex is understood to be the closure of its set of facets.

Definition 5. A pure, d-dimensional simplicial complex is a *combinatorial manifold* if the link of vertex is (piecewise linearly) homeomorphic to a $(d-1)$-dimensional sphere.

Definition 6. A pure (d-dimensional) simplicial complex $\mathcal{S}$ is called a (d-dimensional) *pseudomanifold* if

1. Each $(d-1)$-simplex is a contained in at most two facets of $\mathcal{S}$, (or exactly two facets in case of a pseudomanifold without boundary), and

2. given any two facets $S, S' \in \mathcal{S}$, there exists a sequence of some k facets $S = S_0, S_1, S_2, \ldots, S_k = S'$, such that $|S_{i-1} \cap S_i| = d$ for all $1 \le i \le k$

Any complex satisfying the first of the above conditions is said to be *non-branching*, and a complex satisfying the second condition is said to be *strongly connected.*

Definition 7. Let $\mathcal{S}$ be a pure d-dimensional simplicial complex. Suppcoreose $S \in \mathcal{S}$ has dimension $d-k$ and that the link of S is a $k+1$-vertex S^{k-1} (i.e, the boundary $\partial\Delta$ of a k-simplex Δ). Then we say that the transformation

$$\mathcal{S} \quad \mapsto \quad (\mathcal{S} \setminus \{S * \partial\Delta\}) \cup \{\partial S * \Delta\}$$

is a *bistellar flip* or bistellar k-flip.

If either of two complexes can be obtained from the other by a sequence of bistellar flips, they are PL-homeomorphic. Bistellar flips are also called bistellar moves, or Pachner moves, named after Udo Pachner, whose celebrated theorem proves the converse of the previous statement.

For further definitions, we refer to the survey paper by Datta [5].

1.2 Background

The two combinatorial manifolds that we study here are a 16-vertex triangulation of real projective 4-dimensional space, $\mathbb{R}P^4_{16}$, and a 15-vertex triangulation of an 8-dimensional "manifold like a projective plane", $\sim \mathbb{H}P^2$.

1.2.1 Triangulating real projective spaces

It is well known that the smallest number of vertices required to triangulate $\mathbb{R}P^2$ is 6 and that this triangulation is unique up to relabellings of the vertices. P. Arnoux and A. Marin, in 1991, proved that for $n > 2$, the minimum number of vertices needed to triangulate $\mathbb{R}P^n$ is $\binom{n+2}{2} + 1$ [1]. In 1969, D.W. Walkup constructed a triangulation of $\mathbb{R}P^3$ on eleven vertices[10]. In 1986, W. Kühnel gave a triangulation of $\mathbb{R}P^n$ using $2^{n+1} - 1$ vertices, which takes the barycentric subdivision of all faces of the boundary of the $n+1$-simplex and quotients it by the antipodal map[7]. The Kühnel construction is the only known explicit triangulations of $\mathbb{R}P^n$ for $n > 5$.

The BISTELLAR program written by F.H. Lutz (1999) uses a heuristic search algorithm to reduce the f-vector of a given complex using bistellar flips [3], available through the `GAP` package `simpcomp` [6]. Among the several combinatorial manifolds found by this program was a 16-vertex triangulation of $\mathbb{R}P^4$, called $\mathbb{R}P^4_{16}$. This complex was obtained by applying the BISTELLAR program to the 31-vertex $\mathbb{R}P^4$ due to Kühnel[8]. The automorphism group of this particular triangulation is also calculated in [8], and is found to be S_6, which acts on the 16-element vertex set by splitting it into orbits of size six and 10, and on the set of 150 facets by splitting it into orbits of size 30 and 120. No other triangulation of $\mathbb{R}P^4$ on 16 vertices is known. There does not seem to be a hands-on description of a 16-vertex $\mathbb{R}P^4$ in the literature. We point out the exceptional combinatorial structure of this complex and give three constructions of triangulated $\mathbb{R}P^4$ on 16 vertices.

1.2.2 Triangulating cohomology $\mathbb{H}P^2$

We provide a combinatorial description of a 15-vertex 8-manifold, $\sim \mathbb{H}P^2_{15}$, constructed by U. Brehm and W. Kühnel in 1992. This cohomology $\mathbb{H}P^2$ on 15 vertices is conjectured to be PL-homeomorphic to $\mathbb{H}P^2$. Brehm and Kühnel constructed three simplicial complexes (one of them, $\sim \mathbb{H}P^2_{15}$, vertex-transitive under the $A_5(15)$ -action on its vertices) which triangulate the same manifold, and studied them extensively in [4].

We start with the classical $A_5(6)$ invariant 6-vertex triangulation of $\mathbb{R}P^2$, which has 15 edges and ten triangles. The graph on the set of triangles of $\mathbb{R}P^2_6$ obtained by joining triangles intersecting in an edge (1-simplex), by an edge in the graph, is the Petersen graph. We note that the action of $A_5(15)$ on the vertices of $\sim \mathbb{H}P^2_{15}$ is the same as that induced on the edges of this Petersen graph by the action of $A_5(6)$ on the vertices of $\mathbb{R}P^2_6$, and describe $\sim \mathbb{H}P^2_{15}$ in two ways, one as the edge-sets of subgraphs of the Petersen graph, and one using its bipartite double, the Desargues graph.

Chapter 2

The Witt design on 22 points and a combinatorial description of $\mathbb{R}P^4_{16}$

2.1 Introduction

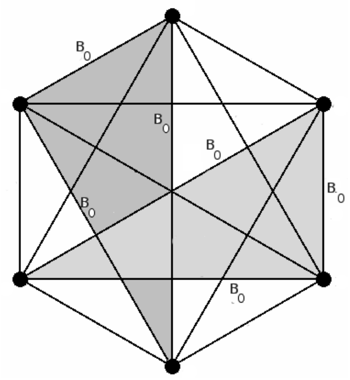
We start by giving an elementary construction of the Witt design on 22 points, starting with the complete graph on six vertices. This construction is related to a 16-vertex triangulation of $\mathbb{R}P^4$. We describe this complex in a purely combinatorial way, and prove that this complex is a triangulated $\mathbb{R}P^4$ by constructing its double cover, a 32-vertex antipodal S^4.

2.2 K_6

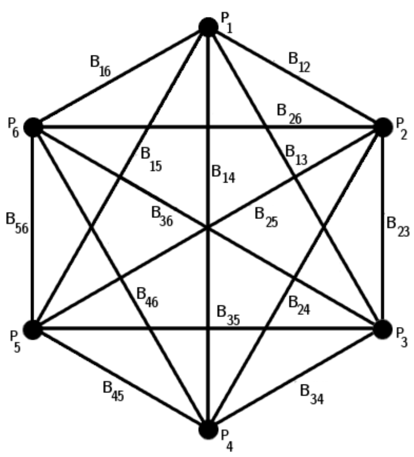
Let K_6 denote the complete graph on six vertices. We label these vertices by members of the set $\mathcal{P} = \{P_1, P_2, \ldots, P_6\}$. ($P$ for point.) K_6 also has 15 edges and 20 triangles. By an *edge* of K_6 we mean a pair of vertices $\{P_i, P_j\}$ where $i \neq j$. By a *triangle* of K_6 we mean a triple of vertices $\{P_i, P_j, P_k\}$, where i, j, k are distinct.

Definition 8. A *bisection* of K_6 is a partition of the vertex set into two triangles, e.g. $\{\{P_1, P_2, P_3\}, \{P_4, P_5, P_6\}\}$.

K_6 has ten bisections. Label them by the set $\mathcal{B} = \{B_0, B_1, B_2, \ldots, B_9\}$.



We can now label the edges of K_6 by the bisections in the following way. Each edge is in four triangles. So we label each edge by the a label of size 4, by the four bisections corresponding to its four incident triangles. We denote the 4-label of the edge joining P_i and P_j by B_{ij}. So our edge-labels form a set of fifteen 4-subsets of a 10-set.



Definition 9. A $t-(v,k,\lambda)$ design $\mathcal{D}$ is a pair consisting of a set V of size v and a set of k-subsets of V called the blocks of $\mathcal{D}$ such that any t-subset of V is contained in exactly λ blocks of $\mathcal{D}$.

The fifteen 4-subsets of $\mathcal{B}$ corresponding to the edges K_6 form the set of blocks of a quasi-symmetric $2-(10,4,2)$ design. To see that this is a 2-design with $\lambda = 2$, note that any pair of distinct bisections intersect in exactly two edges. To see that distinct blocks have two possible intersection sizes, namely one and two, first note that the vertex-set of a pair of adjacent edges in K_6 is a triangle, so is contained in a unique bisection, so their labels intersect in one element, the bisection containing the triangle formed by their edges, (e.g. $\{P_1, P_2\}$ and $\{P_1, P_3\}$ are contained in precisely $\{\{P_1, P_2, P_3\}, \{P_4, P_5, P_6\}\}$.) Now if two edges are not adjacent, they are contained in exactly two bisections, and their labels intersect in those two bisections, (e.g. $\{P_1, P_2\}$ and $\{P_3, P_4\}$ are contained in precisely $\{\{P_1, P_2, P_5\}, \{P_3, P_4, P_6\}\}$ and $\{\{P_1, P_2, P_6\}, \{P_4, P_4, P_5\}\}$.)

Definition 10. A 1-*factor* of a graph (on an even number of vertices) is a partition of its vertex set, where each block of the partition is an edge of the graph.

Being a complete graph on an even number of vertices, K_6 has 1-factors, e.g. $\{\{P_1, P_2\}, \{P_3, P_4\}, \{P_5, P_6\}\}$ is a 1-factor of K_6. Every edge of K_6 is contained in exactly three 1-factors. The number of 1-factors in K_6 is $\frac{6!}{2!2!2!3!} = 15$. The three pairwise disjoint edges in any 1-factor have labels which intersect pairwise in two elements of $\mathcal{B}$ each. But the bisections containing one pair of disjoint edges do not contain the third of these edges. So the union of three 4-labels of the edges in a 1-factor gives a subset of $\mathcal{B}$ of size six. So we can label each 1-factor by the four elements of $\mathcal{B}$ not in the labels of any of its three edges. The set of these 4-labels gives another quasi-symmetric $2-(10, 4, 2)$ design, whose blocks intersect in one point if the corresponding 1-factors are disjoint and in two points if the 1-factors intersect in an edge.

Definition 11. A 1-*factorization* of a graph is a partition of its set of edges, where each block in the partition is a 1-factor of the graph.

A 1-factorization of K_6 can be thought of as an edge-colouring of the graph by five colours, where the 1-factors are the colour-classes.

K_6 is 1-factorable.

We can draw K_6 in the plane with five of its vertices on the vertices of a regular pentagon with the remaining vertex at the centre. Then we colour each radial edge a different colour, and then colour each remaining edge with the same colour as the radial edge perpendicular to it.

More explicitly

$\{\{\{P_1, P_2\}, \{P_3, P_4\}, \{P_5, P_6\}\},$

$\{\{P_1, P_3\}, \{P_2, P_5\}, \{P_4, P_6\}\},$

$\{\{P_1, P_4\}, \{P_2, P_6\}, \{P_3, P_5\}\},$

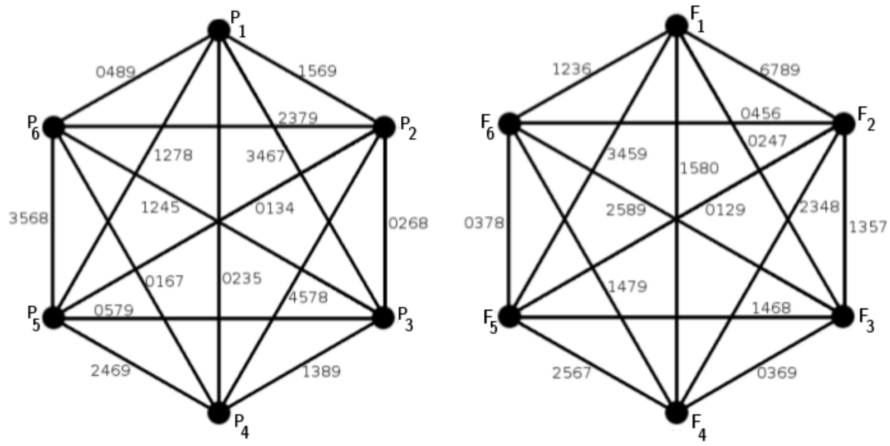
$\{\{P_1, P_5\}, \{P_2, P_4\}, \{P_3, P_6\}\},$

$\{\{P_1, P_6\}, \{P_2, P_3\}, \{P_4, P_5\}\}\}$

We can count the number of 1-factorizations of K_6 as follows. The edge $\{P_1, P_2\}$ is in three possible 1-factors. Now consider the 1-factor containing the edge $\{P_1, P_3\}$. The edge containing P_2 in this 1-factor cannot contain the vertex that was joined to P_3 in the first 1-factor. So we have two choices for the edge containing P_2. Now without loss of generality suppose the first 1-factor is $\{\{P_1, P_2\}, \{P_3, P_4\}, \{P_5, P_6\}\}$ and the second is $\{\{P_1, P_3\}, \{P_2, P_5\}, \{P_4, P_6\}\}$. Then the 1-factor containing $\{P_1, P_4\}$ has to be $\{\{P_1, P_4\}, \{P_2, P_6\}, \{P_3, P_5\}\}$. This fixes the remaining two 1-factors as $\{\{P_1, P_5\}, \{P_2, P_4\}, \{P_3, P_6\}\}$ and $\{\{P_1, P_6\}, \{P_2, P_3\}, \{P_4, P_5\}\}$. So we have $3 \times 2 = 6$ choices for the first two 1-factors which fix the remaining three. So K_6 has six 1-factorizations. We label these from the set $\mathcal{F} = \{F_1, F_2, \ldots, F_6\}$.

We saw that any 1-factor is contained in exactly two 1-factorizations, and that any two disjoint 1-factors determine a unique 1-factorization. Now since any two 1-factorizations can have at most one 1-factor in common, and there are $\binom{6}{2} = 15$ pairs of 1-factorizations, any two 1-factorizations intersect in a unique 1-factor. We denote the 4-label of the edges joining F_i and F_j by B^{ij}. This gives a "dual K_6" with vertices labelled by $\mathcal{F}$ and edges labelled by the elements of B^{ij} where $0 \leq i, j \leq 9$, with $i \neq j$.

The figure below illustrates the two copies of K_6 with their edges labelled by the elements of $\mathcal{B}$. (Here i denotes B_i for $0 \leq i \leq 9$.)



2.3 The Witt design on 22 points

We now describe the blocks in W_{22}, the Witt design on 22 points. This is a $3-(22,6,1)$ design. The 22 points of the design are the elements of $\mathcal{P} \cup \mathcal{B} \cup \mathcal{F}$. The blocks are of five types. The first two blocks are $\mathcal{P}$ and $\mathcal{F}$. Next we include all blocks of the forms $\{P_i, P_j\} \cup B_{ij}$ and $\{F_i, F_j\} \cup B^{ij}$.

Given any pair of bisections B_i, B_j, each triangle in B_i intersects one triangle in B_j in an edge e. This gives a partition of $\mathcal{P}$ into two edges and two points. The edges each correspond to the intersection of two triangles in B_i, B_j. The two points left over determine a third edge disjoint from the other two. Moreover, the 4-label of this edge contains neither B_i nor B_j. Also, since the pair B_i, B_j is contained in the labels of the other two edges, the 4-label of the $1-$factor f composed of the three edges above is disjoint from $\{B_i, B_j\}$. Since $e \in f$, the 4-labels of e and f are disjoint.

Similarly, given an incident edge-$1-$factor pair (e, f), their 4-labels are disjoint, and subtracting the union of these 4-labels from $\mathcal{B}$ leaves two bisections such that the triangles of one intersects the triangles of the other in the two edges of $f \setminus e$. So we have a correspondence between pairs of bisections and incident edge-$1-$factor pairs of K_6.

Now for a block of the fifth type, pick two bisections B_i, B_j, then for the corresponding pair (e, f) take the vertices of e and the two $1-$factorizations intersecting in f. These are all the blocks.

So we have one block each of the first and second kind, and 15 blocks each of the third and fourth kinds corresponding to the 15 edges and 1-factors. There are 15 choices of vertex pairs (edges), and three choices for the 1-factors containing each edge. So there are 45 blocks of the fifth type. This gives 77 blocks in total.

Next we show that the above 77 sets form the blocks of a $3-(22,6,1)$ design.

We show that any 3-subset of the point-set appears in exactly one of the 77 blocks described above.

Theorem 12. *The* 77 *sets described above form the set of blocks of a* $3-(22,6,1)$ *design.*

Proof. Consider the 3-subsets of $\mathcal{P} \cup \mathcal{B} \cup \mathcal{F}$. Any 3-subset of either $\mathcal{P}$ or $\mathcal{F}$ is in exactly one of the blocks of the first or second type. There are $\binom{10}{3} = 120$ subsets of $\mathcal{B}$ of size three. Consider the 3-subsets of sets of the form B_{ij} or B^{ij}. If we can show that no two sets of these forms have a 3-subset in common, it will follow that there are $2 \times 15 \times 4 = 120$ such sets, and that every 3-subset of $\mathcal{B}$ is in exactly one block. Two sets of the form B_{ij} intersect in at most two points of $\mathcal{B}$. Similarly, two sets of the form B^{kl} intersect in at most two points of $\mathcal{B}$. Now consider the intersection of a set of the form B_{ij} with a set of the form B^{kl}. If the edge of K_6 corresponding to B_{ij} is contained in the 1-factor corresponding to B^{kl}, then $B_{ij} \cap B^{kl} = \phi$. Otherwise, the edge corresponding to B_{ij} has its vertices in two different edges of B^{kl}, say $P_iP_{i'}$ and $P_jP_{j'}$, (we omit some brackets from hereon for the sake of convenience) with i, i', j, j' all distinct. Now since the labels of incident edges have one element in common, $|B_{ij} \cap B_{ii'}| = |B_{ij} \cap B_{jj'}| = 1$. Also since $i' \neq j'$, $B_{ij} \cap B_{ii'} \neq B_{ij} \cap B_{jj'}$. Since the labels of the third edge in the 1-factor corresponding to B^{kl} are contained in $B_{ii'} \cup B_{jj'}$, we have $|B_{ij} \cap B^{kl}| = |B_{ij} \cap (\mathcal{B} \setminus (B_{ii'} \cup B_{jj'}))| = 2$. So all the 3-subsets of the 4-labels of edges and 1-factors are distinct, and there are 120 such sets, so each 3-subset of $\mathcal{B}$ is in exactly one block.

Now consider a 3-set consisting of two elements of $\mathcal{P}$ and one element of $\mathcal{F}$, say $P_iP_jF_k$. Since F_k contains exactly one 1-factor containing the edge P_iP_j, the triple $P_iP_jF_k$ is contained in exactly one block of the fifth type. Similarly given a 3-set of the form $P_iF_jF_k$, the 1-factorizations F_j and F_k intersect in a unique 1-factor, which is a partition of the vertex set of K_6. So $P_iF_jF_k$ is contained in exactly one block of the fifth type.

A 3-set of the form $P_iP_jB_k$ can belong to the following cases. If $B_k \in B_{ij}$, then $P_iP_jB_k$ is contained in exactly one block of the third type, namely $\{P_i, P_j\} \cup B_{ij}$. If $B_k \notin B_{ij}$, then, $P_iP_jB_k$ is contained in exactly one block of the fifth type. Say the bisection represented by B_i is $\{\{P_i, P_{i'}, P_{i''}\}, \{P_j, P_{j'}, P_{j''}\}\}$. Then $B_k \in \mathcal{B} \setminus B^{lm}$, where F_l and F_m are the two 1-factorizations which intersect in the 1-factor $\{\{P_i, P_j\}, \{P_{i'}, P_{i''}\}, \{P_{j'}, P_{j''}\}\}$.

Similarly, if $B_k \in B^{ij}$, then $F_iF_jB_k$, is contained in exactly one block of the fourth type, $\{F_i, F_j\} \cup B_{ij}$. If $B_k \notin B^{ij}$, then it is an element of a 4-label of two of the edges in the 1-factor F_i and F_j intersect in. Then the block we need is $\{P_l, P_m, F_i, F_j\} \cup \mathcal{B} \setminus \{B_{lm} \cup B^{ij}\}$.

Now if a 3-set is of the form $P_iB_jB_k$, we have the following two possibilities. Recall that the 4-labels of the edges of K_6 form a quasi-symmetric $2-(10, 4, 2)$ design. So the pair B_jB_k is in the 4-label of two disjoint edges of K_6. If P_i is a vertex in either of these edges, then $P_iB_jB_k$ is in a block of the third type. The two vertices not in either of these edges, form the third edge of the 1-factor containing the two edges whose labels contain B_jB_k. So if P_i is on the third edge of this 1-factor, with $P_{i'}$ the other vertex on this edge, and if this 1-factor is the intersection of F_lF_m, then $P_iB_jB_k$ is in the block $\{P_i, P_{i'}, F_l, F_m, B_j, B_k\}$. Since there are 45 blocks of the fifth type, and $\mathcal{B}$ has $\binom{10}{2} = 45$ subsets of size two, B_jB_k appears in no other set of the fifth type. So $P_iB_jB_k$ will not be contained in any other block.

Similarly a 3-set of the form $F_iB_jB_k$ is in exactly one block of either the fourth or fifth type.

The only remaining type of 3-set is of the form $P_iF_jB_k$. There are $6 \times 6 \times 10 = 360$ of these. These can only be contained in blocks of the fifth type. Now there are 45 blocks of the fifth type, each with $2 \times 2 \times 2 = 8$ subsets of the form $P_iF_jB_k$. So there are at most 360 of these sets, and $P_iF_jB_k$ is in at most one of these blocks. We need to show that $P_iF_jB_k$ is in at least one of these blocks. Let $P_iP_{i'}P_{i''}$ be the triangle containing P_i in the bisection B_k. Say the other triangle is $\mathcal{P}\setminus\{P_i, P_{i'}, P_{i''}\} = \{P_{\bar{i}}, P_{\overline{i'}}, P_{\overline{i''}}\}$. The edge $\{P_{i'}, P_{i''}\}$ is in three 1-factors. Each of these 1-factors is the intersection of two 1-factorizations. No 1-factorization can contain more than one 1-factor containing a given edge, so the six 1-factorizations split into three pairs, each pair intersecting in a different 1-factor containing $\{P_{i'}, P_{i''}\}$. Now consider the pair of which F_j is a member, say $F_jF_{j'}$. The 1-factor they intersect in is $\{\{P_i, P_{\bar{i}}\}, \{P_{i'}, P_{i''}\}, \{P_{\overline{i'}}, P_{\overline{i''}}\}\}$. Then B_k is not contained in the 4-label of the edge $\{P_i, P_{\bar{i}}\}$ or of the 1-factor in the intersection of F_j and $F_{j'}$. So the 3-set $P_iF_jB_k$ is contained in the block $\{P_i, P_{\bar{i}}, F_j, F_{j'}, B_k, B_{k'}\}$, where $B_{k'}$ is the bisection $\{\{P_{\bar{i}}, P_{i'}, P_{i''}\}, \{P_i, P_{\overline{i'}}, P_{\overline{i''}}\}\}$.

This completes our proof. □

2.4 $\mathbb{R}P^4_{16}$

We now construct a combinatorial 4-manifold on 16 vertices homeomorphic to the real projective 4-dimensional space $\mathbb{R}P^4$. We call this complex $\mathbb{R}P^4_{16}$.

The 16 vertices of $\mathbb{R}P^4_{16}$ are the elements of the set $\mathcal{P}\cup\mathcal{B}$. The facets of $\mathbb{R}P^4_{16}$ are of two types. The first type of facet is of the form $\{P_i\}\cup B_{ij}$ where $i \neq j$. The second type of facet is of the form $\{P_i, P_j\}\cup(B_{ik}\setminus B_{ij})$, where i, j, k are distinct. There are $6\times 5 = 30$ facets of the first type, and $6\times 5\times 4 = 120$ facets of the second type.

Observe that the link of an edge (or 1-face) of $\mathbb{R}P^4_{16}$ of the form P_iP_j has eight triangles (2-faces), corresponding to four choices for k and two choices depending on which of the two remaining vertices is i. These triangles form the faces of an octahedron which can also be described as follows. An octahedron on six given points is determined by three pairs of antipodal vertices. Recall our construction of W_{22} above. The three blocks of the fifth type which contain P_iP_j, each contain two points from $\mathcal{B}$. These three pairs of points form the pairs of antipodal vertices of the octahedral link.

We now prove that this simplicial complex is a triangulation of $\mathbb{R}P^4$. We show this by constructing a 32-vertex antipodal 4-sphere whose quotient under the antipodal map is $\mathbb{R}P^4_{16}$.

2.5 A 32-vertex antipodal S^4

Let $\mathbf{e}_i \in \mathbb{R}^n$ denote the i^{th} elementary vector. Consider the following subsets of $\mathbb{R}^6$.

$$V_1 = \{\mathbf{e}_i | 1 \leq i \leq 6\}$$
$$V_3 = \{\frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k) | 1 \leq i < j < k \leq 6\}$$
$$V_5 = \{\frac{1}{5}(\mathbf{1} - \mathbf{e}_i) | 1 \leq i \leq 6\}$$

where $\mathbf{1} = \sum_{i=1}^{6} \mathbf{e}_i$.

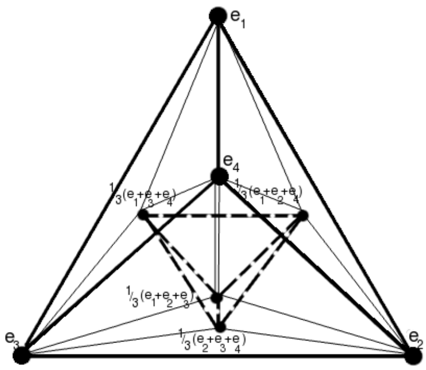
We can visualize the elements of V_1 as the vertices of the standard 5-simplex Δ^5 in $\mathbb{R}^6$. Then the elements of V_3 and V_5 are the barycenters of the triangles and 4-simplices in the boundary of this simplex.

Note that each facet of Δ^5 is a 4-simplex containing five elements of V_1. Suppose this 4-simplex is $\Delta_6 = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5]$. Then introducing the barycenter of Δ_6 allows us to subdivide Δ_6 as the union of five 4-simplices, each of which is the cone over a facet of Δ_6 at the point $\frac{1}{5}(\mathbf{1} - \mathbf{e}_6)$. We can do this for every facet of the boundary of Δ^5. So we transform the boundary of Δ^5, which is PL-homeomorphic to S^4, to a 12-vertex subdivision of S^4. Call this new complex X^4_{12}.

Now every new facet of this manifold is a cone over a tetrahedron with vertices from V_1 at the barycenter of a facet of Δ^5 containing this tetrahedron.

We further subdivide each facet of X^4_{12} in the following way. For each facet of X^4_{12}, $[\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l, \frac{1}{5}(\mathbf{1} - \mathbf{e}_m)]$, the tetrahedron $[\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l]$ can be decomposed into eleven tetrahedra, i.e. six tetrahedra of the form $[\mathbf{e}_i, \mathbf{e}_j, \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_l)]$ corresponding to every pair of elements of $\{i, j, k, l\}$, four tetrahedra of the form $[\mathbf{e}_i, \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_l), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_k + \mathbf{e}_l)]$ corresponding to every of element of $\{i, j, k, l\}$, and the tetrahedron $[\frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_l),$

$\frac{1}{3}(\mathbf{e}_i + \mathbf{e}_k + \mathbf{e}_l), \frac{1}{3}(\mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_l)]$.



We can join $\frac{1}{5}(\mathbf{1} - \mathbf{e}_m)$ to each of these tetrahedra to obtain a decomposition of the facet. This gives us a triangulated S^4 on 32 vertices. We transform this complex, call it X^4_{32}, into an antipodal S^4 using bistellar flips.

Each pair of elements of V_1, say $\{\mathbf{e}_i, \mathbf{e}_j\}$ forms an edge of X^4_{32}, which is contained in four triangles of the form $[\mathbf{e}_i, \mathbf{e}_j, \frac{1}{5}(\mathbf{1} - \mathbf{e}_k)]$. The link of this triangle in X^4_{32} is the boundary of the triangle $[\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_l), \frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_{l'}), \frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_{l''})]$, where $\{i, j, k, l, l', l''\} = \{1, \dots, 6\}$. Also, every triple of vertices of the form $\{\frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_l), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_{l'}), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_{l''})\}$ is the vertex set of the link of a unique triangle of the form $[\mathbf{e}_i, \mathbf{e}_j, \frac{1}{5}(\mathbf{1} - \mathbf{e}_k)]$, since, i, j, k, l, l', l'' are all distinct. So we can apply simultaneous bistellar flips to all triangles of the form $[\mathbf{e}_i, \mathbf{e}_j, \frac{1}{5}(\mathbf{1} - \mathbf{e}_k)]$, replacing the set of facets

$$[\mathbf{e}_i, \mathbf{e}_j, \frac{1}{5}(\mathbf{1} - \mathbf{e}_k)] * \partial[\frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_l), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_{l'}), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_{l''})]$$

with

$$[\frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_l), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_{l'}), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_{l''})] * \partial[\mathbf{e}_i, \mathbf{e}_j, \frac{1}{5}(\mathbf{1} - \mathbf{e}_k)]$$

After this round of flips, the link of $[\mathbf{e}_i, \mathbf{e}_j]$ is the boundary of the tetrahedron

$$[\frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_l), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_{l'}), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_{l''}), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_{l'''})]$$

where i, j, l, l', l'', l''' are distinct. As above, we can perform simultaneous bistellar flips to replace

$$[\mathbf{e}_i, \mathbf{e}_j] * \partial[\frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_l), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_{l'}), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_{l''}), \frac{1}{3}(\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_{l'''})]$$

with

$$[\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_l),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_{l'}),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_{l''}),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_{l'''})]*\partial[\mathbf{e}_i,\mathbf{e}_j]$$

The resulting complex is an antipodal S^4 with 32 vertices, where the antipodal map takes $\mathbf{e}_i$ to $\frac{1}{5}(\mathbf{1}-\mathbf{e}_i)$ and $\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_k)$ to $\frac{1}{3}(\mathbf{1}-\mathbf{e}_i-\mathbf{e}_j-\mathbf{e}_k)$.

It has four types of simplices (or two types under the action of $C_2\times S_6$):

$$[\mathbf{e}_i,\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_l),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_{l'}),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_{l''}),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_{l'''})]$$

$(6\times 5=30$ in number).

$$[\frac{1}{5}(\mathbf{1}-\mathbf{e}_m),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_k),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_l),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_k+\mathbf{e}_l),\frac{1}{3}(\mathbf{e}_j+\mathbf{e}_k+\mathbf{e}_l)]$$

$(i,j,k,l,m$ distinct, 30 in number).

$$[\mathbf{e}_i,\frac{1}{5}(\mathbf{1}-\mathbf{e}_k),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_l),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_{l'}),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_{l''})]$$

$(i,j,k,l,l',l''$ distinct, $6\times 5\times 4=120$ in number).

$$[\mathbf{e}_i,\frac{1}{5}(\mathbf{1}-\mathbf{e}_m),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_k),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_l),\frac{1}{3}(\mathbf{e}_i+\mathbf{e}_k+\mathbf{e}_l)]$$

$(i,j,k,l,m$ distinct, $6\times 5\times 4=120$ in number).

Comparing the two descriptions, it can be seen that the image under the antipodal map of the complex constructed above is the same as $\mathbb{R}P^4_{16}$ described in the previous section.

Chapter 3

Geometric constructions of $\mathbb{R}P^4_{16}$

3.1 Introduction

We give two geometric constructions of a 16-vertex triangulation of $\mathbb{R}P^4_{16}$.

3.2 Constructions using generalized octahedra and cubes

We can construct triangulated real projective n-space, $\mathbb{R}P^n$ in the following way. Take an n-dimensional cross-polytope and triangulate its interior, possibly by adding an extra point $\mathbf{0}$. Say we denote the vertices of the cross-polytope C^n by the vectors $\pm\mathbf{e}_i \in \mathbb{R}^n$, where $1 \leq i \leq n$. Then we add 2^n simplices of the form $[\varepsilon_1\mathbf{e}_1, \varepsilon_2\mathbf{e}_2, \ldots, \varepsilon_n\mathbf{e}_n, \sum_{i=1}^n \varepsilon_i\mathbf{e}_i]$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \{\pm 1\}^n$. Call the new vertex set V. We have $V = C \sqcup Q$, where C is the vertex set of the triangulated C^n and $Q = \{q_\varepsilon | \varepsilon \in \{\pm 1\}^n\} = \{\sum_{i=1}^n \varepsilon_i\mathbf{e}_i | \varepsilon_i = \pm 1, 1 \leq i \leq n\}$.

Remark. It must be borne in mind here that even though our notation represents the vertices of n-dimensional complexes as points in $\mathbb{R}^n$, the objects we construct are purely abstract simplicial complexes, which we do not need to view as embedded in $\mathbb{R}^N$ for any N. Indeed, they most definitely do not embed in $\mathbb{R}^n$. Our choice of notation is motivated by ease of handling and conceptual visualization.

Now we consider subsimplices of ∂C^n going down in dimension, and triangulate the links of each without adding any more vertices. Our goal is to end up with the boundary of the link of a vertex in C as a triangulated 2^{n-1}-vertex $n-2$-sphere. We do this subject to the following conditions. First, for every facet

$$[\varepsilon_{i_1}\mathbf{e}_{i_1}, \varepsilon_{i_2}\mathbf{e}_{i_2}, \dots, \varepsilon_{i_k}\mathbf{e}_{i_k}, q_{\varepsilon^{j_1}}, q_{\varepsilon^{j_2}}, \dots, q_{\varepsilon^{j_{n-k+1}}}]$$

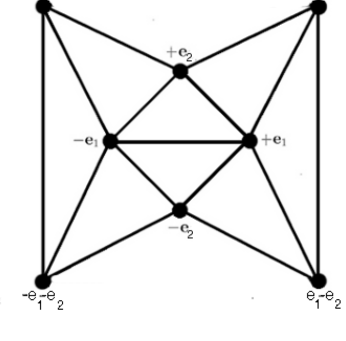
in the complex, its "opposite" facet

$$[-\varepsilon_{i_1}\mathbf{e}_{i_1}, -\varepsilon_{i_2}\mathbf{e}_{i_2}, \dots, -\varepsilon_{i_k}\mathbf{e}_{i_k}, q_{-\varepsilon^{j_1}}, q_{-\varepsilon^{j_2}}, \dots, q_{-\varepsilon^{j_{n-k+1}}}]$$

is also in the complex. Second, no vertex q_ε in Q is joined to its "opposite" vertex $q_{-\varepsilon} = -q_\varepsilon$. In other words $[q_\varepsilon, q_{-\varepsilon}]$ is not an edge of the complex. Third, if a vertex $u \in V$ is joined to another vertex $v \in V$, i.e. if $[u, v]$ is an edge of the complex, then $[u, -v]$ is not an edge of the complex. These conditions allow us to apply the identification map $q_\varepsilon \sim q_{-\varepsilon}$ on Q, leaving us with $2n$ copies of SS^{n-2}. We then triangulate the interior of these $2^{n-1}+2$-vertex $n-1$-spheres, to get a triangulation of $\mathbb{R}P^n$.

Example. It is easily seen that the paradigm outlined above can be used to construct $\mathbb{R}P^2_6$ as follows. We triangulate the square spanned by $\pm\mathbf{e}_1, \pm\mathbf{e}_2$ by joining $+\mathbf{e}_1$ with $-\mathbf{e}_1$. So we have triangles $[\mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2], [\mathbf{e}_1, -\mathbf{e}_1, -\mathbf{e}_2]$.

Next we add the triangles $\pm[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1+\mathbf{e}_2], \pm[\mathbf{e}_1, -\mathbf{e}_2, \mathbf{e}_1-\mathbf{e}_2]$. The links of the vertices $\pm\mathbf{e}_1$ now have boundaries $\{\pm[\mathbf{e}_1+\mathbf{e}_2], \pm[\mathbf{e}_1-\mathbf{e}_2]\}$ respectively and the boundaries of the links of $\pm\mathbf{e}_2$ have boundaries $\{\pm[\mathbf{e}_1+\mathbf{e}_2], \pm[-\mathbf{e}_1+\mathbf{e}_2]\}$. Now we apply the map $q_\varepsilon \sim q_{-\varepsilon}$. We triangulate the 1-sphere containing $\pm\mathbf{e}_2$ by adding the triangles $[+\mathbf{e}_2, -\mathbf{e}_2, \overline{\mathbf{e}_1+\mathbf{e}_2}], [+\mathbf{e}_2, -\mathbf{e}_2, \overline{\mathbf{e}_1+\mathbf{e}_2}]$. Since the link of $[+\mathbf{e}_1, -\mathbf{e}_1]$ is already a 0-sphere in our complex, we triangulate the remaining square as $[+\mathbf{e}_1, \overline{\mathbf{e}_1+\mathbf{e}_2}, \overline{\mathbf{e}_1-\mathbf{e}_2}], [-\mathbf{e}_1, \overline{\mathbf{e}_1+\mathbf{e}_2}, \overline{\mathbf{e}_1-\mathbf{e}_2}]$. This gives us $\mathbb{R}P^2_6$.



3.3 Construction of $\mathbb{R}P^3_{11}$

Start with the octahedron C^3 spanned by the points $\pm\mathbf{e}_i$, where $i = 1, 2, 3$. We can triangulate the interior of the octahedron by taking the cone over its boundary at the point $\mathbf{0}$. This gives us eight tetrahedra of the form

$$[\mathbf{0}, \varepsilon_1\mathbf{e}_1, \varepsilon_2\mathbf{e}_2, \varepsilon_3\mathbf{e}_3].$$

The boundary of the octahedron consists of the eight triangles of the form $[\varepsilon_1\mathbf{e}_1, \varepsilon_2\mathbf{e}_2, \varepsilon_3\mathbf{e}_3]$, where $\varepsilon_i = \pm 1$ for $i = 1, 2, 3$. Now add eight new points $q_\varepsilon = \varepsilon_1\mathbf{e}_1 + \varepsilon_2\mathbf{e}_2 + \varepsilon_3\mathbf{e}_3$ for each $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{\pm 1\}^3$, by taking the eight tetrahedra of the form

$$[\varepsilon_1\mathbf{e}_1, \varepsilon_2\mathbf{e}_2, \varepsilon_3\mathbf{e}_3, \varepsilon_1\mathbf{e}_1 + \varepsilon_2\mathbf{e}_2 + \varepsilon_3\mathbf{e}_3].$$

The boundary of this complex is a triangulated S^2 with f-vector $[14, 36, 24]$.

Now consider the link of an edge of C^3. The link of $[\varepsilon_i\mathbf{e}_i, \varepsilon_j\mathbf{e}_j]$ is the path $[\varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j - \mathbf{e}_k, -\mathbf{e}_k], [-\mathbf{e}_k, \mathbf{0}], [\mathbf{0}, \mathbf{e}_k], [\mathbf{e}_k, \varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j + \mathbf{e}_k]$, where

$\{i, j, k\} = \{1, 2, 3\}$. Its boundary consists of the two points $\varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j \pm \mathbf{e}_k$. Close the boundary of $[\varepsilon_i\mathbf{e}_i, \varepsilon_j\mathbf{e}_j]$ by adding its join with the edge $[\varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j - \mathbf{e}_k, \varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j + \mathbf{e}_k]$. This gives twelve new tetrahedra, and the boundary of the new complex is still a triangulated S^2 with 14-vertices. But the boundary of the link of a vertex $\varepsilon_i\mathbf{e}_i$ of C^3 is now the set of edges of the square $(\mathbf{e}_j + \mathbf{e}_k + \varepsilon_i\mathbf{e}_i, \mathbf{e}_j - \mathbf{e}_k + \varepsilon_i\mathbf{e}_i, -\mathbf{e}_j - \mathbf{e}_k + \varepsilon_i\mathbf{e}_i, -\mathbf{e}_j + \mathbf{e}_k + \varepsilon_i\mathbf{e}_i)$. Also note that the subcomplex induced by Q is the set of edges of the 3-dimensional cube Q^3.

We can now identify $\sum_{i=1}^3 \varepsilon_i\mathbf{e}_i$ with the point $-\sum_{i=1}^3 \varepsilon_i\mathbf{e}_i$, and close the boundary of the link of $\varepsilon_i\mathbf{e}_i$ by taking its cone at the point $-\varepsilon_i\mathbf{e}_i$. That is, for each pair $\pm\mathbf{e}_i$, we take the four tetrahedra

$$[\varepsilon_i\mathbf{e}_i, -\varepsilon_i\mathbf{e}_i, \pm\overline{\mathbf{e}_j + \mathbf{e}_k + \varepsilon_i\mathbf{e}_i}, \overline{\mathbf{e}_j + \mathbf{e}_k + \varepsilon_i\mathbf{e}_i}]$$

The link of the vertex $\varepsilon_i\mathbf{e}_i$ is now the triangulated 8-vertex S^2 with facets

$$[\mathbf{0}, \pm\mathbf{e}_j, \pm\mathbf{e}_k], \pm[\mathbf{e}_j, \mathbf{e}_k, \overline{\mathbf{e}_j + \mathbf{e}_k + \varepsilon_i\mathbf{e}_i}], \pm[\mathbf{e}_j, -\mathbf{e}_k, \overline{\mathbf{e}_j - \mathbf{e}_k + \varepsilon_i\mathbf{e}_i}],$$
$$[\pm\mathbf{e}_j, \overline{\mathbf{e}_j + \mathbf{e}_k + \varepsilon_i\mathbf{e}_i}, \overline{\mathbf{e}_j - \mathbf{e}_k + \varepsilon_i\mathbf{e}_i}], [\overline{\mathbf{e}_j + \mathbf{e}_k + \varepsilon_i\mathbf{e}_i}, \overline{\mathbf{e}_j - \mathbf{e}_k + \varepsilon_i\mathbf{e}_i}, -\varepsilon_i\mathbf{e}_i]$$

The above complex is an 11-vertex triangulation of $\mathbb{R}P^3$. This complex is combinatorially the same as the $\mathbb{R}P^3_1 1$ described by Walkup in [10], as the antipodal quotient of a 22-vertex S^3.

3.4 First Geometric Construction of $\mathbb{R}P^4$

We construct $\mathbb{R}P^4$ by first constructing a 4-dimensional ball B^4 with an antipodal S^3 as boundary, then taking the quotient of the boundary under the antipodal map.

We start with a 4-dimensional (solid) hyperoctahedron C^4, given by the convex hull of $\{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3, \pm\mathbf{e}_4\}$. We triangulate C^4 by joining the vertices $+\mathbf{e}_1$ and $-\mathbf{e}_1$. The resulting complex is a set of eight 4-simplices which can be visualized as the join of the line segment $[-\mathbf{e}_1, +\mathbf{e}_1]$ with the boundary of the octahedron spanned by $\{\pm\mathbf{e}_2, \pm\mathbf{e}_3, \pm\mathbf{e}_4\}$.

The boundary of this triangulated ball is just the boundary ∂C^4 of C^4, which is a triangulated 3-sphere with f-vector $[8, 24, 32, 16]$. In other words, we have triangulated C^4 *internally*, i.e. without changing the simplicial structure at its boundary. Now we join each of the 16 facets of this boundary with a point. That is, for each facet $[\varepsilon_1\mathbf{e}_1, \varepsilon_2\mathbf{e}_2, \varepsilon_3\mathbf{e}_3, \varepsilon_4\mathbf{e}_4]$ of ∂C^4, take the cone over this facet at the point $q_\varepsilon = \sum_{i=1}^4 \varepsilon_i\mathbf{e}_i$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \{\pm 1\}^4$. This gives 16 such 4-simplices, and the boundary now has 24 vertices, $16 \times 4 + 24 = 88$ edges, $16 \times \binom{4}{2} + 32 = 128$ triangles and $16 \times 4 = 64$ tetrahedra. Denote this triangulation of S^3 by $X^{(1)}$.

Now consider the link of each triangle of ∂C^4 in $X^{(1)}$. Consider the triangle $[\varepsilon_i\mathbf{e}_i, \varepsilon_j\mathbf{e}_j, \varepsilon_k\mathbf{e}_k]$. If $1 \notin \{i, j, k\}$, then the link of the triangle is

$$[-\mathbf{e}_1 + \varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j + \varepsilon_k\mathbf{e}_k, -\mathbf{e}_1], [-\mathbf{e}_1, +\mathbf{e}_1], [+\mathbf{e}_1, +\mathbf{e}_1 + \varepsilon_i\mathbf{e}_i + \varepsilon_j\mathbf{e}_j + \varepsilon_k\mathbf{e}_k].$$

Now suppose $1 \in \{i, j, k\}$, then the link of the triangle is

$$[-\mathbf{e}_l + \sum_{i,j,k} \varepsilon_\alpha\mathbf{e}_\alpha, -\mathbf{e}_l], [-\mathbf{e}_l, -\varepsilon_1\mathbf{e}_1], [-\varepsilon_1\mathbf{e}_1, +\mathbf{e}_l], [+\mathbf{e}_l, +\mathbf{e}_l + \sum_{i,j,k} \varepsilon_\alpha\mathbf{e}_\alpha],$$

where $l \in \{1, 2, 3, 4\} \setminus \{i, j, k\}$. In either case the endpoints of the link of the triangle are the two points $-\mathbf{e}_l + \sum_{i,j,k} \varepsilon_\alpha \mathbf{e}_\alpha$ and $+\mathbf{e}_l + \sum_{i,j,k} \varepsilon_\alpha \mathbf{e}_\alpha$ corresponding to the two tetrahedra containing it in C^4.

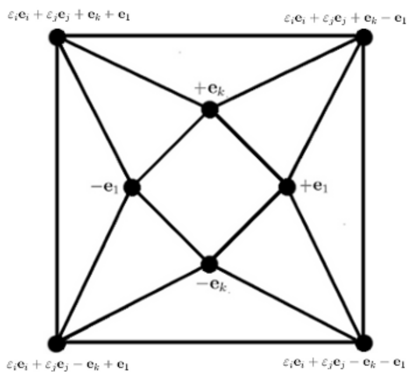
We can now close the links of triangles by adding the 4-simplices

$$[\varepsilon_i \mathbf{e}_i, \varepsilon_j \mathbf{e}_j, \varepsilon_k \mathbf{e}_k, \sum_{i,j,k} \varepsilon_\alpha \mathbf{e}_\alpha - \mathbf{e}_l, \sum_{i,j,k} \varepsilon_\alpha \mathbf{e}_\alpha + \mathbf{e}_l]$$

where $\{i,j,k,l\} = \{1,2,3,4\}$. We have added 32 such 4-simplices, and the boundary of the new complex is a triangulated S^3 with 24 vertices, $88+32 = 120$ edges, $128 - 32 + (32 \times 3) = 196$ triangles, and $64 + 32 = 96$ tetrahedra. Call the boundary $X^{(2)}$.

Observe now that the subcomplex induced by the subset of vertices $Q = \{q_\varepsilon | \varepsilon \in \{\pm 1\}^4\}$ is the 1-skeleton of a 4-dimensional hypercube, which is the dual of C^4.

Now consider the links of edges of C^4. The link of the edge $[\varepsilon_i \mathbf{e}_i, \varepsilon_j \mathbf{e}_j]$, when $1 \notin \{i,j\}$ has 8 vertices, namely $\pm \mathbf{e}_k, \pm \mathbf{e}_1, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j \pm \mathbf{e}_k \pm \mathbf{e}_1$, where $\{i,j,k\} = \{2, 3, 4\}$. These vertices can be seen as forming the corners of a 4-sided antiprism whose opposite (oriented) faces are $(+\mathbf{e}_1, +\mathbf{e}_k, -\mathbf{e}_1, -\mathbf{e}_k)$ and $(\varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_1, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_k - \mathbf{e}_1, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_k - \mathbf{e}_1, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_k + \mathbf{e}_1)$.



The first square is triangulated by the diagonal $[-\mathbf{e}_1, +\mathbf{e}_1]$. The link of the edge $[\varepsilon_1 \mathbf{e}_1, \varepsilon_i \mathbf{e}_i]$ is quite similar, except that the square $[+\mathbf{e}_j, +\mathbf{e}_k, -\mathbf{e}_j, -\mathbf{e}_k]$ is triangulated by taking the cone of its boundary at the vertex $-\varepsilon_1 \mathbf{e}_1$.

In either case, the boundary of the link of the edge $[\varepsilon_i \mathbf{e}_i, \varepsilon_j \mathbf{e}_j]$ is the boundary of the square

$$(\varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_l, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_k - \mathbf{e}_l, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_k - \mathbf{e}_l, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_k + \mathbf{e}_l).$$

We triangulate each of these squares by joining one pair of non-adjacent vertices by a diagonal. Prima facie, we seem to have some amount of choice in this situation. All we have to ensure here is that if we introduce an edge $[q_\varepsilon, q_{\varepsilon'}]$, then we also include the edge $[-q_\varepsilon, -q_{\varepsilon'}]$.

Recall that the 4-dimensional hypercube Q^4 is bipartite, and the vertex partitions divide the vertices $\{q_\varepsilon | \varepsilon \in \{\pm 1\}^4\}$ into two sets,

$$Q_e = \{q_\varepsilon | \varepsilon \in \{\pm 1\}^4, \prod_{i=1}^{4} \varepsilon_i = +1\}$$

and

$$Q_o = \{q_\varepsilon | \varepsilon \in \{\pm 1\}^4, \prod_{i=1}^{4} \varepsilon_i = -1\}.$$

Also note that if q_ε and $-q_\varepsilon = q_{-\varepsilon}$ are always in the same block of the partition. Also any square in Q^4 contains exactly two vertices from Q_o and two from Q_e.

So we can triangulate each square with boundary

$$(\varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_k + \mathbf{e}_l, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_k - \mathbf{e}_l, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_k - \mathbf{e}_l, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_k + \mathbf{e}_l)$$

by joining the elements of either of Q_o or Q_e by a diagonal.

Also, since the link of $[\mathbf{e}_1, -\mathbf{e}_1]$ is already a sphere, we need to triangulate the boundaries of the links of $\pm\mathbf{e}_1$ "internally". Additionally, this triangulation can not introduce a diagonal through the interiors of either sphere, as any point u at (Hamming) distance 3 to a point v is at distance 1 to its antipode $-v$. There is one way of triangulating an 8-vertex 2-sphere with the given partial 1-skeleton without interior diagonals, i.e. the triangulation of the solid cube into 5 tetrahedra. So we triangulate each square in the link of $\pm\mathbf{e}_1$ by joining the elements of say, Q_o, by edges.

Now each element u of Q_o is joined to three other elements v_1, v_2, v_3 of Q_o at distance 2 from it. The other three elements of Q_o at distance 2 from u are $-v_1$, $-v_2$, and $-v_3$. So none of the elements of Q_o can be joined when triangulating the remaining squares. This forces us to triangulate the remaining squares by joining the vertices in Q_e by an edge. This is possible, since for each vertex of Q_e, the three vertices at distance 2 from it, which are across a square in the link of some $[\pm\mathbf{e}_1, \pm\mathbf{e}_i]$, have been ruled out in the previous step.

So for $\{i, j, k\} = \{2, 3, 4\}$, we replace the join of a line segment and the boundary of a square in $X^{(2)}$, i.e.

$$[\varepsilon_i \mathbf{e}_i, \varepsilon_j \mathbf{e}_j] * (\partial[\varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_1 + \varepsilon_i \varepsilon_j \mathbf{e}_k, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_1 - \varepsilon_i \varepsilon_j \mathbf{e}_k] * \\ \partial[\varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_1 + \varepsilon_i \varepsilon_j \mathbf{e}_k, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_1 - \varepsilon_i \varepsilon_j \mathbf{e}_k]),$$

with the join of a new line segment with the boundary of another square, i.e.

$$[\varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_1 + \varepsilon_i \varepsilon_j \mathbf{e}_k, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_1 - \varepsilon_i \varepsilon_j \mathbf{e}_k] * (\partial[\varepsilon_i \mathbf{e}_i, \varepsilon_j \mathbf{e}_j] * \\ \partial[\varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j - \mathbf{e}_1 + \varepsilon_i \varepsilon_j \mathbf{e}_k, \varepsilon_i \mathbf{e}_i + \varepsilon_j \mathbf{e}_j + \mathbf{e}_1 - \varepsilon_i \varepsilon_j \mathbf{e}_k])$$

So we are adding two 4-simplices for every edge in C^4, $24 \times 2 = 48$ in total. The f-vector of the boundary remains $[24, 120, 192, 96]$, the same as that of $X^{(2)}$. Call the boundary of the current complex $X^{(3)}$.

Now consider the link of a vertex from C^4 in $X^{(3)}$. The vertex-set of the link of $\varepsilon_i \mathbf{e}_i$ in $X^{(3)}$ is the set of vertices of the hypercube Q^4 whose i^{th} co-ordinate is ε_i. These vertices span a 3-dimensional cube, and the triangles of the link of $\varepsilon_i \mathbf{e}_i$ in $X^{(3)}$ are "halves" of squares of Q^4.

Now consider opposite pairs of vertices of C^4. The boundaries of the links of $+\mathbf{e}_i$ and $-\mathbf{e}_i$ are opposite (cubical) faces of Q^4. Moreover, the map $\mathbf{x} \mapsto -\mathbf{x}$ swaps the triangulations of these cubes. So X^3 is an antipodal S^3.

In order to triangulate the link of $\varepsilon \mathbf{e}_i, 2 \leq i \leq 4$, we could triangulate the interior of the 8-vertex 2-sphere (or triangulated cube) described above by taking its cone at the point $-\varepsilon_i \mathbf{e}_i$. In other words, we take the join of the edge $\{\pm \mathbf{e}_i\}$ with the 8-vertex S^2 which is now the link of both $\mathbf{e}_i$ and $-\mathbf{e}_i$.

This leaves the pair $\pm \mathbf{e}_1$. The boundaries of the links of either vertex is an 8-vertex S^2, or the boundary of a cube triangulated by joining "every other vertex by an edge". As mentioned above, we triangulate the links of each vertex by splitting it respectively into five tetrahedra, the vertices of four of which have one element each of Q_e and its three neighbouring elements of Q_e, the the vertices of the fifth are four elements of Q_o.

Now we apply the map $x \mapsto -x$ on Q. This gives $\mathbb{R}P^4$.

3.5 Second Geometric Construction of $\mathbb{R}P^4_{16}$

We construct $\mathbb{R}P^4_{16}$ in one more way. Here we work with *polyhedral complexes* instead of the usual simplicial complexes. Again, the idea is to construct a 4-dimensional ball with antipodal boundary, then to quotient via the antipodal map. The treatment in this section sacrifices rigour in the service of intuition. See Appendix A for a more rigorous reworking.

Start with a (solid, 3-dimensional) cube Q^3, embedded in $\mathbb{R}^4$ with vertices $(\pm 1, \pm 1, \pm 1, 0)$. Consider its *suspension* SQ^3 at the points $(0,0,0,\pm 1)$, i.e. the convex hulls of the unions of the each of the faces of Q^3 with each of the aforementioned points. Note that this definition is analogous to that given for the suspension of a simplicial complex in the introduction. The boundary of this object is a 3-dimensional polyhedral complex with ten vertices and $2 \times 6 = 12$ (square-)pyramidal faces. The base of each of these pyramids is a face $S(\pm i)$ of the cube Q^3 given by $x_i = \pm 1, x_4 = 0$, where $1 \leq i \leq 3$. We avoid triangulating the interior of SQ^3 for the time being.

First, we construct a "dual" cell complex D by adding faces of increasing dimension, starting with points.

Corresponding to each face with base square $S(\pm i)$ and apex $(0,0,0,\varepsilon)$ of SQ^3, (where $\varepsilon \in \{\pm 1\}$), take the point $3(\sigma_1, \sigma_2, \sigma_3, \varepsilon)$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector in $\mathbb{R}^3$ taking value ± 1 at the i-th coordinate and 0 elsewhere. So for example, the pyramid with apex $(0,0,0,1)$ and base $S(-2)$ gives the point $(0,-3,0,3)$. Corresponding to each of the twelve faces of the boundary of the suspended cube, we get one point each, for a total of twelve points.

Now join each pair of points in D by an edge if the corresponding facets of SQ^3 intersect in a triangle or square. This gives six edges corresponding to each square of Q^3. Additionally, SQ^3 has $2 \times 12 = 24$ triangles corresponding to each point in $\{(0,0,0,\pm 1)\}$ and each edge of Q^3. This gives 24 more edges.

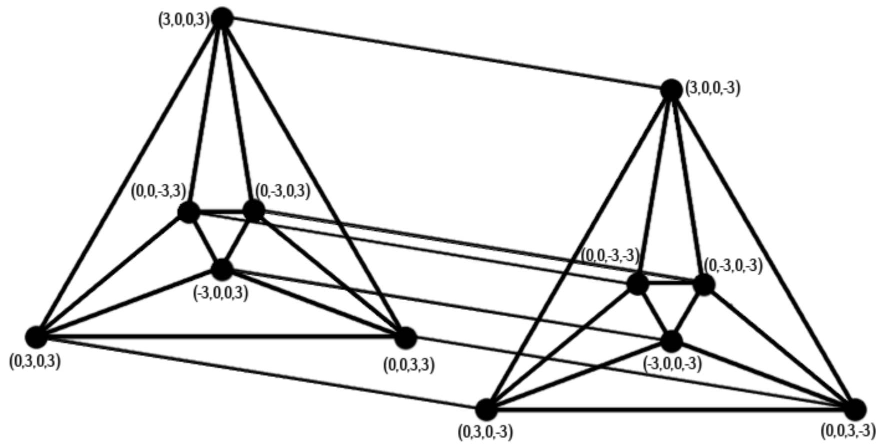
Next, consider the edges of SQ^3, of which there are $12 + 2 \times 8$. The 12 edges of Q^3 correspond to twelve squares in D, of which one pair of opposite edges corresponds to the squares of Q^3 which intersect in this edge, while the other pair corresponds to the two triangles of SQ^3 intersecting this edge. Each of the remaining 16 edges has as endpoints a vertex v of Q^3 and a point in $\{(0,0,0,\pm 1)\}$. Each such edge corresponds to a triangle of D, whose edges are correspond to the three squares in Q^3 intersecting in v.

Now for each of the $8+2=10$ vertices of SQ^3, we add a polyhedron to D. Each of the eight vertices of Q^3 correspond to eight triangular prisms, whose faces are the three rectangles in D corresponding to the three edges intersecting in this vertex, and the two triangles corresponding to the edges joining the vertex to each of $(0,0,0,\pm 1)$. Corresponding to either of the vertices $(0,0,0,\pm 1)$, we have an octahedron whose faces correspond to the eight edges of SQ^3 intersecting at the chosen vertex.

We summarize the above in a table.

SQ^3		D	
Dim	Faces	Faces	Dim
0	$8+2$ points	8 prisms+2 octahedra	3
1	$12+2\times 8$ edges	12 rectangles$+2\times 8$ triangles	2
2	6 squares$+2\times 12$ triangles	$6+2\times 12$ edges	1
3	2×6 pyramids	2×6 points	0

We can visualize the complex D as a prismed octahedron.



Now we can write down some of the 4-simplices in our triangulation. Join each triangle in SQ^3 to its corresponding edge between points of D. For example, we join the triangle $[(0,0,0,1),(1,1,1,0),(1,1,-1,0)]$ to the edge $[(3,0,0,3),(0,3,0,3)]$. This gives 24 facets. Now join each triangle in D to its corresponding edge in SQ^3. For example, the triangle $[(3,0,0,3),(0,3,0,3),(0,0,-3,3)]$ is joined to the edge $[(1,1,-1,0),(0,0,0,1)]$. This gives 16 more facets.

Each point in D is adjacent to five points in D. In the 40 simplices listed above, each point of D is joined to four vertices of Q^3 and one vertex of $\{(0,0,0,\pm1)\}$. Also, each of the vertices $(0,0,0,\pm1)$ is joined to the six points of D, the vertices of the octahedron corresponding to it. Each vertex of Q^3 is joined to three other vertices of Q^3, both vertices $(0,0,0,\pm1)$, and the six vertices of the triangular prism corresponding to it in D. Since the antipodal map we wish to apply to complex we are constructing takes $v \in D$ to $-v$, we can not add any more edges to our complex that contain a point of D. Now consider the links of the 24 triangles in SQ^3.edge in Q^3 with the point $(0,0,0,1)$. The triangle given by the join of $x_i = \varepsilon_i, x_j = \varepsilon_j$ to the point $(0,0,0,\varepsilon)$ is joined to the edge of D with endpoints corresponding to the squares $x_i = \varepsilon_i, x_4 = \varepsilon$, and $x_j = \varepsilon_j, x_4 = \varepsilon$. Now we join each tetrahedra obtained by joining the above triangle to each of the vertices of the edge (in D) above to a third point on the corresponding square in Q^3. Of the two remaining points on each square in Q^3, we choose the point the product of whose first three coordinates is 1. For example, the triangle $[(1,1,-1,0),(1,-1,-1,0),(0,0,0,-1)]$ has as link $[(3,0,0,-3),(0,0,-3,-3)]$. Now the tetrahedron

$$[(1,1,-1,0)(1,-1,-1,0),(0,0,0,-1),(3,0,0,-3)]$$

is joined to the point $(1,1,1,0)$, and the tetrahedron

$$[(1,1,-1,0),(1,-1,-1,0),(0,0,0,-1),(0,0,-3,-3)]$$

is joined to the point $(-1,1,-1,0)$. Now note that any such 4-simplex can be obtained by starting with either of the two edges in Q^3 contained in it. So we get $2 \times 12 \times 2/2 = 24$ simplices.

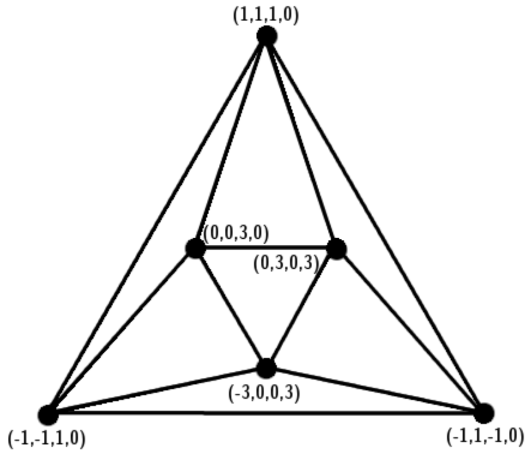
The link of each of the 24 triangles in SQ^3 is now a path of length 3 with endpoints from the vertices of Q^3. Moreover, if the triangle T is obtained by joining an edge E of Q^3 with a point in $(0,0,0,\pm 1)$, with $v \in E$ such that the product of the first three co-ordinates is -1, then the endpoints of the link of T are the neighbours of v in the square in Q^3 containing v but not E. For the next set of 4-simplices, join the endpoints of the link of each triangle in SQ^3 by an edge. For example, the triangle $[(1,1,-1,0),(1,-1,-1,0),(0,0,0,-1)]$ is joined to the edge $[(1,1,1,0),(-1,1,-1,0)]$. So for each choice of $(0,0,0,\pm 1)$ and each of the four vertices $v = (\varepsilon_1, \varepsilon_2, \varepsilon_3, 0)$ of Q^3 such that $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$, we have a simplex whose vertices are v, its three neighbours in Q^3, and one of $(0,0,0,\pm 1)$. This gives eight more simplices. The link of every triangle in SQ^3 is now a circle.

In constructing the previous two sets of simplices, we added one diagonal to each square face of Q^3. Recall that each square in Q^3 given by the equation $x_i = \varepsilon_i$ corresponds to the edge of D spanned by $3(\sigma_1, \sigma_2, \sigma_3, \pm 1)$, where the $\sigma_j = \varepsilon_i$ if $i = j$ and 0 otherwise. Now consider a triangle in our complex with vertices in this square, say $[v_0, v_1, v_2]$, and let the product of the first three coordinates of v_0 be -1. The link of this triangle had four edges. Let v_0' denote the third vertex adjacent to v_0 in Q^3. In the link of the triangle v_0' is joined to $(0,0,0,\pm 1)$, and the latter vertices are respectively joined to $3(\sigma_1, \sigma_2, \sigma_3, \pm 1)$. We join the triangle $[v_0, v_1, v_2]$ to the line segment $[3(\sigma_1, \sigma_2, \sigma_3, 1), 3(\sigma_1, \sigma_2, \sigma_3, -1)]$. Each of the twelve triangles with vertices on a square in Q^3 gives one simplex each, so we get twelve new simplices.

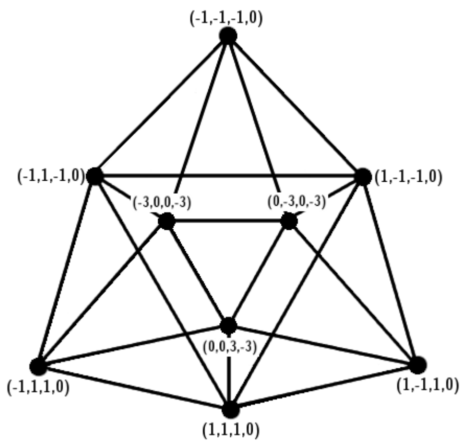
Also, in our last but one set of simplices, we introduced four new triangles, each consisting of three vertices of Q^3, such that the first three coordinates of each have product 1. The link of each such triangle consists of two edges, where the common neighbour of the three vertices is joined to each of $(0,0,0,\pm 1)$. So we have four triangles forming the boundary of a tetrahedron, the boundaries of the links of each being the vertices $(0,0,0,\pm 1)$. We add two new simplices, by taking the tetrahedron consisting of the four vertices of Q^3 and joining it to each of the points $(0,0,0,\pm 1)$.

So now the links of all triangles with vertices from SQ^3 are circles. We consider the links of edges in SQ^3.

The link of an edge of the form $[(\varepsilon_1, \varepsilon_2, \varepsilon_3, 0), (0, 0, 0, \varepsilon)]$ where $\varepsilon_1\varepsilon_2, \varepsilon_3 = -1$ is an octahedron, with one of its faces consisting of the three neighbours of $(\varepsilon_1, \varepsilon_2, \varepsilon_3, 0)$ in Q^3 and its opposite face is the triangle corresponding to the chosen edge in D. The next figure illustrates the link of $[(-1, 1, 1, 0), (0, 0, 0, 1)]$.



The link of an edge of the form $[(\varepsilon_1, \varepsilon_2, \varepsilon_3, 0), (0, 0, 0, \varepsilon)]$ where $\varepsilon_1\varepsilon_2, \varepsilon_3 = 1$ is a 9-vertex S^2 with six vertices of Q^3 and the three vertices of its corresponding triangle in D. Consider, for example the link of $[(-1, -1, 1, 0), (0, 0, 0, -1)]$.



The boundary of the link of an edge in Q^3 is the $1-$skeleton of its corresponding square in D.

The boundary of our complex is now an S^3 with f-vector $[22, 102, 160, 80]$.

Now consider each edge $[v_1, v_2]$ in Q^3 and its opposite edge in Q^3, $[-v_1, -v_2]$. If the boundary of the links of the first edge is $[w_1, w_2], [w_2, w_3], [w_3, w_4], [w_4, w_1]$, then the link of the opposite edge is $[-w_1, -w_2], [-w_2, -w_3], [-w_3, -w_4], [-w_4, -w_1]$.

Now we apply the antipodal map $v \mapsto -v$ on the vertices of D, the two above squares will be identified. So will the two octahedra O^+ and O^-which are the boundaries of the links of $\pm(0,0,0,1)$.

We close the links of the edges $\pm[v_1, v_2]$ by joining each of the tetrahedra containing $[v_1, v_2]$ to the vertex of $[-v_1, -v_2]$ the product of whose first three coordinates is -1. For example, the boundaries of the links of the edges $[(1,1,1,0),(1,1,-1,0)]$ and $[(-1,-1,-1,0),(-1,-1,1,0)]$ is the boundary of the square S with edges $[\overline{(3,0,0,3)}, \overline{(3,0,0,-3)}]$, $[\overline{(3,0,0,-3)}, \overline{(-3,0,0,-3)}]$, $[\overline{(-3,0,0,-3)}, \overline{(3,0,0,-3)}]$, and $[\overline{(3,0,0,3)}, \overline{(3,0,0,-3)}]$. We close the boundary by adding the simplices obtained by joining each of the triangles $[(1,1,1,0),(1,1,-1,0),(-1,-1,-1,0)]$ and $[(-1,-1,-1,0),(-1,-1,1,0),(1,1,-1,0)]$ with each of the edges in S. This gives $2 \times 6 \times 4 = 48$ simplices.

Now we have joined each vertex v in Q^3 to its opposite vertex $-v$. The link of the edge $[v, -v]$ consists of twelve triangles. Suppose $v = (\varepsilon_1, \varepsilon_2, \varepsilon_3, 0)$ where $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$. and let $\overline{P_v}$ be the image under the antipodal map on D of the prism(s) corresponding to v (and $-v$) in D. Then the faces of the link of $[v, -v]$ are the three square faces of $\overline{P_v}$, each subdivided by the corresponding neighbour of v. The boundary of this complex is the set of edges of two disjoint triangles in the image of O^+ (and O^-) under the map $x \mapsto -x$. We add the joins of $[v, -v]$ with each of these triangles. This gives $4 \times 2 = 8$ simplices.

We close the boundaries of $\pm(0,0,0,1)$ by joining each of the faces of the octahedron to the edge $[(0,0,0,1),(0,0,0,-1)]$. This gives eight more simplices.

This gives a complex with f-vector $[16, 120, 330, 375, 150]$. It can be shown that this complex is the same as the one obtained in the previous construction.

3.6 Similarities and Differences

At first glance, it may seem that the automorphism groups of the complexes constructed using the hyperoctahedron and hypercube, and the suspended cube and octahedral prism have automorphism group $C_2 \times S^4$. But the automorphism group is S_6 acting on $10 + 6$ vertices.

In the first construction, the vertex-orbit of S_6 of size six consists of the two points of the hyperoctahedron C^4 used to triangulate it internally, (namely $\pm\mathbf{e}_1$), and the vertices of Q_e.

In the second construction the smaller S_6 orbit consists of the two suspension points $(0,0,0,\pm 1)$, and the four vertices of the cube the product of whose first three coordinates is -1.

If we consider any pair of elements of the orbit $\mathcal{O}_6$ of size six in either construction, we find that the link of the edge joining them is an octahedron consisting of six points of the longer orbit $\mathcal{O}_{10}$, and that the intersection of their links is a solid cube triangulated with 5 tetrahedra, where the vertices of the inner tetrahedron are the remaining vertices of $\mathcal{O}_{10}$ and the other four vertices are the remaining vertices of $\mathcal{O}_6$.

In our first construction, we start with a suspended octahedron (hyperoctahedron) on the inside and a cubical prism (hypercube) on the outside of our 4-dimensional ball. In the second construction we start with a suspended cube on the inside and an octahedral prism on the outside. This gives us a way of visualizing either construction as the other one "turned inside-out".

Also note that Walkup's $\mathbb{R}P^3_{11}$ can also be constructed using the paradigm outlined in the second construction. Recall that the initial object of our construction was an octahedron, surrounded by a cube. Observe that the octahedron and cube are respectively a suspension of a square and a prismed square.

Nonetheless, if instead of constructing antipodal 4-balls, we construct 3-spheres by embedding a point in the barycenters of the faces of the outer objects and triangulating, we get antipodal spheres. It is of interest that in the first case, the 3-sphere has 24 vertices, and the quotient only gives a 12-vertex $\mathbb{R}P^3$, whereas in the second construction, we get the same 3-sphere constructed by Walkup as the double cover of his $\mathbb{R}P^3_{11}$.

The two constructions above also point to different possible generalizations. We have constructions using using one additional point to triangulate a hyperoctahedron placed within a hypercube to obtain an $\mathbb{R}P^5$ using $2^{(5-1)}+2\times 5+1 = 27$

vertices and an $\mathbb{R}P^6$ using $2^5 + 12 + 1 = 45$ vertices following the first paradigm. We do not include the details of these constructions as they are long and tedious, and provide no new insights.

If the second method can be generalized by starting with a double suspension SSQ^3 of the 3-cube, then it may be possible to triangulate $\mathbb{R}P^5$ with only $8 + 4 + \frac{24}{2} = 24$ vertices. The BISTELLAR program of F.H. Lutz has discovered an S_4-invariant 24-vertex triangulation of $\mathbb{R}P^5$, the facet-list of which is available at [9].

Chapter 4

A combinatorial description of Brehm and Kühnel's 15-vertex 8-manifold

4.1 Introduction

Our motivating problem is that of finding a 15-vertex triangulation of the quaternionic projective plane HP^2 as an 8-dimensional real manifold. It is known that 15 is the least number of vertices needed for triangulating HP^2[4].

Viewed as a purely combinatorial object, the abstract simplicial complex $\mathcal{C}$ resulting from such a triangulation consists of a set M of 9-element subsets of a vertex set V of size 15, and together with all subsets of all elements of M. If $S \in M$, then any 8-subset of S is contained in exactly one other element S' of M.

Arnoux and Marin proved that any "cohomology HP^2" on 15-vertices will satisfy the following remarkable *complementarity* property: For any set subset S of the set V of vertices of the complex, either S is a simplex, or $V \setminus S$ is a simplex. Note, in particular that complementarity implies that our complex $\mathcal{C}$ will be 5-*neighbourly*, ie. every 5-subset of V is in $\mathcal{C}$. This enables us to use the idea of special maximal intersecting families, a formulation due to B. Bagchi.

4.1.1 Maximal Intersecting Families

A set of k-element sets (or k-sets) is called an *intersecting family of sets* if any two of its members intersect non-trivially. An intersecting family of sets is called a *maximal intersecting family* of order k (or a MIF(k)) if for any k-set not in the family, there exists a member of the family disjoint from it (i.e. maximality is with respect to inclusion in the family).

Let $\mathcal{F}$ be a MIF(k). The following are immediate consequences of the definition:

1. $\cap_{F \in \mathcal{F}} F = \phi$, since if $a \in F$ for all $F \in \mathcal{F}$, then any set containing a will have to be in $\mathcal{F}$.

2. By the same argument, if $F' \subsetneq F \in \mathcal{F}$, then there exists $G \in \mathcal{F}$ such that $G \cap F' = \phi$.

3. $\mathcal{F}$ is finite. We prove this by backward induction on the possible intersection sizes. In particular, we claim that any set of size $l \leq k$ is contained in at most k^{k-l} sets in $\mathcal{F}$. All sets in $\mathcal{F}$ have size k. Let G be a subset of $\cup_{F \in \mathcal{F}} F$ of size $k-1$, and let $\mathcal{F}_G \subset \mathcal{F}$ denote the set of all $F \in \mathcal{F}$ containing G. Now by the previous assertion, there exists $F' \in \mathcal{F}$ such that $F' \cap G = \phi$. Since F' intersects each element of $\mathcal{F}_G$, we have $|\mathcal{F}_G| \leq k$. Now let G have size $k-l$, and let $\mathcal{F}_G$ be as above. Let $F' \in \mathcal{F}$ be disjoint from G. For each $a \in F'$, $G \cup \{a\}$ is in at most k^{k-l-1} elements of $\mathcal{F}_G$. But every element of $\mathcal{F}_G$ contains $G \cup \{a\}$ for some $a \in F'$. So $|\mathcal{F}_G| \leq k \times k^{k-l-1} = k^{k-l}$.

4. From $l = 0$ in the above we have $|\mathcal{F}| \leq k^k$, and $|\cup_{F \in \mathcal{F}} F| < k^{k+1}$.

We refer to the set $\cup_{F \in \mathcal{F}} F$ as the *base set* of $\mathcal{F}$ and size of the base set of $\mathcal{F}$ as the *size of* $\mathcal{F}$.

A simple example of a MIF(k) is the set of all k-subsets of a $(2k-1)$-set for $k > 1$. The set of all lines in the projective plane of order q forms a $MIF(q+1)$.

Let $\mathcal{M}$ be a complementary pseudomanifold of dimension 8 on a 15-set V. Then, let M_6 denote the set of complements in V of the facets of $\mathcal{M}$. That is,

$$M_6 = \{V \setminus \Delta \quad : \quad \Delta \in \mathcal{M}, |\Delta| = 9\}$$

Now if $S, S' \in M_6$, the complementarity property ensures that $S \notin \mathcal{M}$, so S can not be a subset of $V \setminus S' \in \mathcal{M}$. So S and S' intersect nontrivially, and M_6 forms an intersecting family of 6-sets. Also, any subset of size ten or more of V can not be a simplex in $\mathcal{M}$, so by complementarity, we have that M is $5-$ *neighbourly*. In other words, every 4-simplex is contained in a facet of $\mathcal{M}$, or $\mathcal{M}$ has full 4-skeleton. So for any subset of V of size five or smaller, there exists an element of M_6 disjoint from it. This tells us that M_6 is a MIF(6).

The non-branching property of $\mathcal{M}$ tells us that any 7 subset of V contains at most two elements of M_6. Now if we assume the extra condition that $\mathcal{M}$ has no boundary, we have that any 7 subset of V contains either zero or two elements of M_6.

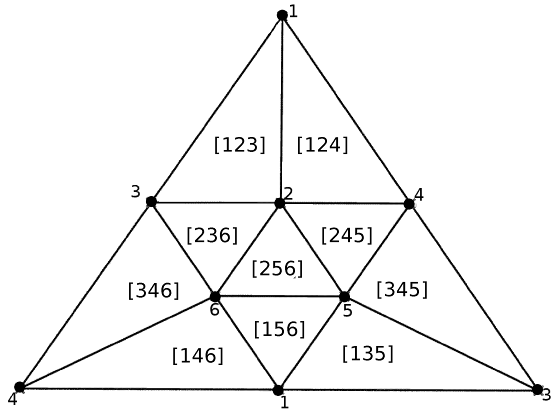
We say that a MIF(k) $\mathcal{F}$ is *special* if every $k+1$ subset of its base set contains either zero or two elements of $\mathcal{F}$.

We construct a special maximal intersecting family (SMIF) of 6-subsets of a 15-element set from the 6-vertex triangulation of $\mathbb{R}\mathrm{P}^2$. Then we prove that this SMIF, which we call $\mathcal{M}_6$ gives one of the combinatorial manifolds constructed by Brehm and Kühnel in their 1994 paper.

4.2 $\mathbb{R}\mathrm{P}^2_6$ and the Petersen Graph

That $\mathbb{R}\mathrm{P}^2$ can be triangulated by six vertices is a classical result. It is also well known that this triangulation, which is generally referred to as $\mathbb{R}\mathrm{P}^2_6$, is unique up to relabellings. Let us revisit this construction.

It is easy to see that the antipodal quotient of the icosahedron, as seen in the following figure, with repeated vertices and edges identified, forms a triangulation of $\mathbb{R}\mathrm{P}^2$.



It is (vertex-)minimal as it satisfies the necessary conditions combined in the Heywood inequality

$$f_0 \geq \frac{7 \pm \sqrt{49 - 24\chi}}{2}$$

where f_0 stands for the number of vertices or 0-faces and χ is the Euler characteristic. It is easy to construct the facets of $\mathbb{RP}^2_6$ as a SMIF(3) with base set $\langle 6 \rangle = \{1, 2, \ldots, 6\}$.

We can assume without loss of generality that our SMIF(3) contains the set [123]. This rules out [456]. Now the set [1234] contains one other member of the $SMIF(3)$. Again we can assume this is [124], ruling out $[234], [134]$ and [356].

Now note that any $S \subset \langle 6 \rangle$ of size three intersects every 3-set in $\langle 6 \rangle$ other than its complement and hence to preserve maximality of the intersecting family, we have to include [156] and [256], which rules out [125] and [126].

Our current selection of subsets is invariant under the permutations generated by $(12), (34), (56) \in S_6$, i.e. the elements of each of the pairs $[12], [34], [56]$ are for now mutually indistinguishable. In addition, every 3-set which remains to be sorted contains exactly one element from each pair above. So we can include, say [135]. This rules out [246] and considering $[1235], [1345]$ and [1356] rules out $[235], [145]$ and [136]. Adding the remaining three sets $[146], [236]$ and [245] to our SMIF(3) completes the construction and gives the required facet list.

We note a few combinatorial properties of $\mathbb{RP}^2_6$.

1. $\mathbb{RP}^2_6$ is complementary and 2-neighbourly. If we divide the twenty 3-subsets of $\langle 6 \rangle$ into disjoint pairs, exactly one 3-set from each pair is a facet of $\mathbb{RP}^2_6$.

Each of the 15 subsets of size two of the set $\langle 6 \rangle$ is an edge or 1-simplex in $\mathbb{R}\mathrm{P}^2_6$.

2. The automorphism group of this triangulation can be seen to be isomorphic to A_5, and its action on the vertex set $\langle 6 \rangle$ gives the degree 6 permutation representation of A_5, or the icosahedral action.

3. The set S_2 of all 2-element subsets of $\langle 6 \rangle$ is partitioned into five cyclically ordered partitions of $\langle 6 \rangle$, also called *amicable partitions* which generate $\mathbb{R}\mathrm{P}^2_6$ [2]. These are

$$[12] \succ [34] \succ [56] \succ [12]$$
$$[13] \succ [25] \succ [46] \succ [13]$$
$$[14] \succ [26] \succ [35] \succ [14]$$
$$[15] \succ [36] \succ [24] \succ [15]$$
$$[16] \succ [45] \succ [23] \succ [16]$$

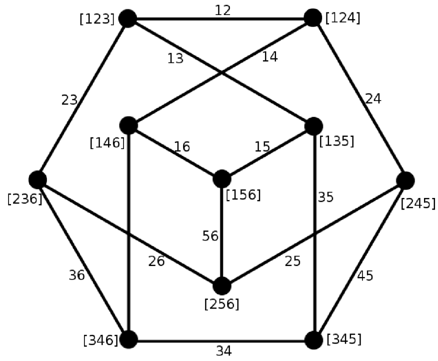
Also note that this naturally gives rise to a partition of S_2 into five subsets of size three, which can be thought of as five mutually disjoint 1-factors of $\langle 6 \rangle$ (or partitions of the vertex set of K_6 into edges) which form a 1-factorization (or partition of the edge set of K_6 into mutually disjoint 1-factors), namely,

$$\begin{aligned} S_2 &= \{[12],[34],[56]\} \cup \{[13],[25],[46]\} \cup \{[14],[26],[35]\} \cup \\ &\quad \{[15],[36],[24]\} \cup \{[16],[45],[23]\} \end{aligned}$$

Also note that any automorphism of $\mathbb{R}\mathrm{P}^2_6$ is an even permutation of the above five 1-factors.

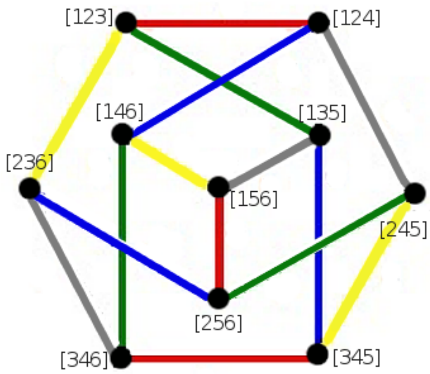
4. We can define a (generalized) dual graph of $\mathbb{R}\mathrm{P}^2_6$ in the following way. A *face* of $\mathbb{R}\mathrm{P}^2_6$ is a 2-simplex in $\mathbb{R}\mathrm{P}^2_6$, of which there are ten. Also, every edge is incident with exactly two faces. So we can define a graph with vertices labelled by the faces of $\mathbb{R}\mathrm{P}^2_6$, and connect each pair of vertices by an edge if the faces corresponding to those vertices share an edge. The edge set of this graph can thus be identified with the set S_2 of edges of $\mathbb{R}\mathrm{P}^2_6$. This graph turns out to be the Petersen graph, which is sometimes defined as the dual

graph of the projective planar embedding of K_6.



We shall denote the Petersen graph with its 6-vertex labelling as above by P_6. Now we note two further combinatorial properties of P_6.

Recall that the automorphism group of the Petersen graph is S_5. So there are 60 automorphisms of P_6 which appear when we move from the combinatorial manifold $\mathbb{RP}^2_6$ to the graph P_6. Note the six pentagons formed by the vertices whose labels contain any given element $i \in \langle 6 \rangle$. Their complementary pentagons are those whose labels do not contain i. The 1-factorization of $\langle 6 \rangle$ partitioning S_2 also gives a 5-edge-colouring of P_6 as shown in the figure below.



Even though this colouring is not minimal with respect to the number of colours needed for an edge colouring, it has some interesting properties.

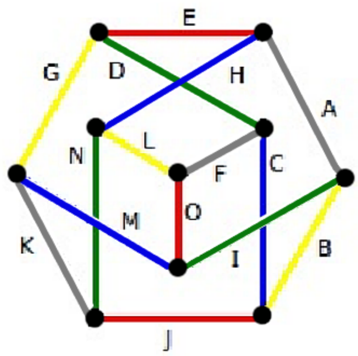
This colouring is a *strong edge-colouring* of the Petersen graph, since any two edges at distance at most two have distinct colours. In particular, we have that two edges in P have the same colour if and only if they are at the maximum possible distance 3 from each other. This gives rise to some neat observations. It is well known that the possible lengths of a cycle in a Petersen graph are $5, 6, 8$

and 9. Since any two edges in a cycle of length five are at distance at most 2 from each other, we have that any two edges of a 5-cycle in P are coloured differently. Therefore, every 5-cycle in P is 5-coloured. Now consider the 6-cycles or hexagons of P. We see that the opposite sides of each hexagon are at distance exactly 3 from each other, since a shorter path between them would violate the girth= 5 condition, so the colour sequence of any hexagon is $1, 2, 3, 1, 2, 3$. Similar arguments show that any 8-cycle has colour sequence $1, 2, 3, 4, 2, 1, 4, 3$, and that all 9-cycles (which are hypo-hamiltonian) have colour sequence $1, 2, 3, 1, 2, 4, 1, 2, 5$.

Also, we can uniquely identify each vertex of this coloured graph by the set of colours assigned to each of its incident edges. Such a colouring is called a *vertex distinguishing edge colouring (vdec)*. This cannot be done if we only use four colours, as $\binom{4}{3} < 10$, and we would have fewer colour-triples than vertices. But $10 = \binom{5}{3}$, so any 3-subset of a set of five colours denotes a unique edge. So the above colouring is an optimal vdec.

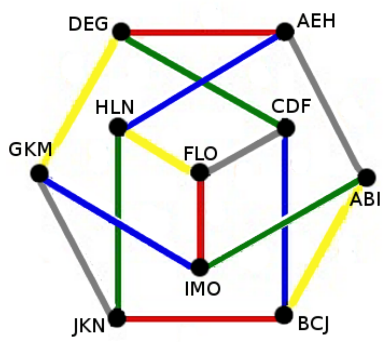
4.3 Construction

We now label the 15 edges of the edge-coloured graph P_6 with 15 symbols. We use the uppercase letters from A through O of the English alphabet.



We label these edges from the set $[A \dots O]$, keeping the 3-sets $[AFK], [BGL]$, $[CHM], [DIN]$, and $[EJO]$ each coloured with the same colour. This particular choice corresponds to the labelling in Brehm and Kühnel's construction.

Now we relabel each of the vertices of P_6 by the set of the three letters denoting its three incident edges. We call the new graph thus labelled and coloured P. See figure below.



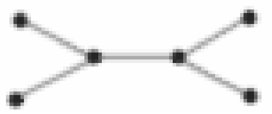
4.3.1 Edge-sets of the Petersen graph

First, we shall construct a set of five amicable partitions, each of which generate 15 elements of $\mathcal{M}_6$. Recall the five amicable partitions of $\mathbb{RP}^2_6$. Each is a partition of $\langle 6 \rangle$ into three 2-subsets. These appear in P as a set of three edges of the same colour. Each such edge join two 3-subsets which intersect in one point, the label of the edge. So we have a 5-subset associated with each edge, given by the set of all its neighbouring edges. For example, if we pick the edge A, together with its neighbours, it gives $[ABEHI]$. Also, note that the three 5-sets obtained by "taking neighbours" of any three edges of the same colours gives a partition of $[A \dots O]$. In our example, taking A, F, K gives $[ABEHI], [CDFLO], [GJKMN]$. Now the 2-subsets of $\langle 6 \rangle$ were $[24], [15]$, and $[36]$ respectively, and they were cyclically ordered as $[15] \succ [36] \succ [24] \succ [15]$, which corresponds to $A \succ K \succ F$. So we take our amicable partition to be $[ABEHI] \succ [GJKMN] \succ [CDFLO]$. Applying the same process to all five colours, we get the amicable partitions

$$[ABEHI] \succ [GJKMN] \succ [CDFLO]$$
$$[ABCIJ] \succ [FHLNO] \succ [DEGKM]$$
$$[AEHLN] \succ [GIKMO] \succ [BCDFJ]$$
$$[ABIMO] \succ [CDEFG] \succ [HJKLN]$$
$$[ADEGH] \succ [FILMO] \succ [BCJKN]$$

which give 75 new 6-subsets.

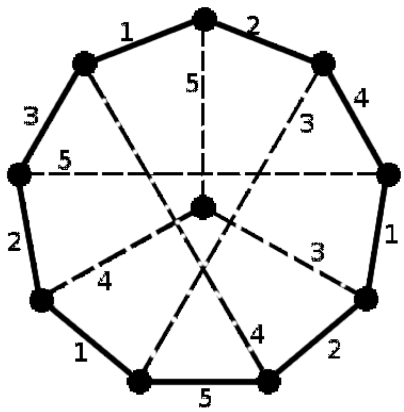
Note that we can also think of the above amicable partitions as partitions of the edge set $[A \dots O]$ into subgraphs of the following "double-Y shape", and their ordering from the orderings within $\mathbb{RP}^2_6$.



Also note that each of these 75 sets is acyclic. Also, they are contained in the complements of cycles of P. For the remaining sets, we consider other 6-subsets of the edge set of P. We pick our other 6-sets from the acyclic subsets of P. Most of them arise from complements of the cycles in P. We consider cycles in decreasing order of length.

9-cycles

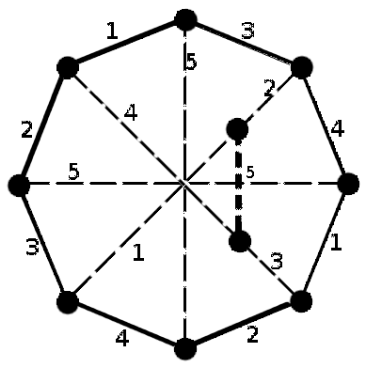
P has 20 cycles of length nine. They are coloured as described above.



We pick the complements of these 9-cycles, of which there are 20.

8-cycles

P has 15 cycles of length eight. They are coloured as described above.



Of the seven edges in the complement, one of the five colours is represented 3 times and the others once each. Of the three edges coloured the same, one is not adjacent to any of the edges in the outer octagon (and is drawn with a thicker line in the figure above), and forms a central edge in some double-Y set of an amicable partition. Its four neighbours are also in the complement of the octagon. Of other two edges,we picked one as a sixth element of a 6-set together with the double-Y set just described. we pick one more 6-set from the complement of he octagon, the one obtained by omitting the edge not adjacent to the octagon.

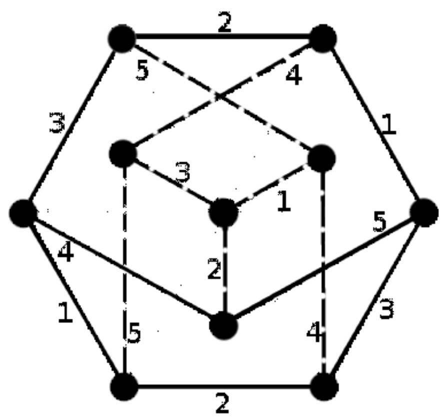
We get 15 such sets.

Also note that none of the above 15 sets are contained in the complement of a pentagon.

The Petersen graph has no 7-cycles.

6-cycles

The ten cycles of length six are coloured as described above.

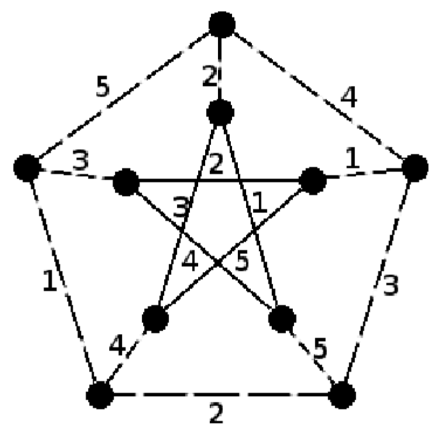


There are nine edges in the complement of each such hexagon. From these we pick the two subsets coloured $1, 2, 3, 4, 4, 4$ and $1, 2, 3, 5, 5, 5$. We get 20 such sets.

Note that none of these sets are contained in the complements of a pentagon.

5-cycles

The Petersen graph has twelve pentagons, coloured as described above.



We see that the complement of a 5-cycle is a graph consisting of a 5-cycle, $\mathcal{C}$ together with a set $\bar{\mathcal{C}}$ of five edges each connecting a vertex $\bar{c_i}$ to a vertex c_i of $\mathcal{C}$, for $1 \leq i \leq 5$. Now we take the following subsets of the above 10-edges.

1. Take any path (of edges) of length 3 in $\mathcal{C}$, say $\{e_1, e_2, e_3\}$. Of the five edges of $\bar{\mathcal{C}}$, pick the three edges of the same colour as e_1, e_2 and e_3.

2. Take any path (of edges) of length 3 in $\mathcal{C}$, say $\{e_1, e_2, e_3\}$. Consider a non-central edge, say e_1 in the above path. Of the five edges of $\bar{\mathcal{C}}$, pick the two edges in $\bar{\mathcal{C}}$ adjacent to e_1, and the edge in $\bar{\mathcal{C}}$ of the same colour as e_2.

So for each pentagon, we get $(5 + 5 \times 2) = 15$ sets of size six.

Perfect matchings

Now consider the set $\bar{\mathcal{C}}$. This is a perfect matching of P The remaining sets will be picked from the complements of $\bar{\mathcal{C}}$, a disjoint union of two pentagons $\mathcal{C}$ and $\mathcal{C}'$. We have the following two kinds of sets.

1. Pick an edge path of length 4, say (e_1, e_2, e_3, e_4) in one of the pentagons, say $\mathcal{C}$. For each pair of consecutive edges starting at a terminal edge of the above path, say (e_1, e_2), pick the two edges e_1', e_2' of the same colour as e_1, e_2 in $\mathcal{C}'$.

2. Pick an edge path of length 3, say $(e_1, e_2, e_3,)$ in one of the pentagons, say $\mathcal{C}$. Pick the three edges e_1', e_2', e_3' of the same colour as e_1, e_2, e_3 in $\mathcal{C}'$.

There are six perfect matchings in P. In the complement of each we are picking $(2 \times 5 \times 2) + (2 \times 5)$ sets of size six.

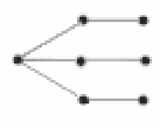
So we get a total of 490 sets of size six.

4.3.2 Edges of the Desargues Graph

We now write down some elements of $\mathcal{M}_6$, which will be 6-element subsets of $[A \ldots O]$. For each vertex given by a set of three colours, we have two colours left over in our palette, which we call its *complementary colours.* Each colour corresponds to a set of three symbols in $[A \ldots O]$. We obtain a 6-set by combining the three symbols denoting a vertex and the three symbols associated with one of its complementary colours.

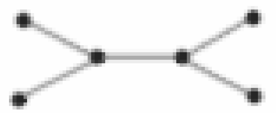
For example, The vertex $[BCJ]$ is coloured *red*, *blue*, and *yellow.* Its complementary colours are *gray* (whose edges are $[AFK]$) and *green* (whose edges are $[DIN]$). So we get the two 6-sets $[BCJAFK]$ and $[BCJDIN]$. Applying this process to all ten vertices, we obtain 20 such sets.

These sets can be thought of as the sets of edges in subgraphs of the form



where the three terminal edges have the same colour.

Note that we can also think of the above amicable partitions as partitions of the edge set $[A \ldots O]$ into subgraphs of the following shape, and their ordering from the orderings within $\mathbb{R}\mathrm{P}^2_6$.



For the remaining sets, we construct a new graph closely related to P.

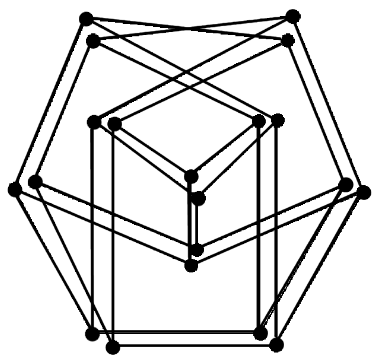
The bipartite double cover of P

Let G be a graph on n vertices with m edges. The *bipartite double cover* of G is a graph G^2 on $2n$ vertices with $2m$ edges. If the vertices of G are labelled $v_1, v_2, \ldots, v_n$, then we label the vertices of G^2 by

$$v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}, v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)}.$$

If $\{v, w\}$ is an edge in G, then $\{v^{(1)}, w^{(2)}\}, \{v^{(2)}, w^{(1)}\}$ are edges in G^2. From construction, it is clear that $\{v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)}\}, \{v_1^{(2)}, v_2^{(2)}, \ldots, v_n^{(2)}\}$ forms a bipartition of the graph. We can think of the graph G^2 as being obtained as the tensor product $G \otimes K_2$.

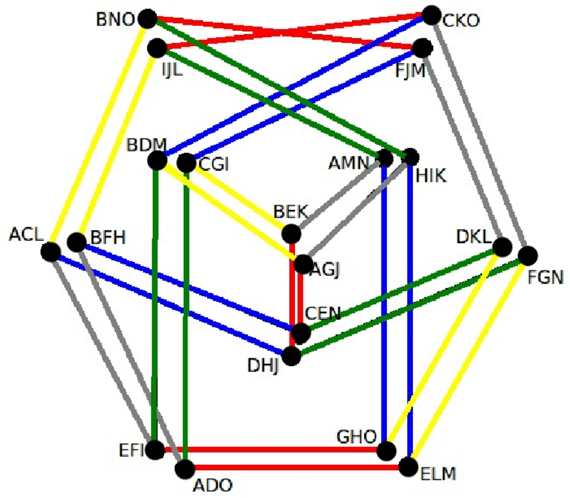
In the case of the Petersen graph, the bipartite double is called the Desargues Graph, and is a rather interesting graph in its own right.



The important property of the Desargues graph worth noting is that it is a distance-transitive graph with maximum distance 5. If we start with any vertex v of this graph, we have a natural partitioning of the vertex set into $\cup_0^5 \Gamma_5(v)$, where $\Gamma_i(v)$ consists of all the vertices of the graph at a distance i from v. We have $|\Gamma_0(v)| = |\Gamma_5(v)| = 1$, $|\Gamma_1(v)| = |\Gamma_4(v)| = 3$, and $|\Gamma_2(v)| = |\Gamma_3(v)| = 6$.

Now we will label the vertices of the Desargues graph as follows. Consider the graph P. Each vertex v of P has three adjacent edges e_1, e_2, e_3 of different colours $\{1, 2, 3\}$. If we delete the vertex v and its three neighbours, we get rid of the edges e_1, e_2, e_3 and every edge coloured with either of its two complementary colours, we are left with a hexagon whose edges are coloured $1, 2, 3, 1, 2, 3,$ as we remarked earlier. We can decompose this hexagon into two disjoint perfect matchings, in exactly one way. For example, if we start with the vertex $[BCJ]$ of P, we delete B, C, J and and the gray edges A, F, K, and the green edges D, I, N and we are left with the hexagon with edges $EHLOMG$ which decomposes into the edge sets $[ELM]$ and $[GHO]$. We can do this with every vertex. Also, note that these hexagons decompose into 3-sets of different coloured edges, and the colour-set of each such 3-set is the same as the colour-set of the vertex v that we started with.

We are going to label one of these vertices $v^{(1)}$, and the other $v^{(2)}$. We can pick an index (1) or (2), without loss of generality. Note that there is nothing in our construction so far that distinguishes $[ELM]$ and $[GHO]$, (or any of the two 3-sets obtained similarly from each vertex)from each other. But if we start with a vertex v of P, and obtain $v^{(1)}$ and $v^{(2)}$ as described above, then if we take another vertex u, the choice of $u^{(1)}$ and $u^{(2)}$ depends on the choice of $v^{(1)}$ and $v^{(2)}$. So we do need to distinguish between them. Now if we go back to the vertex-labelling of P, we see that any two vertices are joined by an edge if their labels (considered as 3-sets) intersect. Also, if we consider any two adjacent vertices u and v of the graph P, we see that the hexagons obtained by deleting either u or v and their respective neighbouring vertices intersect in exactly two edges, the two edges of the same colour as the edge joining u and v. Each of these two edges will be in exactly one of the 3-sets obtained from the hexagons corresponding to each of u, v. That is, if we take any two adjacent vertices u, v of P, and obtain two new 3-sets from each, say v', v'' and u', u'' respectively, then we have natural pairings, say v', u' and v'', u'', such that $v' \cap u'$ and $v'' \cap u''$ each have size one and $v' \cap u'' = v'' \cap u' = \phi$. We index intersecting pairs the same. So if we start with $[BCJ]$ and $[ABI]$, and take $[BCJ]^{(1)} = [ELM]$, and $[BCJ]^{(2)} = [GHO]$, then we have to label $[ABI]^{(1)} = [DKL]$ and $[ABI]^{(2)} = [FGN]$. This gives a vertex-labelling of the Desargues graph, which we will call D.



This gives us a very nice description of the partition of the vertex set of D on the basis of distance from a given vertex of D. Start with a vertex $v^{(i)}$. Then we have $\Gamma_0(v^{(i)}) = \{v^{(i)}\}$ and $\Gamma_5(v^{(i)}) = \{v^{(j)}\}$ where $\{i, j\} = \{1, 2\}$. If u_1, u_2, u_3 are at distance 1 from v in P, then $\Gamma_1(v^{(i)}) = \{u_k^{(j)} | k = 1, 2, 3\}$ and $\Gamma_4(v^{(i)}) = \{u_k^{(i)} | k = 1, 2, 3\}$. If w_k, $1 \leq k \leq 6$ are the 6 vertices at distance 2 from $v^{(i)}$, then $\Gamma_2(v^{(i)}) = \{w_k^{(i)} | 1 \leq k \leq 6\}$ and $\Gamma_3(v^{(i)}) = \{w_k^{(j)} | 1 \leq k \leq 6\}$.

Note that there is no natural way of labelling the edges of D as there was with P. But we can visualize D as "sitting on top of" P as a double cover, with $v^{(1)}$ and $v^{(2)}$ directly above v, and the edges $\{v^{(1)}, u^{(2)}\}$ and $\{v^{(2)}u^{(1)}\}$ above $\{u, v\}$. More formally, we have 2 to 1 maps from the vertex set and edge set of D to the vertex and edge sets of P. But we shall speak of an edge of D as being parallel to an edge of P, as it adds to our geometric picture.

Also, D has certain combinatorial properties worth noting.

We saw that any vertex $v^{(i)}$ of D is disjoint with its neighbouring vertices. Also note that any two neighbours of $v^{(i)}$ have the same index (j) where $j \in \{1, 2\}, j \neq i$. Call them $u_1^{(j)}$ and $u_2^{(j)}$, and let u_1 and u_2 be their corresponding vertices in P. There are two colours common to u_1 and u_2, and any other colour is complementary to one of them. Deleting u_1, u_2 and all their neighbours deletes every edge whose colour is complementary to either of them, and also their incident edges in their common colours. So the deletion process deletes all but a pair of adjacent edges, one each in the two colours common to u_1 and u_2, call them e, e'. So if $u_1^{(j)}$ and $u_2^{(j)}$ do intersect, they intersect in exactly one of these two edges. Say $u_1^{(j)} \cap u_2^{(j)} = \{e\}$. But the vertex $v^{(j)}$ intersects both $u_1^{(j)}$ and u_2^j. And it intersects $u_1^{(j)}$ in an edge of their common colour. This edge is either adjacent to e or e'. But e_1 cannot be adjacent to e, since e_1, e are elements of $u_1^{(j)}$, which is made up of disjoint edges. So, e_1 is adjacent to e'. Arguing similarly with $e_2 \in v^{(j)} \cap u_2^{(j)}$ gives e_2 adjacent to e'. But this means that e_1 and e_2 are adjacent. But since $e_1, e_2 \in v^{(j)}$ and we have a contradiction. So we have proved that $v^{(i)}$ and all its neighbours are mutually disjoint. Also, each of these 3-sets is disjoint from v, since their associated hexagons do not contain any of the edges incident with v. So we have five mutually disjoint 3-sets which form a partition of $[A \ldots O]$. Also note that each of these 3-sets is 3-coloured, ie. if we think of each set as a set of edges of P, the three edges in this set are all coloured differently, as each set is coloured the same as the vertex of P it sits above.

Now we are ready to write down the remaining subsets of our family $\mathcal{M}_6$.

First, for every vertex v of P, $v \cup v^{(i)} \in \mathcal{M}_6$ for $i = 1, 2$.

Then, let $v^{(i)}$ be a vertex of D corresponding to some vertex v of P. Then each 2-subset V_2 of v is a set of two of its neighbouring edges. There is one more edge, say e incident with v in P. There is an edge of D incident with $v^{(i)}$ parallel to e. This joins $v^{(i)}$ to some vertex $u^{(j)}(e)$ of D (where $j \in \{1, 2\}, i \neq j$). For every $x \in u^{(j)}(e)$, the 6-set $V_2 \cup v^{(i)} \cup \{x\}$ is in $\mathcal{M}_6$. Modulo overcounting, we get nine sets for each vertex $v^{(i)}$ in D.

We start again with a vertex $v^{(i)}$ of D, and its corresponding vertex v of P. Each $e \in v$ corresponds to an edge incident with v. The remaining two edges of v each have a parallel edge in P incident with $v^{(i)}$, each of which connect it to two vertices of P. Call them $u_1^{(j)}$ and $u_2^{(j)}$. For each $x_1 \in u_1^{(j)}$ and $x_2 \in u_2^{(j)}$, if the edges $x_1, x_2 \in$ P are neither both adjacent to e in P nor have the same colour, then $v^{(i)} \cup \{e, x_1, x_2\} \in \mathcal{M}_6$. For each edge $e \in v$ of P, there are two pairs $(x_1, x_2), x_1 \in u_1^{(j)}, x_2 \in u_2^{(j)}$ such that x_1 and x_2 are coloured the same, one for each of the common colours of $u_1^{(j)}$ and $u_2^{(j)}$. Also there is exactly one pair (x_1, x_2)

such that $\{e, x_1, x_2\}$ is a vertex of P. So each pair $(v^{(i)}, e)$ gives six 6-sets.

These are all the elements of $\mathcal{M}_6$.

It can be checked that $\mathcal{M}_6$ is equal to the combinatorial manifold M_{15} constructed by Brehm and Kühnel in [4]. It is also possible to prove by elementary counting arguments that the complements of the elements of $\mathcal{M}_6$ are the facets of a combinatorial pseudomanifold.

Appendix A

Addendum to Section 3.5

Here we establish the existence of the polyhedral complex decomposition of the octahedral prism D which we used in Section 3.5. We work in $\mathbb{R}^5$. Place an octahedral prism D in the hyperspace $x_5 = 0$. Let

$$Y_0 = \{\pm e_i \pm e_4 : 1 \leq i \leq 3\}$$

and $D = \text{Conv}(Y_0)$. We place a "dual" object E above D in the hyperspace $x_5 = 1$. Choose $a, b > 0$ and let

$$Y_1 = \{b(\pm e_1 \pm e_2 \pm e_3) + e_5\} \cup \{\pm a e_4 + e_5\},$$

and $E = \text{Conv}(Y_1)$. E is a suspended cube. Let $Y = Y_1 \cup Y_0$ and $K = \text{Conv}(Y)$, which is a 5-dimensional closed and bounded convex set. To each vertex $y \in Y_1$ there is a natural corresponding facet F_y of D. By taking the set of extreme points of F_y we obtain a subset $T(\{y\}) \subseteq Y_0$. Specifically

$$T(\{\lambda a e_4 + e_5\}) = \{\pm e_1 + \lambda e_4, \pm e_2 + \lambda e_4, \pm e_3 + \lambda e_4\}, \quad \lambda \in \{\pm 1\}.$$

$$T(\{b(\sum_{i=1}^{3} \lambda_i e_i) + e_5\}) = \{\lambda_1 e_1 \pm e_4, \lambda_2 e_2 \pm e_4, \lambda_3 e_3 \pm e_4\}, \quad \lambda_1, \lambda_2, \lambda_3 \in \{\pm 1\}.$$

D and E are facets of K. We claim that the other facets of K are constructed as follows: Let S be a subset of Y_1 such that $\text{Conv}(S)$ is a face of E. Let $T(S)$ be the subset of Y_0 defined as

$$T(S) = \bigcap_{y \in S} T(\{y\}).$$

The other facets of K are of the form $\mathrm{Conv}(X)$ where $X = S \cup T(S)$.

The automorphism group of E (the suspended cube) induces the automorphism group of D and hence the automorphism group of K. In order to prove that the sets $\mathrm{Conv}(X)$ above are indeed facets of K we may restrict ourselves to seven types of set X. We show that for each such set X there is a unique linear functional l such that

$$l(x) = 1, \; x \in X.$$

It follows that $\mathrm{Conv}(X)$ has dimension 4. Furthermore by checking the coefficients of $x_i, \; 1 \le i \le 4$, (the coefficient of x_5 is not crucial) in the expansion of l we see that

$$l(x) < 1, \; x \in Y \setminus X.$$

It follows that $\mathrm{Conv}(X)$ is a facet of K.

1. $X = \{ae_4 + e_5\} \cup \{\pm e_1 + e_4, \pm e_2 + e_4, \pm e_3 + e_4\}$.

   This corresponds to the convex hull of a suspension point in E with its dual octahedron in D.

   $$l(x) = x_4 + (1-a)x_5$$

   is the unique linear functional such that $l(x) = 1, \; x \in X$.
   Furthermore $l(x) < 1, \; x \in Y \setminus X$.

2. $X = \{b(e_1 + e_2 + e_3) + e_5\} \cup \{e_1 \pm e_4, e_2 \pm e_4, e_3 \pm e_4\}$.

   This corresponds to the convex hull of a vertex of the cube in E with its dual triangular prism in D.

   $$l(x) = x_1 + x_2 + x_3 + (1-3b)x_5$$

   is the unique linear functional such that $l(x) = 1, \; x \in X$.
   Furthermore $l(x) < 1, \; x \in Y \setminus X$.

3. $X = \{ae_4 + e_5, b(e_1 + e_2 + e_3) + e_5\} \cup \{e_1 + e_4, e_2 + e_4, e_3 + e_4\}$.

   This corresponds to the convex hull of a "supension edge" in E with its dual triangle in D.

   $$l(x) = \frac{a}{a+3b}(x_1 + x_2 + x_3) + \frac{3b}{a+3b}x_4 + \frac{a - 3ab + 3b}{a+3b}x_5$$

is the unique linear functional such that $l(x) = 1,\ x \in X$.

Furthermore $l(x) < 1,\ x \in Y \setminus X$.

4. $X = \{b(e_1 + e_2 + e_3) + e_5, b(e_1 + e_2 - e_3) + e_5\} \cup \{e_1 \pm e_4, e_2 \pm e_4\}$.

   This corresponds to the convex hull of an edge of the cube in E with its dual rectangle in D.

$$l(x) = x_1 + x_2 + (1 - 2b)x_5$$

   is the unique linear functional such that $l(x) = 1,\ x \in X$.

   Furthermore $l(x) < 1,\ x \in Y \setminus X$.

5. $X = \{ae_4 + e_5, b(e_1 + e_2 \pm e_3) + e_5\} \cup \{e_1 + e_4, e_2 + e_4\}$.

   This corresponds to the convex hull of a triangle in E with its dual edge in the corresponding octahedron in D.

$$l(x) = \frac{a}{a+2b}(x_1 + x_2) + \frac{2b}{a+2b}x_4 + \frac{a - 2ab + 2b}{a+2b}x_5$$

   is the unique linear functional such that $l(x) = 1,\ x \in X$.

   Furthermore $l(x) < 1,\ x \in Y \setminus X$.

6. $X = \{b(\pm e_1 \pm e_2 + e_3) + e_5\} \cup \{e_3 \pm e_4\}$.

   This corresponds to the convex hull of a face of the cube in E with its dual "long edge" in D.

$$l(x) = x_3 + (1 - b)x_5$$

   is the unique linear functional such that $l(x) = 1,\ x \in X$.

   Furthermore $l(x) < 1,\ x \in Y \setminus X$.

7. $X = \{ae_4 + e_5, b(\pm e_1 \pm e_2 + e_3) + e_5,\} \cup \{e_3 + e_4\}$.

   This corresponds to the convex hull of a pyramid in E with its dual vertex in D.

$$l(x) = \frac{a}{a+b}x_3 + \frac{b}{a+b}x_4 + \frac{a - ab + b}{a+b}x_5$$

   is the unique linear functional such that $l(x) = 1,\ x \in X$.

   Furthermore $l(x) < 1,\ x \in Y \setminus X$.

It remains for us to show that we have described all of the facets of K. To this end assume that F is a facet of K which is not in the family described above. F intersects E in a face. Let S denote the set $Y_1 \cap F$. F intersects D in a face. Let T denote the set $Y_0 \cap F$. If $T \subseteq T(S)$ then $T = T(S)$, since $\mathrm{Conv}(S \cup T(S))$ is a facet of K. Hence $T \not\subseteq T(S)$. It follows that there are points $p \in S \subseteq Y_1$ and $q \in Y_0 \setminus T(\{p\})$ such that $p, q \in F$.

Note that $\mathbf{0} \in K$. Moreover, since $\mathbf{0}$ is in the relative interior of D, we have $\mathbf{0} \notin F$. So there would exist a linear functional l such that $l(x) = 1$ for all $x \in F$, and $l(x) < 1$ for all $x \in Y \setminus F$.

The automorphism group of K permutes the two suspension points and the six vertices of the cube which together make up Y_1. Hence we may restrict our attention to two choices for p. These are $p = ae_4 + e_5$ and $p = b(e_1 + e_2 + e_3) + e_5$. The stabilizers of each of these points p in turn act transitively on $Y_0 \setminus T(\{p\})$. Therefore for each choice of p we may furthermore restrict out attention to one choice of $q \in Y_0 \setminus T(\{p\}$.

Case 1. Suppose we have a facet F containing vertices $p = ae_4 + e_5$ and $q = e_3 - e_4$. Any linear functional that evaluates to 1 at the two points above is of the form

$$l(x) = a_1x_1 + a_2x_2 + (1 + a_4)x_3 + a_4x_4 + (1 - aa_4)x_5.$$

If $a_4 \neq 0$, then $l(-ae_4 + e_5) = 1 - 2aa_4$ and $l(e_3 + e_4) = 1 + 2a_4$, one of which is greater than 1. Hence $a_4 = 0$, and

$$l(x) = a_1x_1 + a_2x_2 + x_3 + x_5.$$

Let $\lambda_1, \lambda_2 \in \{\pm 1\}$ such that $\lambda_1 a_1, \lambda_2 a_2 \geq 0$. Then $l(b(\lambda_1 e_1 + \lambda_2 e_2 + e_3) + e_5) \geq 1 + b > 1$, which is a contradiction.

Case 2. Suppose we have a facet F containing vertices $p = b(e_1 + e_2 + e_3) + e_5$ and $q = -e_3 + e_4$. Any linear functional that evaluates to 1 at the two points above is of the form

$$l(x) = a_1x_1 + +a_2x_2 + a_3x_3 + (1 + a_3)x_4 + (1 - b(a_1 + a_2 + a_3))x_5.$$

Choose

$$r \in \{b(\pm e_1 \pm e_2 \pm e_3) + e_5\}$$

such that

$$l(r) = b(|a_1|+|a_2|+|a_3|)+1-b(a_1+a_2+a_3) = 1+b(|a_1|-a_1+|a_2|-a_2+|a_3|-a_3).$$

$l(r) \leq 1 \Longrightarrow a_1, a_2, a_3 \geq 0$. Let $M = \max\{a_1, a_2, a_3\}$. Choose

$$s \in \{e_1 + e_4, e_2 + e_4, e_3 + e_4\}$$

such that

$$l(s) = M + 1 + a_3.$$

$$l(s) \leq 1 \Longrightarrow M = 0 \Longrightarrow a_1, a_2, a_3 = 0 \Longrightarrow l(x) = x_4 + x_5.$$

Therefore $l(ae_4 + e_5) = a + 1 > 1$, which is a contradiction.

Thus we can not have any more facets than the 56 previously listed. This establishes the polyhedral decomposition of Y.

To obtain a polyhedral decomposition of D we choose a, b suitably small so that the orthogonal projection of E onto the hyperplane $x_5 = 0$ lies in the relative interior of D. The projections of the facets of K will have the required property. In order to construct a double cover of our simplicial complex we start by taking the union of K with its reflection in $x_5 = 0$. The facets form the initial polyhedral complex of Section 3.5.

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