All-Pay Contests with Constraints

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Abstract

This paper generalizes the results of Siegel (2009) to support contestants who are faced with constraints. It also relaxes the continuity assumptions for some of the players.

Keywords: Liquidity constraint, budget constraint, all-pay, contest, auctions, rent-seeking, lobbying, tounament, cap, limit, ceiling

1. Introduction

Often agents make irreversible investments in order to win a contest with a valuable prize. For instance in job tournaments, in R&D races, in lobbying activities and in political contests the winner takes the prize but both the winners' and the loosers' costs are sunk. Siegel (2009) provides closed form formulae for the players expected payoffs in these environments with multi-prize complete information all-pay auctions under some generic conditions. In this paper we generalize Siegel (2009) to include contests with constraints.

Constraints in all-pay auctions may be induced externally or they may arise naturally. In political contests, Meirowitz (2008) analyzes the repercussions of a campaign spending limit on incumbency advantage. Che and Gale (1998), Kaplan and Wettstein(2008), Pastine and Pastine (2010), Matějka, Onderstal and De Waegenaere (2002) study the effect of political contribution caps in a lobbying game. Caps are also common place in US professional sports leagues (NBA,NFL, NHL, MLS) where teams are constrained with annual salary caps. In Formula 1, cars are restricted to a speed limit of 360km per hour (reference?). In international trade, a potential introduction of capital tax harmonisation in Europe would cap the minimum tax each EU country can impose. Other than institutionally imposed limits, contestants may also naturally face budget or liquidity constraints as in Che and Gale (1996), Gavious, Moldovanu and Sela (2002), Sahuguet (2006), Laffont and Robert (1996) and Pai and Vohra (2009). All-pay auctions are also used to model job tournaments as in Rosen (1986) . Fu (2006) and Pastine and Pastine (2010) model affirmative action in college admissions as an all-pay auction. In these environments constraints arise naturally, as well, since the day has a maximum of twentyfour hours for an employee and there are score ceilings in college admissions. One cannot exceed 2400 in SAT's. Hence in these models one may consider analysing the contests with limits, too.

In this paper we provide generalized payoff results for the contestants in an all-pay auction where the contestants may be constrained. Furthermore we relax some of Siegel (2009) restrictions on continuity of strategy space and on cost. We also provide common features that any contest equilibrium has to posses under generic conditions.

2. The Model

Except where otherwise noted we maintain all the assumptions of Siegel (2009). In cases where we generalize a named assumption or result in Siegel (2009) we append “Generalized” to the name in order to make the changes clear. In cases where we alter the assumption or result but the change is not a strict generalization of the corresponding item in Siegel (2009) we append “Modified” to the name. Once the assumptions or results are established, and where no confusion will result, we drop the epithet.

n players compete for m homogeneous prizes where $0 < m < n$. Each of the players simultaneously and independantly choose a score s_i from their set of feasable scores S_i and each of the m players with the highest scores wins one prize. In the case of ties any tie-breaking rule can be used to allocate the prizes among the tied players.

We maintain Siegel’s assumptions on player utilities which mean that given a profile of scores $s = (s_1, \cdots, s_n)$, $s_i \epsilon S_i$, player i’s payoff is

$$u_i(s) = P_i(s)\, v_i(s_i) - (1 - P_i(s))\, c_i(s_i) \tag{1}$$

where $P_i(s)$ is player i’s probability of winning at profile s, $v_i(s_i)$ is his payoff if he wins, and $c_i(s_i)$ is his payoff if he loses. v_i and c_i are defined on $s_i \epsilon [a_i, \infty)$. We will be able to relax the continuity assumptions on v_i and c_i for some, but not all, of the players. We maintain Siegel’s other assumptions on these functions.

$a_i \epsilon [0, \infty)$ is the initial score.of contestant i before he makes any effort to improve his score. A positive initial score captures a headstart advanatge of the contestant. In Siegel (2009) player i’s set of feasable scores $S_i = [a_i, \infty)$. The primary goal in this paper is to allow for the possibility that players may be constrained in their choices. Thispossiblity can be incorperated by imposing a maxium feasable score such that $s_i \leq k_i$ where $k_i \epsilon [a_i, \infty)$. This is without loss of generality as elimination of stricly dominated strategies implies that no player will choose a score so high that $v_i(s_i) < 0$, and hence any k_i high enough so that $v_i(k_i) < 0$ will have no effect on the equilibrium.[1]

A secondary goal of this paper is to relax the continuity restrictions in Siegel (2009) for as many players as possible. So we will permit the possibility that some of the scores in $[a_i, k_i]$ are *infeasable*. For example, donations to a politician below a certain threashold may not be recoreded with the donor’s name, and hence small donations may not influence the politician’s behaviour with regard to the doner. Or the Olympic committee may be considering only the number of stadiums promised by potential host cities, so it may not be possible to increase a city’s score by less than the cost of a stadium. We require that $a_i,\ k_i \epsilon S_i$ but scores between those values may or may not be in S_i.

Unfortunately it will not be possible to maintain this level of generality for all players so we define s*core continuity on* $[b,\ d]$ to mean that $[b,\ d] \bigcap S_i^c$ has Lebesgue measure zero, *i.e.* scores almost everywhere on $[b,\ d]$ are in S_i.

GENERALIZED ASSUMPTION A1: v_i *and* $-c_i$ *are nonincreasing on* $s_i \epsilon \mathrm{S}_i$.

ASSUMPTION A2: $v_i(a_i) > 0$ *and* $\lim_{s_i \to \infty} v_i(s_i) < c_i(a_i)$.

ASSUMPTION A3: $c_i(s_i) > 0$ *if* $v_i(s_i) = 0$.

Note in particular that this gerneralization of Assumption A1 allows for the possibility that the payoffs may be discontinous. For example applying for a loan may incur a fixed cost for the paper-work, hence v_i and $-c_i$ would both decrease by that fixed cost at the level where the player's own funds ran out.

The limit on feasable scores, k_i, and the possibility that v_i may be discontinous introduces the possibility that a player may be constrained. A player will be said to be *constrained at x* if $x \in S_i$, $v_i(x) > 0$ and either $x = k_i$ or $v_i(\min\{s_i \epsilon \mathrm{S}_i \mid s_i > x\}) < 0$, that is a player is constrained at x if he has a positive value from winning at score x but he is either unable to exceed x or at his next highest feasable score he would have a negative payoff due to discontinuities in his valuation or feasable scores.

3. Payoff Characterization

The four main concepts from Siegel (2009) continue to be key to the analysis. The definition of *reach* must be altered to permit the possibility that a player may be constrained, but conceptually it captures the same idea.

DEFINITIONS:

(i) Player i's *generalized reach*, r_i, is the highest feasable score at which his valuation for winning is non-negative. That is, $r_i = \max\{s_i \epsilon \mathrm{S}_i \mid v_i(s_i) \geq 0\}$. Re-index players in any decreasing order of their reach, so that $r_1 \geq r_2 \geq \cdots \geq r_n$.

(ii) Player $m + 1$ is the *marginal player*.

(iii) The *threshold*, T, of the contest is the reach of the marginal player: $T = r_{m+1}$.

(iv) Player i's *power,* w_i, is his valuation for winning at the threshold. That is, $w_i = v_i(\max\{a_i, T\})$. For players other than the marginal player it is possible that $\mathrm{T} \notin \mathrm{S}_i$ but nevertheless we can leave this definition unaltered. Note however that unlike in Siegel (2009) there is no guarantee that the marginal player's power will be zero. If the marginal player is constrained at T, $w_{m+1} > 0$.

ASSUMPTION A4: If the marginal player is not constained at T, he has score continuity on $[T-\varepsilon,\ T]$. The first m players have score continuity and continuity of v_i and c_i on $[\max\{a_i,\ \min\{a_{-i}\}\},\ T+\varepsilon]$ where $\min\{a_{-i}\}$ is the lowest initial score for all players other than i.

Note that these continuity conditions are weaker than the continutity requirements in Siegel (2009) where continuity was imposed for all players and for all scores. To illustrate the usefulness of the results considder an example from the literature. Note that for exposition the examples are all-pay auctions but the results apply to non-seperable contests as well, see Siegel (2009) for a full dicussion and examples.

EXAMPLE 1: Meirowitz (2008) analyzes the sources of incumbecy advantage with a political contest in campaign spending where the incumbant (candidate I) and the challanger (candidate C) have a common valuation of the prize normalized to 1. The candidates have potentially different maginal utility cost of raising funds, $\beta_i\ \forall i \in \{I, C\}$ and the marginal benefit of campaign spending is one: One dollar of campaign spending raises a candidate's score by one. Meirowitz (2008) considers a positive headstart advantage α for the incumbent in the contest without spending limits, when studying the effect of spending limits the analysis only presents the case without a headstart advantage. For this subcase he shows that whether campaign expenditure limits benefit the incumbent or the challanger depends cruitially on the tie-breaking rule. However for this the analysis is limited to fundraising advantage alone. At the end of this sub-section, Theorem 1 will be applied to complete the analysis and show that the Meirowitz result is not general: with any α>0 expenditure limits always benefit the incumbent regardless of the tie-breaking rule.

Note that in this example both players face a common restriction on thier actions: they cannot spend more than a specified amount. This common restrictions on actions frequently occur: prepperations by litigatinig attornies are constrained by a common trial date, in the United States lobbyists face common maximum political donations, in many sports, teams face common cap on total salaries. However, as Example 1 illistrates, a common constraint on actions does not imply that there is a common constraint on scores. Because of his head-start advantageif both candidates spend the maximum permisable amount the incumbent will win. Likewise the effect of players' actions on their scores may differ, as illistrated in the second example.

EXAMPLE 2: Example 1 fits the U.S. institutional framework well but campaign spending limits were declared unconstitutional by the U.S. Supreme Court. However spending limits are used in Canada and in most of Europe. These countries have parlimentary systems where it is common to have more than two competative political parties.[2] Because a full derivation of the equilibrium was required for any results, Meirowitz (2008) was restricted to two contestants and a single prize and hence it is difficult to judge whether the results can be generalized to more than two candidates. However, the payoff charictarization in Theorem 1 does not require the derivation of the equilibrium and we can easilly add more candidates. Add a Third-party candidate (candidate T) to the model in Example 1. With three candidates the incumbent's head-start advantage cannot be summerized in a single parameter so define the initial scores $a_I > a_c > a_T$ where in Example 1 $\alpha = a_I - a_c$. Suppose that the Third-party candidate is charismatic so that one dollar of campaign spending increases his score by $\eta_T > 1$.We can also incorporate some more realistic fundraising issues as well. The Third-party candidate can raise up to [condidtion] dollars from his core supportors at marginal utility cost $\beta_T = \beta_C$. After that he must get a loan. The banks will not lend to his campaign unless he hires a professional campaign manager which is expensive and requires a substantial loan. He can borrow a minimum of [condition] and each dollar must be paid back with interest so the marginal utility cost of raising the funds is $(1+r)\beta_T$ if he wins the seat. At the end of this sub-section Theorem 1 will be used to show that a moderate cap will benefit the charismatic but finacially challanged Third-party candidate, but that a very restrictive cap will benefit the Incumbent.

In addition to the ability to add more players or more prizes, note that in Example 2 the bank's reluctance to loan to a half-hearted campaign results in a range of scores being infeasable. Note also that the utility cost of paying back the loan if the candidate is not elected was not specified. In reality this is likely to be higher than the cost if he is elected since office holders have more fundraising opportunities than private citizens. An important implication of Theorem 1 is that, although these costs will have a significant effect on the equilibrium of the game, they will not effect the expected payoffs and hence we do not need to specify them here.

GENERALIZED LEAST LEMMA: *In any equilibrium of a contest, the expected payoff of each player who is not constrainted at T is at least the maximum of the player's power and zero.*

PROOF: In equilibrium no player would choose a score higher than his reach since this would result in negative payoff. Players with powers less than or equal to zero can guarantee a zero payoff by simply chosing a_i. By the definition of a player's power, at most m players have positive power and are not constrained at T. Since the players with positive power and are not constrained at T and are able to exceed the threshold by ε by Assumption A4, they can at least guarantee an expected payoff equal to their power. *Q.E.D.*

In Siegel the Least Lemma establishes that for every player the expected payoff is the least of max(w_i, 0) when there are no constraints. The Generalized Least Lemma, which is valid even when there are constraints, is different for two reasons. First, for players with strictly positive power but score and/or value discontinuty on $[T,T+\varepsilon]$ the expecpeted payoff argument does not follow. And the proof of the Least Lemma in Siegel does not go through if a player has a positive power but is constrained at T, hence the change in the lemma. See the example below.

EXAMPLE 3: Consider the contest in Che and Gale (1998). where two players {1,2} compete for one prize. $v_i(s_i) = V_i - s_i$ and $c_i(s_i) = -s_i$ $\forall i \in \{1,2\}$. The players have different valuations of the prize, $V_1 > V_2 > 0$. Players face a common constraint $k_1 = k_2 = k$ so $S_i = [0, k]$. Any ties are resolved by coin toss. Che and Gale (1998) shows that for a sufficiently restrictive constraint, $k < V_2/2$, in any equilibrium both players must choose $s_i = k$ with probability 1 and they each have a 50% chance of winning. Hence the expected payoff of each player is given by $V_i/2 - k$ which is greater than zero and less than the power of the player.

Example 3 also demostrates that the Tie Lemma in Siegel does not generalize to contests with constraints.The Tie Lemma in Siegel shows that if two or more players choose x with strictly positive probability, those player either all win with certainty or they all loose with certainty. The Tie Lemma relies on the fact that if a player's rival has an atom at x and the player has a probability of winning at x less than one but greater than zero, that player would increase his score slightly to avoid the chance of a tie. However if the player is constrained at x this is not possible.[3] We must proceed by an alternative but related method of establishing the equilibrium payoffs that does not require the Tie Lemma.

In our effort to establish the expected payoffs of players in a contest with or withour constaints it will be necessary to assume the generic conditions presented below.

GENERALIZED GENERIC CONDITIONS:

(i) *Generalized Power Condition* —The marginal player is the only player with reach at the threshold.

(ii) *Generalized Cost Condition* —If the marginal player is not constrained at the threshold then for every $x \epsilon S_{m+1} \cap [a_{m+1}, T)$, $v_{m+1}(x) > v_{m+1}(T)$, that is the marginal player's valuation of winning is strictly decreasing at the threshold.

The Generalized Power Condition parallels Siegel's requirement that the marginal player is the only player with power of 0. However with constraints or discontinuities the marginal player may be constrained at the threshold so there may be no player with zero power. Therefore with constraints the conditions are not equivelant.

Define $N_w = \{1, \cdots, m\}$. In a generic contest these are the players who have reaches strictly greater than the threshold. Define $N_L = \{m+1, \cdots, n\}$. In a generic contest these are the players who have reaches less than or equal to the threshold.

Equilibrium may be in mixed strategies so as in Siegel (2009) define for each player G_i as a cumulative probablilty distribution that assigns probability one to his set of feasable pure strategies S_i. For a strategy profile $G = (G_1, \cdots, G_n)$, $P_i(x)$ is player i's probability of winning when he chooses $x \epsilon S_i$ and all other players play according to G, and similarly define $u_i(x)$.

To establish the players' payoffs in the contest we need two more lemmas.

MODIFIED ZERO LEMMA: *In any equilibrium of a generic contest all players in N_L must have best responses with which they win with probability 0 or arbitrarilly close to zero. These players have expected payoff of zero.*

PROOF: Denote by J a set of players including the m players in N_w plus any one other player $j \epsilon N_L$. Let $\tilde{S}$ be the union of the best-response sets of the players in J and let $s_{\inf}$ be the infimum of $\tilde{S}$. Consider three cases: (i) two or more players in J have an atom at $s_{\inf}$, (ii) exactly one player in J has an atom at $s_{\inf}$, and (iii) no players in J have an atom at $s_{\inf}$.

Case i. Initially denote $N' \subseteq J$ as the set of all players in J with an atom at $s_{\inf}$ where $|N'| > 1$. Every player in $J \setminus N'$ chooses scores greater than $s_{\inf}$ with probability 1. Therefore even if every player that is not in J chooses scores strictly below $s_{\inf}$ with probability 1 that leaves one too few prizes to be divided between $|N'|$ players, so $P_i(s_{\inf}) = 1 \ \forall \ i \epsilon N'$ is not possible.

If there are any players in N' with $P_i(s_{\inf}) = 1$ remove them from N' so that $P_i(s_{\inf}) < 1 \ \forall \ i \epsilon N'$. If $|N'|=1$ then that player i loses with certainty with score $s_{\inf}$ and i's expected payoff cannot be positive. From the Generalized Least Lemma and the Generalized Power Condition this player cannot be in N_w so it must be player j. If $|N'| > 1$ let H be the set $N' \cap N_w$. Since there is only one player in $J \setminus N_w$, $|H| \epsilon \{|N'|-1, |N'|\}$. $P_i(s_{\inf})=0$ is not possible for any $i \epsilon H$ since i would have $u_i(s_{\inf}) \leq 0$ and he must have a positive payoff by the Generalized Least Lemma and the Generalized Power Condition. Likewise if player i loses ties with other players in N' with positive probability $P_i(s_{\inf}) \epsilon (0,1)$ is not possible for any $i \epsilon H$ since i can do better by increasing his score slightly above $s_{\inf}$ to avoid ties. Hence every player in H must win every tie with other players in N' at $s_{\inf}$. This is not possible if $|H| = |N'|$ since there are not enough prizes for all the players in N'. Hence $|H| = |N'| - 1$ so $j \epsilon N'$ and j loses all ties with members of N' at $s_{\inf}$. Therefore $P_j(s_{\inf}) = 0$. Since $j \epsilon N'$ and $j \epsilon N_L$ and $P_j(s_{\inf}) = 0$, so $u_i(s_{\inf}) \leq 0$. By the Generalized Least Lemma his expected payoff must be zero.

Cases ii and iii. The corresponding proofs in Siegel (2009) apply without modification and establish that in both cases one player $i \epsilon J$ has a best response in which he wins with probability 0 or arbitrarilly close to 0 and has a payoff of at most 0. By the Generalized Least Lemma i must have a payoff of 0, and by the Generalized Power Condition $i \epsilon N_L$ and so $i = j$.

The above applies for each player $j \epsilon N_L$. *Q.E.D.*

GENERALIZED THRESHOLD LEMMA: *In any equilibrium of a generic contest, the players in N_w have best responses that approach or exceed the threshold and, therefore, the players in N_w have an expected payoff of at most their power.*

PROOF: The proof in Siegel (2009) applies directly as written, and hence is ommited here. However in the course of the proof Siegel considers the possibility that a player $i \epsilon N_w$ has $G_i(s) = 1$ for some $s < T$. This possibility is rejected since player $m+1$ can have a profitable deviation to a score in $(\max\{a_{m+1}, s\}, T)$. However the continuity requirements for player $m+1$ can be relaxed to those in the Gernealized Continuity Condition by considdering instead a profitable deviation to $T - \varepsilon$ if player $m+1$ is not constrained at T, or to T if he is constrained at T. *Q.E.D.*

From these intermediate results we can establish the main result of the paper.

GENERALIZED THEOREM 1: *In any equilibrium of a generic contest, the expected payoff of every player except player m+1 equals the maximum of his power and zero. The expected payoff of player m+1 is zero which will be less than his power if he is constrained at T.*

PROOF: The Generalized Least Lemma and the Generalized Threashold Lemma establish that players in N_w have expected payoffs equal to their power which is greater than zero by the Generalized Power Condition. The Generalized Zero Lemma establishes that the players in N_L have expected payoffs equal to 0. By the Generalized Power Condition this is greater than their power for all players in $N_w \backslash \{m+1\}$. If player $m+1$ is not constrained at T his power is 0. If he is constrained at T his power is greater than zero so his expected payoff is less than his power. *Q.E.D.*

COROLLARY 1: *In any equilibrium of a generic contest, only the constraint of the marginal player will effect expected payoffs:*

- *The derivitive of all players' expected payoffs with respect to k_j is zero for all $j \neq m+1$.*
- *For all $i \in N_L$, the derivitive of player i's expected payoff with respect to to k_{m+1} is zero.*
- *If $T=k_{m+1}$, then for all $i \in N_W$ the derivitive of player i's expected payoff with respect to k_{m+1} is $-\frac{\partial v_i(s_i)}{\partial s_i}\Big|_{s_{i=T}} \square \leq 0$. If $T \neq k_{m+1}$ then the derivitive of all players' expected payoffs with respect to k_{m+1} is zero.*

PROOF: The first and second points follow directly from the Theorem 1 and the definition of a player's power. If $T = k_{m+1}$ marginal decreases in k_{m+1} directly decrease T. Marginal increases in k_{m+1} increase T if player $m+1$ is constrained at T. If $T \neq k_{m+1}$ then marginal changes in k_{m+1} do not alter T. The third point then follows from Theorem 1, the definition of reach and the definition of power. *Q.E.D.*

These results can now be applied to the first two examples.

SOLUTION TO EXAMPLE 1: The monetary limit on campaign spending is denoted by $\overline{m}$ and is common to both players. However since the incumbent has a headstart advantage of α the constraint on scores is asymetric: $k_C=\overline{m}$ and $k_I=\alpha+\overline{m}$. Notice that the challenger's constraint is lower than the incumbent's. The challenger's payoff functions $v_C(s_C) = 1 - \beta_C s_C$ and $c_C(s_C) = \beta_C s_C$ for $s_C \epsilon [0,k_C]$. Since the incumbent starts with ascore of α his payoff functions are $v_I(s_I) = 1 - \beta_I(s_I - \alpha)$ and $c_I(s_I) = \beta_I(s_I - \alpha)$ for $s_I \epsilon [0,k_I]$. Therefore $r_C=\min\{k_C,1/\beta_C\}$ while $r_I=\min\{k_I,\alpha+1/\beta_I)$.

In if the expenditure limits are high enough that they are not binding in a generic game one of two possibilities will occur. Either $1/\beta_C < \alpha + 1/\beta_I$ in which case the challenger will be the marginal player, or $1/\beta_C > \alpha + 1/\beta_I$ in which case the incumbent will be the marginal player (if the expressions are equal the game does not satisfy the generic conditions). The marginal player will have expected payoff of zero while the other player will have a postive expected value by Theorem 1.

In the first case, even after the imposition of an expenditure limit $r_C<r_I$ since each term in r_C is less than the corresponding term in r_I. Therefore when the cap is binding the challenger will be the marginal player and his expected payoff will remain zero. However the limit will reduce the challenger's reach and hence increase the expected payoff of the incumbent, .

Using Genralized Theorem 1, it is straightforward to find the expeced payoffs of constrained game with α>0. Let $\tilde{S}$ be the union of the best-response sets of the players and let s_{sup} be the supremum of $\tilde{S}$. Without spending limits the reach of the incumbant is $\alpha + 1/\beta_1$ and the reach of the challanger is $1/\beta_2$ and Meirowitz (2008) already establishes that $s_{\text{sup}} = \min(\alpha + 1/\beta_1,\ 1/\beta_2)$. A limit $k < s_{\text{sup}}$ is binding. If $\alpha + 1/\beta_1 > 1/\beta_2$, then the challanger is the *m+1*th player. The challanger has zero expected payoff (less than his power) and the incumbent has expected payoff equal to his power, $1 - \beta_1 k$. If $\alpha + 1/\beta_1 < 1/\beta_2$, the incumbant has zero expected payoff (less than his power) and the challanger has expected payoff equal to 1-$\beta_2 k$. It is then easy to derive equilibrium distributions of the players as well as results on expected spending and probablitity of winning.

SOLUTION TO EXAMPLE 2:

3.1. **Discussion of the Payoff Characterization.**

3.2. **Contests That Are Not Generic.**

For non-generic contests with constraints neither Corollary 2 nor Corollary 3 from Siegel (2009) continue to hold. This can be seen in Example 3 which is a non-generic contest because the Power Condition does not hold. There is more than one players with reach at the threshold. When $k < V_2/2$ both players have reaches of k and hence player i has a power of $w_i = V_i - k$. Che and Gale (1998) shows that in any equilibrium each player choose a score at the common constraint with certainty and the allocation of the single prize is decided by coin toss. Hence the expected payoff for player i is $0 < (V_i/2) - k < w_i$. This is a violation of a conjectured extension of Siegel's Corollary 2.

While the results in Che and Gale (1998) are for players with different valuations, the same logic carries over to identical players facing a common constraint and non-zero probabilities of winning a tie. If the common constraint is sufficiently restrictive there will be an equilibrium were both players choose scores at the constraint with probability one. This will yield positive expected payoffs for both players, a violation of a conjectured extension of Siegel's Corollary 3. When constrained, in equilibrium players can put probability mass points at scores where they do not win or lose with certainty. This drives the refutation of Siegel's Tie Lemma in the context of constrainted contests and and the extentions to Siegel's Corollaries 2 and 3.

4. Conclusion

Some kind of conclusion here.

Notes

[1] Conceptually there are two possible types of constraints: constraints on effort and constraints on scores. Both types of constraint can be captured by this specification. Examples of constraints on effort include liquidity constraints or the maximum permissable donation to a candidate's political campaign in the U.S. Since effort translated directly into scores and the most restrictive possible constraint is zero effort $k_i \geq a_i$ captures all possibilities. Constraints placed directly on permissable scores are also possible. For example, by construction the maximum possible score that can be achieved on the SAT university enterence exam is 2400. With constraints directly on scores, an initial score higher than the maximum possible score is nonsensical so $k_i \geq a_i$ can be assumed without loss of generality.

[2] And in at least one case, Ireland, more than one prize is possible. In many Irish political districts the two candidates with the highest vote totals each take a seat in parliment.

[3] If we revert to Siegel's strong continuity assumptions we can get this Generalized Tie Lemma: "*If two or more players who are not constrained at x have an atom at x then all the players with the atom that are not constrained at x either win with certainty or lose with certainty at x.*" While this is potentially useful, it is not sufficient to proceed as players may well be constrained.

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