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Information diffusion on the iterated local transitivity model of online social networks

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ABSTRACT

We study a recently introduced deterministic model of competitive information diffusion on the Iterated Local Transitivity (ILT) model of Online Social Networks (OSNs). In particular, we show that, for 2 competing agents, an independent Nash Equilibrium (N.E.) on the initial graph remains a N.E. for all subsequent times. We also describe an example showing that this conclusion does not hold for general N.E. in the ILT process.

1. Introduction and motivation

Understanding how information and rumours spread is a key issue for modern society. Malicious or inaccurate rumours can lead to unnecessary panic and generate social and economic instability. From another perspective, understanding how information propagates through a population is a necessary first step in the design of viral marketing campaigns [12]. Recent advances in communications and the emergence of social networking sites such as Facebook and Twitter have greatly increased the power of individual agents to disseminate information. This provides strong motivation for analysing the mechanisms of information propagation and the role played by the individual in the process.

In this paper, we build on recent work in [1] in which a simple model of competitive information diffusion was introduced. The model in [1] considers the diffusion process as a competitive game taking place on a graph, which captures the underlying social structure. The model considers interested parties (agents) $\{1, \ldots, n\}$ who wish to propagate their idea or innovation through the network. The agents are initially assigned vertices $\mathbf{x} = (x_1, \ldots, x_n)$, which they "colour" at the first time-step in the diffusion process. At each following time-step, uncoloured vertices adjacent to vertices that are already coloured are coloured according to the following rules. If two vertices of different colour neighbour an uncoloured vertex, then in the next time-step this vertex is coloured grey. Grey nodes are treated differently to other colours and do not propagate; they represent individuals who choose to adopt neither idea and who do not pass on either idea. If an uncoloured vertex is adjacent to vertices of only one colour, then the uncoloured vertex takes this colour. All other uncoloured vertices remain uncoloured. This represents the spread of an idea through a social network. The diffusion process ends when no further vertices can be coloured. The utility $U_i(\mathbf{x})$ of agent i is then the number of vertices coloured i when the process ends.

A central theme in the study of games is the existence of Nash equilibria. A Nash Equilibrium (N.E.) occurs if no agent benefits from unilaterally changing its starting vertex. Formally, this means that $U_i((x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_n)) \leq U_i(\mathbf{x})$ for all i and $v \neq x_i$. The authors of [1] considered the question of when N.E. exist for graphs of diameter 2. In [14], an example

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was presented to show that even in the case of a graph of diameter 2, with 2 agents, a N.E. need not exist. It is possible to ensure the existence of N.E. for graphs of diameter 2 under additional technical assumptions. In keeping with the wish of the authors of [1] that their results be extended to other models of social networks, we consider their model on a recently proposed model of online social networks (OSNs); namely, the Iterated Local Transitivity (ILT) model [2]. We present a result describing conditions in which a N.E. for a 2-agent game on the initial graph in the ILT process remains a N.E. throughout the process. The advantage of this is that it allows us to establish the existence of N.E. for quite complicated graphs, starting from a simple initial network.

The layout of the paper is as follows. In Section 2, we review related work on mathematical modeling of diffusion and social networks. We introduce the main notation used in the paper in Section 3 and present some preliminary technical results in Section 4. Then, in Section 5 we consider the diffusion process of [1] on the ILT model of social networks. We show that if an independent N.E. exists in the initial (seed) graph, then it remains a N.E. for all subsequent graphs in the ILT process. An example is described to illustrate that this conclusion will not necessarily hold for general, non-independent Nash equilibria. We also highlight differences between the diffusion process of [1] and that based on Voronoi games in this section. Finally, in Section 6 we present our conclusions and outline possible directions for future work.

2. Related work

We now give a very brief review of some recent models of OSNs; for more details consult [3]. In [13] a simple deterministic model for network growth based on the matrix theoretic Kronecker product was introduced. This model generates graphs whose diameter decreases over time and also obeys a *densification power law*; both of these properties have been observed in data on real OSNs. Moreover, as highlighted in [13], the simplicity of the model renders it more amenable to rigorous analysis than, typically more complex stochastic models for network growth. The so-called forest fire model also possesses a densification power law and shrinking diameter; however its definition is more complicated than the Kronecker model and involves more parameters. The ILT model we adopt here is based on two fundamental observations concerning the nature of social interactions: transitivity of connections and local growth rules (the update rule for Kronecker graphs is essentially global in nature). The ILT model also reflects the community structure of social networks, as reflected in its spectral and expansion properties. As argued in [2], apart from its theoretical interest, it provides a simple mathematical metaphor that can be used to obtain insights into complex processes taking place on social networks. Given the complexity of the processes behind competitive diffusion, having a graph model of the network that is sufficiently simple to make analysis tractable is a distinct advantage. The same arguments can be advanced for the Kronecker model, and while we do not study these models here, it is hoped that similar analysis can be done for this model class in future work.

While several authors have considered the problem of innovation and information diffusion through networks, relatively little has been done on competitive diffusion. The *threshold model* considered in [11] considers a single innovation and the question of how best to select an initial set of seed nodes to propagate the innovation. The work of [5] is more closely related to our results as it is game-theoretic in nature. However, the models considered in these papers differ fundamentally from the one analysed here. In particular, they are concerned with a single innovation and treat each node in the network as an *agent* in the game, who must choose between adopting or not adopting the innovation. The payout depends on the choices made by an agent's neighbours. In contrast, the model we study views the agents as existing outside the network and competing against each other. Each agent selects a seed node in the network with the aim of maximising the number of nodes adopting their innovation or idea.

As highlighted by the work of [5,9,10], the analysis of network games is far from straightforward. For the related Voronoi game on graphs, the computation or identification of N.E. for a fixed number of agents involves exhaustively enumerating all possible strategy profiles. While this provides a polynomial algorithm, it is nonetheless a computational bottleneck, particularly as network size increases. Motivated by this, the authors of [10] sought graphical properties that guarantee the existence of a N.E. for the Voronoi game. Even for the simple case of 2 agents, their results are restrictive, and apply only to a subclass of *strongly transitive* graphs. For the simple model considered here, providing conditions for the existence of Nash equilibria has also proven troublesome. In [14], an example is given of a graph of diameter 2 for which no N.E. exists even for the case of 2 agents. Our approach is to take advantage of the iterative nature of many graph models for social networks. If it is possible to show that the existence of a N.E. on a graph at any time in the process implies the existence of one at the next time step, then a simple inductive argument can be used to establish the existence of N.E.s for quite complex graphs, starting from simple initial networks. We hope that this simple paradigm may prove useful in deriving results for more realistic and complex models than those we consider.

3. Notation and background

Our graph theoretical terminology and notation is standard [7]. The graphs we consider are finite and undirected; formally, a graph G consists of a set of nodes $V(G)$ and a set of edges $E(G)$ of the form $\{v, w\}$ for $v, w \in V(G)$, $v \neq w$. For notational simplicity, we denote the edge $\{v, w\}$ by vw (or wv). The neighbourhood $N(v)$ of $v \in V(G)$ is defined as

$$N(v) = \{u \in V(G) : uv \in E(G)\}.$$

For a set X, we denote the cardinality of X by $|X|$.

3.1. The diffusion process **D** *[1]*

Consider a graph G with vertex set $V(G)$, $|V(G)| = N$ and a set of agents indexed as $[1, n] = \{1, \ldots, n\}$. At time 0, each agent $i \in [1, n]$ selects a seed node, x_i, in $V(G)$, which is labelled (or coloured) i. The n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$ is known as a *strategy profile*. Throughout the paper, we shall only consider strategy profiles in which all of the x_i are distinct. All other nodes at time 0 are labelled 0 (corresponding to white nodes in [1]). In addition to the labels $0, 1, \ldots, n$ we also use the label -1 to denote grey nodes. In keeping with the original model of [1], grey nodes *do not propagate*. For $v \in V(G)$ and $t \geq 0$, we use $l^t(v)$ to denote the label of v at time t.

The labelling map l^0 is given by

$$l^0(v) = \begin{cases} i & \text{if } v = x_i \text{ where } 1 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

At each subsequent time $t \geq 1$, we define l^t as follows. If $l^{t-1}(v) \neq 0$, then $l^t(v) = l^{t-1}(v)$ (so only vertices labelled 0 can change their label). For nodes v with $l^{t-1}(v) = 0$, let $\mathbb{L}(v) = \{l^{t-1}(w) : w \in N(v)\}$ denote the set of labels of the neighbours of v. Then:

- if $\mathbb{L}(v) \cap [1, n] = \{i\}$, then $l^t(v) = i$;
- if $|\mathbb{L}(v) \cap [1, n]| \geq 2$ then $l^t(v) = -1$;
- $l^t(v) = l^{t-1}(v)$ otherwise.

For $i \in [1, n]$, we denote by $L_i(t)$ those nodes labelled i at exactly time t. Formally $L_i(t) = \{v \in V(G) : l^{t-1}(v) = 0, l^t(v) = i\}$. The process *terminates* at some time t if $L_i(t + 1)$ is empty for all $i \in [1, n]$. Thus no new vertices are labelled $i \in \{1, \ldots, n\}$ in the time step $t \to t + 1$. Note however, that it is possible for nodes to be labelled -1 in the step $t \to t + 1$. As we are only interested in vertices ultimately labelled $i \in [1, n]$, this is unimportant. Clearly, the process must terminate before time $t = N - n$.

3.2. Iterated local transitivity (ILT) graphs

In [2], the *Iterated Local Transitivity (ILT)* model for online social networks was introduced. Let an initial, connected graph G_0 be given. For each time $t \geq 1$, G_t is formed by adding a *clone* v' of every node v in $V(G_{t-1})$, an edge vv' and an edge $v'w$ between v' and every neighbour w of v in G_t. Several basic properties of this model were derived in [2] and a stochastic extension was also introduced.

4. Preliminary results

We are interested in the following question for the ILT model discussed in Section 3. When does the existence of a N.E. in the initial graph G_0 imply the existence of a N.E. in G_t for all $t \geq 0$?

We adopt the following notation. For a connected graph G, $\hat{G}$ denotes the graph obtained through applying one step of the ILT process to G. For $1 \leq i \leq n,\ t \geq 0,\ \hat{L}_i(t)$ denotes the set of nodes in $\hat{G}$ labelled i at *exactly* time t. We also use $W(t)$ $(\hat{W}(t))$ to denote the *white* nodes in G $(\hat{G})$ at time t. Formally, $W(t) = \{v \in V(G) : l^t(v) = 0\}$ $(\hat{W}(t) = \{v \in V(\hat{G}) : l^t(v) = 0\})$.

For a set $U \subseteq V(G)$ $(U \subseteq V(\hat{G}))$, $N(U)$ $(\hat{N}(U))$ denotes the neighbours of U in G $(\hat{G})$.

If a strategy profile $\mathbf{x} = (x_1, \ldots, x_n)$ consists entirely of vertices from $V(G)$, we say that it is a strategy profile in G. If the set $\{x_1, \ldots, x_n\}$ is an independent set, we say that the strategy profile $\mathbf{x}$ is independent.

Definition 4.1. Let a connected graph G, a set of agents $[1, n] = \{1, \ldots, n\}$ and a strategy profile $\mathbf{x}$ be given. If the diffusion process **D** terminates at time T on G, the *Utility* $U_i(\mathbf{x})$ of agent $i \in [1, n]$ is given by

$$U_i(\mathbf{x}) = |\{v \in V(G) : l_T(v) = i\}|.$$

Informally, the utility of agent i is the total number of nodes in G labelled i when the process terminates. We use $\hat{U}_i$ to denote utilities in $\hat{G}$.

Given a strategy profile $\mathbf{x}$, a node $v \notin \{x_1, \ldots, x_n\}$ and $i \in [1, n]$, we denote by $\mathbf{x}_{-i}(v)$ the profile given by

$$\mathbf{x}_{-i}(v) = (x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_n).$$

We next recall the definition of N.E.

Definition 4.2. Let a connected graph G and a set of agents $[1, n] = \{1, \ldots, n\}$ be given. A strategy profile $\mathbf{x}$ is a N.E. for the process **D** on G if $U_i(\mathbf{x}) \geq U_i(\mathbf{x}_{-i}(v))$ for all $i \in [1, n]$ and all $v \in V(G) \setminus \{x_1, \ldots, x_n\}$.

Note the following simple facts, which follow immediately from the definition of the process **D**.

Lemma 4.1. *A vertex v is in $L_i(t + 1)$ if and only if:* (i) $v \in W(t)$; (ii) $v \in N(L_i(t))$; (iii) $v \notin N(L_j(t))$ *for all* $j \in [1, n] - i$.

Lemma 4.2. *A vertex v is in $W(t + 1)$ if and only if:* (i) $v \in W(t)$; (ii) $v \notin N(L_j(t))$ *for all* $j \in [1, n]$.

Proposition 4.1. *Let $(x_1, \ldots, x_n)$ be a strategy profile in a connected graph G. Then for $1 \le i \le n$*

$$\hat{L}_i(1) = \begin{cases} L_i(1) \cup (L_i(1))' \cup \{x_i'\} & \text{if } \mathbf{x} \text{ is independent} \\ L_i(1) \cup (L_i(1))' & \text{otherwise}. \end{cases}$$

$$\hat{W}(1) = W(1) \cup W(1)'.$$

Proof. Note that $\hat{W}(0) = W(0) \cup (W(0))' \cup \{x_1', \ldots, x_n'\}$ and that $\hat{N}(x_j) = N(x_j) \cup (N(x_j))' \cup \{x_j'\}$ for $1 \le j \le n$. As $x_i' \in \hat{N}(x_j)$ if and only if $x_i \in N(x_j)$, the result follows readily from Lemma 4.1. □

The following proposition clarifies the relationship between $\hat{L}_i(t)$ and $L_i(t)$ for $t = 2, 3, \ldots$.

Proposition 4.2. *Let $(x_1, \ldots, x_n)$ be a strategy profile in a connected graph G. Then for $1 \le i \le n$, $t \ge 2$,*

$$\hat{L}_i(t) = L_i(t) \cup (L_i(t))'.$$

$$\hat{W}(t) = W(t) \cup W(t)'.$$

Proof. We prove the result by induction on t. First, we use Proposition 4.1 to establish the result for the case $t = 2$. Lemma 4.1 implies that a vertex $w \in V(\hat{G})$ is in $\hat{L}_i(2)$ if and only if: $w \in \hat{W}(1)$; $w \in \hat{N}(\hat{L}_i(1))$; $w \notin \hat{N}(\hat{L}_j(1))$ for $j \in [1, n] - i$. Using Proposition 4.1, we can see that $\hat{N}(x_i') \cap \hat{W}(1)$ is empty (this follows from Lemma 4.2 as $\hat{N}(x_i')$ is given by $\{x_i\} \cup N(x_i)$). Furthermore, for $1 \le k \le n$

$$\hat{N}(L_k(1) \cup L_k(1)') = \hat{N}(L_k(1)) = N(L_k(1)) \cup N(L_k(1))'.$$

It follows from Lemma 4.1 that $\hat{L}_i(2) = L_i(2) \cup L_i(2)'$. The conclusion $\hat{W}(2) = W(2) \cup W(2)'$ follows from Proposition 4.1 and Lemma 4.2.

Now assume that the result is true for some $t \ge 2$. Lemma 4.1 implies that $w \in \hat{L}_i(t + 1)$ if and only if: $w \in \hat{W}(t)$; $w \in \hat{N}(\hat{L}_i(t))$; $w \notin \hat{N}(\hat{L}_j(t))$ for $j \in [1, n] - i$. Using the induction hypothesis we see that for $k \in [1, n]$

$$\hat{N}(\hat{L}_k(t)) = \hat{N}(L_k(t) \cup L_k(t)') = \hat{N}(L_k(t)) = N(L_k(t)) \cup N(L_k(t))'.$$

As in the previous paragraph, it follows that

$$\hat{L}_i(t + 1) = L_i(t + 1) \cup (L_i(t + 1))'.$$

Moreover, combining the induction hypothesis with Lemma 4.2 yields $\hat{W}(t + 1) = W(t + 1) \cup W(t + 1)'$. □

5. Nash equilibria and the ILT model

In [1], the existence of Nash equilibria for the diffusion process was investigated on graphs of low diameter. We shall provide conditions under which such equilibria are guaranteed to exist for G_t in the ILT model for all t.

The following lemma shows that in the 2-agent case, neither agent can improve their utility by unilaterally changing from x_i to its clone x_i'.

Lemma 5.1. *Let (x_1, x_2) be a strategy profile in G. Then*

$$\hat{U}_1(x_1', x_2) \le \hat{U}_1(x_1, x_2).$$

Proof. We use $\bar{l}^t$ to denote the labelling map for the profile (x_1', x_2), and l^t for the labelling map for (x_1, x_2). It is clear that $\bar{l}^1(v) = 1$ implies $l^1(v) = 1$, and that $l^1(v) = 2$ implies $\bar{l}^1(v) = 2$. Suppose that $\hat{U}_1(x_1', x_2) > \hat{U}_1(x_1, x_2)$. Then there must exist some $t > 1$ and v such that $\bar{l}^t(v) = 1$, $l^t(v) \neq 1$. Let t_0 be the minimal $t > 1$ for which this occurs. It is immediate that $l^{t_0-1}(v) \neq 1$. If $t_0 - 1 = 1$, then this implies that $\bar{l}^{t_0-1}(v) \neq 1$. If $t_0 - 1 > 1$, then as t_0 is minimal, it also follows that $\bar{l}^{t_0-1}(v) \neq 1$. We can thus conclude that $\bar{l}^{t_0-1}(v) = 0$.

As $\bar{l}^{t_0}(v) = 1$, there is some $w_1 \in \hat{N}(v)$ with $\bar{l}^{t_0-1}(w_1) = 1$ and there exists no $w \in N(v)$ with $\bar{l}^{t_0-1}(w) = 2$. We know that $l^{t_0}(v) \neq 1$ and $l^{t_0-1}(w_1) = 1$. It follows from this that there must exist some $w_2 \in \hat{N}(v)$ with $l^{t_0-1}(w_2) = 2$. Moreover, we know that $\bar{l}^{t_0-1}(w_2) \neq 2$. This implies that $t_0 - 1 > 1$. Thus if we define t_1 to be the minimum $t > 1$ for which there exists u with $\bar{l}^t(u) \neq 2$, $l^t(u) = 2$, we can see that $1 < t_1 < t_0$. A similar argument to that used above will show that there must exist some w_3 in $\hat{N}(u)$ such that $l^{t_1-1}(w_3) \neq 1$, $\bar{l}^{t_1-1}(w_3) = 1$. As $1 < t_1 < t_0$ (and this cannot happen for $t = 1$ so that $t_1 - 1 > 1$), this contradicts the minimality of t_0. This shows that $\hat{U}_1(x_1', x_2) \le \hat{U}_1(x_1, x_2)$ as claimed. □

The example in Fig. 1 shows that the previous result need not hold for 3 or more agents. If $\mathbf{x} = (v_1, v_2, v_3)$ and $\mathbf{x}_1 = (v_1, v_2', v_3)$, then $\hat{U}_2(\mathbf{x}) < \hat{U}_2(\mathbf{x}_1)$.

The next lemma, which follows from Proposition 4.2, shows how the utility $U_i(\mathbf{x})$ of an agent on G relates to its utility $\hat{U}_i(\mathbf{x})$ on $\hat{G}$ for a strategy profile $\mathbf{x}$ in G.

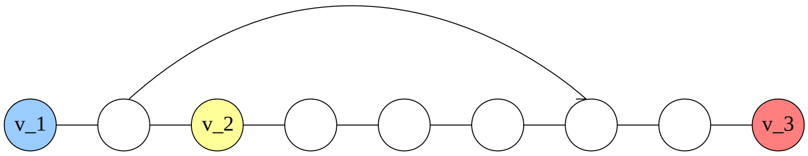

Fig. 1. Utility increases by choosing clone.

Lemma 5.2. *Let* $\mathbf{x} = (x_1, \ldots, x_n)$ *be a strategy profile in G. Then*

$$\hat{U}_i(\mathbf{x}) = \begin{cases} 2U_i(\mathbf{x}) - 1 & \text{if } x_i \text{ is neighboured by some } x_j \\ 2U_i(\mathbf{x}) & \text{if } x_i \text{ is not neighboured by some } x_j. \end{cases}$$

Proof. This result follows immediately from Propositions 4.1 and 4.2. □

For the remainder of this section, we will use the above result to investigate the relationship between Nash equilibria on G and $\hat{G}$.

Proposition 5.1. *Let* $\mathbf{x} = (x_1, \ldots, x_n)$ *be a strategy profile in G. If* $\mathbf{x}$ *is a N.E. for* $\mathbf{D}$ *in* $\hat{G}$ *then* $\mathbf{x}$ *is also a N.E. for* $\mathbf{D}$ *in G.*

Proof. For any strategy profile $\mathbf{y} = (y_1, \ldots, y_n)$ in G, $1 \le i \le n$, let $f_i(\mathbf{y}) \in \{0, 1\}$ be the indicator function of whether y_i neighbours some y_j, $j \neq i$. Let $\mathbf{x}_0$ be a strategy profile $\mathbf{x}_{-i}(v)$ for some $v \in V(G) \setminus \{x_1, \ldots, x_n\}$. As $\mathbf{x}$ is a N.E. in $\hat{G}$, we have

$$2U_i(\mathbf{x}_0) - f_i(\mathbf{x}_0) = \hat{U}_i(\mathbf{x}_0) \le \hat{U}_i(\mathbf{x}) = 2U_i(\mathbf{x}) - f_i(\mathbf{x}).$$

Rearranging the above inequalities, we see that

$$U_i(\mathbf{x}_0) \le U_i(\mathbf{x}) + \frac{f_i(\mathbf{x}_0) - f_i(\mathbf{x})}{2} \le U_i(\mathbf{x}) + \frac{1}{2}.$$

As both $U_i(\mathbf{x}_0)$ and $U_i(\mathbf{x})$ are integers, it follows that $U_i(\mathbf{x}_0) \le U_i(\mathbf{x})$. As this is true for all $i \in \{1, \ldots, n\}$, $v \in V(G)$, it follows that $\mathbf{x}$ is a N.E. in G as claimed. □

The previous proposition shows that in order for a strategy profile $\mathbf{x}$ selected from the vertices of G to be a N.E. in $\hat{G}$, it is necessary for $\mathbf{x}$ to be a N.E. in G. It is tempting to conjecture that the converse will be true: that any N.E. on G will also be a N.E. on $\hat{G}$. The example in Fig. 2 shows that this is not true, even for the simple case of 2 agents. Moreover, for this example, the process $\mathbf{D}$ has no N.E. on $\hat{G}$.

Discussion of example in Fig. 2.

We claim that $\mathbf{x} = (v_1, v_2)$ is a N.E. First note that $U_1(\mathbf{x}) = 8$, $U_2(\mathbf{x}) = 5$. If we remove the edge v_1v_2 then G splits into two connected components. Let C_i denote the component containing v_i for $i = 1, 2$. Consider a strategy profile (v, v_2) with $v \neq v_1$, $v \in C_1$. For such a profile, it is clear that v_1 will either be labelled 2 or -1. In either case, $U_1(v, v_2) \le 7$. On the other hand $U_1(v, v_2) = 1$ for any $v \in C_2$. Thus agent 1 cannot improve their utility by unilaterally changing strategy. Now consider agent 2. Again $U_2(v_1, v) = 1$ for any $v \in C_2$ with $v \neq v_2$. So consider $v \in C_1$. By symmetry, it is enough to consider $v \in \{v_3, v_4, v_5, v_8\}$. For $v \in \{v_4, v_5, v_8\}$, it is easy to see that vertices $\{v_6, v_7, v_9\}$ cannot be labelled 2 so in each of these cases $U_2(v_1, v) \le 4$. Finally, $U_2(v_1, v_3) = 5$. Thus, agent 2 cannot unilaterally improve their utility and (v_1, v_2) is a N.E. as claimed.

Moreover, (v_1, v_2) is the only N.E. in G. Consider any strategy profile (x_1, x_2). If both x_1 and x_2 are in C_2, then one agent has utility as most 1 and this agent can increase their utility by changing to v_1. On the other hand, if both x_1 and x_2 are in C_1, then (as v_1 will either be labelled i for some $i \in \{1, 2\}$ or -1), one agent will have utility at most $\frac{|C_1|}{2} = 4$, while their utility would be at least 5 if they change to v_2. Therefore, any N.E. (x_1, x_2) must have one vertex in each component. Without loss of generality, suppose (x_1, x_2) is a N.E. with $x_1 \in C_1$, $x_2 \in C_2$. We claim that $x_1 = v_1$, $x_2 = v_2$. Suppose this is not the case. First assume $x_1 \neq v_1$. If v_1 is labelled 1 (when the process terminates), then agent 2 will improve their utility by changing x_2 to v_1. On the other hand, if v_1 is not labelled 1, then agent 1 will increase their utility by changing to v_1. This shows that $x_1 = v_1$. A similar argument shows that $x_2 = v_2$ so (v_1, v_2) is the only N.E. in G as claimed.

The N.E. (v_1, v_2) is not independent so it follows from Lemma 5.2 that $\hat{U}_2(v_1, v_2) = 9$. However, for this example, we have seen that $U_2(v_1, v_3) = 5$ and as $v_1v_3 \notin E(G)$, it follows that $\hat{U}_2(v_1, v_2) = 10$. Thus (v_1, v_2) is not a N.E. in $\hat{G}$. Using very similar arguments to those employed above, we can show that any N.E. (x_1, x_2) in $\hat{G}$ must have $x_1 \in \{v_1, v_1'\}$, $x_2 \in \{v_2, v_2'\}$ or $x_2 \in \{v_1, v_1'\}$, $x_1 \in \{v_2, v_2'\}$. Without loss of generality, consider the former case. For both (v_1, v_2'), $(v_1'v_2')$, the utility of agent 2 is 9; in both cases, the utility of agent 2 increases to 10 if they change their seed vertex to v_3. The only remaining possibility for a N.E. is (v_1', v_2). In this case, agent 2 has utility 10. However, if agent 2 changes their seed vertex to v_1, we find $U_2(v_1', v_1) = 14$ so this is not a N.E. either. Thus we see that $\hat{G}$ has no N.E., while G does.

It is instructive to highlight some key factors involved in the construction of the above example. First, G has a *unique* (up to permutation) N.E. (v_1, v_2) which is not independent, and there exists a vertex $v_3 \notin \{v_1, v_2\}$ with $U_2(v_1, v_3) = U_2(v_1, v_2)$.

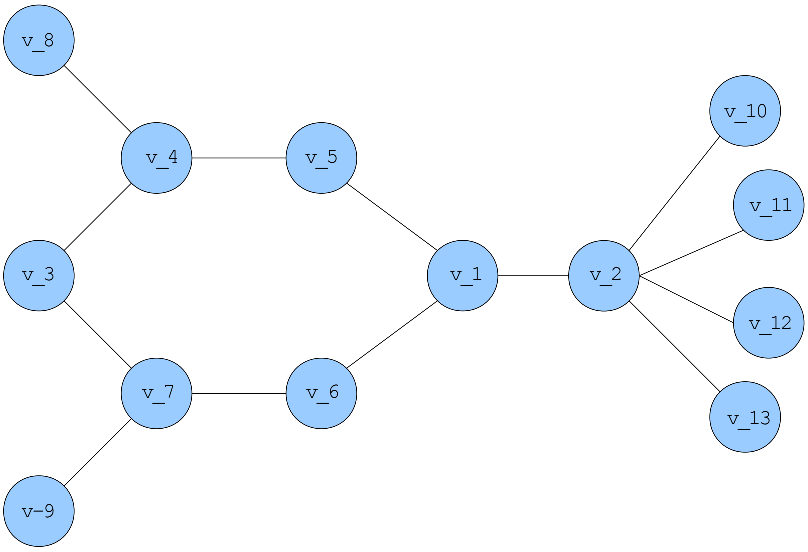

Fig. 2. N.E. on G, no N.E. on $\hat{G}$.

These conditions imply that (v_1, v_2) is not a N.E. in $\hat{G}$. The structural properties of G ensuring a unique N.E., when inherited by $\hat{G}$ rule out the possibility of a N.E. existing in $\hat{G}$. Before finishing our discussion here, note that it is possible for $\hat{G}$ to have a N.E. even if there exists no independent N.E. on G. To see this, consider a simple line graph G with 4 vertices $V(G) = \{v_1, \ldots, v_4\}$, $E(G) = \{v_1v_2, v_2v_3, v_3v_4\}$ and two agents. The only N.E. are (v_2, v_3) or (v_3, v_2). However it is easy to see that these are again N.E. on $\hat{G}$.

We now show that, for two agents, an independent N.E. for **D** on G will also be a N.E. for **D** on $\hat{G}$.

Proposition 5.2. *If $\mathbf{x} = (x_1, x_2)$ is an independent N.E. for the diffusion process* **D** *in G then* **x** *is a N.E. in $\hat{G}$.*

Proof. As **x** is a N.E. in G, we know that for any $i \in \{1, 2\}$, $v \in V(G) \setminus \{x_1, x_2\}$,

$$U_i(\mathbf{x}_{-i}(v)) \leq U_i(\mathbf{x}). \tag{2}$$

As $x_1x_2 \notin E(G)$, we can conclude from Lemma 5.2 that

$$\hat{U}_i(\mathbf{x}) = 2U_i(\mathbf{x}). \tag{3}$$

Furthermore, it follows from Lemma 5.2 that $\hat{U}_i(\mathbf{x}_{-i}(v)) \leq 2U_i(\mathbf{x}_{-i}(v))$ so we can immediately conclude from (2) and (3) that

$$\hat{U}_i(\mathbf{x}_{-i}(v)) \leq \hat{U}_i(\mathbf{x}). \tag{4}$$

To complete the proof, note that for $w = v'$, where $v \in V(G)$, we know from Lemma 5.1 that

$$\hat{U}_i(\mathbf{x}_{-i}(w)) \leq \hat{U}_i(\mathbf{x}_{-i}(v)).$$

Combining this with the previous arguments, we see that for any $v \in V(\hat{G}) \setminus \{x_1, x_2\}$,

$$\hat{U}_i(\mathbf{x}_{-i}(v)) \leq \hat{U}_i(\mathbf{x}).$$

This completes the proof. □

The above result immediately yields the following conclusion concerning the ILT model.

Corollary 5.1. *Consider the ILT graph model with a connected initial graph G_0. Consider the diffusion process* **D** *corresponding to a set of agents $\{1, 2\}$. If there exists an independent N.E. $\mathbf{x} = (x_1, x_2)$ for* **D** *on G_0, then there exists a N.E. for* **D** *(with agents $\{1, 2\}$) on G_t for all $t \geq 0$.*

As a final point, we note that as in the case of Lemma 5.1, the situation becomes more complicated when we consider three or more agents. In Fig. 3, each rectangle represents a set of a or $a/2$ leaves respectively (vertices of degree 1). Choose a to be 20 and assume we have 3 agents. There is a N.E. at $\mathbf{x} = (v_1, v_3, v_4)$ in G. However, it can be verified that this is not a N.E. in $\hat{G}$. In fact, $\hat{U}_2(v_1, v_2', v_4) > \hat{U}_2(v_1, v_3, v_4)$.

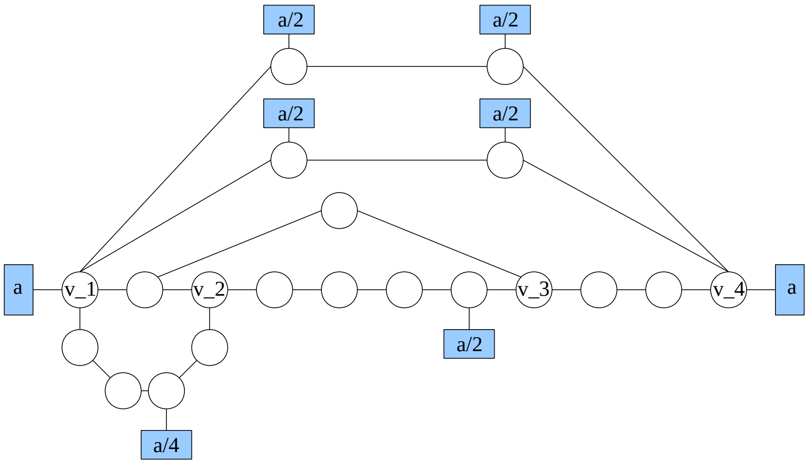

Fig. 3. Independent N.E. on G that is not a N.E. on $\hat{G}$.

6. Conclusions

Motivated by the importance of information dissemination on modern online social networks, we have studied a competitive diffusion process on the iterated local transitivity (ILT) model. Specifically, we have studied the model of [1] for 2 competing agents and shown that an independent N.E. on the initial graph G_0 will remain a N.E. for all $t \geq 0$. We have also described a counterexample to illustrate that this will not hold for non-independent N.E. Proposition 5.2 and Corollary 5.1 allow for the following interpretation. If 2 competing agents select an independent equilibrium strategy at some time, then neither of them can benefit from unilaterally changing their strategy at *any subsequent time* in the evolution of the underlying network. This suggests that, in this case the alterations in the underlying social network do not effect the dynamics of the game. It would be interesting to test the extent to which this conclusion holds for real social networks, for which time-course data can be obtained. The work here can be viewed as a starting point for a significant programme on the timely topic of information diffusion on social networks. There are several possible ways to extend our results. These include investigating alternative models of information diffusion and rumour spread [6,4] on the ILT model in the spirit in which epidemic models have been analysed on graphical models recently [8]. It would also be interesting to investigate the possibility of obtaining similar results for the process **D** on the stochastic ILT models described in [2] or on other models of online social networks such as Kronecker graph models [13].

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