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Electrically charged black branes in $\mathcal{N}=4^+$, $D=5$ gauged supergravity

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ABSTRACT

We analyze the properties of asymptotically AdS electrically charged black brane solutions in a consistent truncation of the $\mathcal{N}=4^+$, $D=5$ Romans' gauged supergravity which contains gravity, $SU(2)$ and $U(1)$ gauge fields, and a dilaton possessing a nontrivial potential approaching a constant negative value at infinity. We find that the $U(1)\times U(1)$ solutions become unstable to forming non-Abelian hair. These configurations emerge as zero modes of the Abelian solutions at critical temperature and a critical (non-vanishing) ratio of the electric charges and can be viewed as holographic p-wave superfluids.

1. Introduction

Recently, considerable effort has been put into extending AdS/CFT correspondence beyond high-energy physics by constructing gravity models that are conjectured to be dual to various condensed matter systems. This has lead to the discovery of holographic superconductors and holographic superfluids, describing condensed phases of strongly coupled, planar, gauge theories. Studying such models involves the construction of electrically charged black holes in an asymptotically AdS spacetime, which below a critical temperature become unstable to forming hair. That is, a phase transition occurs to a superconductor/superfluid state, in which a sufficiently large $U(1)$ charge density triggers the spontaneously breaking of the $U(1)$ symmetry. Then an operator charged under the $U(1)$ acquires a nonzero expectation value (see *e.g.* [1] for a review of these aspects).

For p-wave superconducting black holes, the condensing operator is a vector and hence rotational symmetry is broken. Such black hole solutions have been constructed using either charged non-Abelian vector fields [2] or, alternatively, charged two-forms [3]. However, most of the studies in the literature have assumed an *ad hoc* construction of the Lagrangian of the gravitational system, without a clear connection with a given supergravity model, which makes it rather difficult to describe precisely the application of the AdS/CFT dictionary.

At the same time, the gauged supergravity models generically contain non-Abelian vector fields, which may suggest the existence of p-wave superconducting black hole solutions. The case of $\mathcal{N}=8$, $D=5$ gauged supergravity [4,5] is of particular interest, given its connection with $\mathcal{N}=4$ $U(N)$ super-Yang–Mills theory in $3+1$ dimensions. The bosonic sector of this theory consists of the metric, twenty scalars and fifteen $SO(6)$ Yang–Mills (YM) gauge fields.[1] Solutions of $\mathcal{N}=8$, $D=5$ model have been considered by several authors for various consistent truncations, with subgroups of $SO(6)$ (see *e.g.* [6] and the references therein). However, to our knowledge, to date no attempt has been made to construct non-Abelian superconducting black hole solutions in this context.

This Letter is aimed as a first step in this direction, by taking a consistent truncation of the $\mathcal{N}=8$ model corresponding to $\mathcal{N}=4^+$, $SU(2)\times U(1)$ Romans' gauged supergravity, with a single scalar field ϕ possessing a potential $V(\phi)$ which is the sum of two Liouville terms. The scalar ϕ approaches asymptotically a constant value ϕ_0 corresponding to an extremum of the potential, $dV/d\phi|_{\phi_0}=0$, which yields an effective cosmological constant $\Lambda_{eff}=2V(\phi_0)<0$. It turns out that the basic properties of the $\mathcal{N}=4^+$ solutions with non-Abelian fields are rather similar to those found for pure $D=5$ Einstein–YM–Λ system [8,9]. In particular, we find evidence for the existence, at low temperatures, of a superfluid state with a normalizable non-Abelian condensate.

Since Romans' theory arises as a consistent Kaluza–Klein truncation of the type IIB supergravity on an S^5 [10] and as a consistent compactification of $D=11$ supergravity [11], this shows the existence of holographic superfluids in $D=10,11$ supergravities.

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[1] Note that the field content of the full $\mathcal{N}=8$, $D=5$ gauged supergravity is richer. However, a number of bosonic fields can be consistently set to zero [5].

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2. The $\mathcal{N}=4^+$, $D=5$ Romans' gauged supergravity

The bosonic sector of the $\mathcal{N}=4$, $D=5$ Romans' gauged supergravity [7] consists of gravity, a scalar ϕ, an $SU(2)$ YM potential $A_\mu^{(I)}$ (with field strength $F_{\mu\nu}^{(I)}=\partial_\mu A_\nu^{(I)}-\partial_\nu A_\mu^{(I)}+g_{YM}\epsilon^{IJK}A_\mu^{(J)}A_\nu^{(K)}$ and g_{YM} the $SU(2)$ gauge coupling constant), an Abelian potential B_μ ($f_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu$ being the corresponding field strength), and a pair of two-form fields. These two-form fields can consistently be set to zero, which yields the bosonic part of the action

$$I_{bulk}=\frac{1}{4\pi}\int_{\mathcal{M}} d^5x\sqrt{-g}\bigg(\frac{1}{4}R-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{2a\phi}F_{\mu\nu}^{(I)}F^{(I)\mu\nu}-\frac{1}{4}e^{-4a\phi}f_{\mu\nu}f^{\mu\nu} - \frac{1}{4\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma\tau}F_{\mu\nu}^{(I)}F_{\sigma\tau}^{(I)}B_\tau - V(\phi)\bigg), \tag{2.1}$$

where $a=\sqrt{\frac{2}{3}}$. Here $V(\phi)=-\frac{1}{8}g_{YM}^2(e^{-2a\phi}+2\sqrt{2}\frac{g_M}{g_{YM}}e^{a\phi})$ is the dilaton potential, g_M being the $U(1)$ gauge coupling constant.

As discussed in [7], this theory has three canonical forms, corresponding to different choices of the gauge coupling constant g_M. The case of interest here corresponds to the $\mathcal{N}=4^+$ version, in which $g_M=g_{YM}/\sqrt{2}$ and thus the dilaton potential is

$$V(\phi)=-\frac{1}{8}g_{YM}^2\big(e^{-2a\phi}+2e^{a\phi}\big). \tag{2.2}$$

The field equations are obtained by varying the action (2.1) with respect to the field variables $g_{\mu\nu}$, $A_\mu^{(I)}$, B_μ and ϕ

$$\begin{aligned}
&R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=2T_{\mu\nu},\\
&\nabla^2\phi-\frac{a}{2}e^{2a\phi}F_{\mu\nu}^{(I)}F^{(I)\mu\nu}+ae^{-4a\phi}f_{\mu\nu}f^{\mu\nu}-\frac{\partial V}{\partial\phi}=0,\\
&\partial_\nu\big(e^{-4a\phi}f^{\mu\nu}\big)-\frac{1}{4\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma\tau}F_{\nu\rho}^{(I)}F_{\sigma\tau}^{(I)}=0,\\
&D_\nu\big(e^{2a\phi}F^{(I)\mu\nu}\big)-\frac{1}{2\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma\tau}F_{\nu\rho}^{(I)}f_{\sigma\tau}=0,
\end{aligned} \tag{2.3}$$

where the energy–momentum tensor is defined by

$$\begin{aligned}
T_{\mu\nu}={}&\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}g_{\mu\nu}\partial_\sigma\phi\partial^\sigma\phi-g_{\mu\nu}V(\phi)\\
&+e^{2a\phi}\bigg(F_{\mu\rho}^{(I)}F_{\nu\sigma}^{(I)}g^{\rho\sigma}-\frac{1}{4}g_{\mu\nu}F_{\rho\sigma}^{(I)}F^{(I)\rho\sigma}\bigg)\\
&+e^{-4a\phi}\bigg(f_{\mu\rho}f_{\nu\sigma}g^{\rho\sigma}-\frac{1}{4}g_{\mu\nu}f_{\rho\sigma}f^{\rho\sigma}\bigg).
\end{aligned} \tag{2.4}$$

The scalar potential has exactly one extremum at $\phi=0$, corresponding to the effective cosmological constant $\Lambda_{eff}=-\frac{6}{\ell^2}=2V(0)=-\frac{3}{4}g_{YM}^2$. Then the effective AdS length scale is fixed by the non-Abelian gauge coupling constant, $\ell=2\sqrt{2}/g_{YM}$.

As usual, one supplements (2.1) with a boundary term

$$I_{bound}=-\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^4x\sqrt{-h}K - \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^4x\sqrt{-h}\bigg(\frac{1}{\ell}W(\phi)+\frac{\ell}{4}\mathcal{R}\bigg), \tag{2.5}$$

where apart from the Hawking–Gibbons surface term we include also a counterterm part which is required to regularize the total action and the global charges. In the above relation, $\mathcal{R}$ is the Ricci scalar for the induced metric h of the boundary, K is the trace of the extrinsic curvature, while $W(\phi)=e^{2a\phi}+2e^{-a\phi}$ (this expression of the counterterm was derived in [12], in a more general context).

Then, as in the well known pure-AdS case [13], one can construct a divergence-free boundary stress tensor T_{ij} from the total action $I=I_{bulk}+I_{bound}$ by defining

$$\mathrm{T}_{ij}=\frac{2}{\sqrt{-h}}\frac{\delta I}{\delta h^{ij}}=\frac{1}{8\pi}\bigg(K_{ij}-Kh_{ij}-\frac{1}{\ell}h_{ij}W(\phi)+\frac{\ell}{2}E_{ij}\bigg), \tag{2.6}$$

where E_{ij} is the Einstein tensor of the boundary metric, $K_{ij}=-1/2(\nabla_i n_j+\nabla_j n_i)$ is the extrinsic curvature, with n^i being an outward pointing normal vector to the boundary.

Thus, a conserved charge

$$\mathfrak{Q}_\xi=\oint_\Sigma d^3S^a\,\xi^b\mathrm{T}_{ab} \tag{2.7}$$

can be associated with a surface Σ (with normal n^a), provided the boundary geometry has an isometry generated by a Killing vector ξ^a. For example, if $\xi=\partial/\partial t$ is a timelike Killing vector, then $\mathfrak{Q}_\xi$ is the conserved mass $\mathcal{M}$.

3. The uncondensed phase

3.1. The solutions

We start with a discussion of the basic properties of the Abelian black brane solutions of the $\mathcal{N}=4^+$ Romans' model. They can be found as a particular limit of the black holes obtained in [14] in the so-called STU model. In the general case these black holes possess three different $U(1)$ charges and two independent scalars. After setting one scalar to zero and taking two gauge fields to be equal, one finds after a suitable field redefinition, the following black brane solution of Eqs. (2.3)–(2.4):

$$ds^2=\mathcal{H}(r)^{1/3}\bigg(\frac{dr^2}{f(r)}+r^2\big(dx^2+dy^2+dz^2\big)\bigg) - \mathcal{H}(r)^{-2/3}f(r)\,dt^2, \tag{3.8}$$

with

$$\mathcal{H}(r)=H^2(r)K(r),\qquad H(r)=1+\frac{2Q^2}{Mr^2},$$
$$K(r)=1+\frac{4q^2}{Mr^2},\qquad f(r)=-\frac{M}{r^2}+\frac{1}{8}g_{YM}^2r^2\mathcal{H}(r) \tag{3.9}$$

and the matter fields

$$\phi(r)=\frac{1}{\sqrt{6}}\log\bigg(\frac{H(r)}{K(r)}\bigg),\qquad B=B_t(r)\,dt,\qquad A^{(I)}=A_t(r)\delta^{I3}\,dt,$$
$$\text{with}\quad B_t(r)=\Phi^a-\frac{Mq}{4q^2+Mr^2},\quad A_t(r)=\Phi^A-\frac{MQ}{2Q^2+Mr^2}. \tag{3.10}$$

This solution is written in terms of three parameters (M,Q,q), corresponding (up to some factors) to the global mass and two electric charges.

In what follows, to avoid cluttering our expressions with complicated factors of g_{YM}, we use the observation that the above solution is left invariant by the transformation $r\to\lambda r$, $g_{YM}\to g_{YM}/\lambda$, $(q,Q)\to\lambda(q,Q)$ and $(x,y,z)\to\lambda(x,y,z)$, and we set $g_{YM}=1$ without any loss of generality.

The horizon is located $r=r_H$, with r_H the largest positive root of the equation $f(r)=0$, which reduces to $r_H^6+4(\frac{q^2+Q^2}{M})r_H^4+\frac{4}{M^2}(Q^4+4q^2Q^2-2M^3)r_H^2+\frac{16q^2Q^4}{M^3}=0$. Although one can write

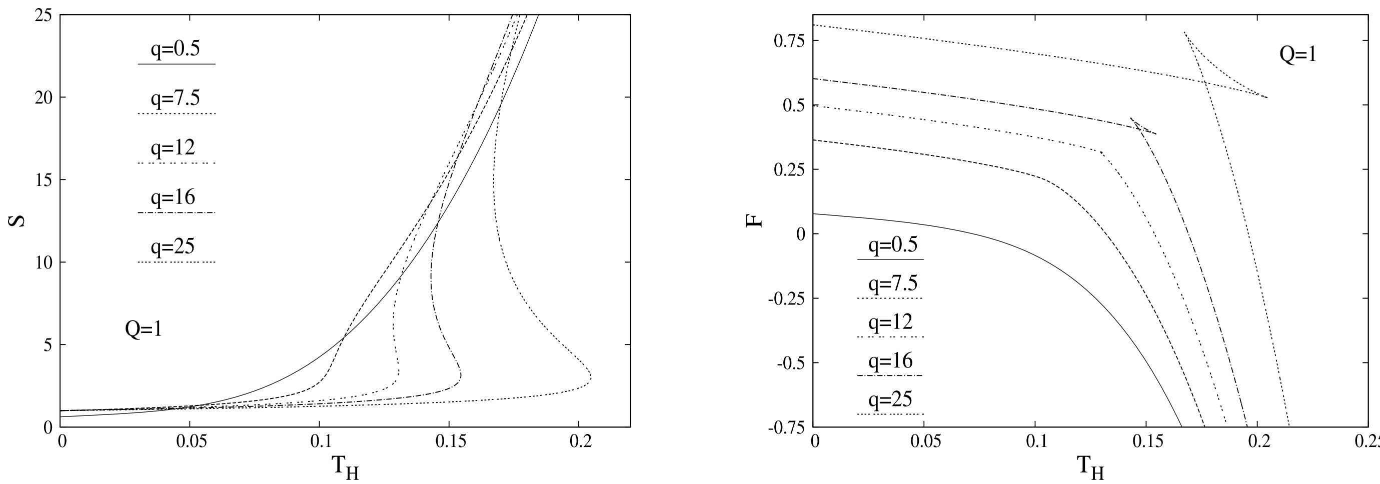

Fig. 1. The entropy and free energy of the Abelian black brane solutions is shown for several values of the electric charges.

an expression for $r_H(M, q, Q)$, it turns out to be more convenient to express q in terms of r_H, Q, M:

$$q = \frac{r_H}{2\sqrt{M}}\sqrt{\frac{8M^3}{(2Q^2+Mr_H^2)^2} - 1}. \tag{3.11}$$

As usual, the constants Φ^A, Φ^a in the expressions of $A_t(r), B_t(r)$ are found by imposing the regularity of the one-forms A, B on the horizon, which implies

$$\Phi^A = \frac{MQ}{2Q^2 + Mr_H^2}, \qquad \Phi^a = \frac{Mq}{4q^2 + Mr_H^2}. \tag{3.12}$$

Thus, physically they correspond to the two chemical potentials associated with the system.

A straightforward computation leads to the following expressions for the mass $\mathcal{M}$, electric charges $\mathcal{Q}_A$ and $\mathcal{Q}_a$ for the $SU(2)$ and $U(1)$ fields, entropy S and Hawking temperature T_H:

$$\mathcal{M} = \frac{3M}{16\pi}\mathcal{V}, \qquad \mathcal{Q}_A = \frac{1}{2\pi}Q\mathcal{V},$$
$$\mathcal{Q}_a = \frac{1}{2\pi}q\mathcal{V}, \qquad S = \frac{1}{2}\sqrt{2M}r_H\mathcal{V},$$
$$T_H = \frac{1}{32\sqrt{2}\pi}\frac{8Q^6+8M^4r_H^2+12MQ^4r_H^2+6M^2Q^2r_H^4+M^3(-16Q^2+r_H^6)}{M^{5/2}r_H(2Q^2+Mr_H^2)},$$

with $\mathcal{V} = \int d^3x$; however, for the rest of this work, to simplify the expressions, we set $\mathcal{V} = 1$, *i.e.* we shall work with mass, entropy and electric charge densities.

A straightforward computation shows that the solutions satisfy the first law of thermodynamics, $d\mathcal{M} = T_H\,dS + \Phi^A\,d\mathcal{Q}_A + \Phi^a\,d\mathcal{Q}_a$, and the Smarr law, $\mathcal{M} = \frac{3}{4}(T_H S + \Phi^A\mathcal{Q}_A + \Phi^a\mathcal{Q}_a)$.

3.2. Thermodynamic properties

These $U(1) \times U(1)$ solutions possess a relatively complicated thermodynamics. Restricting for simplicity to a canonical ensemble, we study black branes holding the temperature T_H, and the charges $\mathcal{Q}_A, \mathcal{Q}_a$ fixed. The associated thermodynamic potential is the Helmholz free energy $F[T_H; \mathcal{Q}_A, \mathcal{Q}_a] = \mathcal{M} - T_H S$. Thermodynamic stability requires the positivity of the specific heat at constant electric charges, $C = T_H(\partial S/\partial T_H)$. A useful relation here is $T_H = \frac{1}{\pi 2^{17/3}}\frac{S^6+(Q^2+q^2)S^4-Q^4q^2}{S^{5/3}((Q^2+S^2)^2(2q^2+S^2))^{2/3}}$, which defines $S(T_H; \mathcal{Q}_A, \mathcal{Q}_a)$, although an explicit formula cannot be written in the general case.

Analytic results are found only when discussing the limiting cases with a vanishing Q or q. The properties of the solutions with $Q = 0$ (*i.e.* a consistent truncation of the model with a $U(1)$ field only, $F_{\mu\nu}^{(I)} = 0$) are discussed at length in [3]. No extremal configurations are found in this case, since the temperature is bounded from below, $T_H^{(min)} > 0$. For any given $T_H > T_H^{(min)}$, there are two branches of solutions, one of them being thermally stable.

By contrast, the solutions with $q = 0$ ($SU(2)$ gauge fields only, $f_{\mu\nu} = 0$), admit an extremal limit which is approached for $M = Q^{4/3}/2^{1/3}$. The entropy of the extremal solutions vanishes, a number of invariant quantities diverging in that limit. For non-extremal configurations one finds a single branch of solutions, with

$$S = 4\pi T_H Q^{2/3}\Bigg[\frac{32\pi^2 T_H^2}{3Q^{2/3}} + \left(1 + \frac{32768\pi^6 T_H^6}{27Q^2} - \sqrt{1 + \frac{65536\pi^6 T_H^6}{27Q^2}}\right)^{1/3} + \left(1 + \frac{32768\pi^6 T_H^6}{27Q^2} + \sqrt{1 + \frac{65536\pi^6 T_H^6}{27Q^2}}\right)^{1/3}\Bigg],$$

which possesses a positive specific heat.

The solutions with two $U(1)$ charges exhibit a complicated picture, which is governed by the value of the relative ratio q/Q. The picture in Fig. 1 appears to be generic: for any $Q \neq 0$, the solutions with small enough q are thermally stable, the entropy increasing with the temperature (note that $S(T_H = 0) \neq 0$, the geometry remaining regular in this limit).

However, when increasing q we notice the occurrence of three branches of solutions for some intermediate range of T_H. The physically relevant branch (which has less free energy) is the third one, which continues to $T_H \to \infty$ (the large temperature behaviour is $S = 128\pi^3 T_H^3 + O(T_H)$). Also, the second branch is unstable since it possesses a negative specific heat.

For a more systematic discussion of the properties of the generic Abelian solutions, it turns out convenient to work with the following scaled quantities[2]

$$q_A = c_1\frac{\mathcal{Q}_A}{\mathcal{M}^{3/4}}, \qquad q_a = c_4\frac{\mathcal{Q}_a}{\mathcal{M}^{3/4}}, \qquad s = c_2\frac{\mathcal{S}}{\mathcal{M}^{3/4}},$$

[2] The $U(1) \times U(1)$ exact solution has an extra scaling symmetry $\mathcal{M} \to \lambda^4\mathcal{M}$, $\mathcal{Q}_k \to \lambda^3\mathcal{Q}_k$, $T_H \to \lambda T_H$, $S \to \lambda^3 S$ and $\Phi^k \to \lambda\Phi^k$, with $k = (a, A)$ and $\lambda > 0$ an arbitrary constant. The quantities in (3.13) are left invariant by this transformation.

$$t_H = c_3 \frac{T_H}{\mathcal{M}^{1/4}}, \qquad \varphi^A = c_5 \frac{\Phi^A}{\mathcal{M}^{1/4}}, \qquad \varphi^a = c_6 \frac{\Phi^a}{\mathcal{M}^{1/4}}, \tag{3.13}$$

with $c_1 = \frac{3^{3/4}}{4}(\frac{\pi}{2})^{1/4}$, $c_2 = \frac{1}{2^{13/4}}(\frac{3}{\pi})^{3/4}$, $c_3 = 4 \times 6^{1/4}\pi^{3/4}$, $c_4 = \frac{3^{3/4}}{2\sqrt{2}}(\frac{\pi}{2})^{1/4}$, $c_5 = (\frac{6}{\pi})^{1/4}$, and $c_6 = 2^{3/4}(\frac{3}{\pi})^{1/4}$ being constant factors which have been chosen such that the expressions below take a simpler form.

All the relevant quantities can then be expressed in terms of q_A and s only:

$$t_H = \frac{(q_A^2+s^2)^3 + s^2 - q_A^2}{s(q_A^2+s^2)}, \qquad q_a = s\sqrt{\frac{1}{(q_A^2+s^2)^2} - 1},$$
$$\varphi^A = \frac{q_A}{q_A^2+s^2}, \qquad \varphi^a = \frac{(q_A^2+s^2)^2}{s}\sqrt{\frac{1}{(q_A^2+s^2)^2} - 1}. \tag{3.14}$$

It is clear that all solutions satisfy the condition

$$s^2 \leqslant 1 - q_A^2,$$

the upper bound being approached for solutions with $SU(2)$ fields only. Moreover, the condition $t_H \geqslant 0$ imposes a lower bound for the reduced entropy:

$$s^2 \geqslant U(q_A) - q_A^2, \tag{3.15}$$

where

$$U(q_A) = \left(\sqrt{\frac{1}{27} + q_A^4} + q_A^2\right)^{1/3} - \left(\sqrt{\frac{1}{27} + q_A^4} - q_A^2\right)^{1/3}. \tag{3.16}$$

One can also show that the scaled $U(1)$ charge q_a has a finite range, with

$$0 \leqslant q_a^2 \leqslant \frac{(1-U^2(q_A))^2}{2U(q_A)}. \tag{3.17}$$

Solutions with a maximal value of q_a correspond to extremal black holes, with $t_H = 0$ and $s^2 = U(q_A) - q_A^2$. From (3.14), the reduced entropy of the extremal solutions can also be written as

$$s^2 = \frac{1}{4}\left(-q_a^2 + q_a\sqrt{q_a^2 + 8q_A^2}\right), \tag{3.18}$$

which is a non-vanishing quantity for $q_A \neq 0$.

4. The superfluid phase

It is clear that the $U(1) \times U(1)$ solutions should possess non-Abelian generalizations. These configurations are found when enlarging the $SU(2)$ ansatz to include a nonzero magnetic potential such that the gauge potential $A^{(I)} = A_t(r)\delta^{I3}\,dt$ is approached only asymptotically.

Following previous works [8,9] on pure Einstein–Yang–Mills (EYM) solutions with vector hair, we choose an $SU(2)$ gauge fields ansatz possessing both electric and magnetic potentials, while the $U(1)$ ansatz is still purely electric:

$$A^{(I)} = w(r)\delta^{I1}\,dx + A_t(r)\delta^{I3}\,dt, \qquad B = B_t(r)\,dt. \tag{4.19}$$

Also, as before, the dilaton field will depend on the r-coordinate only. This leads to a diagonal energy–momentum tensor and thus it is consistent to choose a diagonal metric ansatz.

4.1. Zero modes for the $U(1) \times U(1)$ black brane

Before discussing the general solutions, it is instructive to consider the perturbative limit of the problem. Then $w(r)$ is treated as a small perturbation around the $U(1) \times U(1)$ solutions, $w(r) = \epsilon W(r)$. After substituting into the linearized YM equations, one finds that $W(r)$ solves

$$W'' + \left(\frac{1}{r} - \frac{K'}{K} + \frac{f'}{f}\right)W' + \frac{A_t^2 H^2 K}{f} W = 0. \tag{4.20}$$

For $Q = 0$ one finds the following exact solution of the above equation

$$W(r) = c_0 + c_1\left(\log\left(1 - \left(\frac{r_H}{r}\right)^2\right) - \frac{r_H^8}{64M^2}\log\left(1 + \frac{8M}{r_H^2 r^2}\right)\right) \tag{4.21}$$

(where c_0, c_1 are arbitrary constants). As one can see, this solution possesses an essential logarithmic singularity at the horizon and thus cannot be treated as a perturbation. Thus we conclude that only solutions which are charged with respect to the $SU(2)$ fields may posses an instability.

Although for $Q \neq 0$ Eq. (4.20) does not appear to be solvable in terms of known functions, one can construct approximate solutions near the horizon and at infinity. As $r \to r_H$, the function $W(r)$ behaves as $W(r) = b + O(r - r_H)^2$, while, for large r, the approximate form of $W(r)$ is $W(r) = J/r^2 + O(1/r^4)$, with b and J free parameters. Solutions interpolating between these asymptotics are constructed numerically.[3]

The mechanism triggering the instability is similar to the pure EYM-Λ case [2], the magnetic gauge potential acquiring a tachyonic mass term near the horizon. Interestingly, the picture found for $q = 0$ is rather similar to that valid for $Q = 0$ since in this case too no solutions of (4.20) with correct asymptotics are found. We conclude that, somehow unexpectedly, both electric charges (associated with the $SU(2)$ and $U(1)$ fields) should be non-vanishing for the existence of a normalizable zero mode.

Some results of the numerical integration are shown in Fig. 2. There, the part of the parameter space above the curves corresponds to the unbroken phase, where only Abelian solutions exist. In Fig. 2 (left) we show the critical curve in the (q_A, q_a) plane corresponding to configurations unstable with respect to the non-Abelian perturbations. One can see that the reduced $SU(2)$ charge $q_A = c_1\mathcal{Q}_A/\mathcal{M}^{3/4}$ has a finite range, $0 < q_A < 0.618$, an extremal configuration (with $T_H^{(c)} \to 0$) being approached for the maximal value of q_A and $q_a \to 0.629$ (corresponding to $\Phi^a/\Phi^A \simeq 0.704$). In Fig. 2 (right) we show the critical temperature $T_H^{(c)}$ as a function of the $U(1)$ chemical potential Φ^a (both quantities are normalized *w.r.t.* the $SU(2)$ chemical potential Φ^A); note that the critical temperature is monotonically decreasing as we increase the ratio Φ^a/Φ^A.

4.2. Black holes with non-Abelian hair

4.2.1. The equations and global charges

The instability of the $U(1) \times U(1)$ solution pointed out in the previous section can be viewed as an indication of the existence of a branch of non-Abelian solutions with nontrivial magnetic non-Abelian fields outside the horizon.

[3] In this work we restrict our study to solutions with a monotonic behaviour of $W(r)$.

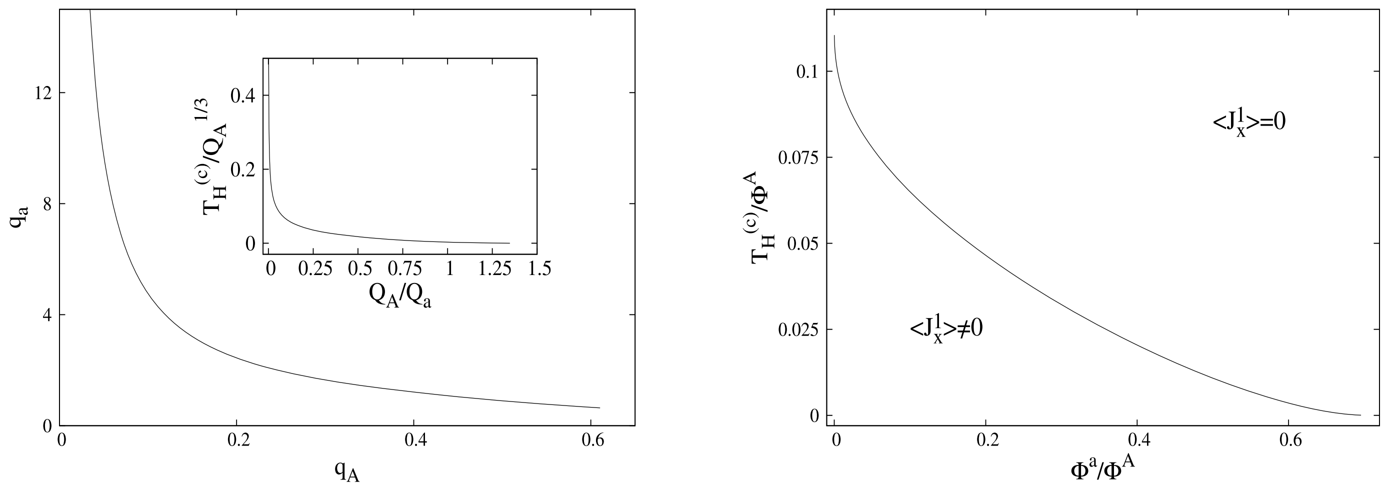

Fig. 2. Critical curves in the parameter space where static linear normalizable non-Abelian perturbations arise.

In the numerical construction of these solutions, we adopt the following metric ansatz,[4] which was first proposed in [8] for the case of the pure EYM-Λ system:

$$ds^2 = \frac{dr^2}{N(r)} + r^2\left(\frac{dx^2}{f^4(r)} + f^2(r)\left(dy^2+dz^2\right)\right) - N(r)\sigma^2(r)\,dt^2,$$
$$\text{with } N(r) = -\frac{4m(r)}{3r^2} + \frac{r^2}{\ell^2}. \tag{4.22}$$

Plugging (4.22) and (4.19) into (2.3)–(2.4) results in the equations of motion[5]:

$$\begin{aligned}
m' &= \frac{3r^3Nf'^2}{2f^2} + \frac{re^{2a\phi}}{2g_{YM}^2}\left(f^4Nw'^2 + \frac{r^2A_t'^2}{\sigma^2} + \frac{f^4A_t^2w^2}{N\sigma^2}\right) \\
&\quad + \frac{e^{-4a\phi}r^3B_t'^2}{2\sigma^2} + \frac{1}{2}r^3N\phi'^2 \\
&\quad + \frac{g_{YM}^2}{8}r^3\left(3g_{YM}^2 - e^{-2a\phi} - 2e^{a\phi}\right), \\
\frac{\sigma'}{\sigma} &= \frac{2r}{3}\left(\frac{3f'^2}{f^2} + \phi'^2 + \frac{1}{g_{YM}^2}\frac{e^{2a\phi}}{r^2}\left(f^4w'^2 + \frac{A_t^2w^2f^4}{N^2\sigma^2}\right)\right), \\
\left(r^3N\sigma\frac{f'}{f^2}\right)' &= \frac{2e^{2a\phi}}{3g_{YM}^2}r\sigma f^3N\left(w'^2 - \frac{A_t^2w^2}{N^2\sigma^2}\right) - r^3N\sigma\frac{f'^2}{f^3}, \\
\left(r^3N\sigma\phi'\right)' &= ar\sigma\left[\frac{e^{2a\phi}}{g_{YM}^2}\left(f^4Nw'^2 - \frac{r^2A_t'^2}{\sigma^2} - \frac{f^4A_t^2w^2}{N\sigma^2}\right)\right. \\
&\qquad \left. + \frac{2e^{-4a\phi}}{\sigma^2}r^2B_t'^2 + \frac{g_{YM}^2}{8}\left(e^{-2a\phi} - e^{a\phi}\right)r^2\right], \\
\left(e^{2a\phi}r^3\frac{A_t'}{\sigma}\right)' &= \frac{e^{2a\phi}rf^4}{N\sigma}w^2A_t, \qquad \left(e^{-4a\phi}r^3\frac{B_t'}{\sigma}\right)' = 0, \\
\left(e^{2a\phi}Nf^4r\sigma w'\right)' &= -\frac{e^{2a\phi}rf^4}{N\sigma}A_t^2w.
\end{aligned} \tag{4.23}$$

These equations possess the following scaling symmetries (invariant functions are not shown)

$$\begin{aligned}
&\text{(i)} \quad \sigma \to \lambda\sigma, \quad A_t \to \lambda A_t, \quad B_t \to \lambda B_t, \\
&\text{(ii)} \quad f \to \lambda f, \quad w \to \frac{w}{\lambda^2}, \\
&\text{(iii)} \quad r \to \lambda r, \quad g_{YM} \to \frac{g_{YM}}{\lambda}, \quad A_t \to \frac{A_t}{\lambda}, \\
&\text{(iv)} \quad r \to \lambda r, \quad m \to \lambda^4 m, \\
&\qquad A_t \to \lambda A_t, \quad B_t \to \lambda B_t, \quad w \to \lambda w,
\end{aligned} \tag{4.24}$$

with $\lambda > 0$ an arbitrary number. The symmetries (i) and (ii) are used to set $\sigma(\infty) = 1$, $f(\infty) = 1$, while (iii) is used to set $g_{YM} = 1$ without any loss of generality, which fixes the AdS length scale, $\ell = 2\sqrt{2}$. Note also that the last equation in (4.23) implies the existence of the first integral

$$B_t' = \frac{2e^{4a\phi}\sigma q}{r^3}, \tag{4.25}$$

with q a constant fixing the $U(1)$ electric charge.

We consider again black branes with a horizon at $r = r_h$, where $N(r_h) = 0$. The non-extremal solutions have the following expansion as $r \to r_H$:

$$\begin{aligned}
&m(r) = \frac{3}{4}\frac{r_H^4}{\ell^2} + O(r - r_H), \qquad \sigma(r) = \sigma_h + O(r - r_H), \\
&f(r) = f_h + O(r - r_H)^2, \qquad \phi(r) = \phi_0 + O(r - r_H), \\
&w(r) = w_h + O(r - r_H)^2, \qquad A_t(r) = V_1(r - r_H) + O(r - r_H)^2, \\
&B_t(r) = v_1(r - r_H) + O(r - r_H)^2,
\end{aligned} \tag{4.26}$$

with the independent parameters $\{\sigma_h, f_h, \phi_0, w_h, v_1, V_1\}$ which fix the coefficients of all higher order terms.

We are interested in solutions approaching the $U(1) \times U(1)$ configurations asymptotically. We assume[6] that, as $r \to \infty$, $w(r)$ vanishes and $\phi(r)$ decays as $1/r^2$. A systematic analysis then reveals the following expansion of the solutions at large r:

$$m(r) = M + O\left(1/r^2\right), \qquad \sigma(r) = 1 - \frac{2}{3}\frac{\alpha^2}{r^2} + O\left(1/r^4\right),$$

---

[4] The line element (3.8) of the $U(1) \times U(1)$ solution can also be written in the form (4.22) (with $f(r) = 1$) by defining a new radial coordinate. However, this results in rather complicated expressions.

[5] One can see that the Chern–Simons term in (2.1) does not contribute to the equations of motion so that the gauge fields do not interact directly.

[6] The generic solutions have a more complicated asymptotic behaviour, with $\omega(r) = \omega_0 - \ell^2\omega_0^3\frac{\log r}{r^2} + \cdots$, $\phi(r) = \frac{\alpha}{r^2} + \beta\frac{\log r}{r^2} + \cdots$, which implies the existence of log terms also in the expression of the metric functions, e.g. $m(r) = M + \frac{1}{16}\beta(\beta - 4\alpha)\log r - \frac{\beta^2}{8}\log^2 r + \frac{3}{2}\omega_0^4\log r + \cdots$.

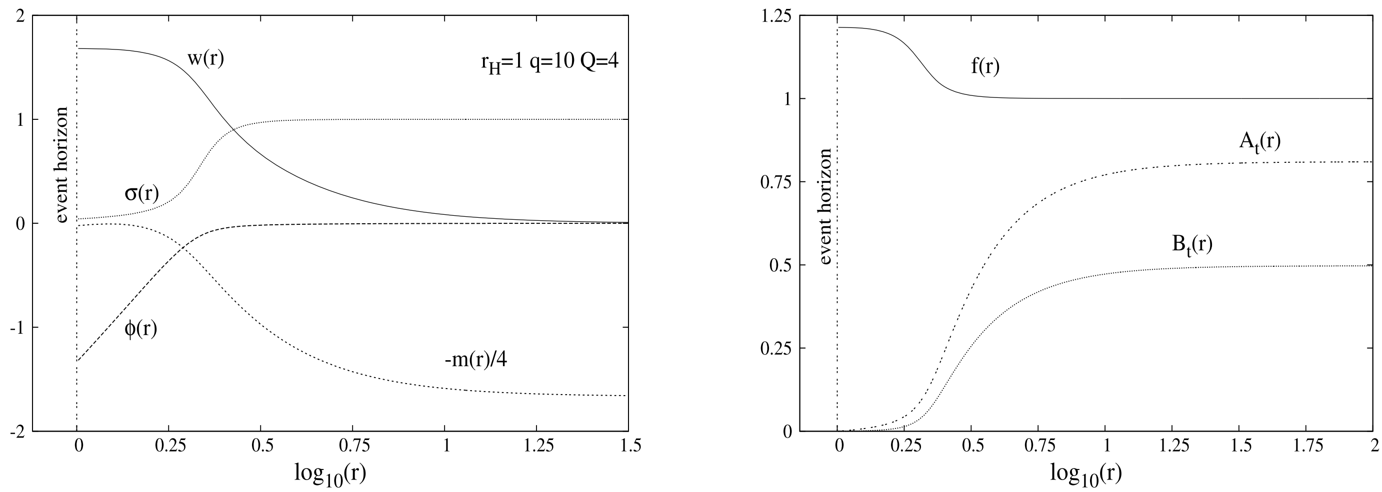

Fig. 3. The profile of a typical non-Abelian solution is shown as a function of the radial coordinate.

$$\phi(r) = \frac{D}{r^2} + O\left(1/r^4\right), \qquad f(r) = 1 + \frac{f_4}{r^4} + O\left(1/r^6\right),$$
$$w(r) = \frac{J}{r^2} + O\left(1/r^4\right), \qquad A_t(r) = \Phi^A - \frac{Q}{r^2},$$
$$B_t(r) = \Phi^a - \frac{q}{r^2}, \tag{4.27}$$

with $\{M, J, Q, q, D, f_4, \Phi^A, \Phi^a\}$ arbitrary coefficients.

All physical quantities are fixed by the data at the horizon and at infinity. As in the Abelian case, the global charges are the mass and the $SU(2)$ and $U(1)$ electric charges, with[7]

$$\mathcal{M} = \frac{1}{4\pi}\left(M + \frac{D^2}{\ell^2}\right), \qquad \mathcal{Q}_A = \frac{1}{2\pi} Q, \qquad \mathcal{Q}_a = \frac{1}{2\pi} q, \tag{4.28}$$

while Φ^A, Φ^a are chemical potentials associated with the two gauge fields. The entropy and Hawking temperature of the solutions are given by

$$S = \frac{1}{4} r_H^3,$$
$$T_H = \frac{\sigma_h}{2\pi}\left[\frac{r_H}{3}\left(\frac{1}{2}\left(2e^{a\phi_0} + e^{-2a\phi_0}\right) - \frac{2e^{2a\phi_0} v_1^2}{\sigma_h^2}\right) - \frac{8e^{4a\phi_0} q^2}{3r_h^5}\right]. \tag{4.29}$$

For completeness, we mention that the boundary stress-tensor T_i^j as defined by (2.6) is diagonal, with the nonzero components:

$$\mathrm{T}_x^x = \frac{1}{8\pi}\frac{2}{3\ell}\left(D^2 - 12 f_4 + M\ell^2\right),$$
$$\mathrm{T}_y^y = \mathrm{T}_z^z = \frac{1}{8\pi}\frac{2}{3\ell}\left(D^2 + 6 f_4 + M\ell^2\right),$$
$$\mathrm{T}_t^t = -\frac{1}{8\pi}\frac{2}{\ell}\left(D^2 + M\ell^2\right), \tag{4.30}$$

such that $\mathrm{T}_i^i = 0$.

4.2.2. Numerical solutions

Eqs. (4.23) with boundary conditions (4.26) and (4.27), respectively, have been solved numerically using a standard shooting method. In addition to using this algorithm, some solutions were also constructed by employing a collocation method for boundary-value ordinary differential equations equipped with an adaptive mesh selection procedure. We have confirmed that there is good agreement between the results obtained with these two different methods.

As expected, some basic properties of these black branes are rather similar to those found in [8,9] in the case of the purely EYM-Λ model. However, the solutions in the present work feature a second control parameter, which is the $U(1)$ electric charge q_a (or equivalently, the chemical potential Φ^a).

For all solutions, the functions $\sigma(r)$ and $A_t(r), B_t(r)$ always increase monotonically with growing r. However, $m(r), f(r), \phi(r)$ and $w(r)$ may feature a more complicated behaviour, with local extrema. For sufficiently small ω_h, all field variables remain close to their values for the Abelian configuration with the same (r_H, Q, q). Significant differences occur for large enough values of ω_h and the effect of the magnetic fields on the geometry becomes increasingly more pronounced. The profiles of a typical solution illustrating these features are presented in Fig. 3.

In the numerical approach, we make use of the existence of the first integral (4.25) to fix the value of the electric charge associated with the $U(1)$ field, which implies $v_1 = 8e^{4a\phi_h} q\sigma_h/r_H^3$ in the near horizon expansion (4.26). The scaling symmetry (iv) in (4.24) is used to set $r_H = 1$, such that the only remaining control parameters are $w(r_H)$ and V_1.

We have studied in a systematic way families of solutions with fixed values of q between 0.5 and 7, the following picture being generic. First, the behaviour of solutions for arbitrary data on the horizon is such that at large r one finds $w \to w_0 \neq 0$ and $\phi(r) \to \log r/r^2$ (in which case the total mass as defined according to (2.7) diverges), or else there is a singularity at finite r. Given $(w_h, q; r_H)$, solutions with the correct asymptotic behaviour[8] are found only for a discrete set of values of (V_1, ϕ_0). Also, all solutions possess a non-vanishing electric Q_A associated with the $SU(2)$ field.

Moreover, for fixed (r_H, q), one finds a branch of non-Abelian solutions for $0 < w_h < w_h^{max}$. Along this branch, the Hawking temperature decreases, an extremal configuration being approached[9] for the maximal value of w_h. The numerical construction of the

[7] Note that, different from the pure EYM-Λ case, the total mass is not given by the asymptotic value of $m(r)$, acquiring a contribution from the scalar field.

[8] These solutions here are also indexed by the node number of the magnetic potential $w(r)$. It turns out that the configurations with nodes represent excited states whose energy is always greater than the energy of the corresponding nodeless configurations, and are therefore ignored in what follows.

[9] Our numerical code usually provided good quality solutions for $T_H \gtrsim T_H^{(c)}/10$.

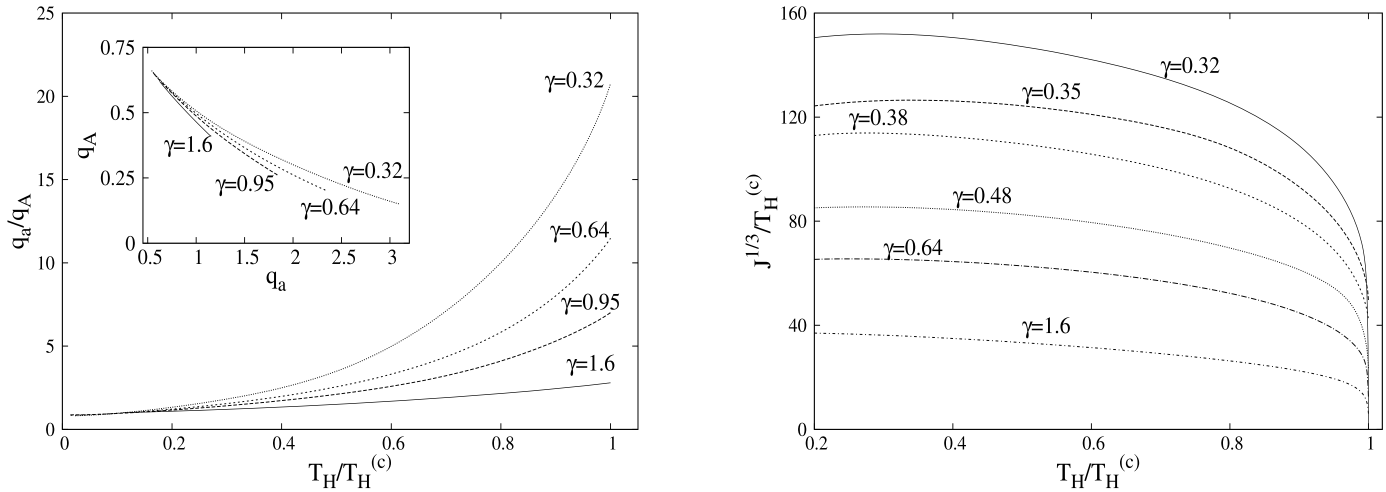

Fig. 4. Various parameters of the non-Abelian solutions are shown for several values of the scale-free ratio $\gamma = \mathcal{Q}_a/S$.

solutions with $T_H = 0$ requires a different metric ansatz than (4.22) and is beyond the scope of this work. However, based on the results in the near-extremal case, we expect the extremal solutions to share the basic properties of their general Abelian counterparts, possessing a regular horizon with non-vanishing entropy.

Some results of the numerical integration are shown in Fig. 4. There we employ scale-free quantities defined in (3.13), which are invariant under the scaling transformation (iv) in (4.24); also, we have found it convenient to define $\gamma = \mathcal{Q}_a/S$ as a second scale-free control parameter. One can see that, for all cases we considered, non-Abelian solutions exist only for values of the Hawking temperature smaller than a critical temperature $T_H^{(c)}$. This is the temperature at which the $U(1) \times U(1)$ solution admits a static linearized perturbation, with non-vanishing but infinitesimally small w. Moreover, the dependence of the order parameter J on the Hawking temperature is similar to that found in the literature for the $\gamma = 0$ case (*i.e.* an EYM-Λ model). Also, as expected, we have found that the difference in the free energy density, F, between a non-Abelian solution and the $U(1) \times U(1)$ solution with the same temperature and electric charges is negative, and thus the non-Abelian solution is thermodynamically favored.

5. Further remarks

In this Letter we have studied electrically charged black branes of the $\mathcal{N} = 4^+$ $SU(2) \times U(1)$ gauged supergravity model with AdS_5 asymptotics. Apart from the Abelian $U(1) \times U(1)$ configurations, we have given numerical evidence that this model possesses also solutions with a non-vanishing magnetic $SU(2)$ fields. Remarkably, these emerge as perturbations of the Abelian configurations at some finite temperature depending on the values of the electric charges, and can be viewed as p-wave superfluids. Moreover, by using the relations in [10,11], one can uplift these configurations to ten dimensional type IIB supergravity and $D = 11$ supergravity. This provides an explicit stringy construction of holographic superfluids.

Our study should be viewed only as a preliminary investigation of the simplest non-Abelian solutions of the $\mathcal{N} = 4^+$ model featuring superfluid properties. Various properties of these black branes remain to be investigated. For example, it would be interesting to compute the conductivity of the solutions or to explore the connection with the unbalanced mixtures discussed recently in [15].

Moreover, we expect the $\mathcal{N} = 4^+$ model to possess a variety of other electrically charged black brane solutions. They would be found for a different (and more complicated) matter field ansatz than (4.19). In particular, the two-form fields which are set to zero in (2.1) may also be present [3]. Similar to the case in this work, we expect the more general solutions to emerge typically as zero modes of the Abelian configurations (3.8)–(3.10), at some critical temperature.

A particularly interesting class of instabilities of the electrically charged solutions of the Romans' model leading to holographic helical p-wave superconductors has been considered in the recent work [18]. The unstable modes studied in that reference are outside the simple ansatz (4.19) (since they possess a dependence of one of the coordinates x, y or z in (3.8)); however, they are still within the truncation (2.1) of the $\mathcal{N} = 4^+$ model. These instabilities are expected to occur at higher temperature than the zero modes discussed in Section 4.1. However, it is likely that the non-Abelian solutions in this work possess as well this kind of spatially modulated instabilities (although the construction of such solutions beyond the linearized level would be a difficult task).

Therefore we conclude that the question of the ultimate ground state of the $\mathcal{N} = 4^+$ Romans' theory is rather intricate, with a variety of possible configurations still to be studied.

We close this work with some remarks on a version of the matter fields Ansatz related to (4.19), which leads to solutions in which the Chern–Simons term enters the dynamics.[10] Interestingly, different from other cases discussed above, these solutions do not emerge as zero modes of the Abelian configurations. To this end, we have considered non-Abelian black branes possessing a purely magnetic $SU(2)$ field, with

$$A^{(I)} = w(r)\big(\delta^{I1}\,dx + \delta^{I2}\,dy + \delta^{I3}\,dz\big), \tag{5.31}$$

and an electric $U(1)$ field, $B = B_t(r)\,dt$. This leads to an isotropic energy–momentum tensor, $T_x^x = T_y^y = T_z^z$, in which case a suitable metric Ansatz is given by (4.22) with $f(r) = 1$. Then (2.3)–(2.4) yield five equations of motion for m, σ, w, B_t and ϕ which were solved numerically. Our results show that the properties of these solutions differ substantially from those found in the

[10] As seen in the case of the pure EYM-Λ model [17], the properties of the spherically symmetric non-Abelian solutions are very different once one switches on a Chern–Simons term in the action.

anisotropic case, discussed in Section 4. First, when treating $w(r)$ as a perturbation around the Abelian solution (which is (3.8)–(3.10) with $Q=0$), the linearized equation can be solved in closed form. However, the solution looks very similar to (4.21) (with a $\log(r-r_H)$ term), with the result that no normalizable zero mode is found. Also, different from the case of anisotropic non-Abelian black branes, we could not find non-perturbative solutions with $w(r)\to 0$ as $r\to\infty$. As a result, the mass of the solutions computed according to the counterterm prescription given in Section 2, diverges.[11] We therefore conclude that these isotropic black brane non-Abelian solutions cannot be interpreted as holographic superfluids.

However, the situation is likely to be different for a more general case featuring an anisotropic $SU(2)$ field and a purely magnetic $U(1)$ field (thus beyond the simple Ansatz (4.19)). Superconducting black brane solutions of this type have been studied recently in a truncation of $\mathcal{N}=4^+$ Romans' model with a vanishing dilaton and an arbitrary Chern–Simons coupling constant [16].

We hope to return elsewhere with a systematic study of these aspects.

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