



# On the topology of $G$ -manifolds with finitely many non-principal orbits

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## ABSTRACT

We study the topology of compact manifolds with a Lie group action for which there are only finitely many non-principal orbits, and describe the possible orbit spaces which can occur. If some non-principal orbit is singular, we show that the Lie group action must have odd cohomogeneity. We pay special attention to manifolds with one and two singular orbits, and construct some infinite families of examples. To illustrate the diversity within some of these families, we also investigate homotopy types.

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## 1. Introduction

In this paper we study  $G$ -manifolds, that is, smooth manifolds admitting a smooth effective action from a Lie group  $G$ . We will further assume that the manifolds and groups we consider are always compact. The local nature of such actions is then determined by the Slice Theorem ([9, p. 32] or [7, Corollary VI.2.4]). In particular, there is a unique maximal orbit type: the principal orbit. Other orbits are either ‘exceptional’, that is, non-principal but having the same dimension as a principal orbit, or singular. Singular orbits have a strictly lower dimension than a principal orbit (see [7, Theorems IV.3.1 and IV.3.2]). In this paper we will study manifolds with Lie group action for which the non-principal orbits are finite in number, and pay special attention to the case where all the non-principal orbits are singular. The motivation for studying this family of manifolds comes primarily from Geometry, which we will now explain.

The most studied families of (Riemannian) manifolds are almost certainly those which display a great deal of symmetry. The homogeneous spaces (equipped with homogeneous metrics) are the most symmetric family of all. These are manifolds admitting a smooth (isometric) Lie group action which is transitive. Put another way, a homogeneous space is a manifold admitting a Lie group action for which the space of orbits consists of a single point.

The next most symmetric families of manifolds are those which admit a smooth action from a compact Lie group for which the dimension of the space of orbits, that is, the cohomogeneity of the  $G$ -manifold, is one. Such manifolds have a simple topological description. For a compact cohomogeneity one manifold, the space of orbits is either a circle or an interval. In the first case, the manifold is just a homogeneous space bundle over the circle, and all orbits are principal. In the second case, there are two non-principal orbits corresponding to the ends of the interval, and the manifold is a union of two disc bundles for which the non-principal orbits form the zero-section. The boundary of each disc bundle (indeed every distance sphere, given an invariant metric) is a principal orbit, and therefore a homogeneous space. The entire manifold

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can be described by a group diagram, involving the main group, the principal isotropy and the two non-principal isotropy subgroups (see [10, Section 1] or [1]).

The topology and geometry of manifolds of cohomogeneity 0 or 1 are for the most part well understood. For instance, by [2] and [11], a compact  $G$ -manifold of cohomogeneity 0 or 1 admits an invariant metric of positive Ricci curvature if and only if its fundamental group is finite. In a previous paper [5], the authors studied the topology and geometry of manifolds of cohomogeneity two, three and four, in the case where the  $G$ -action is asystatic and the singular orbits are fixed points. In [6] we have a construction of positive Ricci curvature metrics on  $G$ -manifolds with finitely many non-principal orbits as studied here.

The present work begins with the observation that a cohomogeneity one manifold has at most finitely many non-principal orbits, namely zero or two. Thus it seems natural to ask about manifolds of higher cohomogeneities which share the feature of having finitely many non-principal orbits. The current paper can therefore be viewed as an outgrowth of the study of cohomogeneity one manifolds.

We will show that the behaviour of cohomogeneity one manifolds is quite different from that of  $G$ -manifolds with finitely many non-principal orbits in higher cohomogeneity, in that the isotropy of the latter manifolds is much more tightly constrained. It is a consequence of this constraint that we are able to describe the possible orbit spaces which arise.

The construction in Definition 1 below is fundamental. In this definition, and throughout this paper, we will use the notation  $\mathbb{P}$  to denote a real, complex or quaternionic projective space, the complex projective space quotient defined in (12), or the quotient of an odd-dimensional sphere by a free linear action of a finite group.

**Definition 1.** For a compact Lie group  $W$ , let  $P_W \rightarrow B$  be a  $W$ -principal bundle over a manifold  $B$  with boundary  $\partial B = \mathbb{P}_1 \cup \dots \cup \mathbb{P}_s$ ,  $s \in \mathbb{N}$ . For  $i = 1, \dots, s$  let  $\alpha_i : L_i \rightarrow W$  be an injective homomorphism from a compact Lie group  $L_i$  which acts freely and linearly on the sphere  $S^{r_i}$  with quotient  $\mathbb{P}_i$ . Also assume that the restriction of  $P_W$  to a boundary component  $\mathbb{P}_i$  is associated to the  $L_i$ -principal bundle  $S^{r_i} \rightarrow \mathbb{P}_i = L_i \backslash S^{r_i}$ ,

$$P_W|_{\mathbb{P}_i} = S^{r_i} \times_{\alpha_i} W = \partial(D^{r_i+1} \times_{\alpha_i} W)$$

where  $S^{r_i} \times_{\alpha_i} W$  (respectively  $D^{r_i+1} \times_{\alpha_i} W$ ) is the quotient of  $S^{r_i} \times W$  (respectively  $D^{r_i+1} \times W$ ) obtained by identifying  $(x, w) \sim (zx, w\alpha_i(z)^{-1})$  for all  $z \in L_i$ ,  $x \in S^{r_i}$  (respectively  $x \in D^{r_i+1}$ ),  $w \in W$ .

We then define a  $W$ -manifold

$$M(P_W, \alpha_1, \dots, \alpha_s) := P_W \cup_{\partial B} \left( \bigcup_{i=1}^s D^{r_i+1} \times_{\alpha_i} W \right) \tag{2}$$

by gluing the principal bundle  $P_W$  with the disc bundles  $D^{r_i+1} \times_{\alpha_i} W$  along their common boundary components.

It is immediate that  $M(P_W, \alpha_1, \dots, \alpha_s)$  is a  $W$ -manifold with  $s$  isolated non-principal orbits, with  $W$  acting freely on the principal orbits. The non-principal orbits are the homogeneous spaces  $W/\text{image}\alpha_i$  and the disc bundles  $D^{r_i+1} \times_{\alpha_i} W$  are  $W$ -invariant tubular neighbourhoods. In Section 2 we will show that all  $G$ -manifolds of cohomogeneity at least 2 with isolated non-principal orbits arise from this construction:

**Theorem 3.** Let  $G$  be a compact Lie group and  $K \subset G$  be a closed subgroup. Let  $N_G K$  be the normaliser of  $K$  in  $G$  and  $W = N_G K/K$  be the Weyl group. Let  $s \in \mathbb{N}$ , and let  $M$  be a  $G$  manifold of cohomogeneity at least 2 with  $s$  non-principal orbits and principal isotropy group  $K$ . Then there is a  $W$ -principal bundle  $P_W \rightarrow B$ , injective homomorphisms  $\alpha_i : L_i \rightarrow W$ , and a manifold  $M(P_W, \alpha_1, \dots, \alpha_s)$  as in Definition 1, for which there is a  $G$ -equivariant diffeomorphism

$$M \cong M(P_W, \alpha_1, \dots, \alpha_s) \times_W G/K.$$

Here, the right hand side is the quotient of  $M(P_W, \alpha_i) \times G/K$  by identifying  $(m, gK) \sim (wm, gKw^{-1})$  for all  $m \in M(P_W, \alpha_i)$ ,  $g \in G$ ,  $w \in W$ .

Furthermore, for each  $i = 1, \dots, s$ , the following hold.

1. The group  $L_i$  is isomorphic to  $U(1)$ ,  $SU(2)$ ,  $N_{SU(2)}U(1)$  or is finite.
2. The bundle  $S^{r_i} \rightarrow \mathbb{P}_i$  is one of the universal bundles  $S^{2k+1} \rightarrow \mathbb{C}P^k$ ,  $S^{4k+3} \rightarrow \mathbb{H}P^k$ ,  $S^{4k+3} \rightarrow \mathbb{X}^k = \mathbb{C}P^{2k+1}/\tau$  with an involution  $\tau$  defined in (12), or a finite covering.

The orbit space is homeomorphic to the space obtained by gluing  $B$  with cones over each boundary component,

$$G \backslash M \approx B \cup_{\partial B} \bigcup_{i=1}^s c\mathbb{P}_i.$$

Note the significance of the Weyl group  $W$  here: the  $G$ -equivariant diffeomorphisms of  $G/K$  are precisely the maps defined by right multiplication by elements of  $W$ .

If one of the non-principal orbits is singular, the corresponding boundary component is even-dimensional. Hence the space of orbits must be odd-dimensional. As this dimension is precisely the cohomogeneity, we deduce:

**Corollary 4.** *If  $M$  is a compact  $G$ -manifold with at least one isolated singular orbit, then the cohomogeneity of the  $G$ -action must be odd.*

We believe that this class of manifolds is both rich and interesting, and worthy of further study from both a topological and a geometric perspective. To illustrate this richness, we construct several infinite families of examples whose non-principal orbits are certain Aloff–Wallach spaces. Firstly, we study manifolds with precisely two singular orbits, as this is the situation which most closely resembles cohomogeneity one. The Aloff–Wallach spaces are a 2-parameter family of 7-dimensional homogeneous  $SU(3)$ -manifolds, whose construction and basic properties we review in Section 3.1. These spaces are a particularly important family in Riemannian geometry, as almost all admit homogeneous metrics with positive sectional curvature (see [23, p. 82]). Our results include the following theorem:

**Theorem 5.** *Given any two Aloff–Wallach spaces  $W_{p_1, p_2}$  and  $W_{q_1, q_2}$ , there is an 11-dimensional  $SU(3)$ -manifold  $M_{p_1 p_2 q_1 q_2}^{11}$  of cohomogeneity three with two singular orbits equal to the given Aloff–Wallach spaces.*

To show the diversity within this family we also prove:

**Theorem 6.** *From within the family  $M_{p_1 p_2 q_1 q_2}^{11}$ , there is an infinite sequence of pairwise non-homotopy equivalent manifolds each of which has two non-homotopy equivalent isolated singular orbits. There is also an infinite sequence of pairwise non-homotopy equivalent ‘doubles’, that is, manifolds with two identical singular orbits.*

Unlike cohomogeneity one, it is possible to have manifolds in higher cohomogeneities which have more than two, or precisely one non-principal orbit. Given that there are plentiful examples of manifolds with precisely one and two non-principal orbits, this suggests the question of which numbers of non-principal orbits are possible.

**Theorem 7.** *For each  $m, c \in \mathbb{N}$ ,  $c \equiv 0 \pmod{2}$ , given compact Lie groups  $G \supset K$  with  $U(1) \subset N_G K/K$ , there is a  $G$ -manifold of cohomogeneity  $c$  with precisely  $m$  exceptional orbits.*

*For each  $m, c \in \mathbb{N}$ ,  $c \equiv 3 \pmod{4}$ , given compact Lie groups  $G \supset K$  with  $SU(2) \subset N_G K/K$ , there is a  $G$ -manifold of cohomogeneity  $c$  with precisely  $m$  singular orbits.*

*For each  $m, c \in \mathbb{N}$ ,  $c \equiv 1 \pmod{4}$ ,  $m \equiv 0 \pmod{2}$ , given compact Lie groups  $G \supset K$  with  $U(1) \subset N_G K/K$ , there is a  $G$ -manifold of cohomogeneity  $c$  with precisely  $m$  singular orbits.*

Taking  $G = SU(n)$  with  $n \geq 2$  and  $K$  trivial in the above theorem, we deduce:

**Example 8.** For any  $m, c, n \in \mathbb{N}$  with  $c \equiv 3 \pmod{4}$  and  $n \geq 2$ , there is an  $SU(n)$ -manifold of cohomogeneity  $c$  with precisely  $m$  singular orbits.

This paper is laid out as follows. In Section 2 we prove Theorem 3 and establish the basic structure of compact  $G$ -manifolds with finitely many singular orbits, focusing on the space of orbits and on issues which arise when we try to construct examples. In Section 3 we study manifolds with two singular orbits and construct two explicit infinite families in Theorems 5 and 26. We show that each of these families contains infinitely many homotopy types. In Section 4 we study the existence of unique singular orbits and construct the manifolds for Theorem 7. We also list the possible non-principal orbits of  $G$ -manifolds with one non-principal orbit and cohomogeneity from 2 to 7.

## 2. The structure of $G$ -spaces with finitely many non-principal orbits

Let  $M$  be a compact smooth  $G$ -manifold,  $G$  a compact Lie group and  $K \subset G$  the principal isotropy group. We assume that all non-principal orbits of the  $G$ -action are isolated, in the sense that within any suitably small  $G$ -invariant tubular neighbourhood of a non-principal orbit, all other orbits are principal.

Let  $q \in M$  be such that  $Gq$  is a non-principal orbit and let  $H = G_q$  be the isotropy at  $q$ . The restriction  $TM|_{Gq}$  of the tangent bundle  $TM$  to  $Gq$  contains the tangent bundle  $TGq$  as a subbundle. By Theorem VI.2.1 in [7] we can endow  $M$  with a  $G$  invariant Riemannian metric. The normal bundle of  $Gq$  in  $M$  is then the orthogonal complement of  $TGq$  in  $TM|_{Gq}$ , i.e. its fibre  $\nu_x(Gq, M)$  at a point  $x \in Gq$  is the orthogonal complement of  $T_x Gq$  in  $T_x M$  with respect to the scalar product on  $T_x M$  coming from the Riemannian metric (see [7, Chapter VI.1]).

Now since the group  $H$  fixes  $q$  and acts on  $Gq$ , we have an isometric linear action  $H \rightarrow O(\nu_q(Gq, M))$  of  $H$  on the vector space  $\nu_q(Gq, M)$ , the slice representation. In particular  $H$  acts on the normal sphere

$$S\nu_q(Gq, M) := \{v \in \nu_q(Gq, M) \mid \|v\| = 1\}$$

where  $\|v\|$  denotes the norm of a vector  $v \in \nu_q(Gq, M) \subset T_qM$  induced from the Riemannian metric on  $M$ . If  $r + 1$  is the codimension of  $Gq$  in  $M$ , then there is a linear isometry  $\mathbb{R}^{r+1} \cong \nu_q(Gq, M)$  mapping the standard sphere  $S^r$  to  $S\nu_q(Gq, M)$ . By means of this isometry we obtain an orthogonal representation  $H \rightarrow O(r + 1)$ .

The crucial observation is that in our situation all  $H$  orbits on  $S\nu_q(Gq, M) \cong S^r$  have the same type.

2.1. Lie group actions on spheres with only one orbit type

Group actions with only one orbit type on a sphere are rather restricted.

**Theorem 9.** ([7, p. 192, Thm. 6.2]) *Let  $L$  be a compact Lie group of positive dimension acting locally smoothly, effectively and with one orbit type on  $S^n$ . Then  $L$  acts transitively or freely. If  $L$  acts freely, we must have  $L \cong U(1)$ ,  $N_{SU(2)}U(1)$  or  $SU(2)$ . (If  $\dim L = 0$  then  $S^n \rightarrow L \backslash S^n$  is the universal covering, so  $L$  must also act freely.)*

A non-principal isotropy group  $H$  acts transitively on the normal sphere  $S^r \cong S\nu_q(Gq, M)$  if and only if the cohomogeneity of the  $G$ -action on  $M$  is one. As this situation is well understood, let us assume that  $G$  acts with cohomogeneity  $\geq 2$ . Under this assumption, Theorem 9 has an immediate corollary for our situation.

**Corollary 10.** *Let  $H \rightarrow O(r + 1)$  be a representation of a compact Lie group  $H$  with only one orbit type on  $S^r$ . Assume that  $H$  does not act transitively on the sphere  $S^r$ . Then the kernel of this action coincides with the isotropy group  $K \subset H$  (so  $K \triangleleft H$ ) and we have one of the following cases:*

1.  $r = 2k + 1$  and the action is via complex multiplication,  $H/K \cong U(1)$  and the quotient space  $H \backslash S^r$  is  $\mathbb{C}P^k = S^{2k+1}/U(1)$ ;
2.  $r = 4k + 3$  and the action is via complex multiplication,  $H/K \cong N_{SU(2)}U(1)$  the normaliser of the maximal torus in  $SU(2)$ , and the quotient space  $N \backslash S^r$  is

$$\mathbb{X}^{2k+1} := \mathbb{C}P^{2k+1}/\mathbb{Z}_2 = \mathbb{C}P^{2k+1}/\tau = S^{4k+3}/N_{SU(2)}U(1) \tag{11}$$

where  $\tau$  is the involution of  $\mathbb{C}P^{2k+1}$  given by

$$\tau([z_1 : z_2 : z_3 : z_4 \cdots]) = [-\bar{z}_2 : \bar{z}_1 : -\bar{z}_4 : \bar{z}_3 \cdots]; \tag{12}$$

3.  $r = 4k + 3$  and the action is via quaternionic multiplication,  $H/K \cong SU(2)$  and the quotient space is  $\mathbb{H}P^k = S^{4k+3}/SU(2)$ ;
4.  $H/K$  is finite and acts freely on  $S^r$ . If  $r$  is even, the quotient must be  $\mathbb{R}P^r$ .

**Proof.** By Theorem 9 we have  $L := H/K \cong U(1)$ ,  $N_{SU(2)}U(1)$ ,  $SU(2)$  or finite, and  $L$  acts freely on  $S^r \subset \mathbb{R}^{r+1}$ . We will show that  $\mathbb{R}^{r+1}$  is the sum of standard representations of  $L$  in the first three cases, and therefore the possible actions that occur must be as listed in 1, 2 and 3. Let  $V \subset \mathbb{R}^{r+1}$  be an irreducible submodule for  $L$ . Since  $L$  acts freely on  $S^r$  it must also act freely on the sphere  $S(V)$  of  $V$ .

In the first case, the real irreducible representations of  $L = U(1) = SO(2)$  are equivalent to  $\mathbb{R}$  with the trivial action, or to  $\mathbb{R}^2 = \mathbb{C}$  with  $z \in U(1)$  acting by complex multiplication with  $z^k$ , for some  $k \in \mathbb{N}$ , which is effective only in the case  $k = 1$ . Thus we must have  $V$  equivalent to  $\mathbb{C}$  with  $U(1)$  acting by complex multiplication.

In the second case,  $U(1) \triangleleft N_{SU(2)}U(1)$  acts freely on  $S(V)$  and as above, we must have  $V \cong \mathbb{C}^s$  with  $U(1)$  acting by complex multiplication. Now,  $N_{SU(2)}U(1)$  is generated by  $U(1)$  (seen as diagonal matrices in  $SU(2)$ ) and an element  $\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  subject to the relations  $\tau^2 = -1 \in U(1)$  and  $z\tau = \tau z^{-1}$  for all  $z \in U(1)$ . If  $x \in V$ ,  $x \neq 0$ , we must have  $\tau x \notin \text{span}\{x\}$ . By irreducibility,  $V = \text{span}\{x, \tau x\} \cong \mathbb{C}^2 = \mathbb{H}$  with the standard action of  $N_{SU(2)}U(1)$ .

In the third case, by the classification of the irreducible complex modules for  $SU(2)$  in [3, Propositions II, 5.1 and 5.2], the only irreducible complex representation of  $SU(2)$  with free action on the sphere is the standard one, i.e. the one with  $A \in SU(2) = S(\mathbb{H})$  acting on  $\mathbb{C}^2 = \mathbb{H}$  by quaternionic multiplication. It follows that  $V \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{H}^{2s}$ . By the equivalence between real modules and complex modules with structure homomorphism as in [3, p. 94],  $V$  is isomorphic to the  $+1$ -eigenspace of the structure homomorphism of  $\mathbb{H}^{2s}$  (which is  $SU(2)$ -equivariant). Since  $V$  is irreducible, and the irreducible submodules of the standard representation of  $SU(2)$  on  $\mathbb{H}^{2s}$  are all isomorphic to  $\mathbb{H}$ , we must have  $V \cong \mathbb{H}$  as an  $SU(2)$ -module.

In the last case,  $L = H/K \subset O(r + 1)$  is finite and acts freely on  $S^r$ . If  $\gamma \in L \setminus \{1\}$  then  $\gamma$  cannot have  $+1$  as an eigenvalue. If  $r = 2k$  is even, then  $\gamma \in O(2k + 1)$  must have  $-1$  as an eigenvalue, as can be seen from writing the matrix in canonical block diagonal form, given that  $1$  is not an eigenvalue. In this case  $\gamma^2$  has an eigenvalue  $+1$  and therefore  $\gamma^2 = \text{id}_{S^{2k}}$ , since  $\gamma^2 \in L$  and  $\gamma^2 \notin L \setminus \{1\}$  by the above. It follows that  $\gamma = -\text{id}_{S^{2k}}$ .  $\square$

From this the quotient  $\mathbb{P} = H \setminus S^r$  is a projective space or a discrete quotient. We have the following possibilities for  $\mathbb{P}$ :

dim $\mathbb{P}$	singular	exceptional
$4k$	$\mathbb{H}\mathbb{P}^k, \mathbb{C}\mathbb{P}^{2k}$	$\mathbb{R}\mathbb{P}^{4k}$
$4k + 2$	$\mathbb{C}\mathbb{P}^{2k+1}, \mathbb{X}^{2k+1}$	$\mathbb{R}\mathbb{P}^{4k+2}$
$2k + 1$	none	$L \setminus S^{2k+1}$

(13)

We can now prove Theorem 3.

**Proof of Theorem 3.** Let  $Gq_i, i = 1, \dots, s$  be the non-principal orbits and let  $H_i = G_{q_i}$ . Since the principal isotropy group lies in suitable conjugates of any isotropy group [7, Theorem IV.3.1], there are  $x_i \in G$  such that  $K \subset x_i H_i x_i^{-1} = G_{x_i q_i}$ . Replacing  $q_i$  by  $x_i q_i$  and  $H_i$  by  $x_i H_i x_i^{-1}$ , we can therefore assume that the  $K \subset H_i = G_{q_i}$ .

By the Slice Theorem ([9, p. 32] or [7, Corollary VI.2.4]) the normal bundle of the orbit  $G_{q_i}$  is associated to the principal  $H_i$ -bundle  $G \rightarrow G_{q_i}$ ,

$$\nu(G_{q_i}, M) \cong \nu_{q_i}(G_{q_i}, M) \times_H G.$$

Furthermore, the non-principal orbit  $G_{q_i}$  has an invariant tubular neighbourhood

$$N_i \cong D\nu(G_{q_i}, M) \cong D\nu_{q_i}(G_{q_i}, M) \times_H G$$

equivariantly diffeomorphic to the disc bundle of the normal bundle. Since the non-principal orbits are isolated, all  $G$ -orbits in  $N_i \setminus G_{q_i}$  are principal. It follows that all  $H_i$ -orbits on the normal sphere  $S\nu_{q_i}(G_{q_i}, M)$  are principal and equivariantly diffeomorphic to  $H_i/K$ . In particular  $K$  contains the kernel of the slice representation,

$$\ker[H_i \rightarrow O(\nu_{q_i}(G_{q_i}, M))] \subset K \subset H_i.$$

Since by Theorem 9 the quotient  $H_i/\ker[H_i \rightarrow O(\nu_{q_i}(G_{q_i}, M))]$  acts freely on  $S^{r_i}$ , we must have

$$K = \ker[H_i \rightarrow O(\nu_{q_i}(G_{q_i}, M))] \triangleleft H_i \subset N_G K.$$

Define the Weyl group  $W := N_G K/K$ .

We first study the structure of the manifold away from the non-principal orbits. Let  $N_i$  be  $G$ -invariant tubular neighbourhoods of the singular orbits as before and  $M^0 = M \setminus \bigcup_{i=1, \dots, s} N_i$ . The quotient  $B = G \setminus M^0$  is a manifold whose boundary is a disjoint union of sphere quotients  $\mathbb{P}_i$  as listed in (13). The  $G$ -invariant self-diffeomorphisms of  $G/K$  are precisely those maps defined by right multiplication by elements of  $W$ . As a consequence,  $M^0$  is the total space of a  $G/K$ -bundle over  $B$  with a global  $G$ -action and structure group  $W$ . In other words we have

$$M^0 = P_W \times_W G/K$$

for some  $W$ -principal bundle  $P_W \rightarrow B$ .

We next study the structure of non-principal orbit neighbourhoods. From Corollary 10 we have isomorphisms

$$\alpha_i : L_i \rightarrow H_i/K,$$

where  $L_i = U(1), N_{SU(2)}U(1), SU(2)$  or a finite subgroup of  $O(r_i + 1)$ . Therefore the boundaries  $T_i$  of the  $G$ -invariant tubular neighbourhoods  $N_i$  are  $G$ -equivariantly diffeomorphic to  $G/K$ -bundles associated to the standard  $L_i$ -bundle  $S^{r_i} \rightarrow \mathbb{P}_i$ , where the action of  $L_i$  on  $G/K$  is given by  $(z, gK) \mapsto gK\alpha_i(z^{-1})$  for  $z \in L_i$ . We will write this as

$$T_i \cong S^{r_i} \times_{\alpha_i} G/K.$$

Similarly, we can write

$$N_i \cong D^{r_i+1} \times_{\alpha_i} G/K. \tag{14}$$

(Equivalently, we could view  $T_i$  and  $N_i$  as being the  $S^{r_i}$ -bundle (respectively  $D^{r_i+1}$ -bundle) associated via  $\alpha_i^{-1}$  to the  $H_i/K$ -bundle  $G/K \rightarrow G/H_i = G_{q_i}$ .)

It follows from (14) that the space of orbits in  $N_i, G \setminus N_i$  is simply the cone  $c\mathbb{P}_i$ , and thus  $G \setminus M = B \cup_{\partial B} \bigcup_{i=1}^s c\mathbb{P}_i$ .

Composing the isomorphism  $\alpha_i$  with the inclusion  $H_i/K \hookrightarrow W$  gives an injection  $L_i \rightarrow W$ , which by abuse of notation we will also call  $\alpha_i$ . Using this new map we can re-express  $T_i$  as

$$T_i \cong (S^{r_i} \times_{\alpha_i} W) \times_W G/K,$$

that is, the  $G/K$ -bundle associated to the  $W$ -principal bundle  $S^{r_i} \times_{\alpha_i} W$ , where as before, the action of  $W$  on  $G/K$  is given by right multiplication. In particular, this means that

$$P_W|_{\mathbb{P}_i} \cong S^{r_i} \times_{\alpha_i} W.$$

We similarly have

$$N_i \cong (D^{r_i+1} \times_{\alpha_i} W) \times_W G/K.$$

Finally, we are now in a position to deduce the global structure of  $M$ . Combining the last two observations we obtain

$$M \cong M(P_W, \alpha_1, \dots, \alpha_s) \times_W G/K,$$

where

$$M(P_W, \alpha_1, \dots, \alpha_s) := P_W \cup_{\partial B} \left( \bigcup_{i=1}^s D^{r_i+1} \times_{\alpha_i} W \right).$$

The  $W$ -manifold  $M(P_W, \alpha_1, \dots, \alpha_s)$  has the same orbit space as the  $G$ -manifold  $M$ , and  $W$  acts freely away from the non-principal orbits.  $\square$

### 2.2. Prescribing the non-principal orbits—cobordism

The question of which non-principal orbits can occur for  $G$ -manifolds with principal isotropy group  $K$  and cohomogeneity  $k$  can now in principle be decided by calculations in the non-oriented cobordism group of maps into the classifying space  $BW$  for  $W = N_G K/K$ -bundles. General references for cobordism are [4,19] or [21] for instance.

Recall that two maps  $f_i : M_i \rightarrow Y$ ,  $i = 0, 1$  from  $n$ -manifolds  $M_i$  to a topological space  $Y$  are bordant if there is a map  $F : W \rightarrow Y$  defined on an  $(n + 1)$ -manifold  $W$  with boundary  $\partial W = M_0 \amalg M_1$  which extends  $f_0 \amalg f_1$ . The cobordism group  $\mathfrak{N}_n(Y)$  of  $n$ -manifolds in  $Y$  is the set of such bordism classes with group structure defined by disjoint union. These groups are well understood. We have isomorphisms

$$\mathfrak{N}_n(Y) \cong \bigoplus_{i=0}^n H_i(Y; \mathbb{Z}_2) \otimes \mathfrak{N}_{n-i}(\ast)$$

where  $\mathfrak{N}_j(\ast)$  is the bordism group of  $j$ -dimensional manifolds (see the second theorem on p. 107 of [19]). Also we have that  $f_i : M_i \rightarrow Y$ ,  $i = 0, 1$ , are bordant if and only if all twisted Stiefel–Whitney numbers coincide, i.e. if

$$w_I(M_0) f_0^*(y)[M_0] = w_I(M_1) f_1^*(y)[M_1] \tag{15}$$

for all  $y \in H^k(Y; \mathbb{Z}_2)$  and all partitions  $I$  of  $n - k$  (see the second corollary on p. 108 of [19]).

Let  $t_i : \mathbb{P}_i \rightarrow BL_i$  be the classifying map of the standard  $L_i$ -bundle  $S^i \rightarrow \mathbb{P}_i$ . Then given embeddings  $\alpha_i : L_i \hookrightarrow W$ ,  $i = 1, \dots, s$ , a manifold  $M(P_W, \alpha_1, \dots, \alpha_s)$  exists if the map

$$f_{\alpha_1, \dots, \alpha_s} : \prod_i \mathbb{P}_i \xrightarrow{\prod_i t_i} \prod_i BL_i \xrightarrow{\prod_i B\alpha_i} BW \tag{16}$$

is a boundary, that is, if all its twisted Stiefel–Whitney numbers vanish.

A necessary condition for (16) to be a boundary is of course that  $\prod_i \mathbb{P}_i$  be a boundary. By [16, Lemma 5],  $\mathbb{H}\mathbb{P}^n$  is non-oriented cobordant to  $\mathbb{C}\mathbb{P}^n \times \mathbb{C}\mathbb{P}^n$  and by [8, proof of Theorem 22.3], or [20],  $\mathbb{C}\mathbb{P}^n$  is non-oriented cobordant to  $\mathbb{R}\mathbb{P}^n \times \mathbb{R}\mathbb{P}^n$ . Now  $\mathbb{R}\mathbb{P}^{2k+1}$  is naturally homeomorphic to the (real) projectivisation of the universal bundle over  $\mathbb{C}\mathbb{P}^k$ , hence it is a bundle with fibre  $\mathbb{R}\mathbb{P}^1 = S^1$  and bounds the corresponding disc bundle. Similarly,  $\mathbb{C}\mathbb{P}^{2k+1}$  is naturally homeomorphic to the (complex) projectivisation of the universal bundle over  $\mathbb{H}\mathbb{P}^k$ , and therefore a bundle with fibre  $\mathbb{C}\mathbb{P}^1 = S^2$  bounding the corresponding disc bundle of a 3-dimensional vector bundle  $E_k \rightarrow \mathbb{H}\mathbb{P}^k$ . Finally,  $\mathbb{X}^{2k+1}$  is the (real) projectivisation of  $E_k$ , hence an  $\mathbb{R}\mathbb{P}^2$ -bundle over  $\mathbb{H}\mathbb{P}^k$ . By the Leray–Hirsch Theorem and the corresponding definition of Stiefel–Whitney classes (see [12, Theorem 2.5 and Definition 2.6 on p. 248]), its  $\mathbb{Z}_2$ -cohomology ring is

$$H^*(\mathbb{X}^{2k+1}, \mathbb{Z}_2) = H^*(\mathbb{H}\mathbb{P}^k, \mathbb{Z}_2)[x] / \langle w_3(E_k) + w_2(E_k)x + w_1(E_k)x^2 + x^3 \rangle$$

where  $x$  corresponds to the generator of  $H^*(\mathbb{R}\mathbb{P}^2, \mathbb{Z}_2)$  and  $w_i(E_k)$  are the Stiefel–Whitney classes of  $E_k$ . Since  $H^*(\mathbb{H}\mathbb{P}^k, \mathbb{Z}_2) = \mathbb{Z}_2[u]/u^{k+1}$  with  $u$  of degree 4, we have

$$H^*(\mathbb{X}^{2k+1}, \mathbb{Z}_2) = \mathbb{Z}_2[x, u] / \langle x^3, u^{k+1} \rangle, \quad \deg u = 4, \quad \deg x = 1$$



which is isomorphic to that of the product  $\mathbb{H}\mathbb{P}^k \times \mathbb{R}\mathbb{P}^2$ . The Stiefel–Whitney classes of the tangent bundle  $T_F$  along the fibre of  $\pi : \mathbb{X}^{2k+1} \rightarrow \mathbb{H}\mathbb{P}^k$  restrict to the Stiefel–Whitney classes of (the tangent bundle of)  $\mathbb{R}\mathbb{P}^2$ , i.e. to  $w(T_F) = (1+x)^3 = 1+x+x^2$ . Now  $T\mathbb{X}^{2k+1} = T_F \oplus \pi^*T\mathbb{H}\mathbb{P}^k$  has Stiefel–Whitney classes

$$w(\mathbb{X}^{2k+1}) = w(T_F) \cup w(\pi^*T\mathbb{H}\mathbb{P}^k) = (1+x)^3(1+u)^{k+1}.$$

Thus there is an isomorphism  $H^*(\mathbb{X}^{2k+1}, \mathbb{Z}_2) \cong H^*(\mathbb{H}\mathbb{P}^k \times \mathbb{R}\mathbb{P}^2, \mathbb{Z}_2)$  preserving the Stiefel–Whitney classes. It follows that  $\mathbb{X}^{2k+1}$  and  $\mathbb{H}\mathbb{P}^k \times \mathbb{R}\mathbb{P}^2$  have the same Stiefel–Whitney numbers and are therefore non-oriented cobordant. Since the even-dimensional real projective spaces  $\mathbb{R}\mathbb{P}^{2k}$  are not boundaries, among the above, precisely  $\mathbb{H}\mathbb{P}^{2k+1}$ ,  $\mathbb{C}\mathbb{P}^{2k+1}$ ,  $\mathbb{X}^{4k+3}$  and  $\mathbb{R}\mathbb{P}^{2k+1}$  are boundaries and can appear in  $G$ -manifolds with precisely one non-principal orbit.

### 3. Manifolds with precisely two singular orbits

As noted in the Introduction, studying manifolds in higher cohomogeneities with precisely two singular orbits is of interest as it can be viewed as a direct generalisation of the most interesting case in cohomogeneity one.

As before, let  $M$  denote the manifold. Removing a small tubular neighbourhood of the two singular orbits gives a manifold with two boundary components, which is the total space of a  $G/K$ -bundle over some manifold  $B$ . The manifold  $B$  also has two boundary components  $\partial B = \mathbb{P}_1 \cup \mathbb{P}_2$  with  $\mathbb{P}_1, \mathbb{P}_2$  as in (13). Computing the Stiefel–Whitney numbers shows that there are the following possibilities for  $\partial B$ , up to interchanging the components:

$$\mathbb{P}_1 = \mathbb{P}_2, \quad L_1 = L_2, \quad (17)$$

$$\partial B = \mathbb{C}\mathbb{P}^{2k+1} \cup \mathbb{X}^{2k+1}, \quad k \text{ odd}, \quad L_1 = S^1, \quad L_2 = N_{\text{SU}(2)}\text{U}(1), \quad (18)$$

$$\partial B = \mathbb{R}\mathbb{P}^2 \cup \mathbb{R}\mathbb{P}^2, \quad L_1 = N_{\text{SU}(2)}\text{U}(1), \quad L_2 = \mathbb{Z}_2, \quad (19)$$

$$\partial B = L_1 \setminus S^{2k+1} \cup L_2 \setminus S^{2k+1}, \quad L_1, L_2 \subset O(2k+2) \text{ discrete.} \quad (20)$$

The third case (19) is the only possibility of mixing an isolated singular with an isolated exceptional orbit due to the coincidence  $\mathbb{X}^1 = \mathbb{R}\mathbb{P}^2$ . A simple example in the fourth case (20) is the join

$$M = S^3 = S^1 * S^1 = [0, 1] \times S^1 \times S^1 / \sim$$

where we identify  $(0, z_0, z_1) \sim (0, z'_0, z_1)$  and  $(1, z_0, z_1) \sim (1, z_0, z'_1)$  for all  $z_0, z'_0, z_1, z'_1 \in S^1$ . For  $m, n \in \mathbb{Z}$ ,  $m, n \geq 2$ , coprime, let  $G = \mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ . We define a  $G$ -action on  $M$  by

$$([k], (t, z_0, z_1)) \mapsto (t, e^{2\pi i \frac{k}{m}} z_0, e^{2\pi i \frac{k}{n}} z_1), \quad k \in \mathbb{Z}, t \in [0, 1], z_0, z_1 \in S^1.$$

Now  $G$  has two exceptional orbits on  $M$  corresponding to  $t = 0$  respectively  $t = 1$ , with isotropy groups isomorphic to  $\mathbb{Z}_m$  respectively  $\mathbb{Z}_n$ .

The simplest case of (17) is that where  $B = \mathbb{P} \times I$  and the quotient space  $G \backslash M = \Sigma\mathbb{P}$  is the suspension of  $\mathbb{P}$ . For a given pair  $(G, K)$  the manifold  $M$  is then

$$M = M(P_W(\phi), \alpha_1, \alpha_2)$$

where

$$\alpha_1, \alpha_2 : L = L_1 = L_2 \rightarrow W$$

are such that there exists an isomorphism of  $W$ -bundles

$$\phi : S_r \times_{\alpha_1} W \rightarrow S_r \times_{\alpha_2} W$$

and  $P_W(\phi)$  is the mapping cylinder of  $\phi$ . Equivalently, the induced maps  $\mathbb{P} \rightarrow BL \xrightarrow{B\alpha_i} BW$  must be homotopic and  $P_W$  is induced from such a homotopy  $\mathbb{P} \times I \rightarrow BW$ .

We will focus on this case, in part because of its simplicity, and in part because it is the case which most closely resembles the cohomogeneity one situation. It is important to note that unlike the cohomogeneity one case, these products are not the only candidates for the manifold  $B$ . Given a choice of product, take any manifold without boundary of the same dimension. Now form the connected sum between the latter manifold and the product (avoiding the boundary components). The resulting manifold is clearly also a candidate for  $B$ .

The simplest example of this is the double

$$M(P_W(\text{id}), \alpha, \alpha) = D^{r+1} \times_{\alpha} G/K \cup_{\text{id}} D^{r+1} \times_{\alpha} G/K$$

obtained by gluing a tubular neighbourhood of a non-principal orbit with itself.

The next examples illustrate that many non-doubles are possible.

### 3.1. Aloff–Wallach spaces

Let  $G = \text{SU}(3)$  and let  $K$  be trivial. Then  $H_1$  and  $H_2$  can be any subgroups of  $\text{SU}(3)$  isomorphic to  $U(1)$  or  $\text{SU}(2)$ , as  $N_G(K) = \text{SU}(3)$ . For  $p_1, p_2, q_1, q_2 \in \mathbb{Z}$  with  $(p_1, p_2)$  and  $(q_1, q_2)$  coprime, let  $\alpha_1, \alpha_2 : U(1) \hookrightarrow \text{SU}(3)$  be given by

$$\begin{aligned} \alpha_1(z) &= \text{diag}(z^{p_1}, z^{p_2}, z^{-p_1-p_2}); \\ \alpha_2(z) &= \text{diag}(z^{q_1}, z^{q_2}, z^{-q_1-q_2}) \end{aligned}$$

where  $z \in U(1)$ . The resulting homogeneous spaces  $\text{SU}(3)/\alpha(U(1))$  are the 7-dimensional Aloff–Wallach spaces  $W_{p_1, p_2}$  and  $W_{q_1, q_2}$ . The topology of Aloff–Wallach spaces is well understood. They have been classified up to homeomorphism and diffeomorphism in [14]. We also refer to [14] for the computation of the cohomology of the  $W_{a,b}$ . In particular, we have

$$H^4(W_{a,b}; \mathbb{Z}) \cong \mathbb{Z}_{a^2+ab+b^2}$$

which already shows that among the Aloff–Wallach spaces there are infinitely many homotopy types. We can therefore choose  $p_1, p_2, q_1, q_2$  so that  $W_{p_1, p_2}$  and  $W_{q_1, q_2}$  are not homotopy equivalent.

For the proof of Theorem 6 we quote the homotopy classification of the Aloff–Wallach spaces.

**Theorem 21.** ([13, Theorem 0.1]) *The Aloff–Wallach spaces  $W_{p_1, p_2}$  and  $W_{q_1, q_2}$  have the same homotopy type if and only if  $p_1^2 + p_1 p_2 + p_2^2 = q_1^2 + q_1 q_2 + q_2^2$  and  $p_1 p_2 (p_1 + p_2) \equiv \pm q_1 q_2 (q_1 + q_2) \pmod{p_1^2 + p_1 p_2 + p_2^2}$ .*

**Corollary 22.** *If  $p_1^2 + p_1 p_2 + p_2^2 \neq q_1^2 + q_1 q_2 + q_2^2$ , then  $W_{p_1, p_2}$  and  $W_{q_1, q_2}$  have different homotopy types.*

### 3.2. Some infinite series of $G$ -manifolds with 2 non-principal orbits

We will now construct some infinite series of  $G$ -manifolds of cohomogeneity 3 and 5, whose singular orbits are Aloff–Wallach spaces. First we show that any two Aloff–Wallach spaces can be equivariantly embedded as singular orbits in an  $\text{SU}(3)$ -manifold of cohomogeneity 3.

**Proof of Theorem 5.** Let  $\phi : S^3 \times_{\alpha_1} \text{SU}(3) \rightarrow S^3 \times_{\alpha_2} \text{SU}(3)$  be any  $W = \text{SU}(3)$ -bundle isomorphism. To see that such an isomorphism exists, note that  $S^3 \times_{\alpha_i} \text{SU}(3)$  is an  $\text{SU}(3)$ -bundle over  $\mathbb{C}P^1 = S^2$ , and these bundles are classified by  $\pi_2 B\text{SU}(3) \cong \pi_1 \text{SU}(3) = 0$ . Now set  $M_{p_1 p_2 q_1 q_2}^{11} = M(P_{\text{SU}(3)}(\phi), \alpha_1, \alpha_2)$ .  $\square$

As non-double examples exist, this suggests investigating the conditions under which non-double examples can arise. This is of course a very broad question. So as to give further examples, and in particular to indicate the richness of the non-double family, we study one situation in some detail. The situation in question is the case where the space of orbits is the suspension of  $\mathbb{C}P^m$ , and where the Weyl group  $W = \text{SU}(n)$ .

Recall that the cohomology of  $\mathbb{C}P^m$  is  $H^*(\mathbb{C}P^m; \mathbb{Z}) = \mathbb{Z}[x]/x^{m+1}$ , where  $x$  is the first Chern class of the universal  $U(1)$ -bundle  $S^{2m+1} \rightarrow \mathbb{C}P$ . Since maximal tori in  $W$  are conjugate, any injective homomorphism  $\alpha : U(1) \rightarrow W$ ,  $W = U(n)$  or  $\text{SU}(n)$ , is conjugate to

$$\alpha(p) : U(1) \rightarrow W, \quad z \mapsto \text{diag}(z^{p_1}, z^{p_2}, \dots, z^{p_n}) \tag{23}$$

for some  $p = (p_1, \dots, p_n) \in \mathbb{Z}^n$  coprime. The total Chern class of  $\alpha(p)$  (i.e. the total Chern class of the  $W$ -bundle pulled back via the composition  $\mathbb{C}P^m \rightarrow \mathbb{C}P^\infty = BU(1) \xrightarrow{B\alpha(p)} BW$ ) is

$$c = \sum_{k=0}^n \sigma_k(p) x^k \tag{24}$$

where  $\sigma_k(p)$  denotes the elementary symmetric polynomial of degree  $k$  in  $p$  (see [15, §8]). In the case  $W = \text{SU}(n)$  we have  $p_1 + \dots + p_n = 0$  and the first Chern class vanishes.

**Proposition 25.** *Let  $W = U(n)$  or  $\text{SU}(n)$  and  $\alpha_1, \alpha_2 : U(1) \rightarrow W$  be injective homomorphisms. Then the  $W$ -bundles  $S^{2m+1} \times_{\alpha_1} W$ ,  $S^{2m+1} \times_{\alpha_2} W$  over  $\mathbb{C}P^m$  are isomorphic if and only if they have the same Chern classes.*

**Proof.** In the “stable range”  $n > m$  this holds for general  $W$ -bundles (i.e. bundles not necessarily associated to the universal bundle), and if  $n \leq m$  then the Chern classes of  $S^{2m+1} \times_{\alpha(p)} W$  determine  $p$  up to permutation (see [17, p. 114] or [22]).  $\square$



We now use the above analysis to construct non-double examples of  $G$ -manifolds with two singular orbits. We will again take  $G = \text{SU}(n)$  and  $K$  to be trivial, so that  $N_G(K) = G = \text{SU}(n) = W$ .

**Theorem 26.** *Given Aloff–Wallach spaces  $W_{p_1, p_2}$  and  $W_{q_1, q_2}$ , there is a 13-dimensional  $\text{SU}(3)$ -manifold  $M_{p_1 p_2 q_1 q_2}^{13}$  of cohomogeneity 5, orbit space  $\Sigma \mathbb{C}P^2$ , and two singular orbits equal to the given Aloff–Wallach manifolds if and only if  $p_1^2 + p_1 p_2 + p_2^2 = q_1^2 + q_1 q_2 + q_2^2$ .*

**Proof.** Since  $-\sigma_2(p_1, p_2, -p_1 - p_2) = p_1^2 + p_1 p_2 + p_2^2$ , the condition guarantees that the second Chern classes of  $\alpha(p_1, p_2, -p_1 - p_2)$  and  $\alpha(q_1, q_2, -q_1 - q_2)$  coincide. Since the first Chern class of an  $\text{SU}(3)$ -bundle vanishes and the third is in  $H^6(\mathbb{C}P^2; \mathbb{Z}) = 0$ , we have from Proposition 25 that there is an  $\text{SU}(3)$ -bundle isomorphism

$$\phi : S^5 \times_{\alpha(p_1, p_2, -p_1 - p_2)} \text{SU}(3) \rightarrow S^5 \times_{\alpha(q_1, q_2, -q_1 - q_2)} \text{SU}(3).$$

Now let  $M_{p_1 p_2 q_1 q_2}^{13} := M(P_{\text{SU}(3)}(\phi), \alpha(p_1, p_2, -p_1 - p_2), \alpha(q_1, q_2, -q_1 - q_2))$ .  $\square$

We now have two infinite families of  $\text{SU}(3)$ -manifolds with precisely two singular orbits, one family in dimension 11 and the other in dimension 13. As remarked earlier, there are infinitely many homotopy types of Aloff–Wallach manifolds, so it follows that both our families contain infinitely many equivariant diffeomorphism classes. However, if we ignore equivariance, this still leaves the question of how many diffeomorphism or homeomorphism or homotopy types occur in these families.

For the proof of Theorem 6 we will need two number-theoretic results.

**Theorem 27.** (*[18, 3.4]*) *A positive integer  $n$  is representable in the form  $n = a^2 + ab + b^2$  with  $(a, b) = 1$  if and only if the following conditions hold:*

1. *if  $3^t$  divides  $n$  then  $t \leq 1$ , and*
2. *if  $r \neq 3$  is prime and  $r$  divides  $n$ , then  $r \equiv 1 \pmod 3$ .*

The second of these number-theoretic results is a classical theorem of Dirichlet about arithmetic sequences:

**Theorem 28.** *Given integers  $a$  and  $d$  with  $(a, d) = 1$ , there exist infinitely many natural numbers  $n$  such that  $a + nd$  is prime.*

**Proof of Theorem 6.** By Theorem 28 there is an infinite monotonically increasing sequence of primes  $r_1, r_2, r_3, \dots$  all of which are congruent to 1 modulo 3. From Theorem 27 we deduce that there is a sequence of integers  $a_1, b_1, a_2, b_2, \dots$  such that for all natural numbers  $i$ :

1.  $(a_i, b_i) = 1$ ;
2.  $a_{2i-1}^2 + a_{2i-1} b_{2i-1} + b_{2i-1}^2 = r_{2i-1}$ ;
3.  $a_{2i}^2 + a_{2i} b_{2i} + b_{2i}^2 = r_{2i-1} r_{2i}$ .

By Corollary 22, we see that the resulting Aloff–Wallach spaces  $W_{a_i, b_i}$  are pairwise non-homotopy equivalent. To complete the proof, we will show that the manifolds  $M_i := M_{a_{2i-1} b_{2i-1} a_{2i} b_{2i}}^{11}$  are pairwise non-homotopy equivalent. We will show that the fourth cohomology groups of the  $M_i$  are non-isomorphic for different  $i$ .

We begin this analysis by observing that  $M_i$  is the union of two disc bundles (specifically  $D^4$ -bundles over  $W_{a_{2i-1} b_{2i-1}}$  respectively  $W_{a_{2i} b_{2i}}$ ) along their common boundaries  $S^2 \times \text{SU}(3)$ . The Mayer–Vietoris sequence for this union includes the following portion:

$$\begin{aligned} \dots &\rightarrow H^3(W_{a_{2i-1} b_{2i-1}}) \oplus H^3(W_{a_{2i} b_{2i}}) \rightarrow H^3(S^2 \times \text{SU}(3)) \rightarrow H^4(M_i) \\ &\rightarrow H^4(W_{a_{2i-1} b_{2i-1}}) \oplus H^4(W_{a_{2i} b_{2i}}) \rightarrow H^4(S^2 \times \text{SU}(3)) \rightarrow \dots \end{aligned}$$

The cohomology of the Aloff–Wallach spaces has been computed in [14, p. 466], and the cohomology of  $S^2 \times \text{SU}(3)$  follows from the Künneth formula. Filling in these groups in the sequence yields the following short exact sequence:

$$0 \rightarrow \mathbb{Z} \rightarrow H^4(M_i) \rightarrow \mathbb{Z}_{r_{2i-1}} \oplus \mathbb{Z}_{r_{2i-1} r_{2i}} \rightarrow 0.$$

Taking tensor products with  $\mathbb{Z}_{r_i}$  is right-exact, hence

$$\rightarrow \mathbb{Z}_{r_i} \rightarrow H^4(M_i) \otimes_{\mathbb{Z}} \mathbb{Z}_{r_i} \rightarrow \left\{ \begin{array}{ll} 0 & l < 2i - 1 \text{ or } l > 2i \\ \mathbb{Z}_{r_i}^2 & l = 2i - 1 \\ \mathbb{Z}_{r_i} & l = 2i \end{array} \right\} \rightarrow 0$$

is exact. In particular, for  $l$  odd, we have

$$\dim_{\mathbb{Z}_{r_l}} H^4(M_i) \otimes_{\mathbb{Z}} \mathbb{Z}_{r_l} \begin{cases} \geq 2, & l = 2i - 1, \\ \leq 1, & l \neq 2i - 1. \end{cases} \quad \square$$

**Theorem 29.** Consider the family of manifolds  $M_{p_1 p_2 q_1 q_2}^{13}$  in Theorem 26. If  $M_{abcd}^{13}$  and  $M_{a'b'c'd'}^{13}$  are two members of this family for which

$$r = a^2 + ab + b^2 = c^2 + cd + d^2 \neq r' = a'^2 + a'b' + b'^2 = c'^2 + c'd' + d'^2,$$

then these manifolds have different homotopy types. Consequently, the family contains infinitely many homotopy types.

**Proof.** We show that  $H^4(M_{abcd}^{13}) \cong \mathbb{Z}_{a^2+ab+b^2}$ . As in the proof of Theorem 6, we decompose  $M = M_{abcd}^{13}$  into two disc bundles, this time  $D^6$ -bundles over Aloff–Wallach spaces  $W_{ab}$  and  $W_{cd}$  and apply the Mayer–Vietoris sequence. The common boundary  $X$  of these bundles is an  $SU(3)$ -bundle over  $\mathbb{C}P^2$ . The relevant portion of the Mayer–Vietoris sequence is

$$\dots \rightarrow H^3(X) \rightarrow H^4(M) \xrightarrow{\chi^*} H^4(W_{ab}) \oplus H^4(W_{cd}) \xrightarrow{\pi^* + \tilde{\pi}^*} H^4(X) \rightarrow \dots \quad (30)$$

Since  $X$  also is the total space of an  $S^5$ -bundle over  $W_{ab}$  the Gysin sequence shows that the projections  $\pi : X \rightarrow W_{ab}$ ,  $\tilde{\pi} : X \rightarrow W_{cd}$  induce isomorphisms  $H^4(X) \cong H^4(W_{ab}) \cong H^4(W_{cd}) \cong \mathbb{Z}_r$  and  $H^3(X) \cong H^3(W_{ab}) \cong 0$ . Under these isomorphisms the homomorphism  $\pi^* + \tilde{\pi}^*$  corresponds to the addition map  $\mathbb{Z}_r \times \mathbb{Z}_r \rightarrow \mathbb{Z}_r$ . Thus  $\chi^*$  in (30) induces an isomorphism  $H^4(M) \cong \ker(\pi^* + \tilde{\pi}^*) \cong \mathbb{Z}_r$ .  $\square$

#### 4. Manifolds with one or many non-principal orbits

It is interesting to compare manifolds with finitely many singular orbits in cohomogeneity at least two with those of cohomogeneity one. Compact cohomogeneity one manifolds are of two basic types: those with no non-principal orbits, in which case the manifold is the total space of a bundle over  $S^1$  with the principal orbit as fibre; and those with precisely two non-principal orbits (see [10, Section 1]). In this latter case, the space of orbits is an interval. The non-principal orbits correspond to the end-points in the orbit space. No cohomogeneity one manifold with precisely one non-principal orbit can exist, because a point is not a boundary. For similar reasons, no connected cohomogeneity one manifold with more than two non-principal orbits can exist. In cohomogeneity at least two, however, the situation is very different. As noted in Section 2,  $\mathbb{H}P^{2k+1}$ ,  $\mathbb{C}P^{2k+1}$ ,  $\mathbb{X}^{4k+3}$  and  $\mathbb{R}P^{2k+1}$  are all boundaries. Thus in these higher cohomogeneities, unique non-principal orbits are possible.

The orbit space of a  $G$ -manifold  $M$  with precisely one non-principal orbit must be of the form

$$G \setminus M = B \cup_{\mathbb{P}} c\mathbb{P}$$

where  $B$  is a manifold with boundary  $\partial B = \mathbb{P} = L \setminus S^f$ ,  $\mathbb{P}$  is one of manifolds listed in (13) and  $c\mathbb{P}$  is the cone over  $\mathbb{P}$ . As discussed at the end of Section 2, the condition for a  $G$ -manifold with principal isotropy  $K$  with only one non-principal orbit  $G/H$ ,  $K \subset H \subset G$ , to exist is that the classifying map

$$f_\alpha : \mathbb{P} \xrightarrow{\iota} BL \xrightarrow{B\alpha} BW \quad (31)$$

extends to a map  $B \rightarrow BW$ , that is, if and only if the bordism class of  $f_\alpha$  vanishes. For this we need to look at the twisted Stiefel–Whitney numbers (15). Note that the map  $\iota : \mathbb{P} \rightarrow BL$  induces an injection in cohomology  $H^q$ ,  $q = 0, 1, \dots, r$ , because its homotopy fibre is  $S^f$ .

##### 4.1. $G$ -manifolds with one non-principal orbit and low cohomogeneity

**Cohomogeneity 2.** In cohomogeneity 2, a non-principal orbit must be exceptional. For such a manifold we must have  $\mathbb{P} = S^1$  and an injective homomorphism  $\alpha : L = \mathbb{Z}_k \rightarrow W$ . The map (31) is the composition

$$f_\alpha : S^1 \xrightarrow{\iota} B\mathbb{Z}_k \xrightarrow{B\alpha} BW,$$

and by considering twisted Stiefel–Whitney numbers (see Section 2.2) together with the fact that  $w_1 S^1 = 0$ , we see that  $f_\alpha$  bounds if and only if it induces the zero map on  $H^1(\cdot; \mathbb{Z}_2)$ . By the Hurewicz Theorem and the Universal Coefficient Theorem, the functors  $H^1(\cdot; \mathbb{Z}_2)$  and  $\text{Hom}(\pi_1(\cdot), \mathbb{Z}_2)$  are naturally equivalent. Because of the long exact sequence of the homotopy groups of the fibration  $W \rightarrow EW \rightarrow BW$ , we have a natural isomorphism  $\pi_1(BW) \cong \pi_0(W) \cong W/W_0$  where  $W_0$  is the connected component of the identity in  $W$ . Under these equivalences the map induced by  $B\alpha$  in  $H^1(\cdot; \mathbb{Z}_2)$  corresponds to the map

$$\text{Hom}(W/W_0, \mathbb{Z}_2) \rightarrow \text{Hom}(\mathbb{Z}_k, \mathbb{Z}_2),$$

$$\beta \mapsto \beta \circ q \circ \alpha$$

where  $q : W \rightarrow W/W_0$  is the quotient map.

Thus the map  $f_\alpha$  bounds (equivalently, there is a manifold  $M(P_W, \alpha)$ ) if and only if for all  $\beta$  the map

$$\mathbb{Z}_k \xrightarrow{\alpha} W \xrightarrow{q} W/W_0 \xrightarrow{\beta} \mathbb{Z}_2$$

is zero. This always holds if  $k$  is odd, as  $\text{Hom}(\mathbb{Z}_k, \mathbb{Z}_2) = 0$ . In this case, the  $k$ -fold covering  $S^1 \xrightarrow{z \mapsto z^k} S^1$  bounds.

**Cohomogeneity 3.** In cohomogeneity 3, by (13) we can have  $\mathbb{P} = S^2$  and  $L = H/K = U(1)$  or  $\mathbb{P} = \mathbb{X}^1 = \mathbb{R}P^2$  with  $H/K = U(1)$  or  $\mathbb{Z}_2$ . However, since  $\mathbb{R}P^2$  does not bound a 3-manifold, the last two cases are ruled out. The map (31) bounds if and only if the map  $U(1) \rightarrow W$  induces the zero map on  $H^1(\cdot; \mathbb{Z}_2)$ . This is automatic if, for instance, the fundamental group of  $W$  has odd order.

**Cohomogeneity 4.** The non-principal orbit must be exceptional and we must have  $\mathbb{P} = L \setminus S^3$  with  $L \subset O(4)$  finite. Since  $L$  acts freely, it preserves the orientation and therefore  $w_1(\mathbb{P}) = 0$ . Note that  $w_2(\mathbb{P}) = 0$  and  $w_3(\mathbb{P}) = 0$ , since these Stiefel–Whitney classes vanish for all 3-dimensional manifolds. It follows that the map  $f_\alpha$ , with  $\alpha : L \rightarrow W$ , bounds if and only if  $B\alpha : BL \rightarrow BW$  induces 0 in  $H^3(\cdot; \mathbb{Z}_2)$ .

**Cohomogeneity 5.** The non-principal orbit must be singular and we must have  $\mathbb{P} = \mathbb{H}P^1 = S^4$  with  $L = SU(2)$ . Since all Stiefel–Whitney classes of  $\mathbb{P}$  vanish, the map  $f_\alpha$ , with  $\alpha : L \rightarrow W$ , bounds if and only if  $B\alpha : BSU(2) \rightarrow BW$  induces 0 in  $H^4(\cdot; \mathbb{Z}_2)$ .

**Cohomogeneity 6.** The non-principal orbit must be exceptional and we must have  $\mathbb{P} = L \setminus S^5$  with  $L \subset SO(6)$  finite, and therefore  $w_1(\mathbb{P}) = 0$ . In order that the map  $f_\alpha$ , with  $\alpha : L \rightarrow W$ , bounds we must have that  $B\alpha : BL \rightarrow BW$  induces 0 in  $H^5(\cdot; \mathbb{Z}_2)$  but this is generally not sufficient as the example  $\mathbb{P} = \mathbb{R}P^5$  shows: in this case, the non-trivial Stiefel–Whitney classes of  $\mathbb{P}$  are  $w_2$  and  $w_4 = w_2^2$ . It follows that  $f_\alpha$ , with  $\alpha : \mathbb{Z}_2 \rightarrow W$ , bounds if  $B\alpha : B\mathbb{Z}_2 \rightarrow BW$  induces 0 in  $H^q(\cdot; \mathbb{Z}_2)$  for  $q = 1, 3, 5$ .

**Cohomogeneity 7.** The non-principal orbit must be singular and we must have  $\mathbb{P} = \mathbb{X}^3$ ,  $L = N_{SU(2)}U(1)$  or  $\mathbb{P} = \mathbb{C}P^3$ ,  $L = U(1)$ . The cohomology of  $\mathbb{X}^3$  is  $H^*(\mathbb{X}^3; \mathbb{Z}_2) = \mathbb{Z}_2[x, u]/(x^3, u^2)$  with  $\deg x = 1$ ,  $\deg u = 4$ . (This can be computed via the Leray–Hirsch spectral sequence, using the fact that  $\mathbb{X}^3$  is an  $\mathbb{R}P^2$ -bundle over  $\mathbb{H}P^1$ .) The non-trivial Stiefel–Whitney classes are  $w_1 = x$ ,  $w_2 = x^2$ . It follows that the map  $f_\alpha$ , with  $\alpha : N_{SU(2)}U(1) \rightarrow W$ , bounds if and only if  $B\alpha : BL \rightarrow BW$  induces 0 in  $H^q(\cdot; \mathbb{Z}_2)$ ,  $q = 4, 5, 6$ .

In the second case,  $\mathbb{P} = \mathbb{C}P^3$ ,  $L = U(1)$ , all Stiefel–Whitney classes of  $\mathbb{P}$  vanish and therefore the map  $f_\alpha$ , with  $\alpha : U(1) \rightarrow W$ , bounds if and only if  $B\alpha : BU(1) \rightarrow BW$  induces 0 in  $H^6(\cdot; \mathbb{Z}_2)$ .

#### 4.2. Proof of Theorem 7

**Proof of Theorem 7.** We first construct  $W$ -manifolds for  $W = U(1)$  and  $W = SU(2)$  with a single non-principal orbit.

In the case  $W = U(1)$ , pick  $p \in \mathbb{N}$ ,  $p > 1$ , and take the join

$$M_{U(1)} := S^{2k+1} = S^{2k-1} * S^1$$

where  $U(1)$  acts freely on  $S^{2k-1} \subset \mathbb{C}^k$  via the standard representation, and via  $U(1) \ni z \mapsto z^p$  on  $S^1$ . This gives a  $U(1)$ -manifold of cohomogeneity  $2k$  having a single exceptional orbit with isotropy  $\mathbb{Z}_p$ .

In the case  $W = SU(2)$  put

$$M_{SU(2)} := S^{4k+2} = S^{4k-1} * S^2$$

where  $SU(2)$  acts freely on  $S^{4k-1} \subset \mathbb{C}^{2k} = (\mathbb{C}^2)^k$  via the standard representation and on  $S^2 = \mathbb{C}P^1$  with isotropy  $U(1)$ . This gives an  $SU(2)$ -manifold of cohomogeneity  $4k - 1$  having a single singular orbit with isotropy  $U(1)$ .

For the general case, given  $G \supset K$  such that  $U(1) \subset N_G K/K = W$  respectively  $SU(2) \subset W$ , we can form the manifolds  $M_{U(1)} \times_{\iota_1} G/K$  and  $M_{SU(2)} \times_{\iota_2} G/K$ , where  $\iota_1$  respectively  $\iota_2$  denotes the inclusion of  $U(1)$  or  $SU(2)$  into  $W$ . These are  $G$ -manifolds of cohomogeneity  $2k$  and  $4k - 1$ , with precisely one exceptional respectively singular orbit. By taking fibre connected sums of copies of these  $G$ -manifolds with one non-principal orbit one can realise any number of non-principal orbits as required for the first two claims of Theorem 7.

For the final statement, take  $m/2$  copies of any  $G$ -manifold with space of orbits  $\Sigma \mathbb{C}P^c$ , as described in Section 3. Now perform fibre connected sums as before. Note that although the  $\mathbb{C}P^{\text{even}}$  are not boundaries, any even number of disjoint copies bound.  $\square$

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