On G-manifolds with finitely many non-principal orbits

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We consider a compact Lie group G acting smoothly on a compact manifold M. The *cohomogeneity* of such an action is the dimension of the space of orbits M/G.

In recent years, the geometry of cohomogeneity-one manifolds has been extensively studied. (See for example [7].) Our motivating aim is to ask what can be said about topology and geometry in cohomogeneity greater than one? This question is deliberately vague, and allows many possible interpretations depending on the additional assumptions one makes in order to create a reasonable problem. Our approach is to focus on G-manifolds with only finitely many non-principal orbits. Thus each non-principal orbit is isolated in the sense that it has a tubular neighbourhood (with respect to some G-invariant background metric) in which all other orbits are principal. For an alternative interpretation of the above question, see [2].

In the following, K will denote the principal isotropy of the G-action, and H_1,...,H_p will denote the non-principal isotropy groups. Let N_i denote a tubular neighbourhood of the non-principal orbit G/H_i. Then $M - \cup_{i=1}^{p} N_i$ has the structure of a principal-orbit bundle. Let B denote the base of this bundle, so $B = (M - \cup_{i=1}^{p} N_i)/G$. It is clear that B is a manifold with p boundary components. We note that $T_i := \partial N_i$ has two fibration structures: it is fibered by principal orbits, and is also fibered by normal spheres S^r.

We first consider the structure of the orbit space M/G. The key to understanding this is the following result:

Theorem. *([1]; chapter 4, §6) Let L be a compact Lie group acting locally smoothly, effectively and with one orbit type on S^r. If $\dim L > 0$ then L acts transitively or freely, and if L acts freely, we must have $L \cong U(1)$, $N_{SU(2)}U(1)$ or $SU(2)$. If $\dim L = 0$ then $S^r \to S^r/L$ is the universal covering, so L must also act freely.*

In our case, if any H_i acts transitively then the cohomogeneity must be one. As we wish to study cohomogeneities greater than one, we will assume the H_i-action is not transitive. We deduce:

Corollary. *If the cohomogeneity is greater than one, then K is ineffective kernel of the H_i action on S^r, so K is normal in H_i and $H_i/K \cong U(1)$, $N_{SU(2)}U(1)$, $SU(2)$, or is finite, and acts freely and linearly on S^r.*

In turn we deduce:

Corollary. *If the cohomogeneity is greater than one, then T_i/G is either a complex or quaternionic projective space, or a $\mathbb{Z}_2$ quotient of an odd dimensional complex projective space in the case of a singular orbit, or in the case of an exceptional orbit a real projective or lens space. Also, each N_i/G is a cone over one of these spaces.*

The structure of M/G is then given by:

Theorem. *M/G is the union of a manifold with boundary B, where each boundary component is one of the above listed spaces, together with cones over the boundary components.*

Notice that if there is at least one singular orbit, this forces the cohomogeneity to be odd. For more details about the topology of these objects, see [3].

We now consider the geometry of these objects, and in particular we consider the existence of invariant metrics with positive Ricci curvature. To provide some motivation for this, let us recall the following result for cohomogeneity one:

Theorem. *([5]) A compact cohomogeneity one manifold admits an invariant metric with positive Ricci curvature if and only if its fundamental group is finite.*

There is little possibility of proving a result as strong as this in the current context: the space of orbits in cohomogeneity one is either a circle or an interval. Either way, this makes no contribution to the curvature. However, in higher cohomogeneities, it is to be expected that the geometry of the space of orbits will play some role in determining the global geometric properties.

In the statement of the theorem below, g_i denotes a metric on the appropriate boundary component induced via the standard submersion from the round metric of radius one. For more details, see [4].

Theorem. *Suppose that $\pi_1(G/K)$ is finite. Then if B admits a Ricci positive metric such that*

i) the metric on boundary component i is $\lambda_i^2 g_i$, and

ii) the principal curvatures (with respect to the inward normal) at boundary component i are greater than $-1/\lambda_i$,
then M admits an invariant Ricci positive metric.

Corollary. *All G-manifolds with two singular orbits, orbit space a suspension $\Sigma\mathbb{C}P^m$ or $\Sigma\mathbb{H}P^m$, and principal orbit G/K with $\pi_1(G/K) < \infty$ admit invariant metrics with positive Ricci curvature.*

To illustrate this, given any two Aloff-Wallach spaces W_{p_1,p_2} and W_{q_1,q_2}, there is a 11-dimensional $SU(3)$-manifold $M^{11}_{p_1p_2q_1q_2}$ of cohomogeneity 3 with two singular orbits W_{p_1,p_2} and W_{q_1,q_2} and orbit space $\Sigma\mathbb{C}P^1 = S^3$. This family contains infinitely many homotopy types, and all manifolds in this family admit invariant metrics of positive Ricci curvature.

The above corollary raises the question of whether there are Ricci positive examples with more than two non-principal orbits. The key to answering this is the following result, which relies on a construction in [6].

Proposition. *For each $n \geq 3$, $m \geq 1$ and sufficiently small $\rho > 0$, there is a $\delta_0 = \delta_0(\rho) > 0$ such that for all $0 < \delta < \delta_0$ there is a Ricci positive metric on $S^n - \coprod_{i=1}^m D^n$ such that each boundary component is a round sphere of radius δ, and the principal curvatures at the boundary (with inward normal) are all equal to $-\rho/\delta$.*

Using this Proposition in conjunction with the above Theorem yields the following:

Theorem. *In cohomogeneities 3 and 5, there are G-manifolds with any given number of isolated singular orbits admitting an invariant metric of positive Ricci curvature.*

For example, in cohomogeneity 3, there is an 11-dimensional $SU(3)$-manifold with an invariant Ricci positive metric having isolated singular orbits equal to any given (finite) collection of Aloff-Wallach spaces.

Open question. *Can we find manifolds with more than two non-principal orbits and an invariant Ricci positive metric in cohomogeneities* $\neq 3, 5$*?*

The problem in this case is to understand the geometry of the space of orbits. We know for example that $\mathbb{H}P^{2k+1}$, $\mathbb{C}P^{2k+1}$ and $\mathbb{R}P^{2k+1}$ are boundaries, and so we can create manifolds with boundary (by a connected sum on the interior of the bounding manifolds) having any selection of these spaces as boundary components. These are all candidates for the manifold B. Geometrically, what can be said about such manifolds? Do any admit Ricci positive metrics?

References

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