



FREEDOM IN MATHEMATICS AND ITS EDUCATIONAL VALUE

Fiacre Ó Cairbre

Department of Mathematics and Statistics

NUI, Maynooth

fiacre.ocairbre@nuim.ie

1. Introduction

The notion of freedom in mathematics shocks many people. However, as Cantor (1845–1918) once said:

The essence of mathematics lies in its freedom

In order to appreciate the freedom in mathematics one must first realise that mathematics essentially comprises an abundance of ideas. The fact that mathematics consists of a myriad ideas will be discussed in section 2 below. Freedom is an important feature in mathematics because one is free to conceive of any new ideas one wants in mathematics. These new ideas may or may not lead to anything interesting or useful. Historically (and probably also in the future), the major breakthroughs in mathematics have typically happened because the great mathematicians were free to conceive of any new ideas they wanted even if their wild thoughts broke with conventions and seemed bizarre to other mathematicians and the general public. Six examples (of many), of freedom in mathematics, are the discovery of an irrational number by Hippasus in Ancient Greece, making zero into a number in its own right by the Indians, the acceptance of negative numbers by the Indians, the acceptance of complex numbers by Bombelli, the idea of a heliocentric (i.e. sun-centred) universe by Copernicus in his search for beauty in mathematics, the creation of Quaternions by Hamilton on the banks of the Royal Canal in Dublin in 1843 which liberated algebra from arithmetic and the discovery of Non-Euclidean Geometry in the nineteenth century which liberated geometry.

I believe there is great educational value in students hearing about how freedom in mathematics has led to many crucial breakthroughs in mathematics. One reason for this belief is that many students think there is no freedom in mathematics and when these students subsequently realise that there is great freedom in mathematics, then they may change their perception of mathematics for the better. I have seen this change of perception happen with many students. The examples in section 3 below are just some of many which illustrate the educational value related to freedom in mathematics. I encourage all teachers to come up with other examples of freedom in mathematics and also to discuss the freedom in mathematics with students.

The ideas in mathematics may be motivated by pure human imagination or by the physical world. For example, the idea of a complex number above came from Bombelli's pure imagination (when he was solving cubic equations in the sixteenth century) and there was no motivation from the physical world. However, complex numbers are now fundamental



in the work of some engineers in understanding the physical world. Also, the idea of Non-Euclidean Geometry above was motivated by pure imagination with no relation to the physical world. However, later on Non-Euclidean Geometry was crucial for Einstein's theory of relativity which helps us understand the physical world. These are just two of many such examples.

The notion of freedom in mathematics may lead to major breakthroughs and advances in how we understand the physical world. Non-Euclidean Geometry above is one example. Another example is the idea of a heliocentric universe by Copernicus above which would ultimately revolutionise science and society.

2. Mathematics and ideas

In this section I will discuss how mathematics essentially comprises an abundance of ideas. Number, circle and derivative are just some examples of the myriad ideas in mathematics. Many people are so familiar with number that they are quite surprised when they are told that number is an idea that cannot be sensed with our five physical senses. Numbers are indispensable in today's society and appear practically everywhere from football scores to car number plates to the time of day. The reason number appears practically everywhere is because a number is actually an idea and not something physical. Some people think that they can physically see the number 2 when it's written on the blackboard but this is not so. The number 2 cannot be physically sensed because it's an idea.

Mathematical ideas like number can only be 'seen' with the 'eyes of the mind' because that is how one 'sees' ideas. Think of a sheet of music which is important and useful but it is nowhere near as interesting, beautiful or powerful as the music it represents. One can appreciate music without reading the sheet of music. Similarly, mathematical symbols on a blackboard are like the sheet of music – they are important and useful but nowhere near as interesting, beautiful or powerful as the actual mathematics (ideas) they represent. The number 2 on the blackboard is just a symbol to represent the idea we call two. Many people claim they do not see mathematics in the physical world and this is because they are looking with the wrong eyes. These people are not looking with the eyes of their mind. For example if you look at an aeroplane with your physical eyes you do not really see mathematics, but if you look with the eyes of your mind you will see a lot of advanced mathematical ideas that are fundamental for the design and operation of the aeroplane.

So what is this idea we call two? If one looks at the history of number one sees that the powerful idea of number did not come about overnight. As with most potent mathematical ideas, its creation involved much imagination, creativity and it took a long time for the idea to evolve into its current state. Here is one way to think of what the number two is – Think of all pairs of objects that exist; they all have something in common and this common thing is the idea we call two.

One can think of any positive whole number in the same way. Note that this idea of two is different from two cows, two boats etc. Also note that it is the fact that number is an idea (and not something we can sense with our five senses) that makes it so powerful and indispensable in day's society. The seemingly simple statement that $14+23=37$ is actually



an abstract statement, since it deals with ideas rather than concrete objects, and solves infinitely many problems (since you can pick any object you want to count) in one go. This illustrates the remarkable practical power of abstraction and many people do not realise that they use abstraction all the time (e.g. when adding). Many people tend to think of abstraction as the antithesis of practicality but as the above example of addition shows, abstraction can be the most powerful way to solve practical problems because it essentially means you attempt to solve many seemingly different problems in one go, in the abstract, as opposed to solving all the different problems separately, which is what people did about eight thousand years ago by using different physical tokens for counting different objects. Of course, there are much more advanced examples of abstraction but the $14+23=37$ example captures the essential feature of abstraction. See [1] for more on abstraction.

Freedom is one important feature of mathematics. There are some other important features of mathematics. For example, I believe that beauty is the most important feature of mathematics and you can read [2] to see why I believe this to be the case. Some other important features of mathematics include practical power [1], deductive reasoning [1], abstraction (above) and research [2].

3. Examples of freedom in mathematics

(a) The discovery of an irrational number

Pythagoras (585–500 BC) was the leader of a cult, called the Pythagoreans, in Crotona, which was a Greek city in what is now southern Italy. The motto of the Pythagoreans was *All is number* meaning that everything in the universe could be explained by rational numbers. Furthermore, they assumed all numbers were rational. The number $\sqrt{2}$ was an important number for the Pythagoreans because it arose naturally in their geometry as the length of the diagonal of a unit square. Consequently, they assumed $\sqrt{2}$ was rational and so they tried to find two integers a and b such that $\sqrt{2} = \frac{a}{b}$. They had difficulty finding two such integers.

Hippasus was a Pythagorean and he began to think differently about $\sqrt{2}$. He began to wonder about the possibility that $\sqrt{2}$ might not be rational. The freedom in mathematics plays a crucial role here because Hippasus felt free to consider the possibility that $\sqrt{2}$ is not rational (i.e. is irrational) even though it had always been assumed that all numbers were rational. Hippasus then went on to actually prove that $\sqrt{2}$ is indeed irrational. Hippasus' proof is an elegant piece of mathematics [1]. This discovery of an irrational number led to a crisis in mathematics because it meant that the Pythagoreans now had to check all their previous proofs and delete the assumption that all numbers were rational. One can say that freedom in mathematics led to the first crisis in mathematics. There have been several other crises in mathematics. The last crisis was relatively recent and occurred just over a hundred years ago in 1902 with the so called Russell's Paradox. Pythagoras was furious at the discovery of an irrational number because it went against his assumption that all numbers were rational. Pythagoras was so angry that he went overboard and literally threw Hippasus overboard into the sea where he drowned!



(b) Much ado about nothing – the story of zero

Probably the best book title I have come across is *Zero, the Biography of a Dangerous Idea* by Seife [3]. It tells the fascinating story of the history of zero and it has a memorable beginning with zero hitting a warship like a torpedo! I will return to this drama on the high seas below.

The Ancient Babylonians were the first to use a symbol for what is now called zero. About 300 BC zero was born as a position indicator meaning that the Babylonians created it as a way to indicate a missing power of 60 in their number system (just like the way it's now used to distinguish 5302 from 532 by indicating a missing power of 10). The Babylonians used essentially a base 60 number system which was very different from other number systems at the time and also very different from our base 10 number system today. Notice how the Babylonians were free to create their own number system, just like the Ancient Egyptians were free to create their own number system which was completely different from the Babylonian one. Many different cultures have been free to create many different number systems throughout history. One could say that *Number systems are like hairstyles – they go in and out of fashion.*

The Babylonian symbol for zero contained two small slanted wedges and mysteriously was only used within a number and never at the end of a number. Before 300 BC the Babylonians had left a space to indicate a missing power of 60 but this was often ambiguous and confusing. The birth of zero was a major event in the history of mathematics.

Zero was not considered as a number in its own right by the Babylonians, in the sense that zero did not interact with the other numbers via addition, multiplication etc. It was as if zero was a piece of punctuation like a comma and the other numbers were like letters. Zero was purely a position indicator and was on a much lower level than the other numbers. It's interesting to note that the Egyptian culture at the same time had no need for a zero in their number system.

If someone said to you *I want to make a comma into a letter*, you would probably consider that person crazy. However, one is free to try and make a comma into a letter and see where it takes them. Well, something very similar happened in India around 500 AD because that's when people in southern India were the first to feel free to try and make zero into a number in its own right (i.e. make zero interact with the other numbers via addition, multiplication etc.) Recall that up until then zero was like a comma and the other numbers were like letters. It's fascinating to read how Brahmagupta and others tried to make zero interact with the other numbers via addition, multiplication etc. because it definitely was not obvious how to do it [4]. For example, they had big problems attempting to make sense of division by zero, and we now know why, because it's impossible. Remember that nobody had ever even considered trying to make zero interact with the other numbers before. The attempt by the Indians was something that seemed bizarre and yet they were successful. Mathematics today would be completely different, and not nearly as powerful, if the Indians had not been free to do something bizarre and try to make zero into a number in its own right.



This story will end with a bang by returning to the drama on the high seas above. In 1997, a billion dollar warship was dead in the water on account of zero. This was because the new computer software tried to divide by zero and thus led to the 80,000 horsepower being useless. The warship had to be towed into port and it took two days for the engineers to get rid of the zero. They had protected the warship from all sorts of modern weapons but nobody had thought of protecting it from zero!

(c) Origin of the equality symbol =

The equality symbol = is arguably the most common symbol in mathematics. The = symbol was born in Ireland in 1557. In that year, the Welsh mathematician, Robert Recorde, was living in Ireland and he justified his adoption of a pair of parallel line segments for the symbol of equality

Because no two thynges can be more equal

The page where the equality symbol first appeared with the above justification can be seen in [5]. Notice how Recorde had great freedom in creating whatever symbol he wanted for equality.

The above thirty second story is a good example of how a very short story (involving freedom in mathematics) can completely change some students' perception of mathematics for the better. In one such case a former student of my history of mathematics course here in NUI, Maynooth, was teaching in second level. The teacher told the students the above story about the origin of the equality symbol. The students thought the story was very enlightening. One particular student took it to a different level. She didn't believe the story because she thought mathematical symbols were just there and had no human involvement. She told her parents the story over dinner that evening. Her mother didn't believe the story and mentioned it to the school Principal during the parent-teacher meeting the following day. The Principal didn't believe the story either and discussed it with the teacher the next day. The teacher showed the Principal the page from the 1557 book above and then the Principal (and later the student and her mother) finally believed the story.

The reason that the mother and the Principal didn't believe the story was (similar to the student) that they all felt that mathematical symbols were just there and had no human involvement. Later, they realised that the equality symbol (and also all mathematical symbols) were created by humans and had their own individual stories about their origins. This realisation completely changed their perception of mathematics for the better because the above story humanised mathematics for them.

A variety of interesting questions and topics may arise from the above seemingly innocent thirty second story. For example, a student once asked me what was used before the equality symbol =. It was a revelation to the student that the symbol was not always just there. Now they wished to know how mathematics was done before the equality symbol. I told the student that people typically used words (like *aequatur* which means equal in Latin) before the equality symbol was created. This student was now, for the first time, witnessing part of the evolution of symbolic mathematical notation. This had a profound positive impact on the student. The student also realised that historically mathematicians had great freedom in creating any mathematical notation they wanted.



(d) Why are there twenty-four hours in a day?

Have you ever wondered why there are twenty-four hours in a day? Why is it twenty-four and not some other number like ten or twenty? The answer is not obvious.

Look at your right hand with your palm facing you. Notice that each of your four fingers, excluding your thumb, has three sections. If you never noticed this before, then don't be surprised. Now, using the thumb on your right hand, you can count to twelve by moving your thumb along each of the three sections on the four fingers. This was a convenient way for many ancient cultures to count to twelve on one hand when the other hand was busy. This also explains why twelve was an important number in ancient times. Even to this day we have a special name, dozen, for counting with twelve.

These ancient cultures were free to divide the day into any number of units they wanted. They wished to divide the day into an equal number of light units and dark units. As mentioned above, twelve was a convenient number for counting and so they divided the day into twelve light units and twelve dark units. In this way they divided the day into twenty-four units which are now called hours. Try to imagine what our society would be like if we had some other number of hours in a day. You probably take twenty-four hours for granted and yet it could so easily be a different number. Similarly, have you ever wondered why there are seven days in a week, sixty seconds in a minute and sixty minutes in an hour?

(e) Liberator of Algebra

William Rowan Hamilton (1805–1865) has been called the Liberator of Algebra because he freed algebra from the shackles of arithmetic. He had the freedom to create a whole new system of numbers that seemed bizarre to other mathematicians. We will see how he did this below.

Hamilton is Ireland's greatest mathematician and one of the world's most outstanding mathematicians ever. He was born in Dominick St. in Dublin and then spent his early youth on the banks of the Boyne in Trim. Like most great mathematicians, his motivation for doing mathematics was the search for beauty [2]. He found much beauty in mathematics and also his mathematics has turned out to very powerful when applied to many different areas including physics, engineering, space navigation, computer games, animation, special effects in movies and much more [6]. One example of an application of quaternions that typically appeals to journalists, radio hosts and students, is the fact that Lara Croft in Tomraider was created using quaternions!

Number couples (or complex numbers) had been important in mathematics and science when working in two dimensional geometry. Hamilton was trying to extend his theory of number couples to a theory of Number Triples (or triplets). He hoped these new triplets would give a natural mathematical structure and a new approach for describing the three dimensional world, in the same way that the number couples played a fundamental role in two dimensional geometry.

Hamilton had problems trying to define the multiplication operation in his search for a suitable theory of triplets. We now know why because one can actually prove that it's



impossible to construct the suitable theory of triplets he was looking for. Then on that famous day, October 16, 1843, Hamilton's mind gave birth to a new system of numbers called Quaternions in a flash of inspiration as he walked along the banks of the Royal Canal at Broombridge in Cabra in Dublin. Hamilton realised that if he worked with Number Quadruples and an unusual multiplication operation, then he would obtain everything he wanted. He called his new system of numbers quaternions because each number quadruple had four components. He had created a completely new structure in mathematics. Mathematicians were stunned at his audacity in creating a new system of numbers that did not satisfy the usual commutative rule for multiplication ($ab=ba$). Hamilton was called the Liberator of Algebra because his quaternions shattered the previous accepted convention that any new and useful algebraic number system should satisfy the rules of ordinary numbers in arithmetic. Hamilton had the freedom to create quaternions and in doing so he freed algebra from the shackles of arithmetic.

Hamilton performed a piece of nineteenth century graffiti by scratching his quaternion formulas on the canal bridge. In an act of mathematical vandalism, Hamilton opened up a whole new mathematical landscape where mathematicians could now feel free to conceive new algebraic number systems that were not shackled by the rules of ordinary numbers in arithmetic. Modern algebra was born on October 16, 1843 on the banks of the Royal Canal in Dublin. One could say *One small scratch for man, one giant leap for mathematics!* Hamilton's eureka moment, when he created quaternions, is commemorated by a plaque at Broombridge which was unveiled by the Taoiseach, Eamon de Valera, in 1958.

I organise an annual walk which commemorates Hamilton's creation of quaternions. The walk takes place on October 16 and participants retrace Hamilton's steps by starting at Dunsink Observatory, where Hamilton lived, and then strolling down to meet the Royal Canal at Ashtown train station. The walk then continues along the canal to the commemorative plaque at Broombridge in Cabra. In total, the walk takes about forty-five minutes and is ideal for a mathematics outing for transition year students. Teachers have said that the walk and the Hamilton story have had a very positive impact on students' perception of mathematics. There are typically about 200 people on the walk from a wide variety of backgrounds including staff and students from second level, third level and many from the general public. There is also usually a large media interest in the walk. Consequently, Hamilton's story and the walk have appeared three times on television, many times on a variety of radio stations and in lots of newspaper articles. You can read more about Hamilton and the annual walk in [6], [7] and on www.maths.nuim.ie/hamiltonwalk. If you are interested in coming on the walk, then contact me.

4. References

- [1] A resource for transition year mathematics teachers, by Fiacre Ó Cairbre, John McKeon and Richard Watson. Published by the Department of Education and Science in 2006. Every second level school in Ireland received two free copies from the Department of Education and Science in 2006.
- [2] The importance of being beautiful in mathematics, by Fiacre Ó Cairbre. Irish Mathematics Teachers' Association Newsletter. No. 109, 2009, pp. 29–45.



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- [4] The nothing that is, by Robert Kaplan.
- [5] An introduction to the history of mathematics, by Howard Eves.
- [6] Twenty years of the Hamilton walk, by Fiacre Ó Cairbre. Irish Mathematical Society Bulletin, No. 65, 2010, pp. 33–49.
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Shakuntala Devi

1929 – 2013

Given any two 13-digit numbers, how long do you think it would take you to multiply them? This task was completed by Shakuntala Devi, in 1980, without a calculator! The numbers she was given were 7,686,369,774,870 and 2,465,099,745,779.

The answer she gave was 18,947,668,177,995,426,462,773,730. She was, of course, correct.

Even more remarkable is the fact that she completed the task in 28 seconds earning a place in the Guinness Book of Records. She could do many such phenomenal calculations unaided. She had no formal education.



Shakuntala Devi, Bangalore, India

<http://www.livemint.com/Opinion/H2c9FBxmI7ofysI8RAPH3I/Shakuntala-Devi-RIP.html>

Devi was a household name in India. It was reported that she strove to simplify mathematics for students and help them get over their maths phobia.

Among her reasons for taking an interest in mathematics were::

- ◆ It gives you a purpose, an aim, a focus that insures you against restlessness;
- ◆ It makes you regard yourself with greater respect and in turn invokes respect from others around you;
- ◆ It makes you more aware, more alert, more keen because it is a constant source of inspiration.

(From her list of reasons in her book, *Mathability: Awaken the Maths Genius in Your Child.*)

Neil Hallinan Dublin Branch