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ABSTRACT

There are relatively few known demand systems that are theoretically satisfactory and practically implementable. This paper investigates building more complex demand systems from simpler known ones by considering sums and products of basic utility functions, an approach that does not seem to have been exploited previously in the literature. Some of the systems that result are interesting and usefully extend the range of available functions. Even the simpler systems that are not sufficiently flexible for the analysis of real world consumption data may still be useful for applied general equilibrium studies and for theoretical explication. Although some systems, instead of being new, turn out to be rediscoveries of already known ones, the way in which they arise as combinations of simple components is of interest in itself in showing them as sub sets of wider classes.

I INTRODUCTION

An indirect utility function $U(\mathbf{p}, y)$, where **p** is a vector of prices and y is income, and the demand equations derived from it through Roy's identity

$$q_i = -\frac{\partial U}{\partial p_i} / \frac{\partial U}{\partial y}, \qquad (1)$$

satisfy demand theory, or utility maximisation, provided $U(\mathbf{p}, y)$ meets stringent criteria. These are that U be homogeneous of degree zero in income and prices (**p)**, non-decreasing in y, non-increasing in **p**, and convex or quasi-convex in **p.** Then the demand equations satisfy the required constraints of aggregation, homogeneity, Slutsky symmetry and negativity utilities. These criteria for the validity of indirect utility functions are very restrictive on the choice of functional forms, even with restrictions placed on the parameters occurring in the forms. There are relatively few known functions U that satisfy validity conditions for *all*, or even for *all plausible* values of prices and income and some of them are very basic. This paper investigates building more complex demand systems from simple known ones by considering sums and products of basic utility functions.

The basic combination devices, which will be described in section II, are quite simple, but at least as far as this author knows, they have not been exploited previously in the literature in order to expand the range of valid demand systems. Some of the simpler systems that result and that will be described in section III, may not be as flexible as might be desired for the analysis of real world survey or time series data on consumer expenditures on commodities. However, they may still be useful for applied general equilibrium studies and for theoretical explication. Some more complex systems, to be derived in section IV, are more flexible and perhaps usefully extend the range of available functions. As might be expected, some systems, instead of being new, turn out to be rediscoveries of already known ones. However, even the way in which they arise as combinations of simple components is of interest in itself in showing them as sub sets of wider classes.

II DEMAND EQUATIONS FROM SUMS AND PRODUCTS OF UTILITIES

Suppose we have two (indirect) utility functions U_1 and U_2 satisfying all validity criteria. Then the criteria obviously apply to $U_1 + U_2$ (the sum of two quasi-convex functions is quasi-convex or convex) and indeed to $(1-\lambda)U_1 + \lambda U_2$, where λ is a positive constant, and corresponding demand systems can be derived. Let $w_{1i} = w_{1i}(\mathbf{p}, y)$ and $w_{2i} = w_{2i}(\mathbf{p}, y)$ be the sets of demand equations, in budget share form, resulting from application of Roy's identity to U_1 and U_2 respectively. Then by applying (1) to $(1-\lambda)U_1 + \lambda U_2$ and simplifying, the demand equations corresponding to this sum of utilities turn out to be

$$w_{si} = w_{1i} \frac{(1-\lambda)\dfrac{\partial U_1}{\partial y}}{(1-\lambda)\dfrac{\partial U_1}{\partial y} + \lambda \dfrac{\partial U_2}{\partial y}} + w_{2i} \frac{\lambda \dfrac{\partial U_2}{\partial y}}{(1-\lambda)\dfrac{\partial U_1}{\partial y} + \lambda \dfrac{\partial U_2}{\partial y}}, \tag{2}$$

or the original individual demand formulae weighted by (apart from constants) the derivatives of utilities with respect to income. The sub-script s denotes the utilities were summed.

For the special case of utility functions of the form

$$U_1 = \frac{y}{P_1} \qquad \text{and} \qquad U_2 = \frac{y}{P_2}\ , \tag{3}$$

where P_1 and P_2 are price indices, the validity of utility functions reduces to the validity of the price indices and it is then evident that the utility function

$$U = \frac{y}{(1-\lambda)P_1 + \lambda P_2}$$

is also valid. So we will be interested in the properties of the *weighted* sum of utility functions

$$U = \frac{(1-\lambda)P_1}{(1-\lambda)P_1 + \lambda P_2} U_1 + \frac{\lambda P_2}{(1-\lambda)P_1 + \lambda P_2} U_2 .$$

Applying Roy's identity to this gives

$$w_{wsi} = w_{1i} \frac{(1-\lambda)P_1}{(1-\lambda)P_1 + \lambda P_2} + w_{2i} \frac{\lambda P_2}{(1-\lambda)P_1 + \lambda P_2}\ , \tag{4}$$

the individual demand formulae weighted (apart from constants) by the price indices, or the reciprocals of the derivatives of utilities with respect to income.

For functions of the form (3), the product of utilities $U_1^{1-\lambda}U_2^{\lambda}$ is a valid utility function[1]. Applying (1) to this gives

$$w_{mi} = w_{1i}\frac{(1-\lambda)\frac{\partial \log U_1}{\partial \log y}}{(1-\lambda)\frac{\partial \log U_1}{\partial \log y}+\lambda\frac{\partial \log U_2}{\partial \log y}} + w_{2i}\frac{\lambda\frac{\partial \log U_2}{\partial \log y}}{(1-\lambda)\frac{\partial \log U_1}{\partial \log y}+\lambda\frac{\partial \log U_2}{\partial \log y}}, \quad (5)$$

the individual demand formulae weighted by the elasticities of utilities with respect to income. The sub-script m denotes the utilities were multiplied.

[1] If the logs of utility functions were utility functions, then the fact that the sum of utilities gives a valid utility would suffice for the product of utilities. Convexity is the crucial property. For functions of the form (3), $\log U = \log y - \log P$. Since P is a valid price index, it is concave in prices. The log function is concave and increasing, so $\log P$ is concave and therefore $\log U$ is convex.

III SIMPLE HOMOTHETIC COMPONENT UTILITY FUNCTIONS

Three simple utility functions can be generated by dividing income by price indices corresponding to (weighted) arithmetic, geometric and harmonic means. They are

$$U_a = \frac{y}{\sum \gamma_j p_j} \quad , \tag{6}$$

$$U_g = \frac{y}{\prod p_j^{\alpha_j}} \tag{7}$$

and

$$U_h = y \sum \frac{\delta_j}{p_j} \tag{8}$$

respectively. All satisfy validity conditions provided the $\gamma' s, \alpha' s$ and $\delta' s$ are positive with $\Sigma \alpha_j = 1$. The corresponding demand equation systems are obtained by applying (1) and for (6) this gives

$$w_{ai} = \frac{\gamma_i p_i}{\Sigma \gamma_j p_j} \qquad \text{or} \qquad q_{ai} = \frac{\gamma_i y}{\Sigma \gamma_j p_j}, \tag{9}$$

where q denotes quantity. These are Leontief demands in that the ratios of quantities of commodities are always in fixed proportions, irrespective of prices or income. For the ith commodity the own-price elasticity is $-w_{ai}$ and the cross-price elasticity with respect to price k is $-w_{ak}$. Note there is over-parameterisation in (9) as any one γ could be eliminated by dividing it into numerator and denominator. But the convention $\Sigma \gamma_j = 1$ is more compatible with the average price interpretation. As is well known, application of (1) to (7) leads to the Bergson, or constant budget share, demands, $w_{gi} = \alpha_i$, with own-price elasticity equalling minus one and cross-price elasticity zero. Applying (1) to (8) gives

$$w_{hi} = \frac{\frac{\delta_i}{p_i}}{\sum \frac{\delta_j}{p_j}} \qquad \text{or} \qquad q_{hi} = \frac{y \frac{\delta_i}{p_i^2}}{\sum \frac{\delta_j}{p_j}}, \tag{10}$$

with own-price elasticity –2 + w_{hi} and cross-price elasticity w_{hk}. As for (9), there is parameter redundancy in (10) and a $\Sigma \delta_j = 1$ convention matches with a harmonic mean price index.

Other simple utility functions are easily written down, for example,

$$U_r = \frac{y}{\sqrt{\sum \phi_j p_j^2}} \quad , \tag{11}$$

which is valid if the $\phi's$ are positive and which gives the demand equations

$$w_{ri} = \frac{\phi_i p_i^2}{\Sigma \phi_j p_j^2} \qquad \text{or} \qquad q_{ri} = \frac{\phi_i y p_i}{\Sigma \phi_j p_j^2}, \tag{12}$$

with own-price elasticity 1-2 w_{ri} and cross-price elasticity $-w_{rk}$. Again a redundant parameter can be accounted for by imposing $\Sigma\phi_j = 1$, which also permits interpretation of the denominator of (12) as a price index.

Even with the four utility functions (6), (7), (8) and (11), there are quite a few potential demand systems. Taking the utility functions two at a time, there are six possibilities and the three combination methods via (2), (4) and (5) makes eighteen demand systems. But how much more flexibility do they give? With income appearing as simply as it does in the four starting point utility functions, it is evident that not only have their demand systems unitary income elasticities[2], but so will the combination systems because the weights in (2), (4) and (5) are functions of prices and not income. So we are only considering greater flexibility in response to price changes. Taking as a first example the combination of (6) and (7) by (4), gives the demand system

$$w_{wsi} = \frac{(1-\lambda)\alpha_i \prod p_j^{\alpha_j} + \lambda\gamma_i p_i}{(1-\lambda)\prod p_j^{\alpha_j} + \lambda\Sigma\gamma_j p_j}, \tag{13}$$

which can be written as

$$w_{wsi} = \frac{(1-\lambda)\alpha_i \bar{p}_g + \lambda\gamma_i p_i}{(1-\lambda)\bar{p}_g + \lambda\bar{p}_a} \tag{14}$$

where the $\bar{p}'s$ denote means of prices (which are functions of parameters) with the subscript denoting the type of mean. By dividing numerator and denominator of (13) by $1-\lambda$ and writing

$$\frac{\lambda}{1-\lambda}\gamma_i = \gamma_i$$

it is possible to write (13) as

$$w_{wsi} = \frac{\alpha_i \prod p_j^{\alpha_j} + \gamma_i p_i}{\prod p_j^{\alpha_j} + \Sigma\gamma_j p_j}. \tag{15}$$

This gets rid of λ and removes the need for any constraint on the $\gamma's$, but although neater, it destroys the interpretation of $\Sigma\gamma_j p_j$ as an arithmetic mean price index. However, the device will be used in section IV.

For the demand system (9), corresponding to (6), all goods had to be price inelastic and for the constant budget share model the elasticity had to be –1. It is easily verified that for (14) the price elasticity is

$$-w_{wsi} - \alpha_i(1-\alpha_i)\frac{(1-\lambda)\overline{p}_g}{w_{wsi}[(1-\lambda)\overline{p}_g + \lambda\overline{p}_a]},$$

so that price elastic goods are possible. The cross-price elasticity with respect to price k is

$$-w_{wsk} - \alpha_i\alpha_k\frac{(1-\lambda)\overline{p}_g}{w_{wsi}[(1-\lambda)\overline{p}_g + \lambda\overline{p}_a]},$$

so that cross-price elasticities with respect to k are not constant over commodities, unlike the situation for (9) where they all equalled minus the budget share of good k. So the weighted sum of (6) and (7) does give a system with scope to represent a greater range of economic behaviour. Similar remarks apply to the sum and product combinations via (2) and (5), which are

$$w_{si} = \frac{(1-\lambda)\alpha_i\overline{p}_a + \lambda\gamma_i p_i\dfrac{\overline{p}_g}{\overline{p}_a}}{\lambda\overline{p}_g + (1-\lambda)\overline{p}_a} \qquad \text{and} \qquad w_{mi} = (1-\lambda)\alpha_i + \lambda\frac{\gamma_i p_i}{\overline{p}_a}$$

respectively. For these two systems to become the same and also equal to (14) would require $\overline{p}_g = \overline{p}_a$. But as is well known, a geometric mean is always less than an arithmetic mean unless all commodities have the same price. Although the systems are distinct, they have an evident similiarity – own price appears explicitly and linearly in all, while the other prices (and own price) occur implicitly through the price indices.

The corresponding demand systems for combinations of (7) and (8) are

$$w_{wsi} = \frac{(1-\lambda)\alpha_i\overline{p}_g + \lambda\dfrac{\delta_i}{p_i}\overline{p}_h^2}{(1-\lambda)\overline{p}_g + \lambda\overline{p}_h},$$

[2] A well known related characteristic of these basic demand systems is that they could have been derived from additive direct utility functions, for example, $\Sigma\alpha_j\log q_j$ in the case of constant budget share demands, or $\Sigma q_j^2/\phi_j$ in the case of (12).

$$w_{si} = \frac{(1-\lambda)\alpha_i \bar{p}_h + \lambda \dfrac{\delta_i}{p_i} \bar{p}_h \bar{p}_g}{\lambda \bar{p}_g + (1-\lambda)\bar{p}_h}$$

and

$$w_{mi} = (1-\lambda)\alpha_i + \lambda \frac{\delta_i \bar{p}_h}{p_i}$$

respectively, where the harmonic mean is

$$\bar{p}_h = \frac{1}{\sum \dfrac{\delta_j}{p_j}} .$$

Again, equality of the systems requires $\bar{p}_g = \bar{p}_h$, but a harmonic mean is always less than a geometric mean unless prices are equal. So again the systems are distinct, with the similarity that the reciprocal of own price appears explicitly and linearly in all three, while the other prices feature only through the price indices. As might be expected, the combinations display more flexibility in price elasticities than their components did. For example, for the demand system (10), corresponding to (8), all goods had to be price elastic, but the combinations relax this. The demand systems for combinations of (7) and (8) can be obtained too and similar comments apply. The equations for a commodity are found to explicitly feature both own price and its reciprocal, which has benefits for own-price flexibility, but other prices again feature only through price indices.

Now consider the combination of utility functions (6) and (11), or demand systems (9) and (12), via (4). This leads to

$$w_{wsi} = \frac{(1-\lambda)\gamma_i p_i + \lambda \dfrac{\phi_i p_i^2}{\bar{p}_{rq}}}{(1-\lambda)\bar{p}_a + \lambda \bar{p}_{rq}} , \tag{16}$$

where $\bar{p}_{rq}$ denotes the root quadratic price index

$$\sqrt{\sum \phi_j p_j^2} .$$

As already mentioned, all the demand systems derived in this section, have unitary income elasticity and this could be seen as a serious inflexibility. So it is if we are trying to model observed consumer demand. However, it is often considered a desirable property in applied general equilibrium studies, sometimes along with extreme parsimony in parameters. For example, if (16) is further simplified by taking $\gamma_i = \phi_{ii} = 1/n$, where n is the number of commodities, we get (in quantity rather than budget share form) the single parameter system

$$q_{wsi} = \frac{y(1-\lambda+\lambda\frac{p_i}{\overline{p}_{rq}})}{n[(1-\lambda)\overline{p}_a + \lambda\,\overline{p}_{rq}]}, \qquad (17)$$

with $\overline{p}_a$ and $\overline{p}_{rq}$ now the simplest price mean and the square root of the simple mean of squared prices. But (17) is the new class of demand equations proposed by Datta and Dixon (2000)[3] for general equilibrium models, which they believe will also be useful in a variety of other applications. From the development here it is evident theirs is a sub-class of a much wider one. They see the 'linearity' in explicit own price[4] in (17) as a particular virtue, but that is not unique to their case. The systems resulting from combinations of (6) and (11) via (2) and (5), with the same imposition of $\gamma_i = \phi_{ii} = 1/n$,

$$q_{si} = \frac{y[(1-\lambda)\frac{\overline{p}_{rq}}{\overline{p}_a} + \lambda\frac{\overline{p}_a}{\overline{p}_{rg}^2}\,p_i]}{n[(1-\lambda)\overline{p}_{rq} + \lambda\,\overline{p}_a]} \qquad \text{and} \qquad q_{mi} = \frac{y[(1-\lambda) + \lambda\frac{\overline{p}_a}{\overline{p}_{rg}^2}\,p_i]}{n\overline{p}_a}$$

share the property. Nor are these the only ones. The formulae given earlier for the demand systems from combinations of (7) and (8), if written as equations for quantities rather than budget shares, show that all three have the property. Presumably too, there will be occasions when more than a single parameter is desired, so that the more general formulae are applicable. Perhaps there may even be be situations where 'linearity' in the reciprocal of own price may be desirable instead of, or as well as, 'linearity' in own price. It is not implausible to suspect there are other useful systems for use in general equilibrium modelling besides (17) to be obtained from combinations of simple components systems.

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[3] They use $-\lambda$ instead of λ, defining it to be negative. Of course, they prove the validity conditions directly for their system rather than deducing them from the properties of components.

[4] They argue, citing Dixit and Stiglitz (1977) that sometimes the non-linearity due to p_i implicit in price indices is unimportant.

IV MORE COMPLEX COMPONENTS

In the previous section the combination of simple components with unitary income elasticities generated demand systems also with unitary income elasticities. This need not always be so. Consider combining the two utility functions (7) and

$$U_2 = 1 - \frac{\Sigma\gamma_j p_j}{y}$$

Obviously, applying Roy's identity to U_2 must give the same result as applying it to (6), that is the system (9). The elasticity of U_1 with respect to y is unity and that of U_2 is easily seen to be

$$\frac{\gamma_i y}{y - \Sigma\gamma_j p_j}.$$

Combining via (5) and using the device discussed in the previous section when writing (13) as (15), gives

$$w_{mi} = \alpha_i \frac{y - \Sigma\gamma_j p_j}{y} + \frac{\gamma_i p_i}{\Sigma\gamma_j p_j}\frac{\Sigma\gamma_j p_j}{y}, \tag{18}$$

or

$$p_i q_{mi} = \gamma_i p_i + \alpha_i (y - \Sigma\gamma_j p_j)$$

which is the famous linear expenditure system (LES). The non-unitary income elasticity arises because the weights in (18) are functions of y. Of course, it could be said this is a roundabout way to obtain the LES, since it is straightforward from its indirect utility function. It is also true that textbooks (e.g. Deaton & Muellbauer, 1980, p.145) often interpret the LES as giving a consumer's budget shares as a weighted average of a 'rich' person's and a 'poor' person's budget shares, although that is usually in the context of interpreting the γ_i as essential minimum purchased quantities and $\Sigma\gamma_j p_j$ as 'subsistence' income, and that may be an unnecessarily narrow interpretation given that formulae (2), (4) and (5) are all weighted averages of budget shares. However, the important suggestion from the derivation here is that there may be other interesting non-homothetic demand systems obtainable by considering weighted averages of the component demand systems.

A point concerning validity needs to be mentioned though. Here U_2 is not of the form of Section III and so the argument of footnote 1 showing convexity of the product utility could not be relied on. Of course, it is well known the LES utility function can be shown to satisfy validity conditions except perhaps at low incomes, but the moral is that having employed the combination formula for a product, validity may need further investigation. A related interpretation is that range of validity may change. The demand systems following from utilities (6) and (7) were valid, given the conditions on the α_i and γ_i, for all prices and incomes, but (19) requires y $> \Sigma\gamma_j p_j$.

For combination of utilities via (2) this difficulty does not arise. The sum of the same utilities gives

$$w_{si} = \frac{\alpha_i y^2 + \gamma_i p_i \prod p_j^{\alpha_j}}{y^2 + \Sigma\gamma_j p_j \prod p_j^{\alpha_j}},$$

which may be written

$$w_{si} = \alpha_i \frac{y^2}{y^2 + \Sigma\gamma_j p_j \prod p_j^{\alpha_j}} + \frac{\gamma_i p_i}{\Sigma\gamma_j p_j} \frac{\Sigma\gamma_j p_j \prod p_j^{\alpha_j}}{y^2 + \Sigma\gamma_j p_j \prod p_j^{\alpha_j}}, \qquad (19)$$

showing that is it is a combination of the same components as (18), but with different weights and hence somewhat different properties and elasticities.

However, there are more interesting possibilities. Consider the combination of (7) and

$$U_2 = 1 - \sum \gamma_j \left(\frac{p_j}{y}\right)^{\beta_j}. \qquad (20)$$

The utility function U_2 itself leads to Houthakker's (1960) indirect addilog system (IAD)

$$w_i = \frac{\gamma_i \beta_i \left(\frac{p_i}{y}\right)^{\beta_i}}{\sum \gamma_j \beta_j \left(\frac{p_j}{y}\right)^{\beta_j}}.$$

There are 2n – 1 independent parameters, since numerator and denominator can be divided by any constant. Again, however, we will define a new γ_i as the previous one multiplied by $\lambda/(1-\lambda)$. Employing (2) leads to the system

$$w_{si} = \frac{\alpha_i + \frac{\gamma_i \beta_i}{p_i}\left(\frac{p_i}{y}\right)^{\beta_i + 1} \prod p_j^{\alpha_j}}{1 + \sum \frac{\gamma_j \beta_j}{p_j}\left(\frac{p_j}{y}\right)^{\beta_j + 1} \prod p_j^{\alpha_j}} \qquad (21)$$

and using (5) gives

$$w_{mi} = \frac{\alpha_i \left\{ 1 - \sum \gamma_j \left(\frac{p_j}{y} \right)^{\beta_j} \right\} + \gamma_i \beta_i \left(\frac{p_i}{y} \right)^{\beta_i}}{1 - \sum \gamma_j (1 - \beta_j) \left(\frac{p_j}{y} \right)^{\beta_j}} . \tag{22}$$

Both (21) and (22) are of course expressible as weighted sums of constant budget share and IAD demand systems, differing in the weights, which are functions of prices and income. Taking all $\beta_i = 1$ in (21) and (22) give (19) and (18) as special cases.

Again, combining (11) and (20) via (2) and (5) gives

$$w_{si} = \frac{\frac{\phi_i p_i^2}{\Sigma \phi_j p_j^2} + \frac{\gamma_i \beta_i}{p_i} \left(\frac{p_i}{y} \right)^{\beta_i + 1} \sqrt{\Sigma \phi_j p_j^2}}{1 + \sum \frac{\gamma_j \beta_j}{p_j} \left(\frac{p_j}{y} \right)^{\beta_j + 1} \sqrt{\Sigma \phi_j p_j^2}} \tag{23}$$

and

$$w_{mi} = \frac{\frac{\phi_i p_i^2}{\Sigma \phi_j p_j^2} \left\{ 1 - \sum \gamma_j \left(\frac{p_j}{y} \right)^{\beta_j} \right\} + \gamma_i \beta_i \left(\frac{p_i}{y} \right)^{\beta_i}}{1 - \sum \gamma_j (1 - \beta_j) \left(\frac{p_j}{y} \right)^{\beta_j}} . \tag{24}$$

As before, there are special cases for all $\beta_i = 1$ with (24) taking an LES type form

$$p_i q_{mi} = \gamma_i p_i + \frac{\phi_i p_i^2}{\Sigma \phi_j p_j^2} (y - \Sigma \gamma_j p_j) \cdot$$

The systems (21), (22), (23) and (24), are considerably more complicated than those of the previous section. All involve 3n – 1 parameters as the constraint $\Sigma \alpha_j = 1$ remains operative in (21) and (22) and $\Sigma \phi_j = 1$ in (23) and (24). Detailed assessment of their properties, ranges of validity and value for modelling consumer expenditure or other data lies outside the scope of this paper. However, the demand system (22) has been separately derived and examined by Conniffe (2002). It seems potentially very useful and can be seen as a generalisation of the LES, which relaxes some of the well known inflexibilities of that system, while retaining attractive features. The other systems here, or indeed yet more that could have been derived, may also have promise. But perhaps enough has been presented to suggest the usefulness of examining sums or products of indirect utility functions.

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