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Non-negative versus positive scalar curvature

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ABSTRACT

In this note, we look at the difference, or rather the absence of a difference, between the space of metrics of positive scalar curvature and metrics of non-negative scalar curvature. The main tool to analyze the former on a spin manifold is the spectral theory of the Dirac operator and refinements thereof. This can be used, for example, to distinguish between path components in the space of positive scalar curvature metrics. Despite the fact that non-negative scalar curvature a priori does not have the same spectral implications as positive scalar curvature, we show that all invariants based on the Dirac operator extend over the bigger space. Under mild conditions we show that the inclusion of the space of metrics of positive scalar curvature into that of non-negative scalar curvature is a weak homotopy equivalence.
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RÉSUMÉ

Dans cet article, nous examinons la différence, ou plutôt l'absence d'une différence, entre l'espace des métriques à courbure scalaire positive et des métriques à courbure scalaire non-négative. Les méthodes principales pour analyser le premier pour une variété spin utilisent la théorie spectrale de l'opérateur Dirac et ses améliorations. Avec ça on peut, par exemple, distinguer des components connexes de l'espace des métriques à courbure scalaire positive. En débit du fait que courbure scalaire non-négative à priori n'ont pas les mêmes implications que courbure scalaire positive, nous démontrons que toutes les invariantes qui utilisent l'opérateur Dirac peuvent être étendues sur l'espace plus grand. Sous des conditions très faibles nous démontrons que l'inclusion de l'espace des métriques à courbure scalaire positive dans lequel de la courbure scalaire non-négative est une équivalence d'homotopie faible.
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1. Introduction

The study of the topology of spaces and moduli spaces of Riemannian metrics satisfying some form of curvature condition on a fixed manifold has for many years been an important research subject. Such curvature conditions include positive scalar curvature, positive Ricci curvature, and non-negative sectional curvature. For some recent results concerning closed manifolds, see for example [7], [5], [20], [12], [13], [11], [3], [37], [38], [41], [15], [35], and the book [36].

In this paper all manifolds under consideration will be closed and connected unless otherwise stated, and we will always assume that spaces of metrics are equipped with the C^∞-topology.

The principal theme in this paper is the comparison of (moduli) spaces of non-negative scalar curvature metrics with (moduli) spaces of positive scalar curvature metrics on closed spin manifolds M. In this context the Ricci flat metrics play a special role, and with this in mind we make the following

Definition 1.1. Let $\mathcal{N}$ denote the space of non-negative scalar curvature metrics on M. Similarly, let $\mathcal{P}$ denote the space of positive scalar curvature metrics on M. Denote by $\mathcal{RIC}_{=0}$ the space of Ricci flat metrics, and set $\mathcal{P}^\sharp := \mathcal{P} \cup \mathcal{RIC}_{=0}$.

We have the obvious inclusion relations

$$\mathcal{P} \subset \mathcal{P}^\sharp = \mathcal{P} \cup \mathcal{RIC}_{=0} \subset \mathcal{N}.$$

We claim that if $\mathcal{P} = \emptyset$, then $\mathcal{P}^\sharp = \mathcal{RIC}_{=0} = \mathcal{N}$. To see this we begin by recalling that the Trichotomy Theorem of Kazdan and Warner ([25], [26], compare [31]) implies that if M admits a non-negative scalar curvature metric for which the scalar curvature is not identically zero, then M in fact admits a positive scalar curvature metric. Thus if $\mathcal{P} = \emptyset$ and $g \in \mathcal{N}$, we conclude that $\mathrm{scal}(g) \equiv 0$. A classical result of Bourguignon (compare [4, 4.49]) now asserts that if the only metrics on M with non-negative scalar curvature are scalar flat, then any scalar flat metric on M must be Ricci flat. This establishes the claim.

Using the Ricci flow, we see that in a homotopy theoretic sense the above claim remains true in general:

Theorem 1.2. *The inclusion $\mathcal{P}^\sharp \hookrightarrow \mathcal{N}$ is a weak homotopy equivalence.*

Corollary 1.3. *Let M a closed spin manifold which does not admit a Ricci flat Riemannian metric. Then the inclusion $\mathcal{P} \hookrightarrow \mathcal{N}$ is a weak homotopy equivalence.*

Note that it is rare that a manifold admits a Ricci flat metric. For example, by [10] (see also Theorem 2.5), the fundamental group of a Ricci flat manifold contains a free abelian subgroup (possibly trivial) of finite index.

In view of Corollary 1.3, the interesting case for our investigation now is the complementary case where $\mathcal{P} \neq \mathcal{P}^\sharp$, i.e. $\mathcal{RIC}_{=0} \neq \emptyset$.

Most of the results to date concerning (moduli) spaces of positive scalar curvature metrics are established using the index theory of Dirac operators. We will present some of the relevant details concerning this in Section 3, however for now it suffices to note that one of the key results which makes index theory such an important tool in this context is the classical theorem of Schrödinger-Lichnerowicz. In order to state this, let us first recall that if (M, g) is a Riemannian spin manifold, we can consider the spin Dirac operator $\mathcal{D}$ defined by Atiyah and Singer acting on the space of sections of the spinor bundle over M. This operator depends on the metric and on the spin structure. Sections which belong to the kernel of $\mathcal{D}$ are called harmonic spinors. The basic case of the Schrödinger-Lichnerowicz Theorem then states that a compact spin manifold with positive scalar curvature admits no non-trivial harmonic spinors.

One can extend this result by generalizing the concepts of Dirac operator and harmonic spinor by twisting the spinor bundle (that is, forming the tensor product) with a flat bundle F over the same base, see for example [28, pages 164-165]. This is an important construction, and is frequently used in the case where the flat bundle is not a vector bundle, but a bundle of modules over an auxiliary C^*-algebra A. For us, the most relevant C^*-algebra is the maximal (real) C^*-algebra of the fundamental group of M, called $C^*\pi_1(M)$. In some sense there is a universal case of twisting with a flat bundle, and this involves the so-called Mishchenko line bundle L_M over M. This is a bundle with fibre a free rank one module over $C^*\pi_1(M)$, which comes equipped with a canonical flat connection. We will recall the construction of L_M in Section 3, and explain the claim that this is the "universal" case. The idea is that the spectral theory of the Dirac operator twisted with L_M contains all information which can be obtained using any kind of Dirac operator, formulated in [32] as Conjecture 1.5.

The version of the Schrödinger-Lichnerowicz Theorem which will be crucial for the results in this paper is the following. It is based on the Schrödinger-Lichnerowicz formula (equation (3)).

Theorem 1.4. *If a compact spin manifold M has positive scalar curvature, then the spin Dirac operator twisted by any flat bundle (where the fibres are vector spaces or more generally modules over a C^*-algebra) is invertible. In particular, the Dirac operator twisted with the Mishchenko line bundle is invertible in this case.*

Let us therefore make the following

Definition 1.5. Denote by $\mathcal{R}^{INV}$ the space of Riemannian metrics such that the Dirac operator twisted with the Mishchenko bundle is invertible. We call these the metrics with a *universally invertible Dirac operator*.

By Theorem 1.4 we have $\mathcal{P} \subset \mathcal{R}^{INV}$.

In general, the invertibility of the untwisted Dirac operator depends on the chosen spin structure. However, it is a basic fact that this is not so for the universal case: invertibility of the Dirac operator twisted with the Mishchenko line bundle L_M is independent of the chosen spin structure, so that $\mathcal{R}^{INV}$ is unambiguously defined. The idea is as follows: if s_1 and s_2 are two spin structures on M, then there is a (graded) real metric line bundle $L \to M$ such that the spinor bundles S_1 and S_2 are related by $S_2 = S_1 \otimes L$. The line bundle L has a canonical flat connection, and when we are in addition twisting with the Mishchenko line bundle, L can be absorbed into the latter bundle at the expense of applying an automorphism of the real group C^*-algebra. This implies that the Dirac operator on S_2 twisted by the Mishchenko line bundle is unitarily equivalent to that on S_1. (See [30, Section 3] for details.) The invertibility claim now follows immediately.

As stated above, essentially all the tools known to study $\mathcal{P}$ actually extend to $\mathcal{R}^{INV}$. Indeed, it is a challenging and open problem to understand the difference between $\mathcal{P}$ and $\mathcal{R}^{INV}$ better.

For us, however, the goal is to transfer information about the homotopy type of $\mathcal{P}$ to $\mathcal{N}$ or rather the weakly homotopy equivalent $\mathcal{P}^\sharp$. For example, we want to show that path components of $\mathcal{P}$ which belong to distinct path-components of $\mathcal{R}^{INV}$ remain distinct also in $\mathcal{P}^\sharp$. This would evidently be true if $\mathcal{P}^\sharp \subset \mathcal{R}^{INV}$, i.e. if $\mathcal{RIC}_{=0} \subset \mathcal{R}^{INV}$. However, in general this is not true.

Nonetheless, and this is one of the main results of this paper, those Ricci flat metrics which do not have a universally invertible Dirac operator are completely isolated from all the metrics with positive scalar curvature.

Theorem 1.6. *We have a disjoint union decomposition*

$$\mathcal{RIC}_{=0} = \mathcal{RIC}_{=0}^{INV} \amalg \mathcal{RIC}_{=0}^{s},$$

where $\mathcal{RIC}_{=0}^{INV} := \mathcal{R}^{INV} \cap \mathcal{RIC}_{=0}$, $\mathcal{RIC}_{=0}^{s} := \mathcal{RIC}_{=0} \setminus \mathcal{RIC}_{=0}^{INV}$, and both $\mathcal{RIC}_{=0}^{INV}$ and $\mathcal{RIC}_{=0}^{s}$ consist of a union of path-components of $\mathcal{RIC}_{=0}$. For all metrics in $\mathcal{RIC}_{=0}^{s}$, the universal covering admits a non-

trivial parallel spinor, and (in particular) the metric has special holonomy. On the other hand, no metric in $\mathcal{RIC}^{INV}_{=0}$ has a non-trivial parallel spinor on its universal cover.

Theorem 1.7. *We can write $\mathcal{P}^\sharp = \mathcal{RIC}^s_{=0} \amalg (\mathcal{P}^\sharp \setminus \mathcal{RIC}^s_{=0})$, where the former is a union of path-components of $\mathcal{P}^\sharp$ and the latter embeds into $\mathcal{R}^{INV}$. In particular, all information about the non-triviality of the homotopy type of $\mathcal{P}$ which factors through $\mathcal{R}^{INV}$ (e.g. about path-components of $\mathcal{P}$ which belong to distinct path-components of $\mathcal{R}^{INV}$) extends to $\mathcal{P}^\sharp$.*

Because the inclusion $\mathcal{P}^\sharp \hookrightarrow \mathcal{N}$ is a weak homotopy equivalence (in particular a bijection on π_0), we have an analogous decomposition into unions of path components

$$\mathcal{N} = \mathcal{RIC}^s_{=0} \amalg (\mathcal{N} \setminus \mathcal{RIC}^s_{=0}),$$

and all information about the non-triviality of the homotopy type of $\mathcal{P}$ which factors through $\mathcal{R}^{INV}$ extends to $\mathcal{N}$.

Below, we will give a number of specific examples of the principle described in Theorem 1.7.

It should be noted that the existence of a parallel spinor for some metric does not exclude the possibility that the manifold admits metrics of positive scalar curvature. For example, simply-connected Calabi-Yau 3-folds are known to admit both positive scalar curvature metrics as well as Ricci-flat metrics with parallel spinors. The existence of positive scalar curvature on these objects is automatic as the α-invariant, which is the only obstruction for simply-connected spin manifolds to admit positive scalar curvature by [34], vanishes in real dimension six.

One should remark that no example of a Ricci flat metric without parallel spinor on its universal covering is known. This means that in the decomposition of Theorem 1.6, the part $\mathcal{RIC}^{INV}_{=0}$ might well be empty in all cases. This would, by Theorem 1.7, mean that up to weak homotopy equivalence, we obtain the space of non-negative scalar curvature metrics $\mathcal{N}$ from the space of positive scalar curvature metrics $\mathcal{P}$ by just adding a collection of very special components, consisting of Ricci flat metrics with special holonomy.

The main tools to prove Theorem 1.7 from Theorem 1.6 rely on the theory of special holonomy. To apply them, we have to establish a link between universal non-invertibility and the existence of parallel spinors. This is done via the construction of harmonic spinors. On a closed manifold, it is elementary to see that the (untwisted) Dirac operator is non-invertible if and only if there is a non-trivial harmonic spinor. Moreover, if the metric has non-negative scalar curvature, the Schrödinger-Lichnerowicz formula, Equation (3), implies that a harmonic spinor is parallel, which in turn forces the holonomy to be special.

The more subtle problem is dealing with a non-invertible Mishchenko twisted Dirac operator. In general, this does not imply the existence of a non-trivial kernel, because the spectrum of such an infinite dimensional operator is not in general discrete.

However, we can make use of the fact that the existence of a Ricci flat metric implies that $\pi_1(M)$ is virtually abelian. Using this, we will show that at least for some finite dimensional twist bundle, the twisted Dirac operator has a kernel, i.e. there exists a “twisted harmonic spinor”. This will be sufficient to establish Theorem 1.6 and will be discussed in Section 3.

We now turn our attention to applications of the above results. We reiterate that almost all known invariants which detect topology in the space $\mathcal{P}$ factor through the space $\mathcal{R}^{INV}(M)$. This means that most existing results about the topology of the (moduli) space of positive scalar curvature metrics can be generalized to non-negative scalar curvature. We now present some concrete examples.

The Kreck-Stolz s-invariant is an important tool for studying the path-connectedness of moduli spaces of positive scalar curvature metrics. This was developed and first used in [24]. The s-invariant is defined for spin manifolds M^{4n-1} $(n \geq 2)$ with vanishing real Pontrjagin classes and positive scalar curvature. It is an invariant of the path-component in the space of positive scalar curvature metrics. Moreover, if

$H^1(M;\mathbb{Z}_2)=0$ (which means the spin structure on M is uniquely determined by the orientation), and g is a positive scalar curvature metric on M, then $|s(M,g)|\in\mathbb{Q}$ is an invariant of the path-component in the *moduli* space of positive scalar curvature metrics on M containing g.

Using Theorem 1.7 we can establish:

Theorem 1.8. *For a closed spin manifold (M,g) of dimension $4k-1$, ($k\geq 2$), with positive scalar curvature and vanishing real Pontrjagin classes, the Kreck-Stolz s-invariant is an invariant of the path-component of non-negative scalar curvature metrics containing g. If in addition $H^1(M;\mathbb{Z}_2)=0$, $|s|$ is an invariant of the path-component containing $[g]$ in the moduli space of non-negative scalar curvature metrics.*

From Theorem 1.8 we immediately obtain the following result, which is the non-negative scalar curvature analogue of [24] Corollary 2.15:

Corollary 1.9. *Given any M as in Theorem 1.8 with $H^1(M;\mathbb{Z}_2)=0$, the moduli space of non-negative scalar curvature metrics on M has infinitely many path-components.*

Besides the s-invariant, one can re-visit other types of results for (moduli) spaces of positive scalar curvature metrics established using index theory, and making the required adjustments re-state these as results about non-negative scalar curvature. For example, one can do this with the theorems about the higher homotopy groups of the (observer moduli) space of positive scalar curvature metrics established in [20], as these results rely on the invertibility of a family of Dirac operators which is governed by the existence or otherwise of harmonic spinors. As a sample result, extending [20, Theorem 1.1] and [5, Theorem A], we have

Theorem 1.10. *Given $k\in\mathbb{N}\cup\{0\}$, there is an $N(k)\in\mathbb{N}$ such that for each $n\geq N(k)$ and each closed spin manifold M^{4n-k-1} admitting a metric g_0 with positive scalar curvature, the homotopy group $\pi_k(\mathcal{N},g_0)$, where $\mathcal{N}$ denotes the space of non-negative scalar curvature metrics on M, contains elements of infinite order if $k\geq 1$, and infinitely many different elements if $k=0$. Their images under the Hurewicz homomorphism in $H_k(\mathcal{N})$ still have infinite order.*

Indeed, using [5], $N(k)$ can be chosen to be equal to 6 for the statement on π_k.

We expect that also the statement on H_k holds with $N(k)=6$. However, the corresponding question for $\mathcal{P}$ is not treated in [5].

In precisely the same way, one can generalize to the space $\mathcal{N}$ the classic results of Hitchin on the non-triviality of $\pi_0(\mathcal{P})$ and $\pi_1(\mathcal{P})$ for spin manifolds in dimensions 0 and 1, respectively 0 and 7 modulo 8. See [21] for the full details, or for a synopsis explaining the dependence of these results on the invertibility of the Dirac operator, see IV.7 of [28]. The same can also be said for the more recent results of Crowley-Schick ([12]), Crowley-Schick-Steimle ([13]), Botvinnik-Ebert-Randal-Williams [5] and Ebert-Randal-Williams [16], as the underlying analytic facts are precisely the same as in Hitchin's work.

We also use Theorem 1.7 to derive some new examples involving Ricci non-negative metrics. We remark that the following theorem presents merely one set of examples among many that are possible. Details of the Bott manifold B^8 appearing in this theorem are given in section 4.

Theorem 1.11. *If K^4 denotes the K3 surface, B^8 the Bott manifold, and Σ^{4n-1} is any homotopy $(4n-1)$-sphere ($n\geq 2$) which bounds a parallelisable manifold, then both $\Sigma\times K^4$ and $\Sigma\times B^8$ have infinitely many path-components of non-negative Ricci curvature metrics.*

As far as the authors are aware, Theorem 1.11 is the only result to date concerning the topology of the space of Ricci non-negative metrics in the simply-connected case. It should be noted that we cannot use Theorem 1.8 to establish these examples as the real Pontrjagin classes are not all zero, and so the s-invariant is not defined. The important thing here is that although the manifolds above are known to admit metrics which have both positive scalar and non-negative Ricci curvature, none are known to admit metrics with strictly positive Ricci curvature. There are no known obstructions to positive Ricci curvature for these manifolds: besides admitting positive scalar curvature, they also have finite fundamental group and thus comply with Myers' Theorem.

Since the initial version of this paper was made available, other results concerning the topology of the moduli space of Ricci non-negative metrics have appeared, see [35]. These results rely on the fundamental group being non-trivial, in contrast to Theorem 1.11. Specifically, [35] contains examples of manifolds for which the moduli space of Ricci non-negative metrics has infinitely many path components in both the closed case (in all dimensions ≥ 7) as well as in the complete non-compact case (in all dimensions ≥ 8). It is also established there that the higher homotopy and rational cohomology groups of the moduli space can be non-trivial in certain cases.

This paper is laid out as follows. In Section 2 we collect the geometric results and prove Theorems 1.2 and 1.7. In Section 3 we recall the basic constructions of higher index theory of (twisted) Dirac operators and harmonic spinors and prove Theorem 1.6. In Section 4 we prove the concrete applications of index theory to spaces of metrics with non-negative scalar curvature and non-negative Ricci curvature.

This paper grew out of a paper with the same title by the second-named author, and in relation to this he would like to express his deep gratitude to Bernd Ammann for his interest and extensive correspondence which considerably enhanced the paper. Thanks also go to Anand Dessai, Wilderich Tuschmann, Guofang Wei, Hartmut Weiss and Mark Walsh for their comments. Finally, we thank an anonymous referee for many helpful comments, in particular for pointing out a wrong argument in the proof of Proposition 3.5 and providing Proposition 3.6 with its proof as a remedy.

2. Proofs of the geometric results

We want to start with the proofs of our "geometric" results, which are actually independent of the higher index theory discussed at the end of the introduction.

Our first result is Theorem 1.2, which is a rather direct consequence of the powerful machinery of the Ricci flow. Versions of Theorem 1.2 are certainly known to the experts. As a preliminary, we consider the effects of the Ricci flow on metrics with non-negative scalar curvature.

Lemma 2.1. *[8, 2.18] If M is a closed manifold and g_0 is a metric on M with non-negative scalar curvature, consider the Ricci flow $g(t)$ with $g(0) = g_0$. Suppose that the flow exists for all $t \in [0, T]$. Then $g(t)$ has non-negative scalar curvature for all $t \in [0, T]$. Moreover, $g(t)$ has positive scalar curvature for all $t \in (0, T]$ unless g_0 is Ricci flat, in which case $g(t) = g_0$ for all $t \in [0, T]$.*

Proof of Theorem 1.2. Let $f\colon (D^n, S^{n-1}) \to (\mathcal{N}, \mathcal{P}^\sharp)$ be continuous. By [40, Chapter II, Lemmas (3.1) and (3.2)] (in conjunction with [40, Chapter IV, Section 7]) we have to find a homotopy

$$F\colon (D^n \times [0, T], S^{n-1} \times [0, T]) \to (\mathcal{N}, \mathcal{P}^\sharp) \quad \text{such that } F(D^n \times \{T\}) \subset \mathcal{P}^\sharp.$$

Due to the results on the short time existence of the Ricci flow and the compactness of D^n, there is indeed $T > 0$ such that the Ricci flow defines a map $F\colon D^n \times [0, T] \to \mathcal{N}$ with $F|_{D^n \times \{0\}} = f$. By [6, Theorem A], the Ricci flow depends continuously on the initial data, and thus the map F is continuous. By Lemma 2.1, $F(D^n \times (0, T]) \subset \mathcal{P}^\sharp$. This means that all the required conditions for the above homotopy are satisfied. As f was arbitrary, the assertion follows. □

We now address our second geometric result, Theorem 1.7, stating that Ricci flat metrics which do not have a universally invertible Dirac operator are isolated among metrics with non-negative scalar curvature.

Lemma 2.2. *[18, Satz 2] (See also [19].) If N is a connected Riemannian spin manifold with a non-zero parallel spinor, then N is Ricci flat.*

The existence of a parallel spinor on a compact Riemannian spin manifold has consequences beyond the Ricci flatness of the metric. Indeed, the next result shows that there cannot be positive scalar curvature metrics arbitrarily close-by.

Theorem 2.3. *([14], Theorem 4.2 and subsequent Remark) If (M, g) is a closed Riemannian spin manifold with a non-trivial parallel spinor, then there is no path of metrics g_t, with $g_0 = g$, such that $\mathrm{scal}(g_t) > 0$ for all $t > 0$. More generally, there is no path of non-negative scalar curvature metrics g_s with $g_0 = g$ containing a sequence of positive scalar curvature metrics g_{s_n}, where $s_n \xrightarrow{n\to\infty} 0$.*

The existence of a parallel spinor on a compact Riemannian spin manifold places restrictions on the holonomy group of that manifold. For a discussion about these points and detailed references, see for example [1, Section 1]. Although we will not use holonomy arguments directly, the above results from [14] depend in part on such matters. One might also compare the results in [39]. Holonomy is central to the paper [1], from which we will need the following theorem:

Theorem 2.4. *[1, Corollary 3] Let (M, g_0) be a closed Riemannian spin manifold which admits a parallel spinor on its universal cover. If g_t, $t \in [0, T]$, is a smooth family of Ricci-flat metrics on M extending g_0, then the pull-back of g_t to the universal cover admits a parallel spinor for all $t \in [0, T]$, and the dimension of the space of parallel spinors is independent of t.*

There is one final result from the literature which we will need, and this is the basic structure theorem for Ricci-flat metrics (see [10], or 4.5 of [17]), which also enters crucially in the proof of Theorem 2.4 above.

Theorem 2.5. *(The Ricci-flat structure theorem.) If (M, g) is a closed Ricci-flat manifold, then there is a finite normal Riemannian covering $\pi\colon (\bar{M}, \bar{g}) \times (T^q, h_{fl}) \to (M, g)$, where $(\bar{M}, \bar{g})$ is a simply-connected Ricci-flat manifold and (T^q, h_{fl}) is the q-torus equipped with a flat metric. In particular, $\pi_1(M)$ contains a free abelian subgroup of finite index.*

With this preparation at hand, we are now in a position to prove Theorem 1.7, assuming Theorem 1.6. The essential point is to generalize Theorem 2.3 from closed manifolds with a parallel spinor to closed manifolds whose universal covering has a parallel spinor:

Proposition 2.6. *Let (M, g_0) be a closed Riemannian manifold such that its universal covering is spin with a non-zero parallel spinor. Let $(g_t, 0 \le t \le T)$ be a continuous path of metrics with $g_t \in \mathcal{P}^\sharp$ starting at g_0. Then $g_t \in \mathcal{RIC}_{=0}$ for all $t \in [0, T]$.*

The following result will be used in the proof of Proposition 2.6.

Lemma 2.7. *If (M, g_0) is a closed Riemannian manifold such that its universal covering is spin with a non-zero parallel spinor, then there exists a finite Riemannian covering $(\bar{M}, \bar{g})$ which has a parallel spinor.*

Proof. By Lemma 2.2, the existence of a non-zero parallel spinor on the universal covering of (M, g_0) means that the universal cover is Ricci flat, from which it follows that (M, g_0) is also Ricci flat. By Theorem 2.5, some finite covering $(\bar{M}, \bar{g})$ of (M, g_0) is a Riemannian product $(N, h_N) \times (T^q, h_{fl})$ with simply connected N, so the universal covering of (M, g_0) is the Riemannian product $(N, h_N) \times (\mathbb{R}^q, h_{fl})$. The existence of a non-zero parallel spinor on a Riemannian product is equivalent to the existence of a parallel spinor on each factor individually, compare e.g. [29, Theorem 2.5]. In particular, (N, h_H) admits a parallel spinor. With a suitable spin structure, (T^q, h_{fl}) also has a parallel spinor, and we conclude that the closed manifold $(\bar{M}, \bar{g})$ admits a parallel spinor (with a suitable spin structure). □

Proof of Proposition 2.6. Let $t_1 \in [0, T]$ be maximal such that $g_t \in \mathcal{RIC}_{=0}$ for all $t \in [0, t_1]$. This exists because $\mathcal{RIC}_{=0}$ is closed. Combining Theorem 2.4 with the arguments of the above paragraph, we see that $(\bar{M}, \bar{g}_{t_1})$ has a parallel spinor (for a suitable spin structure). If $t_1 < T$ we could now directly apply Theorem 2.3 to the path $(\bar{g}_t, t_1 \le t \le T)$ of non-negative scalar curvature metrics lifted to $\bar{M}$, to deduce that $\bar{g}_t \notin \mathcal{P}$ for t close to t_1, $t > t_1$. Therefore $\bar{g}_t \in \mathcal{RIC}_{=0}$, and hence $g_t \in \mathcal{RIC}_{=0}$ for such t. This is a contradiction to the maximality of t_1, so $t_1 = T$, and the claim is proved. □

Proof of Theorem 1.7. (Assuming Theorem 1.6.) By Theorem 1.6, every metric in $\mathcal{RIC}^s_{=0}$ is such that its universal covering admits a parallel spinor. Therefore, by Proposition 2.6, a path in $\mathcal{P}^\sharp$ which starts in $\mathcal{RIC}^s_{=0}$ must remain in $\mathcal{RIC}_{=0}$. But then Theorem 2.4 implies that each metric in the path admits a parallel spinor on its universal covering. It now follows from Theorem 1.6 that the path remains in $\mathcal{RIC}^s_{=0}$, i.e. $\mathcal{RIC}^s_{=0}$ is a union of path components of $\mathcal{P}^\sharp$.

The decomposition in Theorem 1.6 shows that $\mathcal{RIC}_{=0} \setminus \mathcal{RIC}^s_{=0} \subset \mathcal{R}^{INV}$, and therefore by the Schrödinger-Lichnerowicz Theorem 1.4 we also have $\mathcal{P}^\sharp \setminus \mathcal{RIC}^s_{=0} \subset \mathcal{R}^{INV}$. □

3. Twisted index theory and harmonic spinors

In this section, we review some facts about the index theory of Dirac operators on a spin manifold M, potentially twisted with a flat Hermitian bundle, where this flat bundle is allowed to be a Hilbert A-module bundle for an auxiliary C^*-algebra A. We will then also study the theory of harmonic and parallel spinors in this context, and prove in particular Theorem 1.6.

However, we will only use A-module bundles in a very special situation. The relevant C^*-algebra always is the group C^*-algebra $C^*\pi$ of the fundamental group $\pi = \pi_1(M)$ of a Ricci flat manifold M. By the structure Theorem 2.5, π then contains a free abelian subgroup of finite index, and in particular is amenable, so there is only one group C^*-algebra: $C^*_{red}\pi = C^*_{max}\pi =: C^*\pi$.

The relevant flat $C^*\pi$-module bundle is the 'Mishchenko line bundle' over M. This is a bundle whose fibre is a free rank one module over $C^*\pi$, constructed as follows. Let $\tilde{M}$ be a universal cover of M. There is a free right action of π on $\tilde{M}$ and a left action on $C^*\pi$, which allows us to form the flat $C^*\pi$-line bundle

$$L_M := \tilde{M} \times_\pi C^*\pi \to M.$$

Despite the terminology, if we choose to view this as a complex vector bundle, its dimension is equal to the order of $\pi_1(M)$.

We note here that a *Hilbert* A-module structure on an A-module generalizes the Hermitian structure in the case $A = \mathbb{C}$; it consists of an A-valued inner product satisfying suitable axioms. The basic concepts about Hermitian structures generalize readily, compare [27].

The Mishchenko line bundle is the "universal" flat Hilbert A-module bundle in a precise sense as follows:

Proposition 3.1. *Let $E \to M$ be any flat Hermitian bundle, or more generally a Hilbert A-module bundle for some C^*-algebra A with fibre a finitely generated projective A-module P (a Hermitian bundle in the special case $A = \mathbb{C}$ and $P = \mathbb{C}^d$). Such a flat bundle corresponds to a (holonomy) representation $\rho\colon \pi \to U_A(P)$. In the special case of a Hermitian bundle this is a unitary representation $\rho\colon \pi \to U(d)$.*

By the universal property of the (maximal) group C^*-algebra, this representation extends to a C^*-algebra homomorphism $\rho\colon C^*\pi \to \mathrm{End}_A(P)$, making P a $C^*\pi$-A-bimodule (in particular, a $C^*\pi$-left module). The flat bundle E is then obtained as an associated bundle from the Mishchenko line bundle by fibrewise tensor product:

$$E = L_M \otimes_{C^*\pi} P. \tag{1}$$

If M has a spin structure then the Dirac operator $\mathcal{D}_E$ twisted by E, acting as an unbounded operator on the Hilbert A-module of L^2-sections of the spinor bundle twisted by E, is obtained from the Mishchenko twisted Dirac operator $\mathcal{D}_{L_M}$ as follows: one tensors its domain $C^*\pi$-module over $C^*\pi$ with P (and completes appropriately), and one tensors the operator with the identity,

$$\mathcal{D}_E = \mathcal{D}_{L_M} \otimes_{C^*\pi} 1_P. \tag{2}$$

Proof. All of this follows directly from the definitions. For (1) observe that $E = \tilde{M} \times_\pi P$. Moreover, $P = C^*\pi \otimes_{C^*\pi} P$, so that finally

$$E = \tilde{M} \times_\pi C^*\pi \otimes_{C^*\pi} P = L_M \otimes_{C^*\pi} P.$$

Tracing the identifications, this holds with π and $C^*\pi$ both acting on P via ρ.

The statement about the Dirac operators follows again directly from the definitions as unbounded Hilbert A-module operators, compare [27] and [33]. □

This can be used to show that invertibility of the Mishchenko-twisted Dirac operator implies invertibility for all Dirac operators twisted with flat bundles.

Theorem 3.2. *Let M be a connected spin manifold, A a C^*-algebra, $L \to M$ a flat bundle with fibres finitely generated projective A-modules, with typical fibre the A-module P. This corresponds to a (holonomy) representation $\rho\colon \pi \to U_A(P)$. As in Proposition 3.1, write L as a bundle associated to the Mishchenko bundle L_M, $L = L_M \otimes_\rho P$.*

The spectrum of the L-twisted Dirac operator $\mathcal{D}_L$ is contained in the spectrum of the Mishchenko-twisted Dirac operator $\mathcal{D}_{L_M}$. In particular, if $\mathcal{D}_{L_M}$ is invertible, i.e. 0 is not in its spectrum, the same is true for $\mathcal{D}_L$.

If the C^*-algebra homomorphism $\rho\colon C^*\pi \to \mathrm{End}_A(P)$ is injective, the spectra of $\mathcal{D}_L$ and $\mathcal{D}_{L_M}$ even coincide.

Proof. By Proposition 3.1, $\mathcal{D}_L = \mathcal{D}_{L_M} \otimes_\rho 1_P$. The statement about the spectra therefore is a direct consequence of the corresponding general and abstract result for spectra of unbounded operators on Hilbert A-modules as presented in [33, 14.25]. □

We now turn to the discussion and application of harmonic spinors. By definition, a harmonic spinor is a section of the spinor bundle belonging to the kernel of the Dirac operator. Similarly, for a finite dimensional flat Hermitian bundle E, we define an E-twisted harmonic spinor as an element in the kernel of $\mathcal{D}_E$.

It is a standard fact in the theory of elliptic self-adjoint operators that, in this situation, $\mathcal{D}$ and $\mathcal{D}_E$ are invertible if and only if there is no non-trivial (twisted) harmonic spinor.

Note that the situation is more complicated for the Mishchenko-twisted Dirac operator $\mathcal{D}_{L_M}$. Typically, if π is infinite, even if 0 is in the spectrum of $\mathcal{D}_{L_M}$, its kernel will be trivial due to the presence of a continuous spectrum in this situation.

For us, harmonic spinors are important because they give rise to parallel spinors, which we need for our special holonomy considerations. We first observe that twisted parallel spinors suffice to guarantee the existence of a regular parallel spinor on the universal covering.

Proposition 3.3. *Suppose that a closed Riemannian spin manifold M admits a non-zero parallel twisted spinor for some finite dimensional twisting bundle. Then the universal cover equipped with the pull-back metric admits a regular non-zero parallel spinor.*

Proof. Let S denote the spinor bundle on M, let $E \to M$ be a flat Hermitian bundle with corresponding (holonomy) representation $\rho\colon \pi \to U(d)$, and $S \otimes E$ the twisted spinor bundle. Suppose that $\nabla\sigma \equiv 0$ for some $\sigma \in \Gamma(S \otimes E)$.

We first observe that the pull-back $\tilde{E}$ of E to the universal cover $\tilde{M}$ is a trivial bundle with trivial flat connection: it is associated to the holonomy representation $\{1\} = \pi_1(\tilde{M}) \to \pi_1(M) \xrightarrow{\rho} U(d)$, which is obviously trivial. Consequently, as a flat bundle $\tilde{E} \cong \tilde{M} \times \mathbb{C}^d$.

The pull-back $\tilde{\sigma}$ of σ to $\tilde{M}$ is a parallel section of $\tilde{S} \otimes \tilde{E}$ with respect to the pull-back connection. This is the usual twisted spinor connection because the covering projection is a local isometry, locally preserving all structures. Using the identification $\tilde{E} = \tilde{M} \times \mathbb{C}^d$ (as flat bundles), we identify $\tilde{S} \otimes \tilde{E}$ with $(\tilde{S})^d$ (as bundle with connection), and $\tilde{\sigma}$ can be identified with a vector of d parallel spinors on $\tilde{M}$. Because σ and therefore $\tilde{\sigma}$ is non-trivial, at least one of these components is non-trivial, providing a regular non-zero parallel spinor on $\tilde{M}$. □

The next lemma is more or less standard. It is a key result for the proof of Theorem 1.4.

Lemma 3.4. *Let (M, g) be a closed connected spin manifold with non-negative scalar curvature. Let $E \to M$ be a flat finite dimensional Hermitian bundle and assume that there is a non-trivial E-twisted harmonic spinor. Then g is Ricci flat and every twisted harmonic spinor is parallel.*

Proof. The main argument needed here is well-known, see for example [28, II.8.10,II.8.17-II.8.18]. It begins with the Schrödinger-Lichnerowicz formula

$$\mathcal{D}_E{}^2 = \nabla^*\nabla + \frac{1}{4}\mathrm{scal}, \tag{3}$$

where $\mathcal{D}_E$ is the twisted Dirac operator and $\nabla^*\nabla$ is the connection Laplacian on spinors twisted by the flat bundle E with its flat connection. Because the connection of E is flat, there is no additional term on the right hand side. Given any non-trivial E-twisted harmonic spinor σ, integrating over M gives the following equation:

$$\int_M \frac{\mathrm{scal} \cdot |\sigma|^2}{4} + |\nabla\sigma|^2 = 0,$$

where the form of the second term uses the definition of the connection Laplacian $\nabla^*\nabla$. Thus in the context of non-negative scalar curvature, we see that $|\nabla\sigma| \equiv 0$ on M and, and thus σ is a non-trivial parallel E-twisted spinor. By Proposition 3.3, there is a non-zero parallel spinor on the universal covering of M. As the existence of a parallel spinor forces the metric to be Ricci flat by Lemma 2.2, the universal covering of M, and therefore M itself, are both Ricci flat. □

The final preparational result provides twisted harmonic spinors if the metric does not have a universally invertible Dirac operator, but only in the case of our very special fundamental group. This is a partial converse to Theorem 3.2, and is probably well known.

Proposition 3.5. *Let M be a closed Riemannian spin manifold such that its fundamental group π has a free abelian subgroup of finite index. If for every finite dimensional flat Hermitian bundle E the twisted Dirac operator $\mathcal{D}_E$ is invertible, then the metric has a universally invertible Dirac operator. Equivalently, if the metric does not have a universally invertible Dirac operator then it admits a non-zero twisted harmonic spinor.*

The proof relies on the following detection principle for the spectrum of Hilbert $C^*\pi$-module operators. This is our interpretation of the classical Floquet-Bloch theory. It was provided to us, together with its proof, by the anonymous referee, and we are grateful for this help.

Proposition 3.6. *Assume that the group π contains the subgroup $\mathbb{Z}^n$ with finite index d. Let a be a possibly unbounded self-adjoint Hilbert $C^*\pi$-module operator on the countably generated Hilbert $C^*\pi$-module E, such that its bounded transform $T := a(a^2+1)^{-1/2}$ satisfies the property that $S := T^2 - 1$ is compact in the sense of Hilbert $C^*\pi$-module morphisms.*

If 0 is in the spectrum of a, then 0 is already in the spectrum of $a \otimes_\rho 1$ for at least one representation $\rho\colon \pi \to U(d)$.

Proof. By the spectral mapping theorem, 0 is in the spectrum of a if and only if 0 is in the spectrum of T if and only if -1 is in the spectrum of S, so we study S instead of a.

Next, we normalize the Hilbert module by taking the direct sum of S with the zero operator on $l^2(C^*(\pi))$. By Kasparov's stability theorem, we then may assume that $E = l^2(C^*(\pi))$.

Assume initially that $\pi = \mathbb{Z}^n$, with Fourier transform isomorphism $C^*\mathbb{Z}^n \to C(T^n)$. Recall that $C^*\mathbb{Z}^n$ is the C^*-algebra of bounded operators on $l^2(\mathbb{Z}^n)$ generated by convolution with z_i, where $z_1, \ldots, z_n$ are generators of the infinite cyclic summands. The Fourier transform isomorphism $l^2(\mathbb{Z}^n) \cong L^2(T^n)$ just reinterprets z_i as a variable of the factor $S^1_i \subset \mathbb{C}$ of the torus T^n. Under this identification, convolution with z_i becomes multiplication by z_i, which is now a continuous function on T^n. In this way, $C^*\mathbb{Z}^n$ is identified with a C^*-subalgebra of $C(T^n)$, and by the Weierstraß approximation theorem is indeed all of $C(T^n)$.

Next, when passing to Hilbert $C(T^n)$-modules we have the isomorphisms $\oplus_{k\in\mathbb{N}} C(T^n) = l^2(C(T^n)) \cong C(T^n, l^2)$, with the $C(T^n)$-valued inner-product defined pointwise. The crucial fact now is that the C^*-algebra of compact $C(T^n)$-Hilbert module operators is identified with $C(T^n, K(l^2))$, where $K(l^2)$ is the algebra of compact operators on the Hilbert space l^2 with the norm topology. We thank the referee for pointing out that the corresponding statement is not true for the bounded operators, when using the norm topology on $B(l^2)$.

For a norm continuous function taking values in compact operators, $S \in C(T^n, K(l^2))$, it is clear that $S - \lambda$ is invertible if and only if for each $\rho \in T^n$ the operator $S(\rho) - \lambda = S \otimes_\rho 1 - \lambda$ is invertible. This uses the fact that the subset of invertible operators on l^2 is open in $B(l^2)$, and we use the interpretation of $\rho \in T^n$ as evaluation homomorphism $\rho\colon C(T^n) \to \mathbb{C}$.

Now we pass to the general situation, i.e. $\mathbb{Z}^n$ is a subgroup of finite index of π. Choose a set $\{g_1, \ldots, g_d\}$ of right coset representatives for $\mathbb{Z}^n$ in π. We obtain the Fourier isomorphism

$$l^2(\pi) = \oplus_{j=1}^d g_j l^2(\mathbb{Z}^n) \cong \oplus g_j L^2(T^n) = L^2(T^n, \oplus_{j=1}^d g_j \mathbb{C}),$$

with $L^2(T^n, \oplus g_j \mathbb{C})$ the space of $\mathbb{C}^d$-valued L^2-functions on T^n.

Left multiplication by an element $g \in \pi$ permutes the right cosets and maps g_j to $g_{\alpha(j)} v_j$ with $v_j \in \mathbb{Z}^n$, (v_j and the permutation α depend on g). Under our Fourier transform isomorphism, this operator becomes the operator which multiplies the j-th component with the Fourier polynomial $v_j \in C(T^n)$, and then applies pointwise the permutation matrix α. In particular, the closure, $C^*\pi$, is identified with a sub-C^*-algebra of the matrix-valued continuous functions $C(T^n, M_d(\mathbb{C}))$, which we interpret as the C^*-algebra of endomorphisms of the Hilbert $C(T^n)$-module $C(T^n)^d$.

The inclusion $C^*\pi \hookrightarrow \mathrm{End}_{C(T^n)}(C(T^n)^d)$ allows us to induce the Hilbert $C^*\pi$-module $l^2(\pi)$ up to the Hilbert $C(T^n)$-module $l^2(C^*\pi) \otimes_{C^*\pi} C(T^n)^d \cong l^2(C(T^n)^d)$. This gives rise to the embedding

$$\mathrm{End}_{C^*\pi}(l^2(C^*\pi)) \hookrightarrow \mathrm{End}_{C(T^n)}(l^2(C(T^n)^d)); S \mapsto S \otimes 1_{C(T^n)^d},$$

which maps compact elements to compact elements. By [33, 14.25], used already in the proof of Theorem 3.2, under this embedding the spectrum is unchanged. Using the special case of $\mathbb{Z}^n$ we already established, the spectrum is then detected by looking at the induced operators $(S \otimes 1_{C(T^n)^d}) \otimes_\rho 1_\mathbb{C}$ for the evaluation homomorphisms $\rho\colon T^n \to \mathbb{C}$, because $l^2(C(T^n)^d) \cong l^2(C(T^n))$.

Composed with the embedding $C^*\pi \hookrightarrow C(T^n, M_d(\mathbb{C})) = \mathrm{End}_{C(T^n)}(C(T^n)^d)$, such an evaluation homomorphism becomes the homomorphism associated to the representation $R\colon \pi \to M_d(\mathbb{C}) = \mathrm{End}(\oplus g_j\mathbb{C})$ induced up from the irreducible representation of $\mathbb{Z}^n$ corresponding to ρ. This is true because, by definition, in this induced representation $g \in \pi$ maps the basis element g_j of $\oplus g_j\mathbb{C}$ to $g_{\alpha(j)}\rho(v_j)$, if $gg_j = g_{\alpha(j)}v_j$ as above. Consequently, the spectrum of S is detected by looking at the spectrum of the operators $S \otimes 1_{C(T^n)^d} \otimes_\rho 1_\mathbb{C} = S \otimes_R 1_{\mathbb{C}^d}$ for the induced representations $R\colon \pi \to M_d(\mathbb{C})$.

To wrap up: if 0 is in the spectrum of a, then -1 is in the spectrum of S, i.e. $S + 1$ is not invertible. This implies that $(S + 1) \otimes_\rho 1$ is not invertible for some representation $\rho\colon \pi \to U(d)$. Because induction is compatible with functional calculus, this implies finally by the spectral mapping theorem that 0 is in the spectrum of $a \otimes_\rho 1$. □

Proof of Proposition 3.5. We now deal with the unbounded self-adjoint operator $\mathcal{D}_{L_M}$ on the Hilbert $C^*\pi$-module of sections of the Mishchenko-twisted spinor bundle. By definition, the metric does not have a universally invertible Dirac operator if 0 is in the spectrum of $\mathcal{D}_{L_M}$.

We can apply Proposition 3.6 to $\mathcal{D}_{L_M}$ because the bounded transform $T := \mathcal{D}_{L_M}(\mathcal{D}_{L_M}^2 + 1)^{-1/2}$ is known by elliptic theory to be a bounded self-adjoint Hilbert $C^*\pi$-module operator such that $T^2 - 1$ is compact in the Hilbert $C^*\pi$-module sense.

Therefore, Proposition 3.6 together with Theorem 3.2 on the identification of $\mathcal{D}_{L_M} \otimes_\rho id$ with $\mathcal{D}_\rho = \mathcal{D}_L$, the Dirac operator twisted with the flat bundle L associated to the representation $\rho\colon \pi \to U(d)$, imply that if 0 is in the spectrum of $\mathcal{D}_{L_M}$ then $\mathcal{D}_L$ is not invertible for at least one finite dimensional flat bundle L. □

Proof of Theorem 1.6. We begin by arguing that all metrics in $\mathcal{RIC}^s_{=0}$ admit a non-trivial parallel spinor on the universal cover. By Proposition 3.5, if $g \in \mathcal{RIC}^s_{=0}$ then it admits a non-zero twisted harmonic spinor. By Lemma 3.4 this twisted spinor is parallel, which implies by Proposition 3.3 the desired (regular) parallel spinor on the universal covering. It is a standard result that the existence of a non-zero parallel spinor forces the holonomy group to be special, compare e.g. [1].

Next, we argue that the existence of a parallel spinor on the universal covering implies that the metric does not have a universally invertible Dirac operator. Observe that by the Schrödinger-Lichnerowicz formula (3), because the Ricci and therefore scalar curvature now vanish identically, every (twisted) parallel spinor which is square integrable lies in the kernel of the Dirac operator, and this applies in particular to every (twisted) parallel spinor on a finite covering of M.

By Lemma 2.7, the parallel spinor on the universal covering gives rise to a parallel spinor on a suitable finite covering. A priori, this is for a spin structure different from the one pulled back from M. But the spinor bundle for this a priori different spin structure equals the spinor bundle for the pull-back spin structure twisted with an appropriate flat line bundle, by [30, Section 3]. Therefore a non-zero parallel spinor on the universal covering produces a twisted parallel spinor on a finite covering which by Theorem 3.2 implies that the metric does not have a universally invertible Dirac operator.

Finally, given the existence, respectively non-existence, of non-trivial parallel spinors on the universal cover for metrics in $\mathcal{RIC}^s_{=0}$, respectively $\mathcal{RIC}^{INV}_{=0}$, we note that by Theorem 2.4 there can be no path within $\mathcal{RIC}_{=0}$ linking $\mathcal{RIC}^{INV}_{=0}$ and $\mathcal{RIC}^{s}_{=0}$. Hence $\mathcal{RIC}^{INV}_{=0}$ and $\mathcal{RIC}^{s}_{=0}$ must each be a union of path-components of $\mathcal{RIC}_{=0}$. □

4. Applications via index theory

In this section we show how to reduce the proof of Theorems 1.8 and 1.11 to their counterparts for the space $\mathcal{P}$ of positive scalar curvature metrics, using Theorems 1.2 and 1.7.

Proof of Theorem 1.8. We have to show that for a given path-component $C_{\mathcal{N}}$ of $\mathcal{N}$ which contains several path components $C_1, \ldots, C_k$ of $\mathcal{P}$, the Kreck-Stolz invariants of $C_1, \ldots, C_k$ coincide.

Let g_t, $t \in [0,1]$ be a path in $C_{\mathcal{N}}$ joining say C_1 and C_2. By Theorem 1.2 we can deform this path slightly to obtain a path $\tilde{g}_t$ now in $\mathcal{P}^\sharp$ joining C_1 with C_2, as C_1, C_2 are open in $\mathcal{P}^\sharp$. Concretely, we obtain $\tilde{g}_t$ by applying the Ricci flow for a short time, starting with the metrics in g_t. The path $\tilde{g}_t$ lies in $\mathcal{P}^\sharp \backslash \mathcal{RIC}^{s}_{=0}$ because its path component contains metrics of positive scalar curvature, in contrast to $\mathcal{RIC}^{s}_{=0}$. The arguments in [24] can now be applied to the path $\tilde{g}_t$, using solely that it runs through the space $\mathcal{R}^{INV}$ of metrics with invertible Dirac operator. This gives the desired invariance properties of s. □

We remark that it is not difficult to show that the (untwisted) Dirac operator is invertible for any metric in $C_{\mathcal{N}}$, and hence the Kreck-Stolz invariant (which makes sense irrespective of the curvature) takes the same value for all metrics in this path-component, not just those with positive scalar curvature.

Turning our attention to examples, we consider two families of products, one involving a K3 surface K^4, and the other involving a Bott manifold B^8 as a factor. Recall that, as a smooth manifold, K^4 can be defined by

$$K^4 := \{(z_0, z_1, z_2, z_3) \,|\, z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{C}P^3.$$

This is known to support a Ricci flat metric, see for example [4, page 128], though since $\hat{A}(K^4) = -2$ there is no metric of positive scalar curvature.

A Bott manifold is a closed simply-connected 8-dimensional spin manifold B^8 with $\hat{A}(B^8) = 1$, which therefore does not admit a metric of positive scalar curvature. We consider here an example constructed by D. Joyce in [22] which has Spin(7)-holonomy, and thus admits a Ricci flat metric.

We also consider the set of homotopy spheres which bound parallelisable manifolds in dimensions $4n-1$, $(n \geq 2)$. Although finite for each n, the order of this family grows more than exponentially with dimension. The moduli space of positive Ricci curvature metrics for each of these spheres was shown to have infinitely many path-components in [41]. This result was established by exhibiting an infinite family of Ricci positive metrics on each sphere, and showing that these metrics can be distinguished by their s-invariants.

Proof of Theorem 1.11. It suffices to consider $\Sigma^{4n-1} \times K^4$ for some choice of homotopy sphere Σ^{4n-1} bounding a parallelisable manifold, as the argument for $\Sigma^{4n-1} \times B^8$ is identical.

In [41] it was shown that we can find a sequence of Ricci positive metrics g_i on Σ such that $s(\Sigma, g_i) \neq s(\Sigma, g_j)$ whenever $i \neq j$, so g_i and g_j belong to different path-components of the moduli space of positive scalar curvature metrics on Σ. For each i there is a parallelisable bounding manifold W_i for Σ such that g_i extends to a positive scalar curvature metric $\bar{g}_i$ over W_i (product near the boundary), see [41, Corollary 6.4].

The W_i are constructed by plumbing D^{2n}-bundles over S^{2n}. If we consider the oriented union $W_i \cup_\Sigma (-W_j)$, it is established for example in [9, page 73] that $\hat{A}(W_i \cup_\sigma (-W_j))$ is a non-zero multiple of the difference of signatures $\mathrm{sig}(W_i) - \mathrm{sig}(W_j)$. As noted in [41, §2], for $i \neq j$ we have $\mathrm{sig}(W_i) \neq \mathrm{sig}(W_j)$, and thus $\hat{A}(W_i \cup_\Sigma (-W_j)) \neq 0$. As the $\hat{A}$-genus is multiplicative for products and $\hat{A}(K^4) \neq 0$, we deduce that

$$\hat{A}((W_i \times K^4) \cup_{\Sigma\times K^4} (-W_j \times K^4)) \neq 0.$$

Let g_K denote a Ricci flat metric on K^4, and consider the product metrics g_i+g_K. These have non-negative Ricci curvature and positive scalar curvature. By the above, these metrics can be extended to positive scalar curvature metrics $\bar{g}_i + g_K$ on $W_i \times K^4$. We can now use standard results about the Atiyah-Patodi-Singer index, compare e.g. the survey [36, Section 5] for background, to obtain

$$\mathrm{ind}(\mathcal{D}^+(W_i \times K^4, \bar{g}_i + g_K)) = \mathrm{ind}(\mathcal{D}^+(W_j \times K^4, \bar{g}_j + g_K)) = 0.$$

For $i \neq j$ suppose the metrics $g_i + g_K$ and $g_j + g_K$ belong to the same path-component of non-negative scalar curvature metrics on $\Sigma \times K^4$, i.e. there is a path h_t, $t \in [0,1]$, with $\mathrm{scal}(h_t) \geq 0$, $h_0 = g_i + g_K$ and $h_1 = g_j + g_K$. By Theorems 1.2 and 1.7 (concretely, via application of the Ricci flow) we can assume that $h_t \in \mathcal{R}^{INV}$, using that $h_0, h_1 \in \mathcal{P} \subset \mathcal{R}^{INV}$ and that $\mathcal{P}$ is open in $\mathcal{N}$. Let $\bar{h}_t$ be any path of metrics on $W_i \times K^4$, starting with $\bar{g}_i + g_K$, which extend h_t (and take the form of a product near the boundary). By standard index theory arguments, the invertibility of the boundary Dirac operator along the path h_t ensures that $\mathrm{ind}(\mathcal{D}^+(W_i \times K^4, \bar{h}_t))$, is independent of t. Moreover, since the path $\bar{h}_t$ begins with a positive scalar curvature metric, we see that in fact $\mathrm{ind}\mathcal{D}^+(W_i \times K^4, \bar{h}_t) = 0$ for all t. It then follows from [2] that

$$\begin{aligned} 0 =& \mathrm{ind}(\mathcal{D}^+(W_i \times K^4, \bar{h}_1)) - \mathrm{ind}(\mathcal{D}^+(W_j \times K^4, \bar{g}_j + g_K)) \\ =& \hat{A}((W_i \times K^4) \cup_{\Sigma\times K^4} (-W_j \times K^4)) \\ \neq& 0, \end{aligned}$$

and we have a contradiction. Thus $g_i + g_K$ and $g_j + g_K$ cannot belong to the same path-component of non-negative scalar curvature metrics, and hence must belong to different path components of Ricci non-negative metrics. □

As remarked in the introduction, one can replace the homotopy spheres in Theorem 1.11 with other manifolds. For example one could use the infinite family of 7-dimensional Einstein manifolds $M_{k,l}$ considered in [24], which were shown to have infinitely many path-components of Ricci positive metrics in [24], and infinitely many path components of non-negative sectional curvature metrics in [23].

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