Irish Math. Soc. Bulletin
Number 81, Summer 2018, 57–95
ISSN 0791-5578

Aspects of Positive Scalar Curvature and Topology II

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Abstract. This is the second and concluding part of a survey article. Whether or not a smooth manifold admits a Riemannian metric whose scalar curvature function is strictly positive is a problem which has been extensively studied by geometers and topologists alike. More recently, attention has shifted to another intriguing problem. Given a smooth manifold which admits metrics of positive scalar curvature, what can we say about the topology of the *space* of such metrics? We provide a brief survey, aimed at the non-expert, which is intended to provide a gentle introduction to some of the work done on these deep questions.

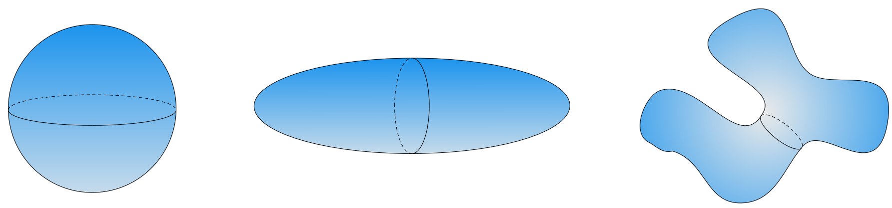

Figure 1. A selection of geometric structures on the sphere

4. The Space of Metrics of Positive Scalar Curvature

We now consider the second of our introductory questions. What can we say about the topology of the space of psc-metrics on a given smooth compact manifold? Before discussing this any further it is worth pausing to consider what we mean by a space of metrics in the first place. Recall Fig. 1, where we depict 3 distinct metrics on the 2-sphere, S^2. Each of the three images represents a point

2010 *Mathematics Subject Classification.* 46A99, 10A10.
Key words and phrases. Riemannian metrics of positive scalar curvature, spin manifolds, surgery, cobordism, Morse functions, loop spaces.
Received on 15-12-2017.
The author acknowledges support from Simons Foundation Collaboration Grant No. 280310.

in the space of metrics. One can now imagine traveling on a path in this space to consist of an animation over time which moves one such picture into another, continuously stretching and warping the sphere. Although, in our minds we usually think of these shapes extrinsically as embedded into $\mathbb{R}^3$, all of these metrics can be thought of as intrinsic geometric structures on S^2. Picturing them this way is a tougher mental exercise but worth doing, especially if one wants to consider the space of metrics on a manifold which does not embed in $\mathbb{R}^3$.

Suppose we add a constraint to our metrics on S^2. We now consider only metrics in this space with positive (Gaussian/scalar) curvature. This subspace is certainly non-empty: the round metric has positive curvature! Consider animations of the round metric which, at every frame, satisfy positive curvature. The second image in Fig.1 seems attainable, but the third metric not so as it undoubtably has some negative curvature. One question we might ask concerns the connectivity of this space. Do there exist positive curvature metrics on S^2 which cannot be connected by a path (through positive curvature metrics) to the round metric? In other words, are there distinct islands of positive curvature metrics? More generally, what can we say about higher notions of connectivity such as the fundamental group (do these islands contain lakes?) or more general homotopy groups? What about the analogous spaces for other manifolds?

It turns out that the answer to all of these questions in the case of S^2 is no. We know from work of Rosenberg and Stolz, making use of the Uniformisation Theorem, that the space of positive curvature metrics on the 2-dimensional sphere is actually a contractible space; see [45]. In a sense this is not too surprising, given the strict limitations placed by positive curvature. When imagining positive curvature geometries on S^2, it is difficult to stray too far beyond objects like the first two pictures on Fig. 1. However, if one sufficiently increases the dimension of the sphere, a great deal of non-trivial topology emerges in these spaces. For example, we know from the work of Carr in [13], that the space of positive scalar curvature metrics on the 7-sphere has infinitely many distinct path components. More generally, there are large numbers of manifolds whose space of positive scalar curvature metrics has non-trivial topological information at multiple levels. We will now attempt to give a taste of this story.

We begin with some preliminary considerations. As before, M denotes a smooth closed (compact with empty boundary) manifold of dimension $n \geqslant 2$. We let $\mathcal{R}(M)$ denote the space of all Riemannian metrics on M, under its usual C^∞-topology; see chapter 1 of [47] for details about this topology. This in fact gives $\mathcal{R}(M)$ the structure of an infinite dimensional Fréchet manifold; see chapter 1 of [47]. This enormous, infinite dimensional space is convex and so is, in a sense, not so interesting topologically. However, by specifying some geometric constraint, C, we can restrict to the subspace $\mathcal{R}^C(M) \subset \mathcal{R}(M)$ of metrics which satisfy this geometric constraint. Of course, depending on the constraint, this subspace may be empty. However, when non-empty, such subspaces may be very interesting indeed from a topological point of view. There are many geometric constraints which are of interest. For example, those interested in positive curvature may wish to study the spaces $\mathcal{R}^{\mathrm{Sec}>0}(M)$, $\mathcal{R}^{\mathrm{Ric}>0}(M)$ or $\mathcal{R}^{s>0}(M)$, the open subspaces of $\mathcal{R}(M)$ consisting of metrics with positive sectional, Ricci or scalar curvature respectively. Alternatively, one might be interested in geometric conditions such as non-negative, constant or negative curvature. Although we will say a few words later about some alternate geometric constraints,[5] our focus here is on positive scalar curvature and on understanding the topology of the space, $\mathcal{R}^{s>0}(M)$. This problem has aroused considerable attention in recent years.

At this point, we should bring up another space which is closely associated with $\mathcal{R}(M)$. This is the moduli-space of Riemannian metrics, denoted $\mathcal{M}(M)$. Before defining it we point out that two Riemannian metrics, g and g' on M, are *isometric* if there is a diffeomorphism $\phi : M \to M$ so that $g' = \phi^* g$. Here, $\phi^* g$ is the "pull-back" of the metric g under the diffeomorphism ϕ and is defined by the formula

$$\phi^* g(u, v)_x = g(d\phi_x(u), d\phi_x(v))_{\phi(x)},$$

where $x \in M$, $u, v \in T_x M$ and $d\phi_x : T_x M \to T_{\phi(x)} M$ is the derivative of ϕ. This determines an action of the group $\mathrm{Diff}(M)$, the group of self-diffeomorphisms $M \to M$, on $\mathcal{R}(M)$. The moduli space $\mathcal{M}(M)$ is then obtained as the quotient of this action on $\mathcal{R}(M)$ and, thus, is obtained from $\mathcal{R}(M)$ by identifying isometric metrics. For some, this is a more meaningful interpretation of the space of "geometries"

[5] It is also interesting to drop the compactness requirement on M, although we will not say much about non-compact manifolds here.

on M, although this is a subject of debate. Restricting the above action to a subspace of $\mathcal{R}(M)$ which satisfies a given curvature constraint leads to the moduli space of Riemannian metrics which satisfy this constraint. In particular, we will consider the moduli space of positive scalar curvature metrics: $\mathcal{M}^{s>0}(M)$. To summarise, we have the following commutative diagram, where the horizontal maps denote projections while the vertical maps are inclusions.

$$\begin{tikzcd}
\mathcal{R}(M) \arrow[r] & \mathcal{M}(M) \\
\mathcal{R}^{s>0}(M) \arrow[r] \arrow[u, hook] & \mathcal{M}^{s>0}(M) \arrow[u, hook]
\end{tikzcd}$$

The earliest results displaying topological non-triviality in the space of Riemannian metrics of positive scalar curvature are due to the landmark work of Hitchin in [29]. Given a closed smooth spin manifold M, Hitchin showed, via the α-invariant, that the action of the diffeomorphism group $\mathrm{Diff}(M)$ could be used to show that, in certain dimensions, $\mathcal{R}^{s>0}(M)$ is not path connected i.e. $\pi_0(\mathcal{R}^{s>0}(M)) \neq 0$. In fact, Hitchin also showed that $\mathcal{R}^{s>0}(M)$ may not be simply connected ($\pi_1(\mathcal{R}^{s>0}(M)) \neq 0$). We will not discuss the details of Hitchin's work here, as it will take us too far afield; for a concise discussion see [47]. One immediate consequence of Hitchin's work is the topological difference between $\mathcal{R}^{s>0}(M)$ and $\mathcal{M}^{s>0}(M)$. As the non-triviality Hitchin exhibits arises from the action of $\mathrm{Diff}(M)$, all of it disappears once we descend to the moduli space. It is important to realise that, as later results show, it is certainly possible for topological non-triviality in $\mathcal{R}^{s>0}(M)$ to survive in $\mathcal{M}^{s>0}(M)$. This is something we will discuss shortly. However it is worth pausing for a moment to consider the implication of two psc-metrics which lie in distinct path components of $\mathcal{R}^{s>0}(M)$, being projected to the same point in $\mathcal{M}^{s>0}(M)$. It is therefore perfectly possible for two Riemannian metrics of positive scalar curvature to be isometric and yet not connected by a path through psc-metrics! Intuitively, one could think of psc-metrics which are "mirror images" of each other (and thus isometric) but for which there exists no continuous animation from one to the other which, at every frame, satisfies the curvature constraint.

In recent years, we have learned that there is a great deal of topological non-triviality in the spaces $\mathcal{R}^{s>0}(M)$ and $\mathcal{M}^{s>0}(M)$. These results usually require that M be a spin manifold, although not always. Hitchin's results in particular have been significantly strengthened to deal with higher connectivity questions; see [17], [26] and [9]. Roughly, these results make use of a particular variation of the Dirac index, introduced by Hitchin in [29], and show that for certain closed spin manifolds, M, and certain psc-metrics, g, there are non-trivial homomorphisms

$$A_k(M,g) : \pi_k(\mathcal{R}^{s>0}(M), g) \longrightarrow KO_{k+n+1}.$$

The latter paper [9] by Botvinnik, Ebert and Randal-Williams contains results which are particularly powerful showing that, when the manifold dimension is at least six, this map is always non-trivial when the codomain is non-trivial. Indeed, the non-triviality detected in this paper captures not simply the non-triviality displayed in Hitchin's work but effectively all of the topological non-triviality known (for spin manifolds) up to this point. Their methods are new, highly homotopy theoretic and make use of work done by Randal-Williams and Galatius on moduli spaces of manifolds; see [23]. At the time of writing it appears that Perlmutter (whose work we will briefly mention a little later) has, using techniques developed in [44], now extended this theorem to hold for dimension 5 also. It is important to point out however that, unlike many of the results it subsumes, this is purely an existence result. In many papers, including those by Crowley and Schick [17] and by Hanke, Schick and Steimle [26], specific non-trivial elements are constructed. Indeed, the latter paper constructs an especially interesting class of examples, something we will say a few words about later on. Before we can continue this discussion however, there are some important concepts we need to introduce.

4.1. **Path Connectivity, Isotopy and Concordance.** As already suggested, the logical first question when studying the topology of the space $\mathcal{R}^{s>0}(M)$ (or $\mathcal{M}^{s>0}(M)$) concerns path connectivity. Is this space path connected? Recall that earlier we mentioned that the space of all psc-metrics on the 2-dimensional sphere S^2, $\mathcal{R}^{s>0}(S^2)$, is a contractible and therefore path-connected space; see [45]. This theorem also implies that $\mathbb{R}P^2$, the only other closed 2-dimensional manifold to admit a psc-metric, has a contractible space of psc-metrics also. This is not so surprising given the constraints positive scalar curvature place at this dimension. Indeed, in dimension 3 the situation may be similar. We know from recent work of Coda-Marques (see [16]) that the space $\mathcal{R}^{s>0}(S^3)$ is path-connected, while a number of experts have suggested that this space may well be contractible also. However, as we increase the dimension n, the scalar curvature becomes more and more flexible and so the possibility for more exotic kinds of geometric structure increases.

In dimension 4, for example, there are examples of 4-dimensional manifolds whose space of psc-metrics (as well as the corresponding moduli space) has many (even infinitely many) path components; see for example work by Ruberman in [46] and recent work by Auckly, Kim, Melvin and Ruberman in [2]. It should be said that the methods used here, such as Seiberg-Witten theory, are specific to dimension 4 and do not apply more generally. Moreover, the manifolds used here are non-trivial in their own right. In particular, they are not spheres. In fact, it is still an open question as to whether the space of psc-metrics on the 4-dimensional sphere, $\mathcal{R}^{s>0}(S^4)$, is path connected or has any non-trivial topology at all. Given that dimension 4 is very often a special case, with features and pathologies all of its own, we will focus on more general techniques for detecting disjoint path components of psc-metrics for manifolds in dimensions $\geqslant 5$.

In order to discuss the problem properly, there are a couple of helpful terms we must define: isotopy and concordance. Given the frequent use of these terms in various mathematical contexts, in particular in studying spaces of diffeomorphisms (a context which overlaps with ours), we will add the prefix "psc." Two psc-metrics g_0 and g_1 in $\mathcal{R}^{s>0}(M)$ are said to be *psc-isotopic* if they lie in the same path component of $\mathcal{R}^{s>0}(M)$, i.e. there is a continuous path $t \mapsto g_t \in \mathcal{R}^{s>0}(M)$, where $t \in [0,1]$, connecting g_0 and g_1. Such a path is called a psc-isotopy. The metrics g_0 and g_1 are said to be *psc-concordant* if there exists a psc-metric $\bar{g}$ on the cylinder $M \times [0,1]$, which near each end $M \times \{i\}$, where $i \in \{0,1\}$, respectively takes the form $g_i + dt^2$. Such a metric on the cylinder is known as a psc-concordance; see Fig. 8.

It is not difficult to verify that both notions determine equivalence relations on the set $\mathcal{R}^{s>0}(M)$, the equivalence classes of psc-isotopy being simply the path components of $\mathcal{R}^{s>0}(M)$. Moreover, it follows from a relatively straightforward calculation that psc-isotopic metrics are necessarily psc-concordant; both [21] and [25] contain

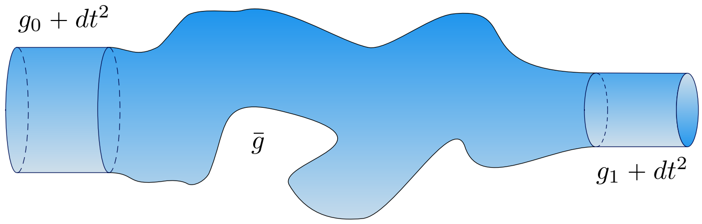

FIGURE 8. A psc-concordance $\bar{g}$ between g_0 and g_1.

versions of this calculation. The idea here is to consider the "warped product" metric $g_t + dt^2$ on the cylinder $M \times [0, 1]$ arising from a psc-isotopy $\{g_t\}_{t\in[0,1]}$ between g_0 and g_1. Although this metric may not have positive scalar curvature, since there may be unwelcome negative curvature arising in the t-direction, it can be rescaled to sufficiently slow down transition in the t-direction so that the positivity of the slices can compensate and so that the resulting metric satisfies a product structure near the ends. Thus, given a psc-isotopy, we can always construct a psc-concordance. This observation suggests at least a strategy for exhibiting distinct path components of the space $\mathcal{R}^{s>0}(M)$. Namely, show that there are psc-metrics in this space which are not psc-concordant. Given the index obstruction discussed earlier, this turns out to be a very reasonable strategy.

Before we discuss how to exhibit distinct psc-concordance classes (and thus distinct path components of $\mathcal{R}^{s>0}(M)$), let us consider the converse to the observation we have just made. Are psc-concordant metrics necessarily psc-isotopic? This is an intriguing question. The short answer is no. We know from work of Ruberman, that the Seiberg-Witten invariant detects psc-concordant metrics which are not psc-isotopic in the case of certain 4-dimensional manifolds; see [46]. But, as we noted earlier, dimension 4 has some very specific features. It might still be the case that in higher dimensions, the two notions coincide. In particular, what about the "reasonable" case of simply connected manifolds of dimension at least five (still a huge class of manifolds)?

For a long time this problem remained completely open, with little hope of progress. In [48], working in this reasonable realm of simply connected manifolds with dimension at least five, this author gave a partial affirmative answer to this question. The result held for a specific kind of psc-concordance arising via the Gromov-Lawson surgery construction, a so-called Gromov-Lawson concordance (something we will look at shortly). But it was still completely unclear as to how one would approach the general problem. We now have, from substantial work by Botvinnik in [7] and [8], the following theorem.

Theorem 4.1. (Botvinnik [8]) *Let M be a simply connected manifold of dimension at least* 5 *which admits psc-metrics. Then two psc-metrics on M are psc-concordant if and only if they are psc-isotopic.*

Botvinnik's work also deals with the non-simply connected case although the story here is a little more complicated. One reason is that, in this case, a certain space of diffeomorphisms on $M \times [0,1]$ may not be path-connected. This allows for the construction of more "exotic" types of psc-concordance. The formulation of the theorem in this case must take into account the fundamental group of the manifold M and something called its Whitehead torsion, a notion we will not discuss here. The proof of Botvinnik's theorem is formidable, incorporating deep theorems in Differential Topology and Geometric Analysis. This is not at all the place for an in-depth discussion of the proof. However it is worth making one remark on why such a proof might be so difficult. Consider the possible complexity of an arbitrary psc-concordance. The sort of psc-concordance discussed earlier, obtained by stretching a warped product metric, is about the tamest kind of psc-concordance imaginable. It does, after all, have a slicewise positive scalar curvature structure. But imagine how such a tame psc-concordance could be made "wild," by the mere act of taking a connected sum, via the Surgery Theorem, with an appropriate psc-metric on the sphere. This would not change the topology of the cylinder, yet, if the psc-metric on the sphere was suitably ugly, could produce a monstrous psc-concordance; see Fig. 9 below.

Indeed, it was observed by Gromov, that the problem of deciding whether or not two psc-concordant metrics are psc-isotopic is, in fact, *algorithmically unsolvable.* Gromov's argument makes use of the well-known fact that the problem of recognising the trivial group from an arbitrary set of generators and relations is algorithmically unsolvable. The idea is to build an arbitrarily complicated "unrecognisable" psc-concordance which represents such an arbitrary set of generators and relations, by means of a geometric construction using appropriate cells and attaching relations to build a representative cellular complex. For details, see Theorem 1.1 of [8]. As Botvinnik

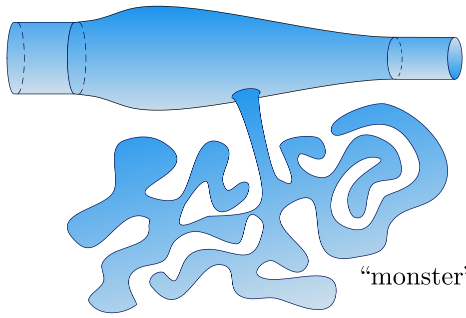

FIGURE 9. A tame psc-concordance made wild by connected sum with a "monster" psc-metric on the sphere.

demonstrates in his theorem, this does not mean that the problem is impossible to solve, just that its solution requires non-algorithmic tools, like surgery.

4.2. **Positive Scalar Curvature Cobordism.** For much of the remainder of this article, we will focus on the problem of demonstrating topological non-triviality in the space of metrics of positive scalar curvature. An important strategy in this endeavour, is to develop tools for constructing "interesting" examples of psc-metrics and, in particular, interesting families of psc-metrics. By interesting, we really mean topologically non-trivial. In the case of individual psc-metrics, this means psc-metrics which lie in distinct path components of the space of psc-metrics. More generally, we want to exhibit families of psc-metrics which represent non-trivial elements of the higher homotopy or homology groups of this space.

One approach is to make use of the action of the diffeomorphism group. As we mentioned earlier, this was part of Hitchin's technique in [29] where he exhibited such non-triviality at the level of path connectivity and the fundamental group for certain spin manifolds. We now consider another method. The principle tool we have for constructing examples of psc-metrics is the surgery technique of Gromov and Lawson. Recall that, given a manifold M and a psc-metric g on M, this technique allows us to construct explicitly a psc-metric g' on a manifold M' which is obtained from M by surgery in codimension at least three (an admissible surgery). In this section, we will consider a useful strengthening of this construction.

By way of motivation, consider a finite sequence of n-dimensional smooth compact manifolds $M_0, M_1, M_2, \cdots, M_k$, where each M_i is obtained from M_{i-1} by an admissible surgery. Thus, any psc-metric g_0 on M_0, gives rise by way of the Gromov-Lawson construction to a collection of psc-metrics $g_0, g_1, \cdots, g_k$ on the respective manifolds. Suppose furthermore that $M_0 = M_k$ (in practice we would need to identify these manifolds via some diffeomorphism $M_0 \cong M_k$, but for the sake of exposition we ignore this). This is a sort of "cyclic" condition on the sequence. It is important to realise that there are many interesting ways in which surgeries can "cancel" and the original smooth topology of M_0 be restored. Indeed it is even possible, when $n = 2p$, that a single surgery on a p-dimensional sphere in M_0 result in a manifold $M_1 = M_0$. In any case, assuming the sequence is such that M_0 is restored at the end, we may now compare the psc-metrics g_0 and g_k. Although $M_0 = M_k$, the psc-metric g_k, having possibly undergone multiple surgeries, may look very different from g_0. So how different are these psc-metrics? Could they now lie in different psc-isotopy classes? As we will shortly see, the answer to this question is yes.

To better understand the effects of surgery on psc-metrics, we need to reintroduce cobordism. Recall that a pair of smooth closed n-dimensional manifolds M_0 and M_1 are cobordant if there is a smooth compact $(n+1)$-dimensional manifold W with $\partial W = M_0 \sqcup M_1$. We now consider the following question.

Question 4. Given a psc-metric, g_0 on M_0, does this metric extend to a psc-metric, $\bar{g}$, on W which takes a product structure near the boundary of W? Thus, if it exists, the resulting metric $\bar{g}$ would satisfy $\bar{g} = g_0 + dt^2$ near M_0 and $\bar{g} = g_1 + dt^2$ near M_1, for some psc-metric g_1 on M_1.

The answer to this question depends on certain topological considerations. Before discussing this further we need to discuss some facts about cobordism and surgery.

Recall that surgery preserves the cobordism type of a manifold. Moreover, cobordant manifolds are always related by surgery. More precisely, the fact that M_0 and M_1 are cobordant means that M_1 can be obtained by successively applying finitely many surgeries to M_0 and vice versa. One way of achieving this is with a Morse function $f : W \to [0, 1]$. This is a certain smooth function satisfying $f^{-1}(i) = M_i$, where $i \in \{0, 1\}$, with only finitely many critical points all in the interior of W and satisfying the condition that, at each critical point w, $\det D^2 f(w) \neq 0$. Here, $D^2 f(w)$ is the Hessian of f at w. This latter condition means that critical points of f are of the simplest possible form and, by a lemma of Morse (see [42]), there is a choice of local coordinates $x = (x_1, \cdots, x_{n+1})$ near w where the function f takes the form

$$f(x) = c - \sum_{i=1}^{p+1} x_i^2 + \sum_{i=p+2}^{n+1} x_i^2,$$

where $f(w) = c$ and $p \in \{-1, 0, 1, \cdots, n\}$. The number $p + 1$ is referred to as the *Morse index* of the critical point w. Specifically, this is the dimension of the negative eigenspace of $D^2 f(w)$ and so is independent of any coordinate choice.

Let us assume for simplicity that w is the only critical point in the level set $f^{-1}(c)$ (this can always be obtained by a minor perturbation of the function). Then, for some $\epsilon > 0$, $f^{-1}[c - \epsilon, c + \epsilon]$ contains only w as a critical point. Moreover the level sets $f^{-1}(c \pm \epsilon)$ are smooth n-dimensional manifolds (since they contain no critical points) and, most importantly, $f^{-1}(c+\epsilon)$ is obtained from $f^{-1}(c-\epsilon)$ via a surgery on an embedded p-dimensional sphere. The cobordism $f^{-1}[c-\epsilon, c+\epsilon]$ is called an *elementary cobordism*, as it involves only one surgery. Equivalently, it is also referred to as the *trace of the surgery* on $f^{-1}(c-\epsilon)$. All of this means that any cobordism W can be decomposed into a collection of elementary cobordisms. Moreover, any surgery has a corresponding cobordism, its trace, associated to it. Thus, the sort of sequence of surgeries we introduced to motivate this section, are more efficiently described using cobordisms.

Let us suppose now that W is given such a decomposition into elementary cobordisms and that each is the trace of a surgery in codimension at least three (the key hypothesis in the Surgery Theorem). This is equivalent to saying that W admits a Morse function $f : W \to [0, 1]$ in which each critical point has Morse index $\leqslant n - 2$; such a Morse function is regarded as *admissible.* Given an admissible Morse function, f, on W then, it follows from a theorem of Gajer in [21] and later work by this author in [48], that the Surgery Theorem can be strengthened to extend a psc-metric g_0 on M_0 to a psc-metric on W satisfying the product structure described above; see Fig. 10.

This provides at least sufficient conditions for an affirmative answer to question 4. Although there are a number of choices made in this construction, the resulting psc-metric which is denoted $\bar{g} = \bar{g}(g_0, f)$ and known as a *Gromov-Lawson cobordism*, depends for the most part only on the initial psc-metric g_0 and the Morse function f. The extent to which the geometry of the psc-metric $\bar{g}$ (and in particular g_1) is affected by different choices of g_0 and f (given a fixed W) is an interesting problem in its own right and something we will return to shortly.

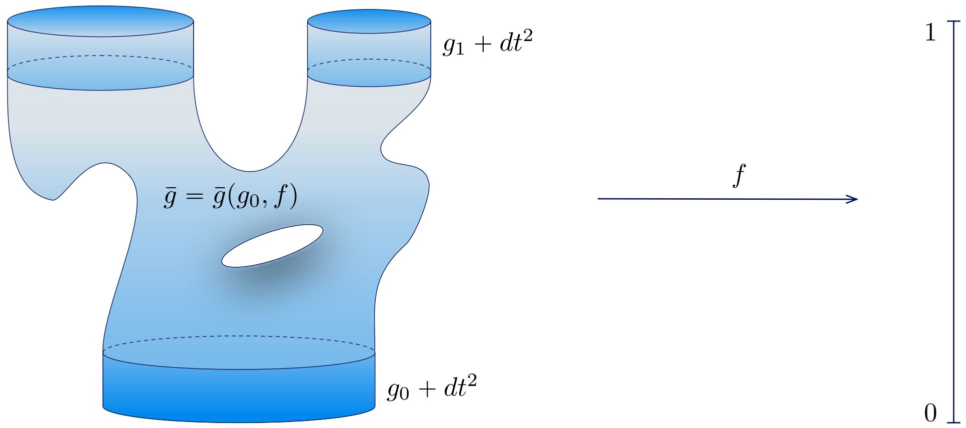

FIGURE 10. The Gromov-Lawson cobordism $\bar{g}$ arising from an admissible Morse function $f : W \to [0, 1]$ and a psc-metric g_0 on $M_0 = f^{-1}(0)$

Let us return to the problem which motivated this section. When the Gromov-Lawson construction is applied to a psc-metric over a finite sequence of admissible surgeries which result in a manifold which is the same as the starting manifold, how different are the starting and finishing psc-metrics? Equivalently, and more succinctly, if W is a cobordism with $M_0 = M_1$, $f : W \to [0, 1]$ is an admissible Morse function and g_0 is a psc-metric on M_0, how "different" can g_0 and $g_1 = \bar{g}(g_0, f)|_{M_1}$ be? In particular, could the psc-metrics g_0 and g_1 be in different path components of $\mathcal{R}^+(M_0)$? The answer to this question is yes, as we shall now demonstrate.

4.3. **Non-isotopic psc-metrics.** As we have mentioned there are many compact spin manifolds whose spin cobordism class does not lie in the kernel of the α-homomorphism and thus do **not** admit metrics of positive scalar curvature. One important example of such a manifold is a Bott manifold, named for its role in generating Bott periodicity. We consider such a manifold, denoted B, an 8-dimensional simply connected spin manifold which satisfies $\alpha([B]) = 1 \in KO_8 \cong \mathbb{Z}$. Importantly, $\alpha([B]) \neq 0$ and so B admits no psc-metrics. The topology of B is well understood; see [33] for a geometric construction. In particular, suppose we remove a pair of disjoint 8-dimensional disks, D_0 and D_1, from B. Then the resulting cobordism of 7-dimensional spheres, $W = B \setminus (D_0 \sqcup D_1)$, admits an admissible Morse function.

Recalling the Gromov-Lawson cobordism construction, we let $f : W \to [0,1]$ denote such an admissible Morse function, S_0 and S_1 denote the 7-dimensional boundary spheres of the cobordism W and $g_0 = ds_7^2$ denote the standard round metric on the S_0-boundary sphere. We now apply the Gromov-Lawson cobordism construction. The resulting psc-metric on W, $\bar{g} = \bar{g}(g_0, f)$ restricts as a psc-metric g_1 on S_1. Suppose now that g_0 and g_1 are psc-concordant and we denote by $\bar{h}$ such a psc-concordance. We now obtain a contradiction: The round psc-metric on S_0 trivially extends as a psc-metric on the disk D_0 with appropriate product structure near the boundary (a so called "torpedo" metric). By attaching the concordance $\bar{h}$ to the other end of W, at S_1, we can similarly extend g_1 to a psc-metric on D_1. But, as Fig. 11 suggests, this results in the construction of a psc-metric on B, something we know to be impossible. Thus, g_0 and g_1 are not psc-concordant and hence not psc-isotopic. Moreover, by "stacking" multiple copies of W and repeating this process, one can obtain infinitely many psc-metrics in $\mathcal{R}^{s>0}(S^7)$ which must all lie in distinct path components.

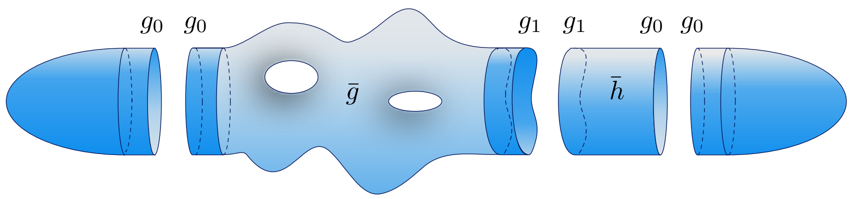

FIGURE 11. The impossible geometric decomposition of B: (left to right) the disk D_0 with torpedo metric, the cobordism W with metric $\bar{g}$, the cylinder $S_1 \times [0,1]$ equipped the proposed concordance $\bar{h}$ between g_1 and g_0 and the disk D_1 with torpedo metric

Although this is not quite the same as Carr's original proof in [13], it is very close and works for the same reasons. The argument generalises as Theorem 4.2 below. Similar arguments, making use of the α-invariant, have been used to show that $\mathcal{R}^{s>0}(M)$ has many, often infinitely many, path components for various compact spin manifolds M; see for example the work of Botvinnik and Gilkey in [12].

Theorem 4.2. (Carr [13]) *The space $\mathcal{R}^{s>0}(S^{4k-1})$ has infinitely many path components when $k \geqslant 2$.*

Interestingly, the fact that $\mathcal{R}^{s>0}(S^{4k-1})$ has *infinitely* many path components (when $k \geqslant 2$) has another important consequence, one which answers an earlier question about topological non-triviality surviving in the moduli space. It is known from work of Milnor, Kervaire [35] and Cerf [14] that the space of self-diffeomorphisms of the sphere, $\mathrm{Diff}(S^n)$, has only finitely many path components, for all n. Thus, at most finitely many of the path components demonstrated by Carr are lost when we descend to the moduli space $\mathcal{M}^{s>0}(S^n)$, meaning that this space also has infinitely many path components. Indeed this fact can be shown to hold for certain other spin manifolds by using an invariant constructed by Kreck and Stolz, called the s-invariant; see chapters 5 and 6 of [47] for a lively discussion of the s-invariant and its applications. We will not define the s-invariant save to say that it assigns a rational number $s(M,g) \in \mathbb{Q}$ to a pair consisting of a smooth closed manifold M with dimension $n = 4k-1$ and a psc-metric g. The manifold itself must satisfy certain other topological conditions concerning the vanishing of particular cohomology classes. Under the right circumstances, $|s(M,g)|$ is actually an invariant of the path component of the moduli space of psc-metrics containing g. Thus, it can often detect when $\mathcal{M}^{s>0}(M)$ is not path connected.

4.4. **Some observations about the Gromov-Lawson Cobordism Construction.** We return once more to the general construction, given an admissible Morse function $f : W \to [0,1]$ on a smooth compact cobordism W with $\partial W = M_0 \sqcup M_1$ and a psc-metric g_0 on M_0, of a Gromov-Lawson cobordism. Recall that this is a certain psc-metric $\bar{g} = \bar{g}(g_0, f)$ on W which extends g_0 and takes the form of a product metric near the boundary ∂W. In the next section we will discuss a "family" version of this construction where g_0 and

f are respectively replaced by certain families of psc-metrics and smooth functions. Here, we preempt this discussion by recalling the question of the dependency of this construction on the choices of psc-metric g_0 or Morse function f. Regarding g_0, it is demonstrated in [48] that the Gromov-Lawson cobordism construction goes through without a hitch for a compact continuously parameterised family of psc-metrics $t \mapsto g_0(t) \in \mathcal{R}^{s>0}(M_0)$, where $t \in K$ and K is some compact space. In particular, this means that if g_0 and g_0' are psc-isotopic metrics on M_0, the resulting psc-metrics $\bar{g} = \bar{g}(g_0, f)$ and $\bar{g}' = \bar{g}(g_0', f)$, as well as $g_1 = \bar{g}|_{M_1}$ and $g_1' = \bar{g}'|_{M_1}$, are psc-isotopic in their respective spaces of psc-metrics.

Perhaps a more interesting question concerns the choice of admissible Morse function. Before considering this, it is important to realise that a given manifold W will admit many many different Morse functions. For simplicity, let us assume that W is the cylinder $M_0 \times [0, 1]$. The projection functions schematically depicted in Fig. 12, which are composed with appropriate embeddings of this cylinder, determine two very different Morse functions. Thus, the fact that the cylinder is the (topologically) simplest cobordism does not prevent the existence of Morse functions with a great many critical points. Such Morse functions in turn lead to highly non-trivial decompositions of the cylinder into many non-cylindrical pieces. For such a non-trivial (and admissible) Morse function, f, the psc-metrics $\bar{g}(g_0, f)$ and g_1, may be very complicated, especially when compared with the analogous psc-metrics for the standard projection with no critical points.

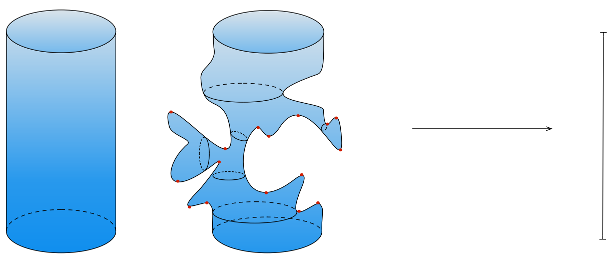

Figure 12. Two Morse functions on the cylinder $M \times [0, 1]$, one with no critical points, one with many "canceling" critical points

Suppose we denote by $\mathcal{M}\mathrm{or}(W)$, the space (under the usual C^∞-Whitney topology) of all Morse functions $W \to [0, 1]$. This space is always non-empty (in fact it is an open dense subspace of the space of all smooth functions $W \to [0,1]$). We will assume that W is such that the subspace $\mathcal{M}\mathrm{or}^{\mathrm{adm}}(W)$, of admissible Morse functions, is non-empty also. It is an important fact that two Morse functions lie in the same path component of $\mathcal{M}\mathrm{or}(W)$ only if they have the same number of critical points of each Morse index. Consequently, the spaces $\mathcal{M}\mathrm{or}(W)$ and $\mathcal{M}\mathrm{or}^{\mathrm{adm}}(W)$ are not path-connected. In fact each has infinitely many path components. The difference between $\mathcal{M}\mathrm{or}(W)$ and $\mathcal{M}\mathrm{or}^{\mathrm{adm}}(W)$ is simply that path components of the former, which contain functions with critical points whose Morse indices are not conducive to Gromov-Lawson surgery, are removed to obtain the latter.

Due to work of Hatcher and Igusa (see [27], [32] and [31]) it is, under reasonable hypotheses on W (assume W is simply connected and has dimension at least six), possible to "connect up" these path components. By this we mean extending the spaces $\mathcal{M}\mathrm{or}(W)$ (and $\mathcal{M}\mathrm{or}^{\mathrm{adm}}(W)$) to obtain path-connected function spaces $\mathcal{GM}\mathrm{or}(W)$ (and $\mathcal{GM}\mathrm{or}^{\mathrm{adm}}(W)$). These path-connected spaces are known respectively as the spaces of *generalised* and *admissible generalised Morse functions* on W and fit into the diagram of inclusions below.

$$\begin{tikzcd}
\mathcal{GM}\mathrm{or}^{\mathrm{adm}}(W) \arrow[r, hook] & \mathcal{GM}\mathrm{or}(W) \\
\mathcal{M}\mathrm{or}^{\mathrm{adm}}(W) \arrow[r, hook] \arrow[u, hook] & \mathcal{M}\mathrm{or}(W) \arrow[u, hook]
\end{tikzcd}$$

A *generalised Morse function* is a smooth function $W \to [0,1]$ which as well as Morse critical points (the ones with non-degenerate Hessian) is allowed to have a certain kind degenerate critical point known as a *birth-death singularity.* Recall we pointed out that near a Morse critical point, the function f took on a "quadratic form." Roughly speaking, a birth-death singularity takes on a cubical form. So, while the function $x \mapsto x^2$ has a Morse critical point at 0, the function $x \mapsto x^3$ has a birth-death critical point at 0. Birth-death singularities are places where certain pairs of regular Morse singularities can cancel along a path through smooth functions called an *unfolding.* A very simple example concerns the family of real-valued functions, $f_t : \mathbb{R} \to \mathbb{R}$ given by the formula

$$f_t(x) = x^3 + tx.$$

When $t < 0$, f_t is Morse with two critical points. When $t > 0$, f_t is Morse with no critical points. The function $f_0(x) = x^3$ is a generalised Morse function, with a lone birth-death singularity at $x = 0$. As t moves from negative to positive the critical points move closer together, collapsing at 0 only to disappear. Thus, from left to right, a death and from right to left, a birth. We see a higher dimensional variation of this in Fig. 13 below, where the projection function on the left hand image moves through a birth-death cancellation in the middle image to obtain the projection function (with no critical points) on the right.

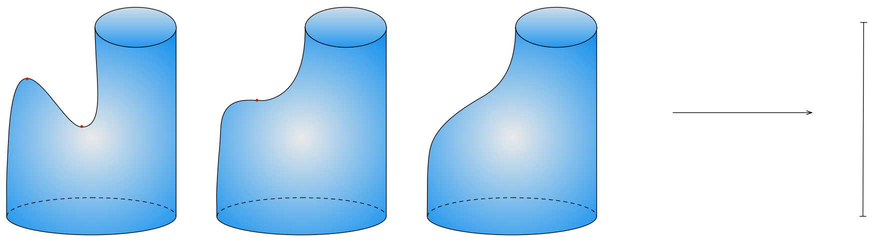

FIGURE 13. The unfolding of a birth-death singularity

We now return to the question of the dependany on the choice of admissible Morse function, f, of a Gromov-Lawson cobordism $\bar{g} = \bar{g}(g_0, f)$ on W. In [49], we make use of results by Hatcher and Igusa, to describe a parameterised version of the Gromov-Lawson cobordism construction which extends the original construction over a birth-death unfolding. In effect, we show that if f_t, with $t \in [0, 1]$, is a path in the space $\mathcal{GM}\mathrm{or}^{\mathrm{adm}}(W)$, connecting two admissible Morse functions on W, there is a corresponding isotopy $\bar{g}_t(g_0, f_t)$ through psc-metrics on W extending the original construction onto generalised Morse functions. In particular, we obtain that $\bar{g}_0 = \bar{g}(g_0, f_0)$ and $\bar{g}_1 = \bar{g}(g_0, f_1)$ are psc-isotopic. This suggests that the psc-isotopy type of the Gromov-Lawson cobordism might be independent of the choice of admissible Morse function. In order to show this of course, one needs to be able to connect with such a path, any arbitrary pair of admissible Morse functions. That is, we need that the space $\mathcal{GM}\mathrm{or}^{\mathrm{adm}}(W)$ be path-connected. Fortunately, there is a powerful theorem of Hatcher, known as the 2-Index Theorem, which sheds considerable light on this issue; see Corollary 1.4, Chapter VI of [32]. The "2-index" in the title refers to the designation of a subspace of the space of generalised Morse functions with upper and lower bounds on the indices of critical points. The theorem itself specifies levels of connectedness for such subspaces, determining that $\mathcal{GM}\mathrm{or}^{\mathrm{adm}}(W)$ is indeed path-connected provided W (along with M_0 and M_1) is simply connected and has dimension at least 6. Thus, under these conditions at least, the Gromov-Lawson cobordism construction is (up to psc-isotopy) independent of the choice of admissible Morse function. Whether or not one can find non-isotopic psc-metrics, by utilising this construction under conditions where $\mathcal{GM}\mathrm{or}^{\mathrm{adm}}(W)$ is not path connected, is an interesting open problem.

One final comment concerns the case when $W = M \times [0,1]$. Here, a Gromov-Lawson cobordism $\bar{g} = \bar{g}(g_0, f)$ is a psc-concordance of psc-metrics g_0 and $g_1 = \bar{g}|_{M\times\{1\}}$ on M. Such a psc-concordance is known as a *Gromov-Lawson concordance*, a specific case of the more general notion. As a consequence of the construction just described, we see that for simply connected manifolds of dimension $\geqslant 5$, Gromov-Lawson concordant psc-metrics are necessarily psc-isotopic. Although the existence part of this result was later subsumed by Botvinnik's solution of the general psc-concordance problem in [7] and [8], it is worth noting that the method used in [49] involves the construction of an explicit psc-isotopy.

4.5. **Family versions of the Gromov-Lawson construction.** The ability to exhibit multiple path components in the space of psc-metrics, by application of the Gromov-Lawson construction, suggests a role for a parameterised or "family" version of this construction in possibly recognising non-trivial *higher* homotopy classes of psc-metrics. Thus, we would apply the construction to families of psc-metrics (and in the cobordism case, families of admissible Morse functions), with the aim of constructing families of psc-metrics which represent non-trivial elements in the higher homotopy groups of the space of psc-metrics.

An important first step in this regard was taken by Chernysh in [15], who makes use of the fact that the original Gromov-Lawson construction works on a compact family of psc-metrics to prove the following fact: *if M and M' are mutually obtainable from each other by surgeries in codimension $\geqslant 3$, then the spaces $\mathcal{R}^{s>0}(M)$*

and $\mathcal{R}^{s>0}(M')$ *are homotopy equivalent.*[6] The implication of this result is analogous to that of the original Surgery Theorem. It hugely increases our pool of examples. In particular, once we obtain information about the topology of the space of psc-metrics for one manifold M, we now have it for a huge class of manifolds which are related to M by appropriate surgery. It is worth mentioning that, in [52], this author proves an analogue of this result for manifolds with boundary.

The idea behind the proof of Chernysh's theorem is to consider subspaces of $\mathcal{R}^{s>0}(M)$ and $\mathcal{R}^{s>0}(M')$, consisting of psc-metrics which are already "standard" near the respective surgery spheres. These subspaces, denoted respectively $\mathcal{R}^{s>0}_{\text{std}}(M)$ and $\mathcal{R}^{s>0}_{\text{std}}(M')$ are easily seen to be homeomorphic via the act of attaching (or removing) the standard handle on individual psc-metrics. It then suffices to show that the inclusion $\mathcal{R}^{s>0}_{\text{std}}(M) \subset \mathcal{R}^{s>0}(M)$ is a homotopy equivalence. From work of Palais in [43], we know that these spaces are dominated by CW-complexes and so by a famous theorem of Whitehead (Theorem 4.5 of [28]), it is enough to show that the relative homotopy groups $\pi_k(\mathcal{R}^{s>0}(M), \mathcal{R}^{s>0}_{\text{std}}(M)) = 0$ for all k. Essentially this means showing that any continuous map $\gamma : D^k \to \mathcal{R}^{s>0}(M)$, which satisfies the condition that $\gamma|_{\partial D^k}$ maps into $\mathcal{R}^{s>0}_{\text{std}}(M)$, can be continuously adjusted (via homotopy) to a map γ_{std}, whose image is contained entirely inside $\mathcal{R}^{s>0}_{\text{std}}(M)$. The important catch is that, at each stage in the homotopy, the restriction to ∂D^k must always be mapped into $\mathcal{R}^{s>0}_{\text{std}}(M)$. Application of the Gromov-Lawson construction to the family of psc-metrics parameterised by γ can be shown to continuously "move" this family into the standard subspace $\mathcal{R}^{s>0}_{\text{std}}(M)$. Unfortunately, along the way, psc-metrics which are already standard such as those parameterised by $\gamma|_{\partial D^k}$ may be temporarily moved out of $\mathcal{R}^{s>0}_{\text{std}}(M)$. As the damage to these metrics is not too severe (the Gromov-Lawson construction displaying a great deal of symmetry) this problem is solved by replacing $\mathcal{R}^{s>0}_{\text{std}}(M)$ with a larger space of "almost standard" psc-metrics which captures all adjustments made to a standard psc-metric by the Gromov-Lawson

[6] The original proof of Chernysh's result was, mysteriously, never published. In [50], this author provided a shorter version of the proof, using Chernysh's method, but making use of the heavy lifting done in [48] regarding a parameterised Gromov-Lawson construction. Although this paper is rather terse, an extremely detailed version of the proof of this theorem is provided in [52].

construction. As this space is not too much larger or more complicated than $\mathcal{R}^{s>0}_{\rm std}(M)$, it is then reasonable to show that the spaces of standard and almost standard psc-metrics are homotopy equivalent.

In [49], this author describes another family construction for positive scalar curvature metrics. The idea is to consider a smooth fibre bundle, the fibres of which are diffeomorphic to a cobordism W (of the type described above), over a smooth compact "base" manifold B. The total space of this bundle, denoted E, is equipped with a smooth function $F : E \to B \times [0,1]$, which restricts on each fibre $E_b \cong W$ over $b \in B$, as an admissible Morse function $F_b = F|_{E_b} : E_b \to \{b\} \times [0,1]$. The function F is known as a *fibrewise admissible Morse function* and is depicted schematically in Fig. 14 below. This figure also includes critical points which, in the picture, form 1-dimensional closed curves in the total space E as do their images, the critical values, in $B \times [0,1]$. Of course in practice the set of critical points (when non-empty) will have dimension the same as the base manifold, B. As such schematic pictures are limited (especially in dimension), we have depicted this bundle as if it were a trivial product bundle $E \cong B \times W$. In practice however, the bundle need not be trivial.

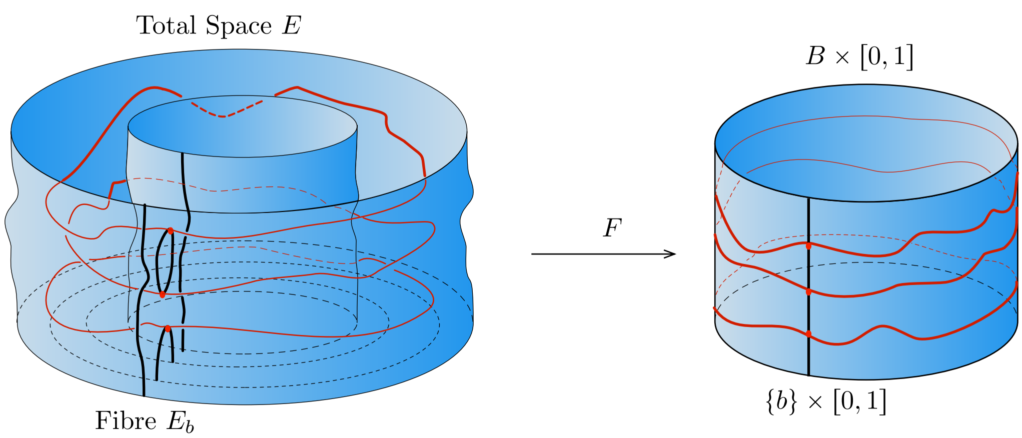

FIGURE 14. The fibrewise admissible Morse function F

Suppose we have a smooth family of psc-metrics $B \to \mathcal{R}^{s>0}(M_0)$, $b \mapsto g_0(b)$. It is then possible to construct a metric $\bar{G}$ on the total space E of the bundle which, for each $b \in B$, restricts on the fibre E_b as the Gromov-Lawson cobordism metric associated to the psc-metric $g_0(b)$ and the admissible Morse function F_b. Thus, $\bar{G}$ can be used to represent a continuous family of psc-metrics on W and by appropriate restriction, on M_1. One important point here is that, as the bundle E may be non-trivial, the fibres are only diffeomorphic to W, and not canonically so. Thus, in a sense this method is more conducive to obtaining families of psc-metrics on the moduli-spaces of psc-metrics on W and M_1. That said, a little later we will consider applying this construction to a bundle whose total space, near its boundary, takes the form of a product $M_i \times [0, \epsilon) \times B$ with $i \in \{0, 1\}$, even though the bundle itself is not trivial. This means that the metric $\bar{G}$ obtained from this construction determines, by restriction, a family of psc-metrics which unambiguously lies in $\mathcal{R}^{s>0}(M_1)$. For now, let us consider an application to the moduli space of psc-metrics.

In [10], Botvinnik, Hanke, Schick and this author exhibit non-triviality in the higher homotopy groups of the moduli space of psc-metrics, $\mathcal{M}^{s>0}(M)$, for certain manifolds M, using this construction. Initially, we work with a variation of the moduli space which is worth discussing. Recall that $\mathcal{M}(M)$, is obtained from the space of Riemannian metrics on M, as a quotient of the pull back action of $\mathrm{Diff}(M)$. We now consider a certain subgroup of $\mathrm{Diff}(M)$, denoted $\mathrm{Diff}_{x_0}(M)$ where $x_0 \in M$ is a fixed base point. This is the subgroup of diffeomorphisms which fix x_0 and whose derivative at x_0 is the identity map on $T_{x_0}M$. Thus, elements of $\mathrm{Diff}_{x_0}(M)$ all leave the point x_0 and directions emanating from this point unaltered. This point can be thought of as a sort of "observer point" on the manifold. After restricting the pull back action to this subgroup of observer respecting diffeomorphisms, we obtain $\mathcal{M}_{x_0}(M) = \mathcal{R}(M)/\mathrm{Diff}_{x_0}(M)$, the *observer moduli space of Riemannian metrics on M*. By replacing $\mathcal{R}(M)$, with $\mathcal{R}^{s>0}(M)$ (or $\mathcal{R}^{\mathrm{Ric}>0}$, $\mathcal{R}^{\mathrm{Sec}>0}(M)$), we obtain the *observer moduli space of Riemannian metrics of positive scalar (Ricci, sectional) curvature*, denoted $\mathcal{M}_{x_0}^{s>0}(M)$ ($\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(M), \mathcal{M}_{x_0}^{\mathrm{Sec}>0}(M)$). Regarding this space, the main result of [10], is stated below.

Theorem 4.3. [10] *For any $k \in \mathbb{N}$, there is an integer $N(k)$ such that for all odd $n > N(k)$, and all manifolds M admitting a psc-metric, g, the group $\pi_i(\mathcal{M}_{x_0}^{s>0}(M^n), [g])$ is non-trivial when $i \leqslant 4k$ and $i \equiv 0 \mod 4$.*

In discussing the proof of this result it is important to realise that, unlike $\mathrm{Diff}(M)$, the subgroup $\mathrm{Diff}_{x_0}(M)$ acts freely on $\mathcal{R}(M)$. This gives us some useful topological information about the resulting moduli space $\mathcal{M}_{x_0}(M)$. Consider first the case when M is the sphere, S^n. We have the following calculation, due to Farrell and Hsiang in [19], which helps explain some of the hypotheses of the theorem above. For any $k \in \mathbb{N}$, there is an integer $N(k)$ such that for all odd $n > N(k)$,

$$\pi_i(\mathcal{M}_{x_0}(S^n)) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } i \equiv 0 \mod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for appropriate i and n, we now have lots of non-trivial elements in the groups $\pi_i(\mathcal{M}_{x_0}(S^n))$ to work with. So how does this calculation help us? We have the inclusion

$$\mathcal{M}_{x_0}^{s>0}(S^n) \hookrightarrow \mathcal{M}_{x_0}(S^n),$$

which induces homomorphisms of rational homotopy groups

$$\pi_i(\mathcal{M}_{x_0}^{s>0}(S^n)) \otimes \mathbb{Q} \longrightarrow \pi_i(\mathcal{M}_{x_0}(S^n)) \otimes \mathbb{Q}.$$

If, for some $i = 4m$, we can show that some of the non-trivial elements in Farrell and Hsiang's calculation are in the image of such a homomorphism, then we have exhibited non-triviality in the observer moduli space by way of non-trivial elements in $\pi_{4m}(\mathcal{M}_{x_0}^{s>0}(S^n)$. Note that the dimension n (which is required to be odd) may be large.

In [10], we show that all of the Farrell-Hsiang elements are in the image of such a homomorphism. The idea is as follows. From work of Hatcher, we know that every element of $\pi_i(\mathcal{M}_{x_0}(S^n)) \otimes \mathbb{Q}$ determines a specific S^n bundle over S^i. These "Hatcher bundles" come naturally equipped with fibrewise admissible Morse functions on their total spaces; a comprehensive description is given by Götte in [24]. The fact that the fibres are spheres (and not manifolds with boundary) is not a problem here. One defines the fibrewise Morse function first on a bundle of "southern" hemispherical disks with global minima on the south poles. On the remainder of the disk, the function has a pair of "canceling" critical points; see Fig. 15 for a depiction of the gradient flow of such a function on the disk.

We then form a fibrewise "doubling" of this disk bundle and its fibrewise Morse function to obtain a sphere bundle, the Hatcher bundle. Away from the polar maxima and mimina, each fibre function has four remaining critical points with appropriate cancellation properties. Regarding the metric construction we equip, in a fibrewise sense, a neighbourhood of each south pole with a "torpedo" metric, before using the Gromov-Lawson cobordism construction to extend past the critical points to a family of psc-metrics on the

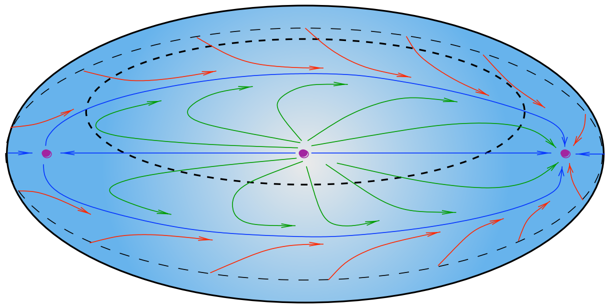

FIGURE 15. The gradient flow of the restriction of the Hatcher fibrewise Morse function to one hemisphere of one spherical fibre

hemispherical disk bundle. This is then "doubled" to obtain the desired fibrewise psc-metric on the total space of the Hatcher bundle, representing a "lift" of a non-trivial element of $\pi_i(\mathcal{M}_{x_0}(S^n))$ to a necessarily non-trivial element of $\pi_i(\mathcal{M}_{x_0}^{s>0}(S^n))$. This proves Theorem 4.3 in the case when M is the sphere. To complete the proof, the authors show that taking a fibrewise connected sum of such a representative bundle with an n-dimensional manifold M (equipped with a psc-metric), gives rise to non-trivial elements in $\pi_i(\mathcal{M}_{x_0}^{s>0}(M))$.

One important observation about the above result is that the technique used to prove it does not make use of the index obstruction α. Indeed, the method applies to both spin and non-spin manifolds, provided they admit psc-metrics and satisfy the appropriate dimension requirements. As far as we are aware, this is the first result displaying such non-triviality in any space of psc-metrics which does this. Returning to the regular moduli-space, we have the following. Using some deep arguments from algebraic topology (which we will not discuss here), the authors show that for certain suitable manifolds M, this non-triviality carries over to the traditional moduli space $\mathcal{M}^{s>0}(M)$. Interestingly, these suitable odd-dimensional manifolds are all *non-spin.*

We close this section with a brief discussion of a useful strengthening of this family Gromov-Lawson cobordism construction. Recall our earlier discussion on *generalised* Morse functions. We now reconsider the earlier smooth fibre bundle, with fibres diffeomorphic to a cobordism W, over a smooth compact "base" manifold B. The total space of this bundle, denoted E, is now equipped with a smooth function $F : E \to B \times [0,1]$, which restricts on each fibre $E_b \cong W$ over $b \in B$, as an admissible *generalised* Morse function $F_b = F|_{E_b} : E_b \to \{b\} \times [0,1]$. The function F is known as a *fibrewise admissible generalised Morse function* and is depicted schematically in Fig. 16 below. This allows for a variation in the numbers of critical points on fibres as certain Morse critical points may now cancel. In this figure, which suppresses the critical points depicted in Fig. 14, the fibres E_a, E_b and E_c depict stages of the unfolding of a birth-death critical point, with two critical points at E_a canceling at E_b. We now consider an application.

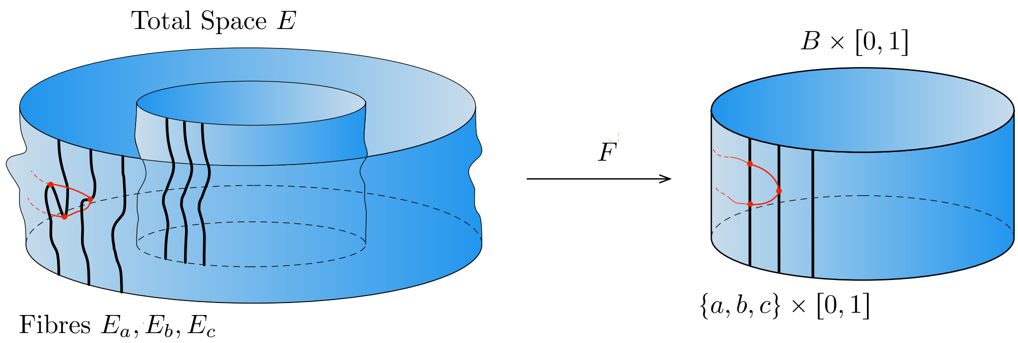

FIGURE 16. The fibrewise admissible generalised Morse function F

In [26], Hanke, Schick and Steimle construct a rather fascinating collection of objects. Recall, for any $4k$-dimensional spin manifold, we can associate a number $\hat{A}(M) \in \mathbb{Z}$, the so-called $\hat{A}$-genus. It is well known that this topological invariant, which depends only on the cobordism type of M, is multiplicative. That is, given a product of manifolds $M \times N$, we know that $\hat{A}(M \times N) = \hat{A}(M)\hat{A}(N)$. What was not clear was whether this multiplicitivity held in manifold bundles. A fibre bundle (unless it is trivial) is of course a "twisted product" and, like the Möbius band, only locally behaves like a regular product. In [26], the authors demonstrate that there exist certain manifold bundles, the fibres and base manifolds of which are compact spin manifolds but where the $\hat{A}$-genus of the total space is **not** the product of the $\hat{A}$-genera of the fibre and base. In particular, they show that there are bundles over the sphere, the total space of which is a spin manifold with non-zero $\hat{A}$-genus, but with fibres having $\hat{A} = 0$. Thus, we have a spin manifold with non-zero $\hat{A}$-genus which decomposes as a "twisted product" of spin manifolds each with zero $\hat{A}$-genus!

This allows for an intriguing analogue of the Bott manifold construction we performed earlier. The authors show that one can construct bundles of the following form. Each bundle has base manifold a sphere, S^k, and fibre a cobordism of n-dimensional spheres which we denote S_0 and S_1 to distinguish the bottom and top ends of the cobordism. There are certain dimension requirements on n and k which we will ignore. The bundle also has the property that, near its boundary, the total space E has a product structure. Thus, the total space has a well-defined bottom and top which take the form $S_i \times [0, \epsilon) \times S^k$ with $i \in \{0, 1\}$. There is thus an obvious way of capping off the ends of the total space with appropriate disk products, $D_i \times S^k$, where D_0 and D_1 are $(n+1)$-dimensional hemispheres, to form a closed manifold $\bar{E}$, from the total space. Crucially, the closed spin manifold $\bar{E}$ is such that $\hat{A}(\bar{E}) \neq 0$.

Now, the fibres of this bundle are such that the total space admits a fibrewise admissible *generalised* Morse function. Thus, given a family of psc-metrics on S_0, parameterised by the base manifold S^k, we can apply the family Gromov-Lawson cobordism construction (strengthened for generalised Morse functions) above to obtain a psc-metric $\bar{G}$ on the total space E, which restricts to a family of psc-metrics on S_1 also parameterised by S^k. For simplicity, let us assume that the family of psc-metrics on S_0 is trivial, i.e. is constantly the round metric. Thus, at the lower end of the total space E we have a standard cylindrical product of round metrics which easily extends as a positive scalar curvature metric on the "southern cap" of $\bar{E}$. As with the Bott manifold example earlier, we know that no such extension is possible at the northern cap of $\bar{E}$. This is because $\hat{A}(\bar{E}) \neq 0$ and so $\bar{E}$ admits no psc-metrics. Hence, the family of psc-metrics obtained on the sphere S_1 is homotopically distinct from the trivial one on S_0 and thus constitutes a non-trivial homotopy class in $\pi_k(\mathcal{R}^{s>0}(S^n))$. The authors go on to obtain a number of very interesting results concerning non-triviality in both the space of psc-metrics and its moduli space for certain manifolds. In particular, they prove the following theorem.

Theorem 4.4. (Hanke, Schick, Steimle [26]) *Given $k \in \mathbb{N} \cup \{0\}$, there is a natural number $N(k)$ such that for all $n \geqslant N(k)$ and* *each spin manifold M^{4n-k+1} admitting a psc-metric g_0, the homotopy group $\pi_k(\mathcal{R}^{s>0}(M); g_0)$ contains elements of infinite order when $k \geqslant 1$ and infinitely many distinct elements when $k = 0$.*

To distinguish some of the non-trivial elements constructed in this paper from those constructed by Hitchin in [29] or by Crowley and Schick in [17] (all of which become trivial in the moduli space), the authors introduce the notion of "geometrically significant" elements. Essentially, elements of $\pi_k(\mathcal{R}^{s>0}(M); g_0)$ are *not* geometrically significant if they can be obtained from a single fixed psc-metric on M via pull-back over an S^k-parameterised family of oriented diffeomorphisms $M \to M$. Otherwise such an element is *geometrically significant.* Obviously, non-geometrically significant elements become trivial in the homotopy groups of the moduli space $\mathcal{M}^{s>0}(M)$. The authors go on to show that many of the elements they construct are in fact geometrically significant. In particular, in the case when the manifold M satisfies the additional hypothesis of being the fibre of an oriented fibre bundle over the sphere S^{k+1} whose total space has vanishing $\hat{A}$-genus, then the groups $\pi_k(\mathcal{M}^{s>0}(M); [g_0])$ also contain elements of infinite order.

In closing this section, we should mention a very significant recent result of Nathan Perlmutter concerning spaces of Morse functions. In [44], he constructs something called a *cobordism category* for Morse functions. We will not attempt to define the term cobordism category here except to say that it (and in particular its associated classifying space) allows us to view the set all manifolds of a particular dimension as a single "space of manifolds" and from this distill aspects of the structure which are stable under certain operations such as surgery. This idea, which is still relatively new, was developed by Galatius, Madsen, Tillmann and Weiss in [22]. Here the authors use it to provide a new proof of a famous problem called the Mumford Conjecture, following the original proof by Madsen and Weiss in [39].

Various versions of cobordism category exist which deal not simply with manifolds, but with manifolds equipped with extra structure, such as Riemannian metrics, complex or symplectic structures. In Perlmutter's case, he considers a category which, very roughly, has as objects n-dimensional manifolds (embedded in a dimensionally large Euclidan space) and as morphisms, $(n + 1)$-dimensional manifolds with boundary which form cobordisms between objects.

Importantly these cobordisms come with a Morse function which arises as the projection, onto a fixed axis, of an embedding of the cobordism in Euclidean space (such as that depicted in Fig. 12). Perlmutter's results extend work of Madsen and Weiss in [39] on understanding the homotopy type of the classifying space of this category, to deal with subcategories where the Morse functions have bounds placed on the indices of their critical points. In particular, Perlmutter sheds a great deal of light on the case of *admissible* Morse functions. This is something which, as Perlmutter explains in his paper, has great significance for positive scalar curvature, particularly in the construction and analysis of a positive scalar curvature cobordism category.

4.6. **H-Space and Loop Space Structure.** We close this section with a very brief discussion regarding another aspect of the topology of the space of positive scalar curvature metrics. Up to now, our search for topological information has essentially meant a search for non-trivial elements of the homotopy groups of the space. We should mention that, although we have not discussed the homology or cohomology of spaces of psc-metrics, there is certainly non-trivial topological information there also, much of it following from that found in the homotopy groups. Indeed, in the paper by Hanke, Schick and Steimle above, [26], the authors explicitly show that the infinite order elements they construct in $\pi_k(\mathcal{R}^{s>0}(M); g_0)$ have infinite order images in the corresponding homology groups, under the Hurewicz homomorphism: $\pi_k(\mathcal{R}^{s>0}(M); g_0) \to H_k(\mathcal{R}^{s>0}(M))$. We now examine the space of psc-metrics for a layer of structure which has substantial homotopy theoretic implications. This concerns the question of whether or not $\mathcal{R}^{s>0}(M)$ admits a multiplicative *H-space* structure or, more significantly, whether this space has the structure of a *loop space*.

We begin with multiplication. A topological space, Z, is an *H-space* if Z is equipped with a continuous multiplication map $\mu : Z \times Z \to Z$ and an identity element $e \in Z$ so that the maps from Z to Z given by $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are both homotopy equivalent to the identity map $x \mapsto x$.[7] An H-space Z is said to be homotopy commutative if the maps μ and $\mu \circ \omega$, where $\omega : Z \times Z \to Z \times Z$ is the "flip" map defined by $\omega(x, y) = (y, x)$, are homotopy

[7] There are stronger versions of this definition (see section 3.C. of [28]), however all of these versions coincide with regard to the spaces we will consider.

equivalent. Finally, Z is a homotopy associative H-space if the maps from $Z \times Z \times Z$ to Z given by $(x, y, z) \mapsto \mu(\mu(x, y), z)$ and $(x, y, z) \mapsto \mu(x, \mu(y, z))$ are homotopy equivalent. The condition that a topological space is an H-space has various implications for its homotopy type and specifically in homology; see section 3.C. of [28]. For example, H-space multiplication gives the cohomology ring the structure of an algebra, a so-called *Hopf algebra.* Another consequence is that the fundamental group of an H-space is always abelian. Hence, it is worth investigating if such structure can be found in our spaces of psc-metrics.

Recall Corollary 3.3of the Surgery Theorem. Manifolds of dimension $\geqslant 3$ admitting psc-metrics may be combined by the process of connected sum to obtain new manifolds admitting psc-metrics. Thus, we have a "geometric connected sum" construction as opposed to a purely topological one. Consider now a pair of psc-metrics on the sphere, S^n, with $n \geqslant 3$. Given that a connected sum of spheres is still a sphere, we now have a way of combining these psc-metrics, via this geometric connected sum, to obtain a new psc-metric on the sphere. This suggests a possible multiplicative structure on the space of psc-metrics on S^n. There are a number technical issues to overcome. In particular, the various choices involved in the connected sum construction mean that, as it stands, this operation is far from well-defined. However, it is shown by this author in [51] that a careful refinement of this geometric connected sum construction leads to the following result.

Theorem 4.5. [51] *When $n \geqslant 3$, the space $\mathcal{R}^{s>0}(S^n)$ is homotopy equivalent to a homotopy commutative, homotopy associative H-space.*

The following corollary follows from a standard fact about H-spaces (mentioned above) when $n \geqslant 3$ and from the fact that $\mathcal{R}^{s>0}(S^n)$ is contractible when $n = 2$.

Corollary 4.6. [51] *When $n \geqslant 2$, the space $\mathcal{R}^{s>0}(S^n)$ has abelian fundamental group.*

The idea behind the proof of Theorem 4.5 is to replace $\mathcal{R}^{s>0}(S^n)$ with a particular subspace, $\mathcal{R}^{s>0}_{\text{std}}(S^n)$, of metrics which take the form of a standard torpedo metric near a fixed base point (the north pole say). As discussed earlier in the proof of Chernysh's theorem, the inclusion $\mathcal{R}^{s>0}_{\text{std}}(S^n) \subset \mathcal{R}^{s>0}(S^n)$ is a homotopy equivalence and so it suffices to work with the subspace. At least now there seems to be a well-defined way of taking connected sums: just remove the caps near the north poles of two such standard psc-metrics and glue. Of course the problem here is that the resulting psc-metric no longer has a standard torpedo at its north pole. We get around this problem by using an intermediary "tripod" metric, as shown in Fig. 17. This metric, as well as having a north pole torpedo, also has two identical torpedo attachments in its southern hemisphere. These southern torpedoes are used to connect distinct psc-metrics in the space, while the northern torpedo ensures that the multiplication is closed. The reader should view Fig. 17 as an equation with the highlighted tripod metric playing the role of a multiplication sign. Verifying that this multiplication is homotopy commutative is not so difficult, the homotopy in question being a simple rotation. Ensuring homotopy associativity is a little more complicated and involves a very careful sequence of geometric manouvres; details can be found in [51].

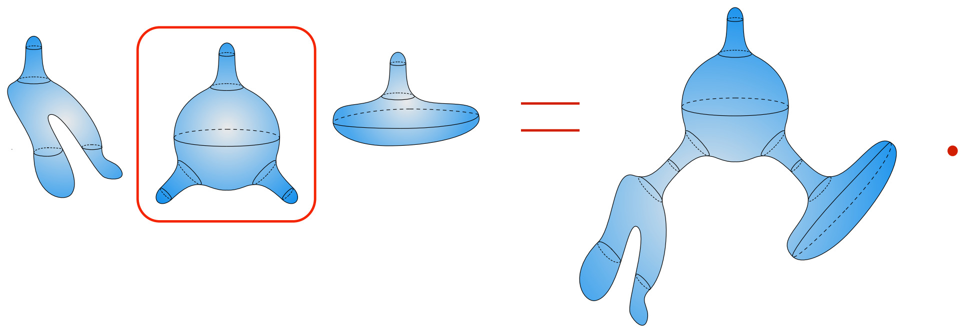

FIGURE 17. Mulitplying two metrics in $\mathcal{R}^{s>0}_{\mathrm{std}}(S^n)$

There is another level of structure, one which goes deeper than H-space structure, which we now consider. Given a topological space Y with a fixed base point $y_0 \in Y$, we define the *loop space of* Y denoted ΩY, as the space of all continuous maps $\gamma : [0,1] \to Y$ so that $\gamma(0) = \gamma(1) = y_0$. Repeated application of this construction yields the k-th iterated loop space $\Omega^k Y = \Omega(\Omega \cdots (\Omega Y))$ where at each stage the new base point is simply the constant loop at the old base point. It is a fact that every loop space is also an H-space with the multiplication determined by concatenation of loops. However the condition of being a loop space is stronger and an H-space may not even be homotopy equivalent to a loop space. As before, such structure has important topological consequences and knowing for example that a topological space Z has the homotopy type of a loop space (that is Z is homotopy equivalent to ΩY for some topological space Y) or better yet an iterated loop space is very helpful in understanding homotopy type.

Bearing in mind the case of $\mathcal{R}^{s>0}(S^n)$, we consider the problem of how to tell if a given H-space has the structure of a loop space. In general, this is a complicated problem concerning certain "coherence" conditions on the homotopy associativity of the multiplication. Roughly speaking, given an H-space Z, we compare the different maps obtainable from a k-fold product, Z^k, to Z by prioritizing (with appropriate brackets) the multiplication of components. For example, in the case when $k = 4$, two such maps are given by:

$$(a, b, c, d) \mapsto (ab)(cd) \quad \text{and} \quad (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}) \mapsto ((\mathrm{ab})\mathrm{c})\mathrm{d},$$

with H-space multiplication now denoted by juxtaposition. These maps may not agree on the nose, but we would like that they are at least homotopic. In the case when $k = 3$, this is precisely the homotopy associativity condition defined above. Returning to the case when $k = 4$, there are 5 maps to consider. These can be denoted in specific fashion by vertices of a pentagonal polyhedron P; see Fig. 18. Suppose we can specify a map

$$P \times Z^4 \longrightarrow Z,$$

which restricts as these various "rebracketing" maps on the vertices and, on the edges, specifies homotopies between specific vertex maps. We then say that Z is an A_4-space. More generally, for each k there is a corresponding polyhedron which describes these higher associativity relations, leading to the notion of an A_k space. This approach was developed by James Stasheff and so these shapes are known as *Stasheff polyhedra* (or sometimes as "associahedra"). A space which is A_k for all k is known as an A_∞-space. It is a theorem of Stasheff that a space is an A_∞-space, if and only if it is a loop space; see Theorem 4.18 in [40].

When it comes to deciding whether or not a space is an iterated loop space, these coherence conditions are more efficiently described using the notion of an *operad.* An operad is not something we will define here except to say that it is a collection of topological spaces with combinatorial data. Operads (potentially) act on topological spaces in a way which captures at a deeper level, the sort of associativity information described above. The idea of an operad arose out

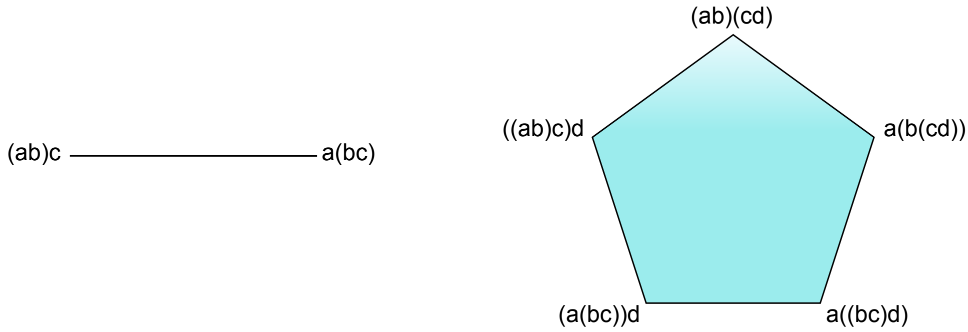

FIGURE 18. Stasheff polyheda for threefold and fourfold multiplication

of work by Boardman, Vogt and May on the problem of recognising whether a given topological space is an iterated loop space and has since seen wide application in various areas including graph theory and theoretical physics; see [40] for a comprehensive guide. Using their so-called *Recognition Principle* (Theorem 13.1 from [41]), this author was able to prove the following strengthening of Theorem 4.5 above.

Theorem 4.7. [51] *When $n \geqslant 3$, $\mathcal{R}^{s>0}(S^n)$ is weakly homotopy equivalent to an n-fold loop space.*

The proof involves demonstrating an action on $\mathcal{R}^{s>0}(S^n)$ of a certain operad known as the *operad of n-dimensional little disks* and makes use of something called the "bar-construction" on operads and in particular Theorem 4.37 of [6]. Roughly, the bar construction allows us to replace the original operad with something which is a little more "flexible" regarding the various geometric manoeuvres on metrics needed to satisfy the associativity conditions required by the operad action. One important point is that, although this works fine for path connected spaces, $\mathcal{R}^{s>0}(S^n)$ is usually not path connected. Thus, initially our result held only for the path component of $\mathcal{R}^{s>0}(S^n)$ containing the round metric. There is however, a way around this. The theorem holds for all of $\mathcal{R}^{s>0}(S^n)$ provided the operad action induces a group structure on the set of path components, $\pi_0(\mathcal{R}^{s>0}(S^n))$; see Theorem 13.1 of [41]. In the end, the proof that such a group structure existed (specifically that elements had inverses) turned out to be a substantially difficult problem. It is true however, but requires as heavyweight a result as the Concordance Theorem of Botvinnik, Theorem 4.1 above.

5. Some Related Work

As we have previously stated, positive scalar curvature is just one of many curvature constraints we might consider. It is important to realise that a good deal of work has been done on understanding the topology of spaces (and moduli spaces) of metrics which satisfy other geometric constraints. We will finish this article by taking a brief look at some of the results concerning these other spaces, before closing with some words on a recent and highly significant result by Sebastian Hoelzel concerning surgery. Much of what we discuss below is drawn from Tuschmann and Wraith's extremely useful book on moduli spaces of Riemannian metrics, [47].

5.1. Spaces of metrics satisfying other curvature constraints.

Early in this article we discussed the general notion of a geometric (in particular curvature) constraint C on Riemannian metrics on a smooth manifold M, leading to a subspace, $\mathcal{R}^{C}(M) \subset \mathcal{R}(M)$, of Riemannian metrics which satisfy C. So far we have only considered the case when C is the condition of having positive scalar curvature. So what about other curvature constraints?

Without even leaving the scalar curvature, one might consider for example the space, $\mathcal{R}^{s<0}(M)$, of all Riemannian metrics of negative scalar curvature on a smooth manifold M. Temporarily, we will drop the compactness assumption on M and allow that M may be compact or not. We know from a previously mentioned result of Lohkamp [38], that for any such M with dimension $n \geqslant 3$, the space $\mathcal{R}^{s<0}(M)$ is non-empty. From the Gauss-Bonnet theorem we know that there are 2-dimensional closed manifolds for which this does not apply: the sphere, projective plane, torus and Klein bottle. Lohkamp goes on to prove a great deal more. In particular, he proves that these non-empty spaces are actually all contractible! Even more amazingly, these results hold just as well if we replace negative scalar with negative *Ricci* curvature and the same is also true of the corresponding negative scalar and negative Ricci curvature moduli spaces. The proofs behind these facts are highly non-trivial but at their geometric heart is the fact that negative Ricci and scalar curvature display a great deal more flexiblity regarding local metric adjustment than do their positive counterparts.

Continuing with the theme of negative curvature, and given the comprehensive results by Lohkamp in the scalar and Ricci case, it remains to consider negative sectional curvature. This case is a little more interesting. In the case when M is a 2-dimensional surface, the story is intimately connected with Teichmüller theory. Without opening a discussion of this subject here, it can be shown that the space, $\mathcal{R}^{\mathrm{Sec}<0}(M)$, of negative sectional curvature metrics on a closed oriented surface, M, with genus at least 2, is homotopy equivalent to an object called the *Teichmüller space* of M, denoted $\mathcal{T}(M)$. This space, which is a complex manifold obtained as a certain quotient of the space of complex differentiable structures on M, is known to be contractible; see section 9 of [47]. In higher dimensions, the story is a little different and requires an analogue of $\mathcal{T}(M)$ known as the *Teichmüller space of Riemannian metrics* on M. There are a number of recent results on this subject, due to Farrell and Ontaneda (see for example [20]), which show a multiplicity of path components and non-triviality in the higher homotopy groups of the space $\mathcal{R}^{\mathrm{Sec}<0}(M)$ (and its moduli space), for certain closed manifolds admitting hyperbolic metrics. Typically, the dimension of these manifolds is at least 10.

Another interesting case concerns that of the positive Ricci curvature. The earliest result concerning topological non-triviality in spaces of such metrics concerns the moduli space of positive Ricci curvature metrics. It is due to Kreck and Stolz and utilises their s-invariant. Recall that, under reasonable circumstances, the s-invariant $s(M, g) \in \mathbb{Q}$ is an invariant of the path component of $[g] \in \mathcal{M}^{s>0}(M)$. Thus, it suffices to find a manifold M which has metrics of positive Ricci curvature (such metrics necessarily have positive scalar curvature) with distinct s-values to demonstrate that $\mathcal{M}^{\mathrm{Ric}>0}(M)$ is not path-connected. In [36], Kreck and Stolz show that there exist closed 7-dimensional manifolds for which the moduli space of positive Ricci curvature metrics has in fact infinitely many path components. The manifolds themselves are part of a class of examples, constructed by Wang and Ziller in [53], of bundles over the manifold $\mathbb{C}P^2 \times \mathbb{C}P^1$ with fibre S^1. Each such manifold admits infinitely many Einstein metrics (metrics of constant Ricci curvature) with positive Einstein constant (thus *positive* constant Ricci curvature). Importantly, the authors compute that for each of these manifolds, every element of this infinite collection of positive Ricci curvature metrics has a different s-invariant.

A further application of the s-invariant was used by Wraith in [58] to extend Carr's theorem on the space of psc-metrics on spheres of dimension $4k-1$, with $k \geqslant 2$, to the positive Ricci case. In particular, Wraith's result applies not just to standard spheres but to certain exotic spheres also. In dimensions $4k-1$ with $k \geqslant 2$, Wraith considers certain collections, denoted bP_{4k}, of smooth spheres which are topologically the same as but not necessarily diffeomorphic to the standard sphere. These spheres are the boundaries of certain $4k$-dimensional manifolds which are *parallelisable* (have trivial tangent bundle), constructed via a process called *plumbing* (a topological construction with certain similarities to surgery). Building on some of his earlier constructive results concerning the existence of positive Ricci curvature metrics in [55], [56] and [57], Wraith proves the following.

Theorem 5.1. (Wraith [58]) *For any sphere $\Sigma^{4k-1} \in bP_{4k}$, with $k \geqslant 2$, each of the spaces $\mathcal{R}^{\mathrm{Ric}>0}(\Sigma^{4k-1})$ and $\mathcal{M}^{\mathrm{Ric}>0}(\Sigma^{4k-1})$ has infinitely many path components.*

Regarding positive Ricci curvature, we should finally mention one very recent result concerning the observer moduli space of positive Ricci curvature metrics. Botvinnik, Wraith and this author have recently proved a positive Ricci version of Theorem 4.3, in the case when the underlying manifold M is the sphere S^n; see [11]. Thus, in appropriate dimensions, the space $\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(S^n)$ has many non-trivial higher homotopy groups. As yet, it is unclear how this result might be extended to other manifolds or to the regular moduli space. The proof works for essentially the same topological reasons as the original, however the geometric construction is very different and relies on certain gluing results of Perelman.

There are a number of results concerning positive and also non-negative sectional curvature for closed manifolds. Regarding positive sectional curvature, Kreck and Stolz in [36] have demonstrated that for a certain class of closed 7-dimensional manifolds known as Aloff-Wallach spaces (see [1] for a description), the corresponding moduli spaces of positive sectional curvature metrics are not-path connected. The strategy here, which again makes use of the s-invariant, is similar to that used in the positive Ricci case for the Wang-Ziller examples described above. Indeed, regarding this positive Ricci result, it was later shown by Kapovitch, Petrunin and Tuschmann in [34] that there are closed 7-dimensional manifolds (of the class constructed by Wang and Ziller) for which the moduli space of *non-negative* sectional curvature metrics has infinitely many path components. A more recent example can be found in [18]. We should also point out that there are a number of interesting results concerning topological non-triviality in the moduli space of non-negative sectional curvature metrics for certain open manifolds; see in particular work by Belegradek, Kwasik and Schultz [5], Belegradek and Hu [4] and very recently Belegradek, Farrell and Kapovitch [3].

5.2. **Extending the Surgery Theorem to other curvatures.** Let us step back once more to consider the general problem of understanding the topology of a space $\mathcal{R}^C(M)$ of Riemannian metrics on a smooth manifold M which satisfy a given curvature constraint C. One of the key tools in understanding this problem in the case where C is positive scalar cuvature, is surgery and, in particular, the Surgery Theorem of Gromov-Lawson and Schoen-Yau. Typically, stronger curvature notions do not behave as well under surgery. The Surgery Theorem as currently stated is simply false if we replace positive scalar curvature with positive Ricci or sectional curvatures. However, this does not mean that there are not certain circumstances under which a curvature condition might be preserved by appropriate surgeries. Indeed, given the power of surgery in constructing examples, understanding how the Surgery Theorem might extend for other curvature notions is an obvious priority.

One example of such an extension is due to Labbi in [37] and concerns a notion called p-curvature. Given a smooth Riemannian manifold M of dimension n and an integer p satisfying $0 \leqslant p \leqslant n-2$, there is a type of curvature s_p defined as follows. For any $x \in M$ and any p-dimensional plane $V \subset T_xM$, we define the p curvature $s_p(x, V)$ to be the scalar curvature at x of the *locally* specified $(n-p)$-dimensional submanifold determined by $V^\perp$. Thus, when $p = n-2$, $V^\perp$ is a 2-dimensional subspace of T_xM and so this is precisely the definition of the sectional curvature. When $p = 0$, $V = \{0\} \in T_xM$ implying that $V^\perp = T_xM$ and $s_0(x, \{0\}) = s(x)$, the scalar curvature at x. The p-curvatures therefore, are a collection of increasingly stronger curvature notions from scalar (when $p = 0$) all the way up to sectional (when $p = n-2$). The Ricci curvature does not appear exactly as one of the p-curvatures but can be described in an equation involving the scalar curvature, s_0, and s_1.[8] In [37], by a careful analysis of the the construction of Gromov and Lawson in [25], Labbi proves the following extension of the Surgery Theorem.

Theorem 5.2. (Labbi [37]) *Let M and M' be smooth manifolds of dimension n with M' obtained from M by a surgery in codimension $\geqslant 3 + p$, where $0 \leqslant p \leqslant n - 2$. Then if M admits a Riemannian metric of positive p-curvature, so does M'.*

Later, an analogous extension of the Surgery Theorem with appropriate codimension hypotheses was worked out by Wolfson in [54] in regard to a curvature constraint called *positive k-Ricci curvature*. As these results share a common origin in the work of Gromov-Lawson and Schoen-Yau, a natural idea would be to extrapolate the general principle and find a theorem which subsumes all of these individual cases. Precisely this was done in a remarkable paper by Sebastian Hoelzel; see [30]. Hoelzel defines the idea of a curvature condition C which is stable under surgeries of a certain codimension and proves a comprehensive generalisation of the original Surgery Theorem covering all the previous curvature extensions and many more. He further goes on to prove an extension of the *classification of simply connected manifolds of positive scalar curvature,* Theorem 3.4above, to this general setting. In better understanding spaces of metrics which satisfy a curvature constraint C, an especially nice next step would be a "family" or paramaterised version of Hoelzel's theorem.

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