A note on the ring axioms

It is well known that if $(R, +, \cdot)$ is a ring with unity 1, then the commutative law of addition $x + y = y + x$ for all $x, y \in R$ is redundant; i.e. it can be deduced from the other ring axioms. In such a ring, every $x \in R$ can be written as a product $x = x \cdot 1$. In fact the stated result holds under this weaker condition.

THEOREM. Let $(R, +, \cdot)$ be a ring in which every element is a product; i.e. given $r \in R$, there exist elements $s, t \in R$ such that $r = s \cdot t$. Then the commutative law of addition, $x + y = y + x$ for all $x, y \in R$, is redundant.

PROOF. Let x and y be arbitrary elements of R and let $x = a \cdot d$ and $y = b \cdot c$. Consider $(a + b) \cdot (c + d)$. Now, using the right distributive law followed by the left distributive law we have

$$(a + b) \cdot (c + d) = a \cdot (c + d) + b \cdot (c + d) = a \cdot c + a \cdot d + b \cdot c + b \cdot d\,.$$

On the other hand, using the left distributive law followed by the right distributive law, we have

$$(a + b) \cdot (c + d) = (a + b) \cdot c + (a + b) \cdot d = a \cdot c + b \cdot c + a \cdot d + b \cdot d\,.$$

Next, using the left and right cancellation laws in the group $(R, +)$ we see that

$$a \cdot d + b \cdot c = b \cdot c + a \cdot d \quad \text{i.e.} \quad x + y = y + x.$$

The following example shows that if we drop the condition that every element of R is a product, then the theorem is no longer valid.

Example. Let $(G, +)$ be a non-commutative group, written additively, with identity element 0. For all $a, b \in G$, define $a \cdot b = 0$, and thus there exists an element of G which is not a product. Also $(G, +, \cdot)$ has all the ring properties except commutativity of $+$.

The final example shows that the condition that every element is a product is weaker than the condition that the ring has a one-sided unity.

Example. Let $R = \{0, a, b, c\}$ and let $R_1 = (R, +, \cdot)$ and $R_2 = (R, +, *)$ be rings defined by

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

$*$	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	b	b	0
c	0	c	c	0

Then it is clear that $(R_1, +, \cdot)$ has left unity but not right unity, whereas R_2 has right unity but no left unity. So their direct sum $R_1 \oplus R_2$ has no left unity or right unity. But $R_1 \oplus R_2$ does have the product property, for if $(x, y) \in R_1 \oplus R_2$ then $(x, y) = (a, y)(x, a)$.

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