ICMA Centre

Henley Business School
UNIVERSITY OF READING

Discussion Paper

A Monte Carlo Multi-asset Option Pricing Approximation for General Stochastic Processes

August 2014

Juan C Arismendi Z
Department of Economics, Federal University of Bahia
ICMA Centre, Henley Business School, University of Reading

Alan De Genaro
Department of Economics, University of São Paulo
Cetip S.A. Mercados Organizados

Discussion Paper Number:
ICM-2014-03

The aim of this series is to disseminate new research of academic distinction. Papers are preliminary drafts, circulated to stimulate discussion and critical comment.

Henley Business School is triple accredited and home to over 100 academic faculty who undertake research in a wide range of fields from ethics and finance to international business and marketing.

admin@icmacentre.ac.uk

www.icmacentre.ac.uk

A Monte Carlo Multi-asset Option Pricing Approximation for General Stochastic Processes

Juan Arismendi[a,b,], Alan De Genaro[c,d]

[a]*Department of Economics, Federal University of Bahia, Rua Barão de Jeremoabo, 668-1154, Salvador, Brazil.*
[b]*ICMA Centre, Henley Business School, University of Reading, Whiteknights, Reading, United Kingdom.*
[c]*Department of Economics, University of São Paulo, Rua Luciano Gualberto, 908, São Paulo, Brazil.*
[d]*Cetip S.A. Mercados Organizados, Av. Brigadeiro Faria Lima, 1663, São Paulo, Brazil.*

Abstract

We derived a model-free analytical approximation of the price of a multi-asset option defined over an arbitrary multivariate process, applying a semi-parametric expansion of the unknown risk-neutral density with the moments. The analytical expansion termed as the *Multivariate Generalised Edgeworth Expansion* (MGEE) is an infinite series over the derivatives of an auxiliary continuous time density. The expansion could be used to enhance a Monte Carlo pricing methodology incorporating the information about moments of the risk-neutral distribution. The efficiency of the approximation is tested over a jump-diffusion and a q-Gaussian diffusion. For the known density, we tested the multivariate lognormal (MVLN), even though arbitrary densities could be used. The MGEE relates two densities and isolates the effects of multivariate moments over the option prices. Results show that a calibrated approximation provides a good fit when the difference between the moments of the risk-neutral density and the auxiliary density are small relative to the density function of the former, and the uncalibrated approximation has immediate implications over risk management and hedging theory. The possibility to select the auxiliary density provides an advantage over classical Gram–Charlier A, B and C series approximations.

Keywords: Multi-asset Option Pricing, Multivariate Risk Management, Edgeworth Expansion, Higher-order Moments

1. Introduction

The distribution of the asset returns in equity markets is 'fat-tailed' and 'skewed' (Kraus and Litzenberger, 1976; Harvey and Siddique, 2000). For this reason, a semi-parametric formula like that of Jarrow and Rudd (1982) profoundly impacted the literature. They approximated an arbitrary continuous risk-neutral density of a univariate asset, using a Generalised Edgeworth Expansion (GEE) over a lognormal density. To obtain the option price, they integrated the resulting approximated density under the risk-neutral measure. To

Email addresses: `j.arismendi@icmacentre.ac.uk` (Juan Arismendi), `adg@usp.br` (Alan De Genaro)

calibrate the approximation, the GEE requires the empirical moments of the unknown density of the asset. By doing this, not only the price is calculated, and the moments of the asset incorporated into the final formula, but also the effects of perturbations over the moments of the distribution on the option price can be easily observed.[1]

There exist popular versions of multi-asset options, one of which is the basket option: Given a vector of weights $\boldsymbol{\omega} = \{\omega_1, \ldots, \omega_n\}$, a strike price K, and a n-variate vector of assets $\mathbf{s}(t) = \{s_1, \ldots, s_n\}$, the payoff of a basket option at maturity t is $\Pi(\mathbf{s}(t), \boldsymbol{\omega}, K) = [\omega_1 s_1(t) + \cdots + \omega_n s_n(t) - K]^+$. Rainbow, quanto, spread and even index options can be regarded in the class of multi-asset options. In general, the payoff of multi-asset options can be specified as a function of the assets: $\Pi(\mathbf{s}(t), \phi, K) = [\phi(s_1(t), \ldots, s_n(t), K)]^+$, where $\phi(\cdot)$ is a multivariate real function. Special case of basket options are spread options, which are highly traded on NYMEX. The compilation made by Carmona and Durrleman (2006), is a very extensive and complete reference of previous attempts and models that addressed the issue of pricing and hedging spread options. Numerical methods like Monte Carlo, binomial and trinomial trees, Fourier transform, had been used; However all methods use an approximation of the univariate density of the sum of the assets. Krekel et al. (2004) did a comparison of different basket option pricing methods, concluding that Ju (2002) and Beißer (1999) are the best performing methods, but both methods approach the pricing through the univariate density of the distribution the payoff.

In this research, an approximate multi-asset option price is provided applying the *Multivariate Generalised Edgeworth Expansion* (MGEE) framework. In other words, we extend the results of Jarrow and Rudd (1982) to the multivariate case. Our formula disentangles the impact of multivariate higher-order moments on the option prices.[2] It is the first time that a formula that disentangles the impact of multivariate higher-order moments on option prices has been provided.[3] The main advantage of our approximation is that it is for arbitrary processes; this means it can be used with discontinuous-time models originated not only from a Wiener diffusion, but also from Lévy processes such as jump-diffusion or q-Gaussian diffusion.[4] In the

[1]As a result of the success of this model, it has been used in a large amount of empirical research, including Corrado and Su (1996), Bhandari and Das (2009) for options on portfolios, Lim et al. (2005) for a parametric option pricing model, Flamouris and Giamouridis (2002) for a semi-parametric model and Aït-Sahalia and Lo (1998) for a non-parametric model for density estimation.

[2]The option price formula is derived using a Fourier inversion method. Nevertheless, the method is applied for the large class of continuous density functions with partial derivatives, resulting in a formula that is on the time domain, and there will be no need of a Fourier inversion method for pricing. In a paper by Níguez and Perote (2008), a density expansion using the moments of the distribution termed General Moments Expansion (GEM) is provided. This expansion generates only positive densities; however, it needs an additional vector of parameters of the same dimension of the distribution dimension; these additional parameters have no economical significance.

[3]Schlögl (2013) provides an multi-asset option approximation using a multivariate Gram–Charlier A series expansion; however, there are assumptions over the risk-neutral density, and an additional methodology is needed to extract the moments inside the expansion from the Hermite polynomials.

[4]Our results complement the results of Filipović et al. (2013), as we provide a thorough study of the higher-order moments

Jarrow and Rudd (1982) formula the value of the European option is equal to the Black and Scholes price plus corrections based on the difference of the moments of the lognormal distribution and the real market distribution. In this paper, the GEE is extended to the multivariate case (MGEE), and then the Black and Scholes price is calculated using a Monte Carlo simulation, as there exists no equivalent closed-form Black and Scholes formula for the multivariate case. Another benefit of our results is that the moments of the risk-neutral density of the assets could be obtained separately through empirical work and, if they are available, then the price of the option is straightforwardly obtained using our formula. As a result, higher-order moment effects like the ones observed during market crashes can be easily modelled into the pricing or the hedging of the option.

The approximation provided allows us to calculate the moments of the distribution of the sum of lognormals in a multivariate setting, and this can be considered an interesting result not only for finance, but in general.[5] In Ju (2002), an univariate approximation of the risk-neutral density is provided, using a Taylor expansion over a univariate lognormal density. Kristensen and Mele (2011) also provide an approximation with an application to asset pricing theory. This approximation is based on a Taylor expansion of a differential operator over the divergence between the Black and Scholes model price and the real price. Consequently, the moments of the distribution are not part of the option pricing formula, making it very difficult to understand how changes over the distribution affect the final price. Our approach for valuing multi-asset options using the multivariate risk-neutral density is novel, since all previous models attempt to price multi-asset options with a function of univariate densities: Li et al. (2008) and Li et al. (2010) developed two new approximations, an original termed *second-order boundary approximation*, and an extension to the multivariate case of the Kirk (1996) formula for spread options termed the *extended Kirk approximation*. Both approximations reduce the dimensions of the problem, from a multivariate integration to a function of an univariate normal standard density. In Alexander and Venkatramanan (2011) the price of a spread option is approximated as the price of the sum of two compound options, and that is extended in Alexander and Venkatramanan (2012) for multi-asset options. The prices of the compound options were calculated by Geske (1979) and by Carr (1988). The final formula will be a function of the product of univariate densities. Working with the multivariate risk-neutral density requires additional notation from multivariate statistics. Nevertheless, the main advantage for empirical research is a more realistic framework, and it provides new tools for hedging and risk management.

The MGEE can be considered another important contribution of our research for other fields of application such as statistics. Although Perote (2004) and Del Brio et al. (2009) produced a *Multivariate Edgeworth*

---

effect over option prices. In Knight and Satchell (2001) a Gram–Charlier expansion is derived for pricing options using the first four moments of a univariate risk-neutral distribution.

[5] Limpert et al. (2001) and Dufresne (2004) review the importance of the distribution of the sum of lognormals in finance, and in physical sciences in general.

Expansion (MEE), this expansion is based on an approximation of the multivariate normal (MVN) distribution, with the complications of negative density values when the empirical density to fit is leptokurtic. We face the same risk, but if we select an appropriate distribution with skewness and kurtosis more similar to the risk-neutral density, this problem is diminished.

The structure of this paper is as follows: Section 2 contains the definitions and the notation used, describes the MGEE, and describes the method used in finding an approximation for multi-asset options. Section 3 presents the multi-asset option approximation.[6] We integrate the resulting density from the MGEE using a Monte Carlo method. In Section 4, a numerical example is presented, where the MGEE is used to price a basket option over multivariate jump-diffusion and multivariate q-Gaussian diffusion, and this section introduce the possible extensions in the use of the MGEE as a tool for risk management. In Section 5 we provide a calibration methodology. In Section 6 we present concluding remarks and further developments with some possible modifications to our approach.

2. The *Multivariate Generalised Edgeworth Expansion* (MGEE): the distribution approximation

In this section we define the *arbitrary processes* that can be approximated using a MGEE. Let $\mathbf{X}(t) = \{X_i(t) \in \mathbb{R}^+, t \geq 0\}, i \in \{1, \ldots, n\}$ be a general n-variate continuous stochastic process. This process is called the *asset price process*. Let Q be the n-variate risk-neutral probability measure. Denote by $f_{\mathbf{X}(t)}$ the existent and unique density of $\mathbf{X}(t)$ under Q. We restrict $\mathbf{X}(t)$ to the class of processes where $f_{\mathbf{X}(t)}$ is a continuous density function, and its partial derivatives $(df_{\mathbf{X}(t)}/dX_i(t))$ exist. Define the filtered probability space $(\Omega, \mathcal{F}, Q)$, where $\mathcal{F}$ is the filtration generated by the sigma-algebra $\{\mathbf{X} : \mathcal{F}_t = \{X_i(t), t \geq 0\}\}$.

Define the n-variate stochastic process $\mathbf{S} = \{S_i(t) \in \mathbb{R}^+, t \geq 0\}, i \in \{1, \ldots, n\}$, described by:

$$\begin{aligned} dS_i(t) &= \mu_i S_i(t)dt + \sigma_i S_i(t) dW_i(t), \\ \langle dW_i(t), dW_j(t) \rangle &= \rho_{i,j} dt, \end{aligned} \tag{1}$$

where $i, j \in \{1, \ldots, n\}$, $W_i(t)$ are Wiener processes under the risk-neutral measure Q, and μ_i, σ_i are the constant drift and the constant volatility of the variable $S_i(t)$, and $\rho_{i,j}$ is the constant correlation between $S_i(t)$ and $S_j(t)$. The process $\mathbf{S}(t)$ will be used to approximate the asset price process $\mathbf{X}(t)$, and has a multivariate lognormal density function $g_{\mathbf{S}(t)}$ under the risk-neutral measure Q (see Section 3.1). To simplify the option pricing formula provided and its derivation, the risk-free interest rate r will be considered constant.

The gist of the model approximation is to use the properties of the well-known distribution $g_{\mathbf{S}(T)}$ of the geometric Brownian motion (GBM) process $\mathbf{S}(T)$, where $t = T$ is the maturity of the option, to fit the

[6] All proofs are provided in the appendix.

unknown distribution $f_{\mathbf{X}(T)}$. In this sense we will have:

$$f_{\mathbf{X}(T)} = H(g_{\mathbf{S}(T)}) + \varepsilon,$$

where H is a function with information about the moments of $f_{\mathbf{X}(T)}$, and ε is a bounded error term. Denote by $\Pi(\mathbf{X}(T))$ a payoff function over the asset price. Then, the price of the European option $C_{\{t=0\}}(\Pi(\mathbf{X}(T)))$ is the expected value of the discounted payoff under the risk-neutral measure:

$$C_0(\Pi(\mathbf{X}(T))) = \exp(-rT)\mathbb{E}_0^Q\left[\Pi(\mathbf{X}(T))\right],$$

and to calculate this expected value, we use $H(g_{\mathbf{S}(T)})$ instead of the unknown risk-neutral density $f_{\mathbf{X}(T)}$:

$$\mathbb{E}_0^Q\left[\Pi(\mathbf{X}(T))\right] = \mathbb{E}_0^{f_{\mathbf{X}}(T)}\left[\Pi(\mathbf{X}(T))\right] \approx \mathbb{E}_0^{g_{\mathbf{S}(T)}}\left[\Pi(\mathbf{S}(T))\right].$$

We introduce tensor notation with the purpose of simplifying the final formula. Attempting to extend Jarrow and Rudd's (1982) results to the multivariate case without this notation would make the task intractable. We use the notation used by Kendall (1947) to provide general results. To simplify the notation, when the time index of an stochastic process is omitted we refer to the random variable at time t: $\mathbf{X} \equiv \mathbf{X}(T)$.[7]

To define the tensor notation we use the summation convention as it is the appropriate notation for working with tensors. This notation is commonly used in physics and is attributed to Einstein. A tensor is a mathematical object similar to a multidimensional array. We use the brackets on the left-hand side to highlight the use of this implicit summation convention:

Definition 2.1. *Let* $\mathbf{a}$ *be a real valued vector of dimension* m *with components* $a_1, \ldots, a_m$*. A tensor product of* $\mathbf{X}$ *and* $\mathbf{a}$ *between* p *of their components is defined as:*

$$a_{[l_1]} \ldots a_{[l_p]} X_{[l_1]} \ldots X_{[l_p]} \quad \equiv \quad \sum_{l_1=1}^{n} \cdots \sum_{l_p=1}^{n} a_{l_1} \ldots a_{l_p} X_{l_1} \ldots X_{l_p},$$

where $l_1, \ldots, l_p \in \{1, \ldots, n\}$ *and the subscript* $[l_p]$ *represents a summation notation used to substitute the summation symbol. The iterated tensor product* $x_{[l_1,[l_2,[l_3,\ldots,[l_p]\ldots]]]} a_{[l_1]} \ldots a_{[l_j]}$ *is defined as:*

$$X_{[l_1,[l_2,[l_3,\ldots,[l_p]\ldots]]]} a[l_1] \ldots a[l_j] \equiv \sum_{l_1=1}^{n} \left(X_{l_1} a_{l_1} + \sum_{l_2=1}^{n} \left(X_{l_1,l_2} a_{l_1,l_2} + \cdots + \sum_{l_j=1}^{n} X_{l_1,\ldots,l_j} a_{l_1,\ldots,l_j} \right) \right).$$

[7] The vector notation $\mathbf{x}(T)$ in lower-case refers to the variable of integration, then $\mathbb{E}(\mathbf{X}) \equiv \int \mathbf{x}(T) dF_{\mathbf{X(T)}}$.

Definition 2.2. *Define the abbreviated integral operator as:*

$$\int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} (\cdot) d\xi_1 \ldots d\xi_n = {}^{(n)}\!\!\int_{a_i}^{\infty} (\cdot) d\xi,$$

for $i = \{1, \ldots, n\}$ *where* ξ *is a vector with components* $(\xi_1, \ldots, \xi_n)$.

The density approximation provided generalises the univariate results of Jarrow and Rudd (1982). Considering the same restrictions as in Jarrow and Rudd (1982), the MGEE method can be applied only to the set of continuous distributions. More general distributions can be included, but a more formal presentation using measure theory outside the scope of this paper will be required. There have been previous attempts to approximate a distribution using other distributions: the multivariate Gram–Charlier and the multivariate Edgeworth expansion with the Edgeworth–Sargan density of Perote (2004). However, in these cases the approximation is done over the multivariate normal distribution. This method has the inconvenience that only a limited set of distributions can be modelled, and heavy-tailed distributions especially can not be approximated with the MVN.

Definition 2.3. *Let* $\mathbf{X}$ *have an absolutely continuous density function* $f_{\mathbf{X}}$. *We assume that* $f_{\mathbf{X}}$ *is differentiable and that the cumulative distribution function* $F_{\mathbf{X}}$ *exists. Let* $I = \{i_1, \ldots, i_p\}$ *be a vector of integer numbers, the* p*-order moment function of* $\mathbf{X}$ *is defined by,*

$$m_{\{i_1,\ldots,i_p\}}(\mathbf{x}) = m_{p,I}(\mathbf{x}) = \mathbb{E}\left[X_{i_1} \times \cdots \times X_{i_p}\right],$$

and these moments can be computed with the integral:

$$m_I(\mathbf{x}) = {}^{(n)}\!\!\int_{-\infty}^{\infty} \frac{x_{i_1} \ldots x_{i_p} f_{\mathbf{x}}}{F_{\mathbf{X}}} dx_1 \ldots dx_n.$$

Another equivalent expression for moments is:

$$m_{\boldsymbol{\alpha}}(\mathbf{x}) = E[X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_n^{\alpha_n}], \tag{2}$$

where $\boldsymbol{\alpha}$ is a vector of integer numbers.

Assumption 2.1. *The cumulants* $k_{l_1,\ldots,l_j}(\mathbf{x})$ *of the unknown risk-neutral density* $f_{\mathbf{X}}$ *con be estimated.*

Definition 2.4. *Denote* $\psi(\mathbf{x}, \xi) = \mathbb{E}\left[\exp\left(\xi_{[l_1]} X_{[l_1]}\right)\right]$ *as the moment-generating function. The cumulant-generating function (CGF) of* $\mathbf{x}$ *is defined as:*

$$K(\mathbf{x}, \xi) \quad = \quad \log \psi(\mathbf{x}, \xi).$$

which is convergent for small ξ.

This function can be expanded into the infinite series:

$$\begin{aligned}\log \psi(\mathbf{x},\xi) &= \xi_{[l_1]}k_{[l_1]}(\mathbf{x}) + \xi_{[l_1]}\xi_{[l_2]}k_{[l_1,l_2]}(\mathbf{x})/2! + \xi_{[l_1]}\xi_{[l_2]}\xi_{[l_3]}k_{[l_1,l_2,l_3]}(\mathbf{x})/3! + \ldots, \\ &= \sum_{j=1}^{\infty} \xi_{[l_1]}\ldots\xi_{[l_j]}k_{[l_1,\ldots,l_2]}(\mathbf{x})/j!,\end{aligned} \tag{3}$$

which is convergent for small ξ where the terms $k_{l_1,\ldots,l_p}(\mathbf{x})$ will be defined as the cumulants. The cumulant $k_{l_1}(\mathbf{x})$ is the mean, $k_{l_1,l_2}(\mathbf{x})$ is the variance, $k_{l_1,l_2,l_3}(\mathbf{x})$ is a measure of skewness and $k_{l_1,l_2,l_3,l_4}(\mathbf{x})$ is a measure of kurtosis. The expansion (3) can be used to find the values of $k_{l_1,\ldots,l_p}(\mathbf{x})$.

The difference of the moments can be expressed in terms of the difference of cumulants of $\mathbf{X}$ and $\mathbf{S}$ as:

$$\begin{aligned}M_0 &= 1, \\ M_{l_1} &= k_{l_1}(\mathbf{x}) - k_{l_1}(\mathbf{s}), \\ M_{l_1,l_2} &= (k_{l_1,l_2}(\mathbf{x}) - k_{l_1,l_2}(\mathbf{s})) + M_{l_1}M_{l_2}, \\ M_{l_1,l_2,l_3} &= (k_{l_1,l_2,l_3}(\mathbf{x}) - k_{l_1,l_2,l_3}(\mathbf{s})) + (M_{l_1}\left(k_{l_2,l_3}(\mathbf{x}) - k_{l_2,l_3}(\mathbf{s})\right) + M_{l_2}\left(k_{l_1,l_3}(\mathbf{x}) - k_{l_1,l_3}(\mathbf{s})\right) + \\ &\quad M_{l_3}\left(k_{l_1,l_2}(\mathbf{x}) - k_{l_1,l_2}(\mathbf{s})\right) + M_{l_1}M_{l_2}M_{l_3}, \\ M_{l_1,l_2,l_3,l_4} &= (k_{l_1,l_2,l_3,l_4}(\mathbf{x}) - k_{l_1,l_2,l_3,l_4}(\mathbf{s})) + \left\{M_{l_1}\left(k_{l_2,l_3,l_4}(\mathbf{x}) - k_{l_2,l_3,l_4}(\mathbf{s})\right)\right\}_{\binom{4}{3}} + \\ &\quad \left\{\left(k_{l_1,l_2}(\mathbf{x}) - k_{l_1,l_2}(\mathbf{s})\right)\left(k_{l_3,l_4}(\mathbf{x}) - k_{l_3,l_4}(\mathbf{s})\right)\right\}_{\mathcal{S}_2(3)} + \\ &\quad \left\{M_{l_1}M_{l_2}\left(k_{l_3,l_4}(\mathbf{x}) - k_{l_3,l_4}(\mathbf{s})\right)\right\}_{\binom{4}{2}} + M_{l_1}M_{l_2}M_{l_3}M_{l_4},\end{aligned} \tag{4}$$

where,[8]

$$\begin{aligned}\left\{M_{l_1}M_{l_2}\left(k_{l_3,l_4}(\mathbf{x}) - k_{l_3,l_4}(\mathbf{s})\right)\right\}_{\binom{4}{2}} &\equiv M_{l_1}M_{l_2}\left(k_{l_3,l_4}(\mathbf{x}) - k_{l_3,l_4}(\mathbf{s})\right) + M_{l_1}M_{l_3}\left(k_{l_2,l_4}(\mathbf{x}) - k_{l_2,l_4}(\mathbf{s})\right) + \\ &\quad M_{l_1}M_{l_4}\left(k_{l_2,l_3}(\mathbf{x}) - k_{l_2,l_3}(\mathbf{s})\right) + M_{l_2}M_{l_3}\left(k_{l_1,l_4}(\mathbf{x}) - k_{l_1,l_4}(\mathbf{s})\right) + \\ &\quad M_{l_2}M_{l_4}\left(k_{l_1,l_3}(\mathbf{x}) - k_{l_1,l_3}(\mathbf{s})\right) + M_{l_3}M_{l_4}\left(k_{l_1,l_2}(\mathbf{x}) - k_{l_1,l_2}(\mathbf{s})\right), \\ \left\{M_{l_1}\left(k_{l_2,l_3,l_4}(\mathbf{x}) - k_{l_2,l_3,l_4}(\mathbf{s})\right)\right\}_{\binom{4}{3}} &\equiv M_{l_1}\left(k_{l_2,l_3,l_4}(\mathbf{x}) - k_{l_2,l_3,l_4}(\mathbf{s})\right) + M_{l_2}\left(k_{l_1,l_3,l_4}(\mathbf{x}) - k_{l_1,l_3,l_4}(\mathbf{s})\right) + \\ &\quad M_{l_3}\left(k_{l_1,l_2,l_3}(\mathbf{x}) - k_{l_1,l_2,l_3}(\mathbf{s})\right) + M_{l_4}\left(k_{l_1,l_2,l_3}(\mathbf{x}) - k_{l_1,l_2,l_3}(\mathbf{s})\right).\end{aligned}$$

The notation,

$$\begin{aligned}\left\{\left(k_{l_1,l_2}(\mathbf{x}) - k_{l_1,l_2}(\mathbf{s})\right)\left(k_{l_3,l_4}(\mathbf{x}) - k_{l_3,l_4}(\mathbf{s})\right)\right\}_{\mathcal{S}_2(3)} &\equiv \left(k_{l_1,l_2}(\mathbf{x}) - k_{l_1,l_2}(\mathbf{s})\right)\left(k_{l_3,l_4}(\mathbf{x}) - k_{l_3,l_4}(\mathbf{s})\right) + \\ &\quad \left(k_{l_1,l_3}(\mathbf{x}) - k_{l_1,l_3}(\mathbf{s})\right)\left(k_{l_2,l_4}(\mathbf{x}) - k_{l_2,l_4}(\mathbf{s})\right) + \\ &\quad \left(k_{l_1,l_4}(\mathbf{x}) - k_{l_1,l_4}(\mathbf{s})\right)\left(k_{l_2,l_3}(\mathbf{x}) - k_{l_2,l_3}(\mathbf{s})\right),\end{aligned}$$

[8] The binomial $\binom{4}{2}$ notation represents the possible partitions of the set $\{l_1,l_2,l_3,l_4\}$ into two sets of two elements each. The binomial $\binom{4}{3}$ notation represents the four possible partitions of the set $\{l_1,l_2,l_3,l_4\}$ into two sets of one and three elements each.

represents the three different possible partitions of the set $\{l_1, l_2, l_3, l_4\}$ into two sets of two elements each. This number is equivalent to the number of partitions of the set of four elements into two sets, or the Stirling number $\mathcal{S}_2(3) = 2^{3-1} - 1 = 3$. Additional moments could be developed following combinatorics rules, and the work of McCullagh (1987) is a good reference for this purpose.

Proposition 2.1. *Define* $\mathbf{X}$ *as an* n*-variate stochastic process with a multivariate continuous density function* $f_{\mathbf{X}}$. *Define* $g_{\mathbf{S}}$ *to be another multivariate continuous distribution defined over the random vector* $\mathbf{s}$. *This density will be the approximate density. Denote* $m_{l_1,\ldots,l_p}(\mathbf{x})$ *as the moment of order* p *of* $\mathbf{X}$ *and* $k_{l_1,\ldots,l_p}(\mathbf{x})$ *the cumulants of order* p *of* $\mathbf{X}$. *Then, the density* $f_{\mathbf{X}}$ *can be expressed in terms of the following expansion:*

$$f_{\mathbf{X}} \quad = \quad g_{\mathbf{S}} + \sum_{j=1}^{n-1} M_{[l_1,[l_2,[\ldots,[l_j]\ldots]} \frac{(-1)^j}{j!} \frac{\partial^j}{\partial s_{[l_1]} \ldots \partial s_{[l_j]}} g_{\mathbf{S}} + \varepsilon(\mathbf{s}, n),$$

where

$$\varepsilon(\mathbf{s}, n) = \frac{1}{2\pi} {}^{(n)}\!\!\int_{-\infty}^{\infty} \exp(i\xi'\mathbf{s}) o(\|\xi\|^n) d\xi.$$

This expansion will be termed the Multivariate Generalised Edgeworth Expansion (MGEE). The tensor notation $M_{[l_1,[l_2,[\ldots,[l_j]\ldots]}$ *refers to:*

$$M_{[l_1,[\ldots,[l_j]\ldots]} \frac{(-1)^j}{j!} \frac{\partial^j}{\partial s_{[l_1]} \ldots \partial s_{[l_j]}} g_{\mathbf{S}} \quad \equiv$$
$$\sum_{l_1=1}^{n} \left(M_{l_1}(-1) \frac{\partial}{\partial s_{l_1}} g_{\mathbf{S}} + \sum_{l_2=1}^{n} \left(M_{l_1,l_2} \left(\frac{1}{2}\right) \frac{\partial^2}{\partial s_{l_1} \partial s_{l_2}} g_{\mathbf{S}} + \cdots + \sum_{l_j=1}^{n} M_{l_1,\ldots,l_j} \frac{(-1)^j}{j!} \frac{\partial^j}{\partial s_{l_1} \ldots \partial s_{l_j}} g_{\mathbf{S}} \right) \right).$$

Proof. See Section A.1 of the appendix.

The MGEE approximation has been presented until the fourth-order moment. Continuous densities with their derivatives are candidates for the auxiliary distribution $g_{\mathbf{s}}$. Although the focus of this research is the approximation of option prices using multivariate densities, the MGEE is useful for any application where the density $f_{\mathbf{x}}$ is unknown, but the moments are available or they could be estimated.

3. Multi-asset option approximation

The result presented in the previous section will be used to approximate the risk-neutral distribution $f_{\mathbf{X}(T)}$, where $t = T$ is the maturity of the contract. It is sufficiently general that it can be used for different distribution approximations. However, if we want to use it for basket option pricing we will need to provide additional approximations. The density approximation is used towards finding the value of a European multi-asset option. A general case is presented for the arbitrary continuous-time price processes $\mathbf{X}(T)$ described in Section 2:

Corollary 3.1. *Denote the n-variate continuous-time stochastic price process $\mathbf{X}(T)$ with a unique continuous density function $f_{\mathbf{X}(T)}$. Define $\mathbf{S}(T)$ as the multivariate lognormal process used to approximate $\mathbf{X}(T)$ and denote by $g_{\mathbf{S}(T)}$ the density function of $\mathbf{S}(T)$. Denote $\Pi(\mathbf{S}(T))$ the payoff function, the value of an option C_t at time $t = 0$ can be approximated as:*

$$
\begin{aligned}
C_0\left(\Pi(\mathbf{x}(T))\right) \quad = \quad & \exp(-rt) \overset{(n)}{\int_0^\infty} \Pi(\mathbf{s}(T)) dG_{\mathbf{S}(T)} + \\
& \exp(-rt) \sum_{j=1}^{n-1} M_{[l_1,[\ldots,[l_j]\ldots]} \frac{(-1)^j}{j!} \overset{(n)}{\int_0^\infty} \Pi(\mathbf{s}(T)) \frac{\partial^j}{\partial s_{[l_1]}(T) \ldots \partial s_{[l_j]}(T)} g_{\mathbf{S}(T)} d\mathbf{s}(T) + \varepsilon(\Pi(\mathbf{s}(T)), n),
\end{aligned}
$$

where $d\mathbf{s}(T) = ds_1(T) \ldots ds_n(T)$ and,

$$
\varepsilon(\Pi(\mathbf{s}(T)), n) = \frac{1}{2\pi} \overset{(n)}{\int_0^\infty} \exp(i\xi' \mathbf{s}(\mathbf{T})) o(\|\xi\|^n) d\xi.
$$

$\Pi(\mathbf{s})$ will be equal to $\Pi(\mathbf{x})$ substituting the components x_i by s_i.

Proof. Using the risk-neutral pricing approach, the value of the option is:

$$
C_0(\Pi(\mathbf{x}(T))) \quad = \quad \exp(-rt)\mathbb{E}_0^Q \left[\Pi(\mathbf{X}(T)) \;\middle|\; \mathcal{F}_0 \right] = \exp(-rt) \overset{(n)}{\int_0^\infty} \Pi(\mathbf{x}(T)) dF_{\mathbf{X}(T)}.
$$

Then the result follows immediately from applying Proposition 2.1.

The option price formula reveals three components:

$$
C_0(\Pi(\mathbf{x}(T))) \quad = \quad C_{0,\mathbb{W}}(\Pi(\mathbf{x}(T))) + \sum_{j=1}^{n-1} C_{0,\mathbb{W},[l_1,\ldots,l_j]}(\Pi(\mathbf{x}(T))) + \varepsilon(\Pi(\mathbf{s}(T)), n). \tag{5}
$$

where,

$$
\begin{aligned}
C_{0,\mathbb{W}}(\Pi(\mathbf{x}(T))) \quad &= \quad \exp(-rt) \overset{(n)}{\int_0^\infty} \Pi(\mathbf{s}(T)) dG_{\mathbf{s}(T)}, \\
\sum_{j=1}^{n-1} C_{0,\mathbb{W},[l_1,\ldots,l_j]}(\Pi(\mathbf{x}(T))) \quad &= \quad \exp(-rt) \sum_{j=1}^{n-1} M_{[l_1,[\ldots,[l_j]\ldots]} \frac{(-1)^j}{j!} \times \\
& \qquad \overset{(n)}{\int_0^\infty} \Pi(\mathbf{s}(T)) \frac{\partial^j}{\partial s_{[l_1]}(T) \ldots \partial s_{[l_j]}(T)} g_{\mathbf{S}(T)} d\mathbf{s}(T),
\end{aligned}
$$

The first part, $C_{0,\mathbb{W}}(\Pi(\mathbf{x}(T)))$, is the value of the option under a simple Black and Scholes world, of a multivariate Wiener process with constant parameters, also known as geometric Brownian motion (GBM). In the univariate case this part will be reduced to a Black and Scholes formula (Jarrow and Rudd, 1982). Given that there still does not exist an equivalent Black and Scholes closed formula for the multivariate case, for numerical applications, or to calibrate the model, an approximation of the first section is required. There are very good approximations for the bivariate case (spread options), including Borovkova et al. (2007), Li et al. (2008) and, for the multivariate case, Li et al. (2010) and Alexander and Venkatramanan (2012).

However, due to the improved precision we use a Monte Carlo simulation method, integrating the payoff over the corresponding lognormal distribution.

The second part, $\sum_{j=1}^{n-1} C_{0,\mathbb{W},[l_1,\ldots,l_j]}(\Pi(\mathbf{x}(T)))$, is the correction given by the MGEE, or the difference between the moments of the asset distribution $f_{\mathbf{X}}$ and the multivariate lognormal distribution $g_{\mathbf{S}}$ times a partial derivative of the lognormal distribution. In the univariate case this second part will be reduced to a lognormal density. In the multivariate case it can be demonstrated that reducing these partial derivatives to a multivariate lognormal is equivalent to finding the density of the sum of lognormal distributions, and this problem is still unsolved. We derive expressions for the partial derivatives, and re-use the simulation paths of the first part of the formula to calculate the integrals.

The third component of the formula, $\varepsilon(\Pi(\mathbf{s}(T)), n)$, is just the error of the approximation. We calculate some bounds for specific cases. These bounds will be determined in the section on the numerical efficiency of the model, for the case of an option defined over jump-diffusion processes.

3.1. Value of $C_{0,\mathbb{W}}(\Pi(\mathbf{x}(T)))$: fitting multivariate Wiener processes

The general structure of the formula to value options over multivariate arbitrary processes was outlined in (5). Before we can find a formula for specific cases to provide numerical applications, we must find an approximation of:

$$C_{0,\mathbb{W}}(\Pi(\mathbf{x}(T))) = \exp(-rt) \overset{(n)}{\int_0^\infty} \Pi(\mathbf{s}(T)) dG_{\mathbf{S}(T)}. \tag{6}$$

Using as a payoff the definition of the basket option,

$$\Pi(\mathbf{X}(T), \boldsymbol{\omega}, K) = [\omega_1 X_1(T) + \cdots + \omega_n X_n(T) - K]^+, \tag{7}$$

the integral becomes,

$$\overset{(n)}{\int_0^\infty} [\omega_1 s_1(T) + \cdots + \omega_n s_n(T) - K]^+ dG_{\mathbf{S}(T)}.$$

This integral can be rewritten as:

$$\overset{(n)}{\int}_{[\omega_1 s_1(T)+\cdots+\omega_n s_n(T)-K]^+} [\omega_1 s_1(T) + \cdots + \omega_n s_n(T) - K] g_{\mathbf{S}(T)} d\mathbf{s}(T),$$

and this is just a function of the first moment of the multivariate density $g_{\mathbf{S}(T)}$, truncated at the line $\omega_1 s_1(T) + \cdots + \omega_n s_n(T) \geq K$. We need to find the density of $g_{\mathbf{S}(T)}$. It is straightforward to demonstrate that the density function $g_{\mathbf{S}(T)}$ will be MVLN. Now, we find the parameters of $g_{\mathbf{S}(T)}$:

Let the process $\mathbf{S}(T)$ be defined as in (1), with initial values $\mathbf{S}(0) = (S_1(0), \ldots, S_n(0))$. Define the vector $\log(\mathbf{S}(T)) = (\log(S_1(T)), \ldots, \log(S_n(T)))$. Applying Itô's lemma to each component $\log(S_i(T))$, and solving this differential equation we have:

$$\log(S_i(T)) = \log(S_i(0)) + \left(r - \frac{1}{2}\sigma_i^2\right) t + \sigma_i W_i(T).$$

The distribution $g_{\mathbf{S}}(T)$ will be n-variate lognormal with parameters:

$$\boldsymbol{\mu}_{\mathbf{s}} = \begin{pmatrix} \log\left(S_1(0)\right) + \left(r - \frac{1}{2}\sigma_1^2\right) t \\ \vdots \\ \log\left(S_n(0)\right) + \left(r - \frac{1}{2}\sigma_n^2\right) t \end{pmatrix} \quad \boldsymbol{\Sigma}_{\mathbf{s}} = \begin{pmatrix} \sigma_1^2 t & \sigma_1\sigma_2\rho_{1,2}t & \cdots \\ \sigma_2\sigma_1\rho_{1,2}t & \sigma_2^2 t & \cdots \\ \vdots & & \ddots \end{pmatrix}. \tag{8}$$

Having found the density parameters of $g_{\mathbf{S}(T)}$, the problem of solving the integral (6) reduces to finding the first moment of the multivariate lognormal with parameters (8), truncated at the semi-plane $\omega_1 S_1(T) + \cdots + \omega_n S_n(T) \geq K$. Assume that we have a bivariate case with payoff $\Pi(\mathbf{S}(T)) = [S_1(T) + S_2(T) - K]^+$; the value of such option will be $\mathbb{E}_0^Q\left[(S_1(T) + S_2(T) - K)^+\right]$.

The multivariate integral is approximated using a Monte Carlo method:

$$\begin{aligned} C_{0,\mathbb{W}}(\Pi(\mathbf{x}(T))) &= \exp(-rt) \overset{(n)}{\int}_{[\omega_1 s_1(T)+\cdots+\omega_n s_n(T)-K]^+} [\omega_1 s_1(T) + \cdots + \omega_n s_n(T) - K] g_{\mathbf{S}(T)} d\mathbf{s}(T) \\ &\approx \exp(-rt)\frac{1}{N}\sum_{p=1}^{N}\left(\sum_{i=1}^{n}\omega_i S_i(0) e^{\left(r-\frac{1}{2}\sigma_i^2\right)t+\sigma_i\sqrt{t}\phi_i^j} - K\right)^+, \end{aligned}$$

where N is the number of path simulations, n the number of assets, and ϕ_i^p is a multivariate normal standard variable generated with correlations ρ_{i_1,i_2} between assets S_{i_1}, S_{i_2} for the sample-path p.

3.2. Value of $C_{0,\mathbb{W},[l_1,\ldots,l_j]}(\Pi(\mathbf{x}(T)))$: corrections of the price by moments of the risk-neutral distribution

For the second part of the formula, the integral will be approximated with a Monte Carlo simulation, although other methods like the Laplace inverse transform are suggested for future extensions of this work for cases of low dimensionality.[9] The sample-paths used to calculate the Wiener part $C_{0,\mathbb{W}}(\Pi(\mathbf{x}(T)))$ could be re-used to calculate the integrals of $C_{0,\mathbb{W},[l_1,\ldots,l_j]}(\Pi(\mathbf{x}(T)))$. We proceed to calculate the partial derivatives.

It turns out that the partial derivatives are functions of the density $g_{\mathbf{S}}$:

$$\frac{\partial^j}{\partial S_{l_1}\ldots\partial S_{l_j}} g_{\mathbf{S}(T)} = g_{\mathbf{S}(T)} h\left(S_{l_1}(T), \ldots, S_{l_j}(T)\right),$$

where $h\left(S_{l_1}(T), \ldots, S_{l_j}(T)\right)$ is a function of $S_{l_1}(T), \ldots, S_{l_j}(T)$. In Section A.2 of the appendix there is a detailed description of the form of the partial derivatives.

[9] As mentioned before, when the time index of the stochastic process is omitted we refer to the random variable at time t: $\mathbf{s} \equiv \mathbf{s}(T)$.

For calculating $C_{0,\mathbb{W},[l_1,\ldots,l_j]}\left(\Pi(\mathbf{x}(T))\right)$, the moments $M_{[l_1,[\ldots,[l_j]\ldots]}$ are given by the cumulants of the risk-neutral density $k_{l_1,\ldots,l_j}(\mathbf{x})$, and the cumulants $k_{l_1,\ldots,l_j}(\mathbf{s})$ of the MLVN$(\boldsymbol{\mu},\boldsymbol{\Sigma})$ distribution are:

$$k_{l_1,\ldots,l_j}(\mathbf{s}) = \mathbb{E}\left[S_1^{\alpha_1} S_2^{\alpha_2} \ldots S_n^{\alpha_n}\right] = \exp\left(\frac{1}{2}\boldsymbol{\alpha}'\Sigma\boldsymbol{\alpha} + \boldsymbol{\alpha}'\boldsymbol{\mu}\right), \tag{9}$$

where $\sum_i \alpha_i = j$. The integrals are approximated using the Monte Carlo path simulations generated before for the calculation of $C_{0,\mathbb{W}}(\Pi(\mathbf{x}(T)))$:

$$\begin{aligned}
&\sum_{j=1}^{n-1} C_{0,\mathbb{W},[l_1,\ldots,l_j]}(\Pi(\mathbf{x}(T))) = \\
= \quad & \exp(-rt)\sum_{j=1}^{n-1} M_{[l_1,[\ldots,[l_j]\ldots]}\frac{(-1)^j}{j!}{}^{(n)}\!\!\int_{[\omega_1 s_1(T)+\cdots+\omega_n s_n(T)-K]^+}[\omega_1 s_1(T)+\cdots+\omega_n s_n(T)-K]\frac{\partial^j}{\partial s_{l_1}\ldots\partial s_{l_j}}g_{\mathbf{s}(T)}d\mathbf{s}(T) \\
\approx \quad & \exp(-rt)\left(\sum_{j=1}^{n-1} M_{[l_1,[\ldots,[l_j]\ldots]}\frac{(-1)^j}{j!}\frac{1}{N}\sum_{p=1}^{N} h\left(\mathbf{s}(T)\right)\left(\sum_{i=1}^{n}\omega_i s_i(0)e^{\left(r-\frac{1}{2}\sigma_i^2\right)t+\sigma_i\sqrt{t}\phi_i^j}-K\right)^+\right).
\end{aligned}$$

3.3. Analysis of the correction term $C_{0,\mathbb{W},[l_1,\ldots,l_j]}\left(\Pi(\mathbf{x}(T))\right)$

The term $C_{0,\mathbb{W},[l_1,\ldots,l_j]}\left(\Pi(\mathbf{x}(T))\right)$ is developed further. With the intention of abbreviating the notation, the time parameter is omitted, therefore $\mathbf{s}(T) \equiv (S_1,\ldots,S_n)$. By definition, the density $g_{\mathbf{S}}$ is:

$$g_{\mathbf{S}} = (2\pi)^{-n/2}\left|\boldsymbol{\Sigma}_{\mathbf{s}}\right|^{-1/2}\left(\prod_{i=1}^{n} S_i^{-1}\right)\exp\left(-\frac{1}{2}\left(\log(\mathbf{s})-\boldsymbol{\mu}_{\mathbf{s}}\right)'\Sigma_{\mathbf{s}}^{-1}\left(\log(\mathbf{s})-\boldsymbol{\mu}_{\mathbf{s}}\right)\right), \tag{10}$$

where $\log(\mathbf{s}) = (\log(S_1),\ldots,\log(S_n))$ and $\boldsymbol{\mu}_{\mathbf{s}},\Sigma_{\mathbf{s}}$ are the mean vector and covariance matrix defined in (8).

The second part of the option approximation (5) up to the second-order is,

$$\begin{aligned}
\sum_{j=1}^{2} C_{0,\mathbb{W},[l_1,\ldots,l_j]}(\Pi(\mathbf{s}(T))) \quad = \quad & \exp(-rt)\sum_{l_1=1}^{n} M_{l_1}(-1)\,{}^{(n)}\!\!\int_0^\infty \Pi(\mathbf{s}(T))\frac{\partial}{\partial S_{l_1}}g_{\mathbf{S}} + \\
& \exp(-rt)\sum_{l_1=1}^{n}\sum_{l_2=1}^{n} M_{l_1,l_2}\frac{1}{2}\,{}^{(n)}\!\!\int_0^\infty \Pi(\mathbf{s}(T))\frac{\partial^2}{\partial S_{l_1}\partial S_{l_2}}g_{\mathbf{S}}.
\end{aligned} \tag{11}$$

Define by $\Sigma_{\mathbf{s}}^{-1}$ the inverse matrix of $\Sigma_{\mathbf{s}}$:

$$\boldsymbol{\Sigma}_{\mathbf{s}}^{-1} = \begin{pmatrix} \varsigma_{1,1} & \varsigma_{1,2} & \cdots \\ \varsigma_{2,1} & \varsigma_{2,2} & \cdots \\ \vdots & & \ddots \end{pmatrix}. \tag{12}$$

After the analysis of the correction terms of $C_{0,\mathbb{W},[l_1,\ldots,l_j]}\left(\Pi(\mathbf{x}(T))\right)$ up to the second-order, developed in the Section A.3 of the appendix, we have that the first-order moment correction of (11) becomes:

$$\exp(-rt)\sum_{l_1=1}^{n} M_{l_1}(-1)\overset{(n)}{\int_0^\infty}\Pi(\mathbf{s}(T))\frac{\partial}{\partial S_{l_1}}g_{\mathbf{S}} \quad = \quad \exp(-rt)\Bigg(\sum_{l_1=1}^{n} M_{l_1}(-1)(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}-1)\exp(-\mu_{l_1})\overset{(n)}{\int_0^\infty}\Pi(\mathbf{s}(T))g_{\mathbf{S}} + \sum_{l_1=1}^{n} M_{l_1}(-1)\exp(-\mu_{l_1})\sum_{j=1}^{n}\varsigma_{l_1,j}\overset{(n)}{\int_0^\infty}\log(S_j)\Pi(\mathbf{s}(T))g_{\mathbf{S}}\Bigg).$$

The term $\log(S_j)=\log(S_j(T))$ is a log-contract, and it is an essential instrument to hedge variance swaps, and moment swaps in general (see Neuberger, 1994; Demeterfi et al., 1999; Schoutens, 2005, for more details). This log-contract is defined by Neuberger (2012) as an entropy contract. The log-contract appears after applying the first-order partial derivative of the MVLN density, and it represents the sensitivity of the risk-neutral density to changes of $S_{l_1}(T)$, and it could be considered a type of 'Delta' of the risk-neutral density, with respect to future changes $S_{l_1}(T)$, and it is related to the classical 'Delta' of changes with respect to the current stock price $S_{l_1}(0)$. The log-contract is essential for variance swaps, and similarly seems to be essential for the sensitivity to changes of $S_{l_1}(T)$.

The second-order moment correction of (11) is (see Section A.3 of the appendix):

$$\exp(-rt)\sum_{l_1=1}^{n} M_{l_1,l_1}\frac{1}{2}\overset{(n)}{\int_0^\infty}\Pi(\mathbf{s}(T))\frac{\partial^2}{\partial S_{l_1}^2}g_{\mathbf{S}} \quad = \quad \exp(-rt)\Bigg(\sum_{l_1=1}^{n} M_{l_1,l_1}\frac{1}{2}\exp(-2\mu_{l_1})\left(2-3\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}+\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)^2-\varsigma_{l_1,l_1}\right)\overset{(n)}{\int_0^\infty}\Pi(\mathbf{s}(T))g_{\mathbf{S}} + \sum_{l_1=1}^{n} M_{l_1,l_1}\frac{1}{2}\exp(-2\mu_{l_1})\left(3-2\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)\sum_{j=1}^{n}(\varsigma_{l_1,j})\overset{(n)}{\int_0^\infty}\Pi(\mathbf{s}(T)\log(S_j)g_{\mathbf{S}} + \sum_{l_1=1}^{n} M_{l_1,l_1}\frac{1}{2}\exp(-2\mu_{l_1})\sum_{j_1=1}^{n}(\varsigma_{l_1,j_1})^2\overset{(n)}{\int_0^\infty}\Pi(\mathbf{s}(T))\log(S_{j_1})^2 g_{\mathbf{S}}\Bigg),$$

and the second-order cross-moment correction is:

$$\exp(-rt)\sum_{l_1=1}^{n}\sum_{l_2=2}^{n} M_{l_1,l_2}\frac{1}{2}\overset{(n)}{\int_0^\infty}\Pi(\mathbf{s}(T))\frac{\partial^2}{\partial S_{l_1}^2}g_{\mathbf{S}} \quad = \quad \exp(-rt)\sum_{l_1=1}^{n}\sum_{l_2=1}^{n} M_{l_1,l_2}\frac{1}{2}\exp(-\mu_{l_1}-\mu_{l_2})\times \Bigg(\left(1-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}-\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}+\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}-\varsigma_{l_1,l_2}\right)\overset{(n)}{\int_0^\infty}\Pi(\mathbf{s}(T))g_{\mathbf{S}} + \sum_{j=1}^{n}\left(\varsigma_{j,l_1}\left(1-\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)+\varsigma_{l_2,j}\left(1-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)\right)\overset{(n)}{\int_0^\infty}\Pi(\mathbf{s}(T))\log(S_j)g_{\mathbf{S}} + \sum_{j_1=1}^{n}\sum_{j_2=1}^{n}(\varsigma_{l_1,j_1}\varsigma_{l_2,j_2})\overset{(n)}{\int_0^\infty}\Pi(\mathbf{s}(T))\log(S_{j_1})\log(S_{j_2})g_{\mathbf{S}}\Bigg).$$

In this case we have calculated second-order sensitivities of the risk-neutral density to changes of $S^2_{l_1}(T)$, or a 'Gamma' equivalent of the risk-neutral density. The quadratic log-contract functions will produce quadratic volatility terms or variance terms, essentials for calculating the sensitivity of the risk-neutral density to $S^2_{l_1}(T)$.

We could extrapolate that the correction terms of higher-order consist of:

1. Functions of Wiener processes related to $C_{0,\mathbb{W}}(\Pi(\mathbf{x}(T)))$ with transformed MVLN densities,
2. Sums of log-contracts times Wiener processes,
3. Sums of cross log-contracts of higher-order times Wiener processes.

The option price approximation could be separated in three terms as in (5). The first term, $C_{0,\mathbb{W}}(\Pi(\mathbf{x}(T)))$, is an integral over a MVLN density as in the Black and Scholes (1973) world. For the second term, $C_{0,\mathbb{W},[l_1,\ldots,l_j]}(\Pi(\mathbf{x}(T)))$, we analysed the expansion up to the second-order and we found that it could be expressed as an integral of shifted MVLN densities times log-contracts and cross log-contracts. The expansions beyond the second-order will have a similar pattern given the nature of the MVLN density. This result could be used to hedge the risk-neutral density using the moments of higher-order, and further important theories could be developed from this result.

3.4. Analysis of the error term $\varepsilon(\Pi(\mathbf{s}(T)), n)$

We note that the error term of the MGEE is:

$$\varepsilon(\Pi(\mathbf{s}(T)), n) = \frac{1}{2\pi} \overset{(n)}{\int_0^\infty} \exp(i\xi'\mathbf{s}(\mathbf{T}))o(\|\xi\|^n)d\xi \quad = \quad \sum_{j=n}^{\infty} M_{[l_1,[\ldots,[l_j,[\ldots]]\ldots]} \frac{(-1)^j}{j!} \frac{\partial^j}{\partial s_{l_1} \ldots \partial s_{l_j}} g_{\mathbf{S}}.$$

Numerical analysis in the univariate GEE for the lognormal case were done by Schleher (1977) and by Jarrow and Rudd (1982), and a deeper analysis of multivariate Edgeworth expansions was done by Skovgaard (1986). In our case, all the cumulants of the MLVN $k_{l_1,\ldots,l_j}(\mathbf{s})$ exist (see Equation 9), and in case all cumulants of the risk-neutral density $k_{l_1,\ldots,l_j}(\mathbf{x})$ exist the difference of cumulants $M_{l_1,\ldots,l_j}$ will be finite. Then, assuming,

$$\lim_{n\to\infty} M_{[l_1,[\ldots,[l_j,[\ldots]]\ldots]} \frac{(-1)^j}{j!} = 0.$$

it can be shown that,

$$\lim_{n\to\infty} \sup \|\varepsilon(\Pi(\mathbf{s}(T)), n)\| = 0,$$

noting by the result of previous section that,

$$\lim_{n\to\infty} \frac{\partial^n}{\partial s_{l_1} \ldots \partial s_{l_n}} g_{\mathbf{S}} \approx \lim_{n\to\infty} \frac{\log^n(S_{l_j})}{S^n_{l_j}} g_{\mathbf{S}} = 0.$$

4. Numerical analysis of multi-asset option pricing: methods comparison

4.1. Multivariate Merton's jump-diffusion

Consider that we are in a jump-diffusion risk-neutral world as in Merton (1976), but an asset manager does not acknowledge the presence of jumps, and actually he prices the options in the market considering only the *Wiener* diffusions (GBM). The mispricing will be related to the size of volatility and the drift of the jumps, but let us assume both are unknown for the asset manager. In this section we developed a set of numerical examples to test the benefits of measuring risk-neutral moments and using a MGEE, against using classical multi-asset options that do not incorporate this information. The analytic approximations of Li et al. (2010) and Alexander and Venkatramanan (2012) were developed and used to compare with the results of the MGEE, additionally to the results of plain vanilla Monte Carlo methodology.

To measure the option pricing corrections with a practical example, we select as the candidate for the risk-neutral density to be approximated $f_{\mathbf{X}(T)}$, the density on which a jump-diffusion (J-D) process of Merton (1976) converges. We extend the definition of Merton processes to the multivariate case:

Definition 4.1. *Denote the multi-asset jump-diffusion (MJ-D) to the n-variate stochastic process $\mathbf{X} = \{X_i(T) \in \mathbb{R}^+, t \geq 0\}, i \in \{1, \dots, n\}$, described by:*

$$dX_i(T) = \mu_i X_i(T)dt + \sigma_i X_i(T)dW_i(T) + (J_i(T) - 1)dP_i(\lambda),$$

where $W_i(T)$ are Wiener processes, $P_i(\lambda)$ is a Poisson process with intensity parameter λ, and $(J_i(T) - 1)$ represents the jump-size. The jump size has a normal distribution: $J_i(T) \sim \phi\left(\delta_i, \nu_i^2\right)$. We assume that the jump's size and the jump's occurrence are independent, therefore uncorrelated between,

$$\langle J_i(T), J_j(T)\rangle = \langle dP_i(T), dP_j(T)\rangle = 0,$$

with $i, j \in \{1, \dots, n\}, i \neq j$, likewise the Wiener processes and the jumps:

$$\langle dW_i(T), dP_j(T)\rangle = \langle dP_i(T) = J_j(T)\rangle = 0.$$

On average, the MJ-D will be similar to a GBM diffusion:

$$dX_i(T) = \mu_i X_i(T)dt + \sigma_i X_i(T)dW_i(T),$$

but every λ times it jumps $J_i(T) - 1$, generating the change in the asset i:

$$dX_i(T) = \mu_i X_i(T)dt + \sigma_i X_i(T)dW_i(T) + (J_i(T) - 1).$$

For this process to be a martingale, the drift needs to be extracted:

$$dX_i(T) = \left(r - \frac{1}{2}\sigma_i^2 - b\right) X_i(T)dt + \sigma_i X_i(T)dW_i(T) + (J_i(T) - 1)dP_i(\lambda),$$

where r is the constant risk-free interest rate and b is the adjustment due to the jump process. Merton found that, if the jumps are i.i.d., the price process will be lognormal distributed. In the multivariate case, $\mathbf{X}(T)$ will have a MVLN(μ, Σ) distribution. Applying the results of Das and Uppal (2004)[10] to the moments of $dX_i(T)/X_i(T)$, we calculate the values of the parameters of μ, Σ:

$$\boldsymbol{\mu} = \begin{pmatrix} \log\left(X_1(0)\right) + \left(r - \frac{1}{2}(\sigma_1^2 + \lambda(\delta_1^2 + \nu_1^2))\right)t \\ \vdots \\ \log\left(X_n(0)\right) + \left(r - \frac{1}{2}(\sigma_n^2 + \lambda(\delta_n^2 + \nu_n^2))\right)t \end{pmatrix} \quad \boldsymbol{\Sigma} = \begin{pmatrix} (\sigma_1^2 + \lambda(\delta_1^2 + \nu_1^2))t & \cdots \\ \sigma_2\sigma_1\rho_{1,2}t & \cdots \\ \vdots & \ddots \end{pmatrix}.$$

Consequently, the value for b that transforms the density of the process in a martingale is:

$$b = \frac{1}{2}\lambda\left(\delta_i^2 + \nu_i^2\right)t.$$

Merton established that δ_i must be equal to zero for the drift of the process to be zero.[11] In Bates (1991) additional expressions for b are derived where $\delta_i \neq 0$ to generate asymmetric jump-diffusion processes.

The cumulants of $f_{\mathbf{X}(T)}$ will be necessary to calculate the option price. In the univariate case, a closed-form density for $\mathbf{X}(T)$ is provided by Merton. For the MVLN moments we use the expression in (9). The first four cumulants are calculated using the expressions in (4).

Despite the fact that we have a closed-form expression for $f_{\mathbf{X}(T)}$, we use a MGEE to approximate the option price. The auxiliary function $g_{\mathbf{S}(T)}$ will be a MVLN$(\tilde{\mu}, \tilde{\Sigma})$ similar to $f_{\mathbf{X}(T)}$, with the total volatility of the assets without the jump effect $(\delta_i = 0, \nu_i = 0)$:

$$\begin{aligned} \hat{\sigma}_i &= \sigma_i + \lambda(\nu^2), \\ \tilde{\sigma}_i &= \sigma_i, \end{aligned}$$

where $\hat{\sigma}_i$ is the total volatility of the jump-diffusion assets, and $\tilde{\sigma}_i$ is the total volatility of the simple diffusion assets. The parameters of the simple diffusion are the same as (8).

[10] In Das and Uppal (2004), the moments of the multivariate returns $dX_i(T)/X_i(T)$ are calculated with the characteristic function. Das and Uppal assume a perfect correlation between the jumps: $\langle J_i(T), J_j(T)\rangle = 1$. In our case to simplify results we assume independent jumps, but jumps' correlations different to 0 and 1 can easily be modelled with the characteristic function.

[11] If δ_i is not zero, the jump-diffusion price process changes $dX_i(T)/X_i(T)$ are not MVN, and the third- and fourth-order cumulants are:

$$\begin{aligned} k_{l_1,l_2,l_3} &= \lambda\left(\nu_{l_1}\nu_{l_2}\mu_{l_3} + \nu_{l_1}\nu_{l_3}\mu_{l_2} + \nu_{l_2}\nu_{l_3}\mu_{l_1} + \mu_{l_1}\mu_{l_2}\mu_{l_3}\right), \\ k_{l_1,l_2,l_3,l_4} &= \lambda\left(\nu_{l_1}\nu_{l_2}\mu_{l_3}\mu_{l_4} + \nu_{l_1}\nu_{l_3}\mu_{l_2}\mu_{l_4} + \nu_{l_2}\nu_{l_3}\mu_{l_1}\mu_{l_4} + \nu_{l_1}\nu_{l_4}\mu_{l_2}\mu_{l_3} + \nu_{l_2}\nu_{l_4}\mu_{l_1}\mu_{l_3} + \nu_{l_3}\nu_{l_4}\mu_{l_1}\mu_{l_2} + 3\nu_{l_1}\nu_{l_2}\nu_{l_3}\nu_{l_4} + \right. \\ &\quad \left. \mu_{l_1}\mu_{l_2}\mu_{l_3}\mu_{l_4}\right), \end{aligned}$$

for $l_1, \ldots, l_4 \in \{1, \ldots, n\}$.

4.2. Pricing basket options over multivariate jump-diffusion processes

Numerical results for pricing basket options are presented in Table A.1 in the Appendix, where the risk-neutral density $f_{\mathbf{X}(T)}$ is generated by a 5-dimensional jump-diffusion process with parameters $\lambda \in \{1, 10\}, \delta_i = 0, \nu_i \in \{0.05, 0.20\}, (X_1(0), \dots, X_5(0)) = (35, 25, 20, 15, 5), r \in \{0.05, 0.10\}, t \in \{0.25, 1\}, \sigma_i = 0.2, i \in \{1, \dots, 5\}$. The payoff of the basket option to be calculated is $\Pi(\mathbf{X}(T)) = \left(\sum_{i=1}^{5} X_i(T) - K\right)^+$ with $K \in \{90, 100, 110\}$. We focus our attention not only on the precision of the MGEE approximation, but on the contribution of the differences in the cumulants of different risk-neutral density states. The columns *AV2012* and *Li2010a* represent the option price of Alexander and Venkatramanan (2012) and Li et al. (2010) methodologies.

Consider a situation where the real market evolves either by $Wiener$ states or *J-D* states. We could estimate λ, ν as in Das and Uppal (2004), but we would have no information about the impact of risk-neutral moments on the price. Additional hedging strategies could be generated with this information. For a risk manager, the differences in the prices between the $Wiener$ and *J-D* columns are the price premium caused by the jumps. The increase of λ and ν will increase the difference between these columns. In the options deep OTM, the price difference is even higher, as an effect of the higher cumulants caused by the jumps, and the wider region on which the payoff will be positive for *J-D*. Despite the higher cumulants, the price difference of the $Wiener$ and *J-D* columns for options deep ITM is small, caused by the narrower region over which the payoff will be zero.

Third- and fourth-order corrections add noise in the extreme case of $\lambda = 10$ and $\nu = 0.20$. Nevertheless, these values are extreme as the reported values in Das and Uppal (2004) with real market data, which were in the range of $\lambda \in (0.0138, 0.0501)$ and $\nu \in (0.0792, 0.1185)$ for equity indices of developed countries. Equity indices of emerging markets report a higher jump-volatility ν, but the jump intensity λ is still much lower than the parameters considered in these examples, and also the multiplication of the jump volatility by the intensity is much lower in the examples considered. For the cases with a lower jump-intensity ($\lambda = 1$), the third- and fourth-order corrections reduce the absolute price difference of $Wiener$ and *J-D* from 14.02% to 4.26% and 13.74%, respectively.

4.3. Multivariate q-Gaussian diffusion

Lévy processes are commonly used in physics to model the behaviour of some complex systems in nature. One example of a Lévy process is the q-Gaussian process that is one of the q-processes derived from maximisation of the *Tsallis entropy* (Tsallis et al., 1995).

The q-Gaussian process is capable of reproducing high levels of third- and fourth-order moments, as the generated distribution can have heavy tails. First applications on which the q-Gaussian processes are used for option pricing were provided by Borland (2002*a*) and Borland (2002*b*), where a closed-form formula for the price of an European call option is provided. The closed-form formula emerge as the solution of a non-linear Fokker-Planck equation that incorporates a Tsallis distribution price dynamic, and the formula is dependent on the parameter q of the q-Gaussian distribution; when $q \to 1$, the Black&Scholes formula is recovered. Some empirical tests conducted in the S&P500 index by Borland (2002*b*) determine $q \approx 1.5$ fits some stylised facts of stock prices such as heavy tails and volatility smiles for the case of index options.

A non-extensive approach for studying the dependence between two non-Gaussian variables was developed by Duarte-Queirós (2005). In his research, Duarte-Queirós (2005) provides a generalisation of the Kullback-Leibler mutual information measure using the Tsallis entropy framework. This new mutual information q-measure is used to test dependency between the NYSE and DJ stock indices. The results show that the q-measure of mutual information confirms the autocorrelation stylised facts observed by calibrating ARCH and GARCH processes as in Engle (1982).

In this research we are concerned with price effects of multivariate moments over multi-asset option prices, as a result we use a multivariate extension of the q-Gaussian distribution presented by Vignat and Plastino (2007):

Definition 4.2. *Denote the multi-asset q-Gaussian to the n-variate stochastic process* $\mathbf{X} = \{X_i(T) \in \mathbb{R}^+, t \geq 0\}, i \in \{1, \dots, n\}$, *described by:*

$$dX_i(T) = \mu_i X_i(T)dt + \sigma_i X_i(T) d\Omega_i(T), \tag{13}$$

where $d\Omega_i(T)$ are Tsallis feedback processes,

$$d\Omega_i(T) = P_i(\Omega_i(T))^{(1-q)/2} dW_i(T),$$

with P_i components of a variable with the q-Gaussian density that satisfies the Fokker-Planck equation:

$$\frac{\partial}{\partial t} P_i \left(\Omega_i, t | \Omega_i', t'\right) = \frac{1}{2} \frac{\partial}{\partial \Omega_i^2} P_i^{2-q} \left(\Omega_i, t | \Omega_i', t'\right). \tag{14}$$

Let $\Omega_i(0) = 0$, a solution for each univariate equation in (14) is:

$$P_q(\Omega_i, t | \Omega_i', t') = \frac{1}{Z(t)} \left(1 - \beta_i(t)(1-q)\Omega_i(t)^2\right)^{1/(1-q)},$$

with,

$$\beta_i(t) = c_i^{(1-q)/(3-q)} \left((2-q)(3-q)t\right)^{-2/(3-q)}, \qquad Z_i(t) = \left((2-q)(3-q)c_i t\right)^{1/(3-q)},$$
$$c_i = \beta_i Z_i^2, \qquad Z_i = \int_{-\infty}^{\infty} \left(1 - (1-q)\beta_i \Omega^2\right)^{1/(1-q)} d\Omega_i.$$

Define $\sigma_i = 1/((n+4) - (n+2)\beta_i)$, and let the covariance of the distribution be $\sigma \mathbf{I}$ ($\mathbf{I}$ = identity matrix), then Vignat and Plastino (2007) offers a re-parametrised multivariate explicit solution of (14):

$$P_{q,\sigma}\left(\Omega, t | \Omega', t'\right) = \frac{1}{\sigma^n K_{q,n}} \left(1 - \frac{(1-q)}{((n+4)-(n+2)q)} \frac{\|\Omega\|^2}{\sigma^2}\right)^{\frac{1}{1-q}}, \tag{15}$$

where,

$$K_{q,n}=\begin{cases}\left(\frac{(n+4)-(n+2)q}{1-q}\right)^{n/2}\frac{\pi^{n/2}\Gamma\left(\frac{2-q}{1-q}\right)}{\Gamma\left(\frac{2-q}{1-q}+\frac{n}{2}\right)} & \text{for } -\infty<q<1,\\ \left(\frac{(n+4)-(n+2)q}{q-1}\right)^{n/2}\frac{\pi^{n/2}\Gamma\left(\frac{1}{q-1}-\frac{n}{2}\right)}{\Gamma\left(\frac{1}{q-1}\right)} & \text{for } 1<q<\frac{n+4}{n+2}.\end{cases}$$

The moments of distribution (15) were calculated by Ghoshdastidar et al. (2014):

$$E_{P_{q,\sigma}}\left[\frac{(\Omega_1)^{b_1}\dots(\Omega_n)^{b_n}}{(\rho(\Omega))^b}\right]=\begin{cases}\bar{K}\left(\frac{(n+4)-(n+2)q}{1-q}\right)^{\Sigma_{i=1}^n\frac{b_i}{2}}\left(\prod_{i=1}^n\frac{b_i!}{2^{b_i}\left(\frac{b_i}{2}\right)!}\right) & \text{if } b_i \text{ is even } \forall i=1,\dots,n,\\ 0 & \text{otherwise.}\end{cases} \tag{16}$$

with,

$$\bar{K}=\begin{cases}\frac{\Gamma\left(\frac{1}{1-q}-b+1\right)\Gamma\left(\frac{1}{1-q}+1+\frac{n}{2}\right)}{\Gamma\left(\frac{1}{1-q}+1\right)\Gamma\left(\frac{1}{1-q}-b+1+\frac{n}{2}+\sum_{i=1}^n\frac{b_i}{2}\right)} & \text{if } -\infty<q<1,\\ \frac{\Gamma\left(\frac{1}{q-1}\right)\Gamma\left(\frac{1}{q-1}+b-\frac{n}{2}-\sum_{i=1}^n\frac{b_i}{2}\right)}{\Gamma\left(\frac{1}{q-1}+b\right)\Gamma\left(\frac{1}{q-1}-\frac{n}{2}\right)} & \text{if } 1<q<\frac{n+4}{n+2}.\end{cases}$$

Thistleton et al. (2007) proposed an algorithm for simulating q-Gaussian variables, nevertheless, we are interested in a multivariate version. Ghoshdastidar et al. (2014) derived an algorithm using some results in Vignat and Plastino (2006) to simulate n-variate q-Gaussian variables. Let $\mathbf{X}_Z$ be a normal standard n-dimensional vector. Let a be a χ^2 random variable:

$$a\sim\begin{cases}\chi^2\left(\frac{2(2-q)}{1-q}\right) & \text{for } -\infty<q<1,\\ \chi^2\left(\frac{n+2-nq}{q-1}\right) & \text{for } 1<q<\frac{n+4}{n+2},\end{cases}$$

and let,

$$\mathbf{Y}=\begin{cases}\sqrt{\frac{n+2-nq}{1-q}}\frac{\mathbf{X}_Z}{\sqrt{a+\mathbf{X}_Z^T\mathbf{X}_Z}} & \text{for } -\infty<q<1,\\ \sqrt{\frac{n+2-nq}{q-1}}\frac{\mathbf{X}_Z}{\sqrt{a}} & \text{for } 1<q<\frac{n+4}{n+2},\end{cases}$$

then the variable $\mathbf{X}$ defined as,

$$\mathbf{X}=\left(\mu_q+\Sigma_q^{1/2}\mathbf{Y}\right), \tag{17}$$

will be distributed multivariate q-Gaussian as in (15) with mean μ_q and covariance Σ_q. We use (16) and (17) for pricing multi-asset options in the next section.

4.4. Pricing basket options over q-Gaussian processes

In Table A.2 in the Appendix we present numerical results for pricing basket options, where the risk-neutral density $f_{\mathbf{X}(T)}$ is generated by a 5-dimensional q-Gaussian process as in (13) with parameters $q\in\{1.05,1.10,1.15,1.20\},(X_1(0),\dots,X_5(0))=(35,25,20,15,5),r\in\{0.05,0.10\},t\in\{0.25,1\},\sigma_i=0.2,i\in$

$\{1,\ldots,5\}$. The payoff of the basket option to be calculated is $\Pi(\mathbf{X}(T)) = \left(\sum_{i=1}^{5} X_i(T) - K\right)^+$ with $K \in \{90, 100, 110\}$. Although Borland (2002*a*) mentions that q-Gaussian with $q \approx 1.5$ fits some stylised facts from stock indices in the univariate case, the equivalent q parameter for the multivariate case has lower bounds due to aggregation of the variables. In this numerical example for $n = 5$, the q value in the density (15) has an upper bound of $1 + \frac{n+4}{n+2} \approx 1.2857$, much lower than the $5/3 \approx 1.6666$ upper bound in the univariate case. In Table A.2 we can observe how for lower values of $q = 1.05$, the fourth-order expansion $MGEE4$ is in general the best approximation, while if we increase the value of q the $MGEE3, MGEE2$ become the best approximations successively. In Figure A.1 we plot the implied volatility surface of a MGEE approximating a q-Gaussian diffusion basket option with $q = 1.10$, where we recognise a convex volatility smile similar to convex volatility smiles observed in stock indices. For a large $q = 1.20$ close to the boundary of 1.2857, none of the $MGEE$ produce a good approximation, and this behaviour of decreasing order of precision $MGEE4 < MGEE3 < MGEE2 < Wiener$ is the result of approximating multivariate q-Gaussians with a MVLN distribution. For a large q parameter, q-Gaussian will have heavier tails, up to the point of not having finite higher-order moments; for example, for univariate q-Gaussian the fourth-order moment exist just when $q < 7/5 = 1.4$. This affects negatively the performance of MGEE. Our methodology is not constrained to approximating the risk-neutral density $f_{\mathbf{X}(T)}$ with MVLN, we suggest as an extension of our work to develop the equations of the first four partial derivatives of a multivariate q-Gaussian distribution as in Section AppendixA.2 of the appendix.

5. Calibration and numerical efficiency of the approximation

In this section we measure the precision and the efficacy of the MGEE approximation, and we compare it with the other three different option pricing methodologies: plain vanilla Monte Carlo, Li et al. (2010) and Alexander and Venkatramanan (2012). This time we developed a test where the four different processes acknowledge the information of the moments of the risk-neutral density. The effects of higher-order moments for $Wiener$-based algorithms (Monte Carlo; Li et al., 2010; Alexander and Venkatramanan, 2012) will be contained in the optimisation of the volatility. In Zhao et al. (2013) it is mentioned that the effect of skewness and kurtosis of the risk-neutral density is incorporated in the volatility structure. A main concern in Section 4 was the possibility of negative MGEE density values, and their effect over the precision of the algorithm. The precision of the expansion depends on the difference of cumulants against the selected density. If the application is to hedge a risk-neutral density $f_{\mathbf{X}(T)}$ with another density $g_{\mathbf{S}(T)}$, the selection of the auxiliary density is based on the future scenarios; and the extent to which $g_{\mathbf{S}(T)}$ can be adjusted to $f_{\mathbf{X}(T)}$ will be limited to the constraints of the risk model. Generally, large deviations from $f_{\mathbf{X}(T)}$ are the typical scenarios to be tested. But when we apply MGEE to price an option, we can select $g_{\mathbf{S}(T)}$ and distort its moments to fit $f_{\mathbf{X}(T)}$ over most of its domain. The calibration algorithm reduces the difference of the cumulants of

$f_{\mathbf{X}(T)}$ and $g_{\mathbf{S}(T)}$.

5.1. Calibration algorithm

Given an unknown density $f_{\mathbf{X}(T)}$ with known moments or cumulants, over which a payoff $\Pi(\mathbf{x}(T))$ is defined, the objective is to select a MVLN$(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ density with moments that are close as possible to the moments of $f_{\mathbf{X}(T)}$. For *Wiener*-based option pricing algorithms (Monte Carlo; Li et al., 2010; Alexander and Venkatramanan, 2012), the calibration algorithm provided the optimal set of diffusion volatilities, incorporating the risk-neutral density moments information. Even though the market risk-neutral density is generally extracted from the market prices, future scenarios can be generated and priced only with changes to the cumulants, then, MGEE could be used for market risk sensitivity analysis.

There are four parameters of the density $g_{\mathbf{S}(T)}$ that can be changed: $S_i(0), t, \rho_{i,j}$ and $\tilde{\boldsymbol{\sigma}} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ for $i \neq j \in \{1, \ldots, n\}$. Changes to $S_i(0)$ and to t could appear more like a hedging exercise. Besides, changes to $\tilde{\boldsymbol{\sigma}}$ are reflected over all the cumulants of the MVLN density. Then, $\tilde{\boldsymbol{\sigma}}$ is selected as the parameter for the calibration. There are three objective functions $h_i(\tilde{\boldsymbol{\sigma}})$: we will minimise each function for a different calibration:

$$\begin{aligned}
h_2(\tilde{\boldsymbol{\sigma}}) &= \|M_{l_1,l_2}\|_2, \\
h_3(\tilde{\boldsymbol{\sigma}}) &= \|M_{l_1,l_2}\|_2 + \|M_{l_1,l_2,l_3}\|_2, \\
h_4(\tilde{\boldsymbol{\sigma}}) &= \|M_{l_1,l_2}\|_2 + \|M_{l_1,l_2,l_3}\|_2 + \|M_{l_1,l_2,l_3,l_4}\|_2.
\end{aligned}$$

Denote $\hat{\boldsymbol{\sigma}}$ to be the optimal volatility. Increments on $\tilde{\boldsymbol{\sigma}}$ result in increments of the moments and the cumulants of $g_{\mathbf{S}(T)}$. If the moments of $f_{\mathbf{X}(T)}$ are lower than the moments of $g_{\mathbf{S}(T)}$, the algorithm will decrease $\tilde{\boldsymbol{\sigma}}$. The norm used is $\|\cdot\|_2$; However, other norms were tested with slower convergence rates towards the optimal value. The density $f_{\mathbf{X}(T)}$ to be tested is the risk-neutral density of the multi-asset jump-diffusion process defined in Section 4, and it will be calibrated against different $\lambda, \nu_i, \sigma_i, r$, and t parameters. The correlation between assets $\rho_{i,j}$ is set equal to zero for all pair of assets.[12] Since the multi-asset jump-diffusion process of Section 4 converges into a MVLN distribution, the optimal volatility value is:

$$\hat{\sigma_i} = \sigma_i + \lambda\nu_i.$$

If the optimal value is reached by the optimisation algorithm, with a low tolerance (10^{-15}) between the optimal parameter and the proposed solution, the objective function $h_i(\tilde{\boldsymbol{\sigma}}), i \in \{2, 3, 4\}$ will be zero. For this reason, a noise effect is added to the algorithm, estimating the moments of $f_{\mathbf{X}(T)}$ with the sample cumulants

[12] For the specific case of calibrating all models, correlations where set to zero to obtain exact minimisation functions in the second-order moment calibration case; nevertheless when a test calibration with correlations set to 0.5 was conducted, similar results to the results of correlations equal to 0 where obtained.

of a Monte Carlo simulation. Additionally, a maximum number of function evaluations is established for the optimisation.

For each case, the MGEE of zero- ($MGEE0$), second- ($MGEE2$), third- ($MGEE3$) and fourth- ($MGEE4$) order moments are calculated. The expansion of order n includes the polynomials of order $n-1, n-2, \ldots, 1$. As the first-order cumulants are equal for any density, the first moment expansion $MGEE(1)$ is always equal to $MGEE0$; this is due to the arbitrage principle. The algorithm used to minimise $h_i(\boldsymbol{\sigma}), i \in \{1, \ldots, 3\}$ is a constrained convex optimisation method denoted as sequential quadratic programming (SQP). The implementation is in MATLAB with the $fmincon$ function. A constraint over the volatility is that the covariance matrix Σ must be positive semi-definite.

In the case $\rho_{1,j} = 0$, the minimisation of the objective function $h_2(\tilde{\boldsymbol{\sigma}})$:

$$h_2(\tilde{\boldsymbol{\sigma}}) \quad = \quad \|M_{l_1,l_2}\|_2 = (k_{1,1}(f_{\mathbf{X}(T)}) - k_{1,1}(g_{\mathbf{x}(T)}))^2 + \cdots + (k_{n,n}(f_{\mathbf{X}(T)}) - k_{n,n}(g_{\mathbf{x}(T)}))^2,$$

has a closed-form solution, where $k_{i,i}(g_{\mathbf{S}(T)})$ is the second central cumulant of $g_{\mathbf{S}(T)}$, and $k_{i,i}(f_{\mathbf{X}(T)})$ is the second cumulant of $f_{\mathbf{X}(T)}$, a parameter given by the initial conditions of the problem. Equate the second-order cumulants of $f_{\mathbf{X}(T)}$ and $g_{\mathbf{S}(T)}$:

$$\begin{aligned} k_{i,i}(f_{\mathbf{X}(T)}) &= k_{i,i}(g_{\mathbf{S}(T)}) \\ &= m_{i,i}(g_{\mathbf{S}(T)}) - m_i(g_{\mathbf{S}(T)})^2 \\ &= \exp\left(2\log(S_i(0)) + 2\tilde{\sigma}_i^2 t + 2\left(r - \frac{1}{2}\sigma^2\right)t\right) - \exp\left(\log(S_i(0)) + rt\right)^2, \end{aligned}$$

where $m_i(g_{\mathbf{S}(T)})$ and $m_{i,i}(g_{\mathbf{S}(T)})$ are the first- and second-order moments of $g_{\mathbf{S}(T)}$. Clear the $\tilde{\sigma}_i$ variable, and the optimal solution yields:

$$\hat{\sigma}_i = \frac{\sqrt{\log(s_i(0)^2 + k_{i,i}(f_{\mathbf{X}(T)})\exp(-2rt)) - 2\log(s_i)}}{t^{1/2}}. \tag{18}$$

This solution is independent of the distribution of $f_{\mathbf{X}(T)}$, and could be used when the correlations between the assets are zero.

5.2. Results

An inspection of the results demonstrates the effectiveness of the calibration method on precision.[13] Table A.3 shows the mean dollar error of the MGEE approximation, in a cross-pairs *objective function used for calibration – higher-order cumulant considered for MGEE approximation*. There is an evident reduction of the mean dollar error when a calibration method is applied. The two best objective functions are $h_2(\tilde{\boldsymbol{\sigma}})$ and $h_3(\tilde{\boldsymbol{\sigma}})$. The optimal order of the cumulants to be considered for the expansion is the second ($MGEE2$).

[13] Table A.1 of results without calibration of the $\tilde{\sigma}_i$ parameter was considered for testing the effectiveness of the calibration algorithm.

In some cases there exists an improvement in precision when the third-order cumulant is included in the expansion. The inclusion of the fourth-order cumulant reports highly noisy results. This is a consequence of the small cross-moments of the density $f_{\mathbf{X}(T)}$ that exacerbate fourth-order cumulant differences.

The *Li2010a* and *AV2012* methodologies underperformed the MGEE, although this time they incorporated the information of skewness and kurtosis with the calibration of the volatility. For the *Uncalibrated* MGEE, the MC column represents option prices over the multi-asset GBM process calculated with the Monte Carlo algorithm. For the calibrated MGEE, the MC, *Li2010a* and *AV2012* columns represent the option prices after the GBM process were adjusted by the optimal volatility. Tables A.6 and A.7 show the improvement in the MGEE when moments of higher-order are added to the expansion; they were calculated subtracting columns *MGEE2*, *MGEE3* and *MGEE4* from the MC column in Tables A.3 and A.4. It is evident an improvement in the use of a second-order MGEE for extreme cases (Table A.3 with $\lambda \in \{1, 10\}$), and for less extreme cases (Table A.4 with $\lambda = 1$) adding moments of second-, third- and fourth-order were beneficial to explain the price of the jumps in the risk-neutral density.

The row $h_2(\tilde{\sigma})$ of Tables A.3, A.4, A.5, A.6, and A.7 represents the results of the option price over a multivariate GBM process with volatility adjusted by (18). These are the results of the best approximating MVLN distribution, and they will be the benchmark. In 56.25% (27 of 48 of the cases) the MGEE approximation price that included the second-order cumulant correction was superior. In 22.91% (11 of 48) of the cases the inclusion of the third-order cumulant, when $h_2(\tilde{\sigma})$ is used for calibration, will produce a better approximation. Table A.5 shows a resumé of the number of best approximations for each pair *objective function used for calibration – higher-order cumulant considered for MGEE approximation*. The MGEE with the second-order expansion is the best approximation in 42.18% (81 of 192) of the total. The algorithms of *AV2012* and *Li2010a* jointly, are just better in 7.81% (9 of 192) of the total cases. The initial volatility for all assets is $\tilde{\sigma}_i = 0.2$. The optimisation algorithm achieves results for the parameters $\tilde{\sigma}_i$ close to the optimal $\hat{\sigma}$. For $\lambda = 10, \nu = 0.2$ the calibration algorithm outweighs the volatility. The small fourth-order cross-moments of the simulation increase the total kurtosis and the methodology the algorithm uses to reduce this is to increase the $\tilde{\sigma}_i$ parameters beyond the optimal theoretical value.

Analysing the precision in Table A.1, the third-order expansion ($MGEE3$) is the best approximation in the majority of the cases (39.58%), while the second-order expansion ($MGEE2$) achieves the best approximation in 27.08% of the cases. The increase in values of the parameters λ, ν, t, and r are reflected as an increase in the cumulants of the risk-neutral density $f_{\mathbf{X}(T)}$. For example, for the parameters $\lambda = 10, \nu = 0.20$, the third- and fourth-order expansions, $MGEE3$ and $MGEE4$, only add noise to the approximation. However, these values are extreme for the real market data values reported by Das and Uppal (2004). In Table A.4 there is the mean dollar error when only processes with $\lambda = 1$ are considered. The improvement in the fourth-order approximation is significant. The optimisation of $h_3(\tilde{\sigma})$ and $h_4(\tilde{\sigma})$ provide similar results. They are ineffective only when the parameters of the jump-diffusion λ, ν are extreme. The aggregation of higher moments in the calibrating function $h_i(\tilde{\boldsymbol{\sigma}})$, produce a ripple effect over the precision of the immediate previous-order MGEE approximation $MGEE(i-1)$, when compared with the $MGEE(i)$, when the calibration is done with $h_i(\tilde{\boldsymbol{\sigma}})$.

Table A.8 displays the percentage of simulated paths that result in a positive value when they are evaluated over the generated MGEE density. The improvement given by the calibration is significant. The best pair is the second column with $(h_2(\tilde{\boldsymbol{\sigma}}), MGEE2)$. This pair is a MVLN with the volatility calculated by (18), and it is the benchmark method. The results on pair $(h_3(\tilde{\boldsymbol{\sigma}}), MGEE3)$ reveal results close to the density function. For $\lambda = 10, \nu = 0.20$ there is a increase of the negative region; this could be used to measure the performance of the results in the case where the market option price is unknown.

5.3. Performance of the MGEE

The MGEE requires an approximating distribution to fit the risk-neutral density. For pricing multi-asset options like spread or basket options, we will need to numerically integrate the expected payoff as it does not exist a closed-form solution for these particular cases. Analytical approximations like Li et al. (2010) were tested for the expansion of the correction terms in Section 3.3; however, results showed that the errors of the closed-form approximation were amplified when they were used with the MGEE, and precision is the most important feature of the expansion. A solution is to generate Monte Carlo sample-paths for pricing all the terms of the expansion. To select an appropriate number of simulations paths, we evaluated time and precision. The precision of the Monte Carlo algorithm increases at a rate of $1/\sqrt{N}$ for any dimension, a favourable attribute for high-dimension problems. In Figure A.2 we plot the standard deviation of the Monte Carlo integration for the increasing number of paths. Valuation of the integral was tested for pricing two different multi-asset options with jump-diffusion defined in Section 4.2.

Valuations with up to 50,000,000 simulations were tested, finding that 20,000,000 simulations would provide an approximate value with an error of approximately 0.3% for jump-diffusions with an intensity of $\lambda = 1$, and of approximately 10% for jump-diffusions with an intensity of $\lambda = 10$. In Table A.9, we have the running time of the Monte Carlo algorithm, and the additional time consumed by the successive MGEE. The second-order MGEE will consume only 30% more time than the MC algorithm, while the fourth-order MGEE will consume approximately eight times the time consumed by the Monte Carlo algorithm. Considering, that the option price precision could be improved in some cases by more than 20% (see Table A.1), the MC option pricing with MGEE for cases when moments of the risk-neutral density are available would be a suitable decision.

6. Conclusions

The theory of multi-asset option pricing has been developed under the concept of approximating the multivariate risk-neutral density of the assets with the univariate density of the payoff function, undermining important information contained in the dependence structure of the multivariate density. There exist several approximations for multi-asset option pricing;[14] however, there is no pricing formula at the time of writing this research that accounts for the effects of the higher-order cross-moments.

In this research, an density approximation termed the *Multivariate Generalised Edgeworth Expansion* (MGEE) is used to fit the unknown risk-neutral density with a known auxiliary continuous density through the difference in the moments of the risk-neutral density. The expansion can enhance a distribution fit over densities with high skewness or high kurtosis. The method is intended for approximating the risk-neutral density of European options. Nevertheless, if a risk-neutral density from a path-dependent option can be estimated, then the methodology can be applied.

The second purpose of the MGEE is the price approximation. When the multivariate lognormal (MVLN) distribution is used as the auxiliary distribution, the option pricing formula reveals three components: a Wiener component, the moments corrections, and the error term caused by including only the first four coefficients in the approximation. We use a Monte Carlo simulation for calculating the integrals, and the MGEE serves as a moment price enhancement of an the option price, initially derived from a Wiener process. The MGEE approximation is related to variance swaps, and a new contract defined as entropy contract by Neuberger (2012) is proposed as a future extension of our work. Likewise, the partial derivatives of the expansion are sensitivities of the risk-neutral density against changes in its form, translation, dispersion, skewness, and heavy-tailedness, results that are important for future topics of research.

The results of pricing the options over jump-diffusion processes show that the mean dollar error for the approximation could be of $\sim 1\% - 1.5\%$ when only second-order moments are included in the expansion. These results include the extreme case[15] of jump-diffusions with $\lambda = 10$. When the set examples are filtered to the case of $\lambda = 1$, the mean dollar error when third- and fourth-order moments are included in the expansion improves substantially. This is still a high parameter value for equity market returns. A major cost in using the MGEE expansion with Monte Carlo simulation for option pricing is the running performance of the algorithm, which can reduce the speed of the plain Monte Carlo algorithm (8 times slower when including the fourth-order moments in the expansion).

Additional examples of basket options pricing with assets following a multivariate q-Gaussian process suggest that for lower values of q, $q \in (1.05, 1.15)$ in a 5-dimensional case, the MGEE produces a good fit of the risk-neutral distribution and by consequence of the option price. For larger values of q, $q = 1.28 \approx$ upper bound, results show that differences in the third- and fourth-order moments generate substantial deviations that accumulate for a divergent price. We propose the use of a multivariate q-Gaussian as an auxiliary distribution for MGEE in such cases with heavier tails.

The results of this research are only the initial step to further investigations. Important extensions to

[14] See Kristensen and Mele (2011); Li et al. (2010) and Alexander and Venkatramanan (2012).

[15] Das and Uppal (2004) estimate a $\lambda < 0.1$ for emerging equity markets.

our work include the development of a new theory for hedging the cross-moments of the risk-neutral density of multi-asset contract. The instruments to achieve this goal are defined as the *entropy contracts*, and they require an extensive study and development. In a similar way, extensions to the theory of the sensitivity of the risk-neutral distribution to changes in the moments is proposed as a considerable area of investigation.

Risk managers and hedgers will benefit from having an option formula that accounts for the moments of the risk-neutral distributions. Although the results are for general multi-assets contracts, the whole set of univariate option contracts could enhance their performance if the theory of moments developed in this work were applied to their pricing and hedging.

Aït-Sahalia, Y. and A. Lo (1998), 'Nonparametric estimation of state-price densities implicit in financial asset prices', *The Journal of Finance* **53**(2), 499–547.

Alexander, C. and A. Venkatramanan (2011), 'Closed form approximations for spread options', *Applied Mathematical Finance* **18**(5), 447–472.

Alexander, C. and A. Venkatramanan (2012), 'Analytic approximations for multi-asset option pricing', *Mathematical Finance* **22**(4), 667–689.

Bates, D. S. (1991), 'The crash of '87: Was it expected? The evidence from options markets', *The Journal of Finance* **46**(3), 1009–1044.

Beißer, J. (1999), Another way to value basket options, Technical report, Johannes Gutenberg-Universitat Mainz.

Bhandari, R. and S. R. Das (2009), 'Options on portfolios with higher-order moments', *Finance Research Letters* **6**, 122–129.

Black, F. and M. Scholes (1973), 'The pricing of options and corporate liabilities', *Journal of Political Economy* **81**, 637–654.

Borland, Lisa (2002*a*), 'Option pricing formulas based on a non-gaussian stock price model', *Physical Review Letters* **89**(9), 98701–98704.

Borland, Lisa (2002*b*), 'A theory of non-gaussian option pricing.', *Quantitative Finance* **2**(6), 415–431.

Borovkova, S., F. J. Permana and H.V.D. Weide (2007), 'A closed form approach to the valuation and hedging of basket and spread option', *Journal of Derivatives* **14**(4), 8–24.

Carmona, R. and V. Durrleman (2006), 'Generalizing the Black–Scholes formula to multivariate contingent claims', *Journal of Computational Finance* **9**(2), 627–685.

Carr, P. (1988), 'The valuation of sequential exchange opportunities', *The Journal of Finance* **43**(5), 1235–1256.

Corrado, C. J. and T. Su (1996), 'S&P 500 index option tests of Jarrow and Rudd's approximate option valuation formula', *Journal of Futures Markets* **16**(6), 611–629.

Das, S. R. and R. Uppal (2004), 'Systemic risk and international portfolio choice', *The Journal of Finance* **59**(6), 2809–2834.

Del Brio, E. B., T.M. Ñíguez and J. Perote (2009), 'Gram–Charlier densities: A multivariate approach', *Quantitative Finance* **9**(7), 855–868.

Demeterfi, K., E. Derman, M. Kamal and J. Zou (1999), More than you ever wanted to know about volatility swaps, Technical report, Goldman Sachs.

Duarte-Queirós, S. M. (2005), 'On non-gaussianity and dependence in financial time series: a nonextensive approach.', *Quantitative Finance* **5**(5), 475–487.

Dufresne, D. (2004), 'The log-normal approximation in financial and other computations', *Advances of Applied Probability* **36**, 747–773.

Engle, R. (1982), 'Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation', *Econometrica* **50**, 987–1007.

Filipović, D., E. Mayerhofer and P. Schneider (2013), 'Density approximations for multivariate affine jump-diffusion processes', *Journal of Econometrics* **176**(2), 93–111.

Flamouris, D. and D. Giamouridis (2002), 'Estimating implied pdfs from american options on futures: A new semiparametric approach', *Journal of Futures Markets* **22**(1), 1–30.

Geske, R. (1979), 'The valuation of compound options', *Journal of Financial Economics* **7**(1), 63 – 81.

Ghoshdastidar, Debarghya, Ambedkar Dukkipati and Shalabh Bhatnagar (2014), 'Smoothed functional algorithms for stochastic optimization using q -Gaussian distributions', *ACM Transactions on Modeling and Computer Simulation* **24**(3), 1–26.

Harvey, C. R. and A. Siddique (2000), 'Conditional skewness in asset pricing tests', *The Journal of Finance* **55**(3), 1263–1295.

Jarrow, R. and A. Rudd (1982), 'Approximate option valuation for arbitrary stochastic processes', *Journal of Financial Economics* **10**(3), 347–369.

Ju, N. (2002), 'Pricing Asian and basket options via Taylor expansion', *Journal of Computational Finance* **5**(3), 79–103.

Kendall, M. G. (1947), *The Advanced Theory of Statistics*, Charles Griffin & Co.

Kirk, E. (1996), 'Correlation in energy markets', *Managing Energy Price Risk* pp. 71–78. Risk Publications.

Knight, J. L. and S. Satchell (2001), 'Pricing derivatives written on assets with arbitrary skewness and kurtosis', *Return Distributions in Finance* pp. 229–250. Butterworth and Heinemann.

Kraus, A. and R. H Litzenberger (1976), 'Skewness preference and the valuation of risk assets', *The Journal of Finance* **31**(4), 1085–1100.

Krekel, Martin, Johan de Kock, Ralf Korn and Tin-Kwai Man (2004), 'An analysis of pricing methods for basket options', *Wilmott Magazine* (3), 82–89.

Kristensen, D. and A. Mele (2011), 'Adding and subtracting Black-Scholes: A new approach to approximating derivative prices in continuous-time models', *Journal of Financial Economics* **102**(2), 390–415.

Li, M., S.J. Deng and J. Zhou (2008), 'Closed-form approximations for spread option prices and Greeks', *Journal of Derivatives* **15**(3), 58–80.

Li, M., S.J. Deng and J. Zhou (2010), 'Multi-asset spread option pricing and hedging', *Quantitative Finance* **10**(3), 305–324.

Lim, G. C., G. M. Martin and V. L. Martin (2005), 'Parametric pricing of higher order moments in S&P500 options', *Journal of Applied Econometrics* **20**(3), 377–404.

Limpert, E., W. A. Stahel and M. Abbt (2001), 'Log-normal distributions across the sciences: Keys and clues', *Bioscience* **51**(5), 341–352.

McCullagh, P. (1987), *Tensor Methods in Statistics*, Chapman & Hall.

Merton, Robert C. (1976), 'Option pricing when underlying stock returns are discountinuous', *Journal of Financial Economics* **3**, 125–144.

Neuberger, A. (1994), 'The log contract', *Journal of Portfolio Management* **20**(2), 74–80.

Neuberger, A. (2012), 'Realized skewness', *Review of Financial Studies* **25**(11), 3423–3455.

Níguez, T.-M. and J. Perote (2008), The general moments expansion: an application for financial risk. Plenary Talk, ERCIM' 08.

Perote, J. (2004), 'The multivariate Edgeworth–Sargan density', *Spanish Economic Review* **6**(1), 77–96.

Schleher, D.C. (1977), 'Generalized gram-charlier series with application to the sum of log-normal variates', *IEEE Transactions on Information Theory* **1**, 275–280.

Schlögl, E. (2013), 'Option pricing where the underlying assets follow a Gram/Charlier density of arbitrary order', *Journal of Economic Dynamics & Control* **37**, 611–632.

Schoutens, W. (2005), 'Moment swaps', *Quantitative Finance* **5**(6), 525–530.

Skovgaard, I.M. (1986), 'On multivariate edgeworth expansions', *International Statistical Review* **54**(2), 169–186.

Thistleton, William J., John A. Marsh, Kenric Nelson and Constantino Tsallis (2007), 'Generalized Box-Müller method for generating q-Gaussian random deviates', *IEEE Transactions on Information Theory* **53**(12), 4805–4810.

Tsallis, Constantino, Silvio V. F. Levy, André M. C. Souza and Roger Maynard (1995), 'Statistical-mechanical foundation of the ubiquity of lévy distributions in nature', *Physical Review Letters* **75**(20), 3589–3593.

Vignat, C. and A. Plastino (2006), 'Poincaré's observation and the origin of tsallis generalized canonical distributions', *Physica A: Statistical Mechanics and its Applications* **365**(1), 167–172.

Vignat, C and A Plastino (2007), 'Central limit theorem and deformed exponentials', *Journal of Physics A: Mathematical and Theoretical* **40**(45), F969–F978.

Zhao, H., J. E. Zhang and E.C. Chang (2013), 'The relation between physical and risk-neutral cumulants', *International Review of Finance* **13**(3), 345–381.

AppendixA. Analytical Proofs

AppendixA.1. Proof of Proposition 2.1

Proof. Let $f_{\mathbf{X}}$ be the continuous-time function density of $\mathbf{X}$ and $\xi \in \mathbb{R}^N$. The characteristic function (CF) of $\mathbf{X}$ is defined as:

$$\psi(\mathbf{x}, \xi) \quad = \quad \mathbb{E}\left[\exp\left(\xi_{[l_1]} X_{[l_1]} i\right)\right].$$

Denote $m_{l_1,\ldots,l_p}(\mathbf{x})$ the p-moment of $\mathbf{X}$. This function can be expanded into the infinite series:

$$\begin{aligned} \psi(\mathbf{x}, \xi) &= 1 + \xi_{[l_1]} m_{[l_1]}(\mathbf{x}) i + \xi_{[l_1]}\xi_{[l_2]} m_{[l_1,l_2]}(\mathbf{x}) i^2/2! + \xi_{[l_1]}\xi_{[l_2]}\xi_{[l_3]} m_{[l_1,l_2,l_3]}(\mathbf{x}) i^3/3! + \ldots, \\ &= \sum_{j=1}^{n-1} \xi_{[l_1]} \ldots \xi_{[l_j]} m_{[l_1,\ldots,l_2]}(\mathbf{x}) i^j/j! + o(\|\xi\|^n), \end{aligned} \tag{A.1}$$

which is convergent for small ξ. We calculate the log function:

$$\log\psi(\mathbf{x}, \xi) \quad = \quad \sum_{j=1}^{n-1} \xi_{[l_1]} \ldots \xi_{[l_j]} k_{[l_1,[\ldots,[l_j]\ldots]}(\mathbf{x}) i^j/j! + o(\|\xi\|^n),$$

where $k_{l_1,\ldots,l_p}(\mathbf{x})$ are the cumulants of order p of $\mathbf{X}$. Now suppose that we have another continuous density function $g_{\mathbf{S}}$ of a stochastic process $\mathbf{s}$. Then we can write:

$$\begin{aligned} \log\psi(\mathbf{x}, \xi) &= \sum_{j=1}^{n-1} \xi_{[l_1]} \ldots \xi_{[l_j]} (k_{[l_1,[\ldots,[l_j]\ldots]}(\mathbf{x}) - k_{[l_1,[\ldots,[l_j]\ldots]}(\mathbf{s})) i^j/j! + \\ &\quad \sum_{j=1}^{n-1} \xi_{[l_1]} \ldots \xi_{[l_j]} k_{[l_1,[\ldots,[l_j]\ldots]}(\mathbf{s}) i^j/j! + o(\|\xi\|^n), \\ &= \sum_{j=1}^{n-1} \xi_{[l_1]} \ldots \xi_{[l_j]} (k_{[l_1,[\ldots,[l_j]\ldots]}(\mathbf{x}) - k_{[l_1,[\ldots,[l_j]\ldots]}(\mathbf{s})) i^j/j! + \log\psi(\mathbf{s}, \xi) + o(\|\xi\|^n), \end{aligned}$$

where $k_{l_1,\ldots,l_p}(\mathbf{s})$ are the cumulants of order p of $\mathbf{s}$. Applying the exponential function on both sides:

$$\psi(\mathbf{x}, \xi) \quad = \quad \exp\left(\sum_{j=1}^{n-1} \xi_{[l_1]} \ldots \xi_{[l_j]} (k_{[l_1,[\ldots,[l_j]\ldots]}(\mathbf{x}) - k_{[l_1,[\ldots,[l_j]\ldots]}(\mathbf{s})) i^j/j!\right) \psi(\mathbf{s}, \xi) \exp\left(o(\|\xi\|^n)\right).$$

It could be demonstrated that: $\exp(o(\|\xi\|^n)) = 1 + o(\|\xi\|^n)$. But the exponential function can be expanded as in (A.1). Then,

$$\psi(\mathbf{x}, \xi) \quad = \quad \left(\sum_{j=0}^{n-1} \xi_{[l_1]} \ldots \xi_{[l_j]} M_{[l_1,[\ldots,[l_j]\ldots]} i^j/j!\right) \psi(\mathbf{s}, \xi) + o(\|\xi\|^n), \tag{A.2}$$

where $M_{l_1,\ldots,l_j}$ are the difference of the moments of distributions $f_{\mathbf{X}}, g_{\mathbf{S}}$.

The Fourier transforms of $f_{\mathbf{X}}$ and $g_{\mathbf{S}}$ are, respectively:

$$f_{\mathbf{X}} = \frac{1}{2\pi}\,{}^{(n)}\!\!\int_{-\infty}^{\infty} \exp(-i\xi'\mathbf{x})\psi(\mathbf{x},\xi)d\xi, \tag{A.3}$$

$$g_{\mathbf{S}} = \frac{1}{2\pi}\,{}^{(n)}\!\!\int_{-\infty}^{\infty} \exp(-i\xi'\mathbf{s})\psi(\mathbf{s},\xi)d\xi, \tag{A.4}$$

and the j-partial derivative of (A.4) is:

$$(-1)^j \frac{\partial^j}{\partial s_{l_1}\dots\partial s_{l_j}} g_{\mathbf{S}} = \frac{1}{2\pi}\,{}^{(n)}\!\!\int_{-\infty}^{\infty} \exp(-i\xi'\mathbf{s}) i^j \xi_{[l_1]}\dots\xi_{[i_j]}\psi(\mathbf{s},\xi)d\xi. \tag{A.5}$$

Applying the inverse Fourier transform to (A.2), we have that,

$$\frac{1}{2\pi}\,{}^{(n)}\!\!\int_{-\infty}^{\infty} \exp(-i\xi'\mathbf{x})\psi(\mathbf{x},\xi)d\xi =$$
$$\frac{1}{2\pi}\,{}^{(n)}\!\!\int_{-\infty}^{\infty} \exp(-i\xi'\mathbf{s}) \left(\left(\sum_{j=0}^{n-1} \xi_{[l_1]}\dots\xi_{[l_j]} M_{[l_1,[\dots,[l_j]\dots]} i^j/j! \right) \psi(\mathbf{s},\xi) + o(\|\xi\|^n) \right) d\xi.$$

Using (A.3), (A.4), and (A.5), it finally yields,

$$f_{\mathbf{X}} = g_{\mathbf{S}} + \sum_{j=1}^{n-1} M_{[l_1,[\dots,[l_j]\dots]} \frac{(-1)^j}{j!} \frac{\partial^j}{\partial s_{l_1}\dots\partial s_{l_j}} g_{\mathbf{S}} + \varepsilon(\mathbf{s},n),$$

where

$$\varepsilon(\mathbf{s},n) = \frac{1}{2\pi}\,{}^{(n)}\!\!\int_{-\infty}^{\infty} \exp(i\xi'\mathbf{s}) o(\|\xi\|^n) d\xi.$$

AppendixA.2. First four partial derivatives of a MVLN distribution

Denote $\Sigma_{\mathbf{s}}^{-1}$ the inverse matrix of $\Sigma_{\mathbf{s}}$ as in (12) and define $\Lambda = -\frac{1}{2}\left(\log(\mathbf{S}) - \boldsymbol{\mu}_{\mathbf{s}}\right)' \Sigma_{\mathbf{s}}^{-1} \left(\log(\mathbf{S}) - \boldsymbol{\mu}_{\mathbf{s}}\right)$. The first four terms of $\frac{\partial^j}{\partial S_{l_1}\dots\partial S_{l_j}} g_{\mathbf{S}(T)}$ are:

The first-order partial derivative is,

$$\frac{\partial}{\partial S_{l_1}} g_{\mathbf{S}} = g_{\mathbf{S}} \left(-\frac{1}{S_{l_1}} + \frac{\partial \Lambda}{\partial S_{l_1}} \right), \tag{A.6}$$

where,

$$\frac{\partial \Lambda}{\partial S_{l_1}} = -\frac{1}{S_{l_1}} \Sigma_{\mathbf{s},(l_1,:)}^{-1} \left(\log(\mathbf{S}) - \boldsymbol{\mu}_{\mathbf{s}}\right)$$

and $\Sigma_{\mathbf{s},(l_1,:)}^{-1}$ is the l_1-th row of $\Sigma_{\mathbf{s}}^{-1}$. The second-order partial derivatives of $g_{\mathbf{S}}$ are,

$$\begin{aligned} \frac{\partial^2}{\partial S_{l_1}^2} g_{\mathbf{S}} &= g_{\mathbf{S}} \left(\frac{2}{S_{l_1}^2} - \frac{2}{S_{l_1}} \frac{\partial \Lambda}{\partial S_{l_1}} + \left(\frac{\partial \Lambda}{\partial S_{l_1}} \right)^2 + \frac{\partial^2 \Lambda}{\partial S_{l_1}^2} \right), \\ \frac{\partial^2}{\partial S_{l_1}\partial S_{l_2}} g_{\mathbf{S}} &= g_{\mathbf{S}} \left(\frac{1}{S_{l_1}S_{l_2}} - \frac{1}{S_{l_1}} \frac{\partial \Lambda}{\partial S_{l_2}} - \frac{1}{S_{l_2}} \frac{\partial \Lambda}{\partial S_{l_1}} + \frac{\partial \Lambda}{\partial S_{l_1}} \frac{\partial \Lambda}{\partial S_{l_2}} + \frac{\partial^2 \Lambda}{\partial S_{l_1}\partial S_{l_2}} \right), \end{aligned} \tag{A.7}$$

where,

$$
\begin{aligned}
\frac{\partial^2 \Lambda}{\partial S_{l_1}^2} &= \frac{1}{S_{l_1}^2}\left(\Sigma_{\mathbf{s},(l_1,:)}^{-1}\left(\log\left(\mathbf{S}\right)-\boldsymbol{\mu}_{\mathbf{s}}\right)-\varsigma_{l_1,l_1}\right), \\
\frac{\partial^2 \Lambda}{\partial S_{l_1}\partial S_{l_2}} &= -\frac{1}{S_{l_1}S_{l_2}}\varsigma_{l_1,l_2},
\end{aligned}
$$

The third-order partial derivatives of $g_{\mathbf{S}}$ are,

$$
\begin{aligned}
\frac{\partial^3}{\partial S_{l_1}^3}g_{\mathbf{S}} &= g_{\mathbf{S}}\left(-\frac{6}{S_{l_1}^3}+\frac{6}{S_{l_1}^2}\frac{\partial \Lambda}{\partial S_{l_1}}-\frac{3}{S_{l_1}}\left(\frac{\partial \Lambda}{\partial S_{l_1}}\right)^2-\frac{3}{S_{l_1}}\frac{\partial^2 \Lambda}{\partial S_{l_1}^2}+3\frac{\partial \Lambda}{\partial S_{l_1}}\frac{\partial^2 \Lambda}{\partial S_{l_1}^2}+\left(\frac{\partial \Lambda}{\partial S_{l_1}}\right)^3+\frac{\partial^3 \Lambda}{\partial S_{l_1}^3}\right), \\
\frac{\partial^3}{\partial S_{l_1}^2\partial S_{l_2}}g_{\mathbf{S}} &= \left(\left(-\frac{1}{S_{l_2}}+\frac{\partial \Lambda}{\partial S_{l_2}}\right)\frac{\partial^2}{\partial S_{l_1}^2}g_{\mathbf{S}}-g_{\mathbf{S}}\left(\frac{2}{S_{l_1}}\frac{\partial^2 \Lambda}{\partial S_{l_1}\partial S_{l_2}}+2\frac{\partial \Lambda}{\partial S_{l_1}}\frac{\partial^2 \Lambda}{\partial S_{l_1}\partial S_{l_2}}+\frac{\partial^3 \Lambda}{\partial S_{l_1}^2\partial S_{l_2}}\right)\right), \\
\frac{\partial^3}{\partial S_{l_1}\partial S_{l_2}\partial S_{l_3}}g_{\mathbf{S}} &= \left(\left(-\frac{1}{S_{l_3}}+\frac{\partial \Lambda}{\partial S_{l_3}}\right)\frac{\partial^2}{\partial S_{l_1}\partial S_{l_2}}g_{\mathbf{S}}+\right. \\
&\quad \left. g_{\mathbf{S}}\left(-\frac{1}{S_{l_1}}\frac{\partial^2 \Lambda}{\partial S_{l_2}\partial S_{l_3}}-\frac{1}{S_{l_2}}\frac{\partial^2 \Lambda}{\partial S_{l_1}\partial S_{l_3}}+\frac{\partial^2 \Lambda}{\partial S_{l_2}\partial S_{l_3}}\frac{\partial \Lambda}{\partial S_{l_1}}+\frac{\partial \Lambda}{\partial S_{l_2}}\frac{\partial^2 \Lambda}{\partial S_{l_1}\partial S_{l_3}}\right)\right),
\end{aligned}
\tag{A.8}
$$

where,

$$
\begin{aligned}
\frac{\partial^3 \Lambda}{\partial S_{l_1}^3} &= \frac{1}{S_{l_1}^3}\left(-2\Sigma_{\mathbf{s},(l_1,:)}^{-1}\left(\log\left(\mathbf{S}\right)-\boldsymbol{\mu}_{\mathbf{s}}\right)+3\varsigma_{l_1,l_1}\right), \\
\frac{\partial^3 \Lambda}{\partial S_{l_1}^2\partial S_{l_2}} &= \frac{1}{S_{l_1}^2 S_{l_2}}\varsigma_{l_1,l_2}.
\end{aligned}
$$

The term $\frac{\partial^3 \Lambda}{\partial S_{l_1}\partial S_{l_2}\partial S_{l_3}}$ is equal to zero.

And the fourth-order partial derivatives of $g_{\mathbf{S}}$ are,

$$
\begin{aligned}
\frac{\partial^4}{\partial S_{l_1}^4} g_{\mathbf{S}} &= g_{\mathbf{S}}\Bigg(\frac{24}{S_{l_1}^4} - \frac{24}{S_{l_1}^3}\frac{\partial\Lambda}{\partial S_{l_1}} + \frac{12}{S_{l_1}^2}\left(\frac{\partial^2\Lambda}{\partial S_{l_1}^2}\right) + \frac{12}{S_{l_1}^2}\left(\frac{\partial\Lambda}{\partial S_{l_1}}\right)^2 - \frac{12}{S_{l_1}}\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial^2\Lambda}{\partial S_{l_1}^2} - \frac{4}{S_{l_1}}\frac{\partial^3\Lambda}{\partial S_{l_1}^3} + \\
&\quad 3\left(\frac{\partial^2\Lambda}{\partial S_{l_1}^2}\right)^2 + 4\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial^3\Lambda}{\partial S_{l_1}^3} + 6\left(\frac{\partial\Lambda}{\partial S_{l_1}}\right)^2\frac{\partial^2\Lambda}{\partial S_{l_1}^2} - \frac{4}{S_{l_1}}\left(\frac{\partial\Lambda}{\partial S_{l_1}}\right)^3 + \left(\frac{\partial\Lambda}{\partial S_{l_1}}\right)^4 + \frac{\partial^4\Lambda}{\partial S_{l_1}^4}\Bigg), \\
\frac{\partial^4}{\partial S_{l_1}^3\partial S_{l_2}} g_{\mathbf{S}} &= \Bigg(\left(-\frac{1}{S_{l_2}} + \frac{\partial\Lambda}{\partial S_{l_2}}\right)\frac{\partial^3}{\partial S_{l_1}^3}g_{\mathbf{S}} + g_{\mathbf{S}}\Bigg(\frac{6}{S_{l_1}^2}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_2}} - \frac{6}{S_{l_1}}\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_2}} - \frac{3}{S_{l_1}}\frac{\partial^3\Lambda}{\partial S_{l_1}^2\partial S_{l_2}} + \\
&\quad 3\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_2}}\frac{\partial^2\Lambda}{\partial S_{l_1}^2} + 3\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial^3\Lambda}{\partial S_{l_1}^2\partial S_{l_2}} + 3\left(\frac{\partial\Lambda}{\partial S_{l_1}}\right)^2\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_2}} + \frac{\partial^4\Lambda}{\partial S_{l_1}^3\partial S_{l_2}}\Bigg)\Bigg), \\
\frac{\partial^4}{\partial S_{l_1}^2\partial S_{l_2}^2} g_{\mathbf{S}} &= \Bigg(\left(-\frac{1}{S_{l_2}} + \frac{\partial\Lambda}{\partial S_{l_2}}\right)\frac{\partial^3}{\partial S_{l_1}^2\partial S_{l_2}}g_{\mathbf{S}} + g_{\mathbf{S}}\Bigg(\frac{2}{S_{l_1}^2 S_{l_2}^2} - \frac{2}{S_{l_1}S_{l_2}^2}\frac{\partial\Lambda}{\partial S_{l_1}} + \frac{2}{S_{l_1}S_{l_2}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_2}} + \\
&\quad \frac{1}{S_{l_2}^2}\left(\frac{\partial\Lambda}{\partial S_{l_1}}\right)^2 - \frac{2}{S_{l_2}}\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_2}} + \frac{1}{S_{l_2}^2}\frac{\partial^2\Lambda}{\partial S_{l_1}^2} - \frac{1}{S_{l_2}}\frac{\partial^3\Lambda}{\partial S_{l_1}^2\partial S_{l_2}} + \frac{2}{S_{l_1}^2}\frac{\partial^2\Lambda}{\partial S_{l_2}^2} - \\
&\quad \frac{2}{S_{l_1}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_2}}\frac{\partial\Lambda}{\partial S_{l_2}} - \frac{2}{S_{l_1}}\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial^2\Lambda}{\partial S_{l_2}^2} + 2\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_2}}\frac{\partial\Lambda}{\partial S_{l_2}} + \left(\frac{\partial\Lambda}{\partial S_{l_1}}\right)^2\frac{\partial^2\Lambda}{\partial S_{l_2}^2} + \\
&\quad \frac{\partial^3\Lambda}{\partial S_{l_1}^2\partial S_{l_2}}\frac{\partial\Lambda}{\partial S_{l_2}} + \frac{\partial^2\Lambda}{\partial S_{l_1}^2}\frac{\partial^2\Lambda}{\partial S_{l_2}^2} - \frac{2}{S_{l_1}}\frac{\partial^3\Lambda}{\partial S_{l_1}\partial S_{l_2}^2} + 2\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial^3\Lambda}{\partial S_{l_1}\partial S_{l_2}^2} + \frac{\partial^4\Lambda}{\partial S_{l_1}^2\partial S_{l_2}^2}\Bigg)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^4}{\partial S_{l_1}^2\partial S_{l_2}\partial S_{l_3}} g_{\mathbf{S}} &= \Bigg(\left(-\frac{1}{S_{l_3}} + \frac{\partial\Lambda}{\partial S_{l_3}}\right)\frac{\partial^3}{\partial S_{l_1}^2\partial S_{l_2}}g_{\mathbf{S}} + g_{\mathbf{S}}\Bigg(\frac{2}{S_{l_1}S_{l_2}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_3}} - \frac{2}{S_{l_2}}\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_3}} - \frac{1}{S_{l_2}}\frac{\partial^3\Lambda}{\partial S_{l_1}^2\partial S_{l_3}} + \\
&\quad \frac{2}{S_{l_1}^2}\frac{\partial^2\Lambda}{\partial S_{l_2}\partial S_{l_3}} - \frac{2}{S_{l_1}}\frac{\partial\Lambda}{\partial S_{l_2}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_3}} - \frac{2}{S_{l_1}}\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial^2\Lambda}{\partial S_{l_2}\partial S_{l_3}} + 2\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_3}}\frac{\partial\Lambda}{\partial S_{l_2}} + \\
&\quad \left(\frac{\partial\Lambda}{\partial S_{l_1}}\right)^2\frac{\partial^2\Lambda}{\partial S_{l_2}\partial S_{l_3}} + \frac{\partial^3\Lambda}{\partial S_{l_1}^2\partial S_{l_3}}\frac{\partial\Lambda}{\partial S_{l_2}} + \frac{\partial^2\Lambda}{\partial S_{l_1}^2}\frac{\partial^2\Lambda}{\partial S_{l_2}\partial S_{l_3}} + \frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_3}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_2}}\Bigg)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^4}{\partial S_{l_1}\partial S_{l_2}\partial S_{l_3}\partial S_{l_4}} g_{\mathbf{S}} &= \Bigg(\left(-\frac{1}{S_{l_4}} + \frac{\partial\Lambda}{\partial S_{l_4}}\right)\frac{\partial^3}{\partial S_{l_1}\partial S_{l_2}\partial S_{l_3}}g_{\mathbf{S}} + g_{\mathbf{S}}\Bigg(\frac{1}{S_{l_1}S_{l_3}}\frac{\partial^2\Lambda}{\partial S_{l_2}\partial S_{l_4}} + \frac{1}{S_{l_2}S_{l_3}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_4}} + \\
&\quad \frac{1}{S_{l_1}S_{l_2}}\frac{\partial^2\Lambda}{\partial S_{l_3}\partial S_{l_4}} - \frac{1}{S_{l_3}}\frac{\partial^2\Lambda}{\partial S_{l_2}\partial S_{l_4}}\frac{\partial\Lambda}{\partial S_{l_1}} - \frac{1}{S_{l_3}}\frac{\partial\Lambda}{\partial S_{l_2}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_4}} - \frac{1}{S_{l_1}}\frac{\partial^2\Lambda}{\partial S_{l_2}\partial S_{l_4}}\frac{\partial\Lambda}{\partial S_{l_3}} - \\
&\quad \frac{1}{S_{l_1}}\frac{\partial^2\Lambda}{\partial S_{l_3}\partial S_{l_4}}\frac{\partial\Lambda}{\partial S_{l_2}} - \frac{1}{S_{l_2}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_4}}\frac{\partial\Lambda}{\partial S_{l_3}} - \frac{1}{S_{l_2}}\frac{\partial^2\Lambda}{\partial S_{l_3}\partial S_{l_4}}\frac{\partial\Lambda}{\partial S_{l_1}} + \frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_4}}\frac{\partial\Lambda}{\partial S_{l_2}}\frac{\partial\Lambda}{\partial S_{l_3}} + \\
&\quad \frac{\partial^2\Lambda}{\partial S_{l_2}\partial S_{l_4}}\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial\Lambda}{\partial S_{l_3}} + \frac{\partial^2\Lambda}{\partial S_{l_3}\partial S_{l_4}}\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial\Lambda}{\partial S_{l_2}} + \frac{\partial^2\Lambda}{\partial S_{l_3}\partial S_{l_4}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_2}} + \\
&\quad \frac{\partial^2\Lambda}{\partial S_{l_3}\partial S_{l_2}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_4}} + \frac{\partial^2\Lambda}{\partial S_{l_2}\partial S_{l_4}}\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_3}}\Bigg)\Bigg),
\end{aligned} \tag{A.9}
$$

where,

$$\begin{aligned}
\frac{\partial^4 \Lambda}{\partial S_{l_1}^4} &= \frac{1}{S_{l_1}^4}\left(6\Sigma^{-1}_{\mathbf{s},(l_1,:)}\left(\log\left(\mathbf{S}\right)-\boldsymbol{\mu}_{\mathbf{s}}\right)-11\varsigma_{l_1,l_1}\right), \\
\frac{\partial^4 \Lambda}{\partial S_{l_1}^3 \partial S_{l_2}} &= -\frac{2}{S_{l_1}^3 S_{l_2}}\varsigma_{l_1,l_2}, \\
\frac{\partial^4 \Lambda}{\partial S_{l_1}^2 \partial S_{l_2}^2} &= -\frac{1}{S_{l_1}^2 S_{l_2}^2}\varsigma_{l_1,l_2}.
\end{aligned}$$

The terms $\frac{\partial^4 \Lambda}{\partial S_{l_1}^2 \partial S_{l_2} \partial S_{l_3}}$ and $\frac{\partial^4 \Lambda}{\partial S_{l_1} \partial S_{l_2} \partial S_{l_3} \partial S_{l_4}}$ are equal to zero.

AppendixA.3. Analysis of the correction term $C_{0,\mathbb{W},[l_1,\ldots,l_j]}\left(\Pi(\mathbf{x}(T))\right)$

The first-order partial derivative of $g_{\mathbf{S}}$ is,

$$\begin{aligned}
\frac{\partial}{\partial S_{l_1}} g_{\mathbf{S}} &= g_{\mathbf{S}}\left(-\frac{1}{S_{l_1}}+\frac{\partial \Lambda}{\partial S_{l_1}}\right) \\
&= g_{\mathbf{S}}\left(-\frac{1}{S_{l_1}}\right)\left(1+\Sigma^{-1}_{\mathbf{s},(l_1,:)}\log\left(\mathbf{s}\right)-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right),
\end{aligned} \tag{A.10}$$

where $\Sigma^{-1}_{\mathbf{s},(l_1,:)}$ is the l_1-th row of $\Sigma^{-1}_{\mathbf{s}}$.

Disentangling the partial derivative we get:

1a. The terms $g_{\mathbf{S}}\left(-\frac{1}{S_{l_1}}\right)$ and $g_{\mathbf{S}}\left(-\frac{1}{S_{l_1}}\right)\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}$ could be re-expressed as MVLN densities:

Proposition AppendixA.1. *Let $\mathbf{S}(T)=(S_1(T),\ldots,S_n(T))$ be a multivariate GBM process defined as in (1), with MVLN distribution and parameters $\boldsymbol{\mu}_{\mathbf{s}},\boldsymbol{\Sigma}_{\mathbf{s}}$, the conditional expected value:*

$$\mathbb{E}_0^Q\left[\log(S_1(T))^{\alpha_1}\ldots\log(S_n(T))^{\alpha_n}S_1(T)^{\beta_1}\ldots S_n(T)^{\beta_n}|S_j(T)\geq K\right],$$

with $j\in\{1,\ldots,n\}$, $\alpha_i,\beta_i\in\mathbb{I}$, is equal to the lower truncated moment of a MVN process $\mathbf{Y}(T)=(Y_1(T),\ldots,Y_n(T))$ times a constant A:

$$A\cdot\mathbb{E}_0^Q\left[Y_1(T)^{\alpha_1}\ldots Y_n(T)^{\alpha_n}|Y_j(T)\geq\log(K)\right],$$

where $\mathbf{Y}(T)\sim N\left(\boldsymbol{\mu}_{\mathbf{s}}+\boldsymbol{\Sigma}_{\mathbf{s}}\boldsymbol{\beta},\Sigma_{\mathbf{s}}\right)$.

Proof. Applying the definition of the MVLN, and after the change of variable $S_i=\exp(Y_i)$, we have the resulting moment over the MVN distribution:

$$
\begin{aligned}
&\mathbb{E}_0^Q\left[\log(S_1(T))^{\alpha_1}\ldots\log(S_n(T))^{\alpha_n}S_1(T)^{\beta_1}\ldots S_n(T)^{\beta_n}|S_j(T)\geq K\right] = \\
&= L_1^{-1}\Bigg(\int_0^{\infty}\!\!{}^{(n-1)}\int_K^{\infty}(2\pi)^{-n/2}|\boldsymbol{\Sigma}_{\mathbf{s}}|^{-1/2}\left(\prod_{i=1}^n\log(s_i)^{\alpha_i}\right)\left(\prod_{i=1}^n s_i^{\beta_i-1}\right)\times \\
&\qquad \exp\left(-\frac{1}{2}\left(\log(\mathbf{s})-\boldsymbol{\mu}_{\mathbf{s}}\right)'\boldsymbol{\Sigma}_{\mathbf{s}}^{-1}\left(\log(\mathbf{s})-\boldsymbol{\mu}_{\mathbf{s}}\right)\right)d\mathbf{s}\Bigg) \qquad \text{(A.11)}\\
&= L_1^{-1}\Bigg(\int_{-\infty}^{\infty}\!\!{}^{(n-1)}\int_{\log(K)}^{\infty}(2\pi)^{-n/2}|\boldsymbol{\Sigma}_{\mathbf{s}}|^{-1/2}\left(\prod_{i=1}^n y_i^{\alpha_i}\right)\exp\left(\mathbf{y}'\boldsymbol{\beta}\right)\times \\
&\qquad \exp\left(-\frac{1}{2}\left(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{s}}\right)'\boldsymbol{\Sigma}_{\mathbf{s}}^{-1}\left(\mathbf{y}-\boldsymbol{\mu}_{\mathbf{s}}\right)\right)d\mathbf{y}\Bigg), \qquad \text{(A.12)}
\end{aligned}
$$

where $L_1 = \mathbb{P}^Q\left(S_j(T)\geq K\right)$. The time parameter of the process $\mathbf{Y}(T)$ is omitted to simplify the notation. The last expression can be transformed as:

$$
\begin{aligned}
&= L_1^{-1}\Bigg(\int_{-\infty}^{\infty}\!\!{}^{(n-1)}\int_{\log(K)}^{\infty}(2\pi)^{-n/2}|\boldsymbol{\Sigma}_{\mathbf{s}}|^{-1/2}\left(\prod_{i=1}^n y_i^{\alpha_i}\right)\times \\
&\qquad \exp\left(\frac{1}{2}\boldsymbol{\beta}'\boldsymbol{\Sigma}_{\mathbf{s}}\boldsymbol{\beta}+\boldsymbol{\beta}'\boldsymbol{\mu}_{\mathbf{s}}-\frac{1}{2}\left(\mathbf{y}-\zeta\right)'\boldsymbol{\Sigma}_{\mathbf{s}}{}^{-1}\left(\mathbf{y}-\zeta\right)\right)d\mathbf{y}\Bigg),
\end{aligned}
$$

with $\zeta = \boldsymbol{\mu}_{\mathbf{s}}+\boldsymbol{\Sigma}_{\mathbf{s}}\boldsymbol{\beta}$. Define $L_2 = \mathbb{P}^Q\left(Y_j(T)\geq\log(K)\right)$, then the last expression becomes:

$$
= \exp\left(\frac{1}{2}\boldsymbol{\beta}'\boldsymbol{\Sigma}_{\mathbf{s}}\boldsymbol{\beta}+\boldsymbol{\beta}'\boldsymbol{\mu}_{\mathbf{s}}\right)\left(\frac{L_2}{L_1}\right)\mathbb{E}_0^Q\left[Y_1(T)^{\alpha_1}\ldots Y_n(T)^{\alpha_n}|Y_j(T)\geq\log(K)\right].
$$

Then the variable $\mathbf{Y}(T)$ is distributed $N\left(\boldsymbol{\mu}_{\mathbf{s}}+\boldsymbol{\Sigma}_{\mathbf{s}}\boldsymbol{\beta},\Sigma_{\mathbf{s}}\right)$, the constant $A = \exp\left(\frac{1}{2}\boldsymbol{\beta}'\boldsymbol{\Sigma}_{\mathbf{s}}\boldsymbol{\beta}+\boldsymbol{\beta}'\boldsymbol{\mu}_{\mathbf{s}}\right)\left(\frac{L_2}{L_1}\right)$ and the result follows.

Corollary AppendixA.2. *Let $g_{\mathbf{S}}$ have the multivariate density as in (10). Denote $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ a vector of integers. The function:*

$$
\hat{g}_{\mathbf{S}} = S_1^{\alpha_1}\ldots S_n^{\alpha_n}g_{\mathbf{S}},
$$

is a MVLN density function and can be re-written as:

$$
\hat{g}_{\mathbf{S}} = \exp(\boldsymbol{\mu}_{\mathbf{s}}'\boldsymbol{\alpha})g_{\mathbf{S}},
$$

Proof. With some algebraic calculations it follows from Proposition (AppendixA.1).

Then, the terms $g_{\mathbf{S}}\left(-\frac{1}{S_{l_1}}\right)$ and $g_{\mathbf{S}}\left(-\frac{1}{S_{l_1}}\right)\Sigma_{\mathbf{s},(l_1,:)}^{-1}\boldsymbol{\mu}_{\mathbf{s}}$ are MLVN densities, setting $\alpha_{l_1} = -1, \alpha_i = 0, i\in\{1,\ldots,n\}, i\neq l_1$ in Corollary (AppendixA.2), and we have the resulting density:

$$
\hat{g}_{\mathbf{S}} = -g_{\mathbf{S}}\left(\frac{1}{S_{l_1}}\right) = \exp(-\mu_1)g_{\mathbf{S}},
$$

The term $\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}$ is an scalar value, it is the dot product of two vectors.

1b. The term $g_{\mathbf{S}}\left(-\frac{1}{S_{l_1}}\right)\Sigma^{-1}_{\mathbf{s},(l_1,:)}\log(\mathbf{s})$ is a sum of MVLN densities times a log-contract.

The first-order moment correction of (11) becomes:

$$\begin{aligned}\exp(-rt)\sum_{l_1=1}^{n}M_{l_1}(-1)\overset{(n)}{\int_0^\infty}\Pi(\mathbf{s}(T))\frac{\partial}{\partial S_{l_1}}g_{\mathbf{S}} \quad &= \\ &\exp(-rt)\Bigg(\sum_{l_1=1}^{n}M_{l_1}(-1)(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}-1)\exp(-\mu_{l_1})\overset{(n)}{\int_0^\infty}\Pi(\mathbf{s}(T))g_{\mathbf{S}}+ \\ &\sum_{l_1=1}^{n}M_{l_1}(-1)\exp(-\mu_{l_1})\sum_{j=1}^{n}\varsigma_{l_1,j}\overset{(n)}{\int_0^\infty}\log(S_j)\Pi(\mathbf{s}(T))g_{\mathbf{S}}\Bigg).\end{aligned}$$

The second-order partial derivative of $g_{\mathbf{S}}$ against the same variable is:

$$\begin{aligned}\frac{\partial^2}{\partial S_{l_1}^2}g_{\mathbf{S}} &= g_{\mathbf{S}}\left(\frac{2}{S_{l_1}^2}-\frac{2}{S_{l_1}}\frac{\partial\Lambda}{\partial S_{l_1}}+\left(\frac{\partial\Lambda}{\partial S_{l_1}}\right)^2+\frac{\partial^2\Lambda}{\partial S_{l_1}^2}\right) \\ &= g_{\mathbf{S}}\left(\frac{1}{S_{l_1}^2}\right)\Bigg(2+2\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\log(\mathbf{s})-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)+\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\log(\mathbf{s})-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)^2+ \\ &\qquad\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\log(\mathbf{s})-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}-\varsigma_{l_1,l_1}\right)\Bigg), \\ &= g_{\mathbf{S}}\left(\frac{1}{S_{l_1}^2}\right)\Bigg(\left(2-3\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}+\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)^2-\varsigma_{l_1,l_1}\right)+ \\ &\qquad\Sigma^{-1}_{\mathbf{s},(l_1,:)}\log(\mathbf{s})\left(3-2\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)+\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\log(\mathbf{s})\right)^2\Bigg).\end{aligned} \tag{A.13}$$

Unfolding we get three terms:

2a. The term $g_{\mathbf{S}}\left(\frac{1}{S_{l_1}^2}\right)\left(2-3\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}+\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)^2-\varsigma_{l_1,l_1}\right)$ could be re-written as a MVLN density:

In this case, set $\alpha_{l_1}=-2, \alpha_i=0, i\in\{1,\ldots,n\}, i\neq l_1$ in the Corollary (AppendixA.2), and the resulting density yields:

$$\begin{aligned}\hat{g}_{\mathbf{S}} &= g_{\mathbf{S}}\left(\frac{1}{S_{l_1}^2}\right)\left(2-3\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}+\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)^2-\varsigma_{l_1,l_1}\right) \\ &= \exp(-2\mu_{l_1})\left(2-3\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}+\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)^2-\varsigma_{l_1,l_1}\right)g_{\mathbf{S}}.\end{aligned}$$

2b. The term $g_{\mathbf{S}}\left(\frac{1}{S_{l_1}^2}\right)\Sigma^{-1}_{\mathbf{s},(l_1,:)}\log(\mathbf{s})\left(3-2\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)$ is a sum of MVLN densities times a log-contract.

2c. The term $g_{\mathbf{S}}\left(\frac{1}{S_{l_1}^2}\right)\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\log(\mathbf{s})\right)^2$ is a sum of MVLN densities times quadratic functions of log-contracts.

Resuming we have the second-order moment correction of (11) with the same index l_1 is:

$$\exp(-rt)\sum_{l_1=1}^{n} M_{l_1,l_1}\frac{1}{2}{}^{(n)}\!\!\int_0^\infty \Pi(\mathbf{s}(T))\frac{\partial^2}{\partial S_{l_1}^2}g_{\mathbf{S}} \quad =$$

$$\exp(-rt)\Bigg(\sum_{l_1=1}^{n} M_{l_1,l_1}\frac{1}{2}\exp(-2\mu_{l_1})\left(2-3\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}+\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)^2-\varsigma_{l_1,l_1}\right){}^{(n)}\!\!\int_0^\infty \Pi(\mathbf{s}(T))g_{\mathbf{S}}+$$
$$\sum_{l_1=1}^{n} M_{l_1,l_1}\frac{1}{2}\exp(-2\mu_{l_1})\left(3-2\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)\sum_{j=1}^{n}\left(\varsigma_{l_1,j}\right){}^{(n)}\!\!\int_0^\infty \Pi(\mathbf{s}(T))\log(S_j)g_{\mathbf{S}}+$$
$$\sum_{l_1=1}^{n} M_{l_1,l_1}\frac{1}{2}\exp(-2\mu_{l_1})\sum_{j_1=1}^{n}\left(\varsigma_{l_1,j_1}\right)^2{}^{(n)}\!\!\int_0^\infty \Pi(\mathbf{s}(T))\log(S_{j_1})^2 g_{\mathbf{S}}\Bigg).$$

The second-order mixed partial derivative of $g_{\mathbf{S}}$ is:

$$\begin{aligned}
\frac{\partial^2}{\partial S_{l_1}\partial S_{l_2}}g_{\mathbf{S}} &= g_{\mathbf{S}}\left(\frac{1}{S_{l_1}S_{l_2}}-\frac{1}{S_{l_1}}\frac{\partial\Lambda}{\partial S_{l_2}}-\frac{1}{S_{l_2}}\frac{\partial\Lambda}{\partial S_{l_1}}+\frac{\partial\Lambda}{\partial S_{l_1}}\frac{\partial\Lambda}{\partial S_{l_2}}+\frac{\partial^2\Lambda}{\partial S_{l_1}\partial S_{l_2}}\right)\\
&= g_{\mathbf{S}}\left(\frac{1}{S_{l_1}S_{l_2}}\right)\Bigg(1+\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\log(\mathbf{s})-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)+\left(\Sigma^{-1}_{\mathbf{s},(l_2,:)}\log(\mathbf{s})-\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)+\\
&\qquad \left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\log(\mathbf{s})-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)\left(\Sigma^{-1}_{\mathbf{s},(l_2,:)}\log(\mathbf{s})-\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)-\varsigma_{l_1,l_2}\Bigg)\\
&= g_{\mathbf{S}}\left(\frac{1}{S_{l_1}S_{l_2}}\right)\Bigg(\left(1-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}-\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}+\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}-\varsigma_{l_1,l_2}\right)+\\
&\qquad \log(\mathbf{s})'\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\left(1-\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)+\Sigma^{-1}_{\mathbf{s},(l_2,:)}\left(1-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)\right)\\
&\qquad +\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\log(\mathbf{s})\,\Sigma^{-1}_{\mathbf{s},(l_2,:)}\log(\mathbf{s})\right)\Bigg).
\end{aligned} \tag{A.14}$$

Re-arranging we get:

2a. The term $g_{\mathbf{S}}\left(\frac{1}{S_{l_1}S_{l_2}}\right)\left(1-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}-\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}+\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}-\varsigma_{l_1,l_2}\right)$ setting $\alpha_{l_1}=-1,\alpha_{l_2}=-1,\alpha_i=0,i\in\{1,\ldots,n\},i\neq l_1\neq l_2$ and applying the Corollary (AppendixA.2), could be transformed as MVLN densities:

$$\begin{aligned}
\hat{g}_{\mathbf{S}} &= g_{\mathbf{S}}\left(\frac{1}{S_{l_1}S_{l_2}}\right)\left(1-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}-\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}+\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}-\varsigma_{l_1,l_2}\right)\\
&= \exp(-\mu_{l_1}-\mu_{l_2})\left(1-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}-\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}+\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}-\varsigma_{l_1,l_2}\right)g_{\mathbf{S}}.
\end{aligned}$$

2b. Log-contracts times MVLN densities,

$$g_{\mathbf{S}}\left(\frac{1}{S_{l_1}S_{l_2}}\right)\log(\mathbf{s})'\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\left(1-\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)+\Sigma^{-1}_{\mathbf{s},(l_2,:)}\left(1-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)\right).$$

2c. Cross log-contracts of second-order, that will appear when calculating cross-sensitivities of the MVLN density:

$$g_{\mathbf{S}}\left(\frac{1}{S_{l_1}S_{l_2}}\right)\left(\Sigma^{-1}_{\mathbf{s},(l_1,:)}\log(\mathbf{s})\,\Sigma^{-1}_{\mathbf{s},(l_2,:)}\log(\mathbf{s})\right).$$

Finally, the second-order mixed moment correction is:

$$\exp(-rt)\sum_{l_1=1}^{n}\sum_{l_1=2}^{n}M_{l_1,l_2}\frac{1}{2}{}^{(n)}\!\!\int_0^\infty \Pi(\mathbf{s}(T))\frac{\partial^2}{\partial S_{l_1}^2}g_{\mathbf{S}} \quad = \quad \exp(-rt)\sum_{l_1=1}^{n}\sum_{l_2=1}^{n}M_{l_1,l_2}\frac{1}{2}\exp(-\mu_{l_1}-\mu_{l_2})\times$$
$$\Bigg(\left(1-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}-\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}+\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}-\varsigma_{l_1,l_2}\right){}^{(n)}\!\!\int_0^\infty \Pi(\mathbf{s}(T))g_{\mathbf{S}}+$$
$$\sum_{j=1}^{n}\left(\varsigma_{j,l_1}\left(1-\Sigma^{-1}_{\mathbf{s},(l_2,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)+\varsigma_{l_2,j}\left(1-\Sigma^{-1}_{\mathbf{s},(l_1,:)}\boldsymbol{\mu}_{\mathbf{s}}\right)\right){}^{(n)}\!\!\int_0^\infty \Pi(\mathbf{s}(T))\log(S_j)g_{\mathbf{S}}+$$
$$\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\left(\varsigma_{l_1,j_1}\varsigma_{l_2,j_2}\right){}^{(n)}\!\!\int_0^\infty \Pi(\mathbf{s}(T))\log(S_{j_1})\log(S_{j_2})g_{\mathbf{S}}\Bigg).$$

Table A.1: Option prices with a MGEE of jump-diffusion processes. The risk-neutral density $f_{\mathbf{X}(T)}$ is generated by a 5-dimensional jump-diffusion process with parameters $\lambda = 10, \delta_i = 0, \nu_i = 0.05, (X_1(0), \ldots, X_5(0)) = (35, 25, 20, 15, 5), r = \{0.05, 0.10\}, t = \{0.25, 1\}, \sigma_i = 0.2, i \in \{1, \ldots, 5\}$. The payoff of the basket option is $\Pi(\mathbf{X}(T)) = \left(\sum_{i=1}^{5} X_i(T) - K\right)^+$ with $K \in \{90, 100, 110\}$. The mean parameters of the auxiliary MVLN density $g_{\mathbf{S}(T)}$ are set by the arbitrage-free constraints, and the volatility parameters are set equal to the jump-diffusion process. The best approximation between AV2012 (Alexander and Venkatramanan, 2012), Li2010 (Li et al., 2010), and MGEE is highlighted .

λ	ν	t	K	r	J-D	Wiener	AV2012	Li2010	MGEE2	MGEE3	MGEE4
1	0.20	0.25	90	0.05	11.256	11.133	11.464	11.124	11.209*	11.092	11.426
				0.10	12.315	12.229	12.457	12.224	12.282*	12.209	12.458
			100	0.05	3.384	2.667	2.779	2.463	3.752	3.655*	2.722
				0.10	4.105	3.446	3.550	3.265	4.470	4.247*	3.259
			110	0.05	0.540	0.110	1.178	0.058	0.356	0.565*	0.931
				0.10	0.699	0.192	1.474	0.115	0.562	0.825*	1.247
		1.00	90	0.05	15.273	14.629	15.023	14.527	15.542	14.682	15.330*
				0.10	18.967	18.628	18.839	18.595	19.039*	18.493	19.716
			100	0.05	8.293	6.818	7.018	6.448	9.089	8.631*	6.731
				0.10	11.395	10.308	10.488	10.079	12.103	11.140*	10.519
			110	0.05	3.865	2.206	3.509	1.790	4.194	4.867	4.098*
				0.10	5.920	4.241	5.577*	3.798	6.750	6.932	4.904
	0.05	0.25	90	0.05	11.134	11.131	11.464	11.124	11.136	11.135*	11.156
				0.10	12.230	12.229	12.457	12.224	12.231	12.230*	12.213
			100	0.05	2.724	2.667	2.779	2.463	2.730	2.728*	2.755
				0.10	3.498	3.446	3.550	3.265	3.506	3.502	3.494*
			110	0.05	0.128	0.109	1.178	0.058	0.124*	0.119	0.207
				0.10	0.218	0.192	1.474	0.115	0.213	0.222*	0.191
		1.00	90	0.05	14.665	14.630	15.023	14.527	14.683	14.653*	14.636
				0.10	18.644	18.625	18.839	18.595	18.649*	18.626	18.572
			100	0.05	6.926	6.817	7.018	6.448	6.949	6.932*	6.987
				0.10	10.382	10.307	10.488	10.079	10.409	10.370*	10.408
			110	0.05	2.327	2.207	3.509	1.790	2.322*	2.344	2.360
				0.10	4.367	4.240	5.577	3.798	4.385*	4.392	4.493
10	0.20	0.25	90	0.05	13.288	11.130	11.464	11.124	12.017*	9.661	33.889
				0.10	14.106	12.229	12.457	12.224	12.660*	11.033	32.494
			100	0.05	7.313	2.666	2.779	2.463	14.064	11.768*	-55.131
				0.10	7.918	3.444	3.550	3.265	14.140	8.409*	-47.872
			110	0.05	3.691	0.110	1.178	0.058	2.692*	7.666	33.744
				0.10	4.068	0.192	1.474	0.115	4.096*	10.746	35.706
		1.00	90	0.05	21.339	14.628	15.023	14.527	25.961*	-23.295	192.038
				0.10	23.930	18.627	18.839	18.595	23.455*	-5.724	289.794
			100	0.05	16.296	6.815	7.018	6.448	34.696	7.941*	-510.066
				0.10	18.613	10.309	10.488*	10.079	32.198	-22.548	-205.431
			110	0.05	12.351	2.206	3.509*	1.790	26.454	65.142	-166.416
				0.10	14.345	4.238	5.577*	3.798	78.379	84.349	-512.313
	0.05	0.25	90	0.05	11.196	11.132	11.464	11.124	11.176	11.157	11.196*
				0.10	12.267	12.230	12.457	12.224	12.255	12.245	12.272*
			100	0.05	3.196	2.666	2.779	2.463	3.299	3.277	3.259*
				0.10	3.934	3.445	3.550	3.265	4.041	3.995	3.935*
			110	0.05	0.318	0.110	1.178	0.058	0.254	0.295	0.331*
				0.10	0.465	0.192	1.474	0.115	0.408	0.462*	0.487
		1.00	90	0.05	15.033	14.625	15.023*	14.527	15.161	14.899	15.110
				0.10	18.829	18.629	18.839*	18.595	18.865	18.691	19.139
			100	0.05	7.806	6.814	7.018	6.448	8.142	8.008*	7.449
				0.10	11.029	10.306	10.488	10.079	11.341	11.051*	10.881
			110	0.05	3.273	2.206	3.509	1.790	3.361	3.563	3.347*
				0.10	5.358	4.242	5.577	3.798	5.705	5.753	5.216*

Table A.2: Option prices with multivariate q-Gaussian processes. The risk-neutral density $f_{\mathbf{X}(T)}$ is generated by a 5-dimensional q-Gaussian process with parameters $q = \{1.05, 1.10, 1.15, 1.20\}, (X_1(0), \ldots, X_5(0)) = (35, 25, 20, 15, 5), r = \{0.05, 0.10\}, t = \{0.25, 1\}, \sigma_i = 0.2, i \in \{1, \ldots, 5\}$. The payoff of the basket option is $\Pi(\mathbf{X}(T)) = \left(\sum_{i=1}^{5} X_i(T) - K\right)^+$ with $K \in \{90, 100, 110\}$. The mean parameters of the auxiliary MVLN density $g_{\mathbf{S}(T)}$ are set by the arbitrage-free constraints, and the volatility parameters are set equal to the jump-diffusion process. The best MGEE approximation is highlighted.

q	t	K	r	q-Gaussian	Wiener	MGEE2	MGEE3	MGEE4
1.05	0.25	90	0.05	11.171	11.129	11.134	11.130	11.142*
			0.10	12.261	12.234	12.234	12.232	12.236*
		100	0.05	2.728	2.668	2.731	2.728*	2.706
			0.10	3.508	3.446	3.505*	3.497	3.478
		110	0.05	0.138	0.109	0.124	0.130	0.140*
			0.10	0.228	0.193	0.214	0.223	0.231*
	1.00	90	0.05	14.781	14.631	14.688*	14.631	14.673
			0.10	18.769	18.627	18.651	18.616	18.674*
		100	0.05	6.982	6.818	6.958*	6.932	6.841
			0.10	10.470	10.300	10.404*	10.348	10.317
		110	0.05	2.356	2.203	2.323	2.367*	2.323
			0.10	4.400	4.238	4.384	4.394*	4.315
1.10	0.25	90	0.05	11.230	11.130	11.145	11.137	11.173*
			0.10	12.320	12.228	12.239	12.233	12.258*
		100	0.05	2.817	2.668	2.829	2.818*	2.727
			0.10	3.596	3.450	3.599*	3.576	3.502
		110	0.05	0.185	0.110	0.147	0.168*	0.202
			0.10	0.284	0.192	0.247	0.274*	0.305
	1.00	90	0.05	15.010	14.622	14.769	14.580	14.777*
			0.10	18.987	18.639	18.708	18.593	18.849*
		100	0.05	7.229	6.816	7.184*	7.093	6.720
			0.10	10.718	10.313	10.589*	10.390	10.332
		110	0.05	2.585	2.208	2.535*	2.677	2.517
			0.10	4.648	4.234	4.620	4.654*	4.320
1.15	0.25	90	0.05	11.330	11.135	11.156*	11.114	11.528
			0.10	12.418	12.230	12.250	12.232	12.317*
		100	0.05	2.966	2.670	3.012	2.981*	2.586
			0.10	3.743	3.445	3.761*	3.690	3.399
		110	0.05	0.270	0.109	0.186	0.252*	0.399
			0.10	0.382	0.193	0.308	0.394*	0.542
	1.00	90	0.05	15.409	14.625	14.974*	14.251	-16.361
			0.10	19.365	18.621	18.758*	18.248	21.859
		100	0.05	7.655	6.830	7.714*	5.483	-593.317
			0.10	11.136	10.312	10.934*	10.076	4.735
		110	0.05	2.967	2.200	2.930*	3.531	0.777
			0.10	5.055	4.239	5.121*	5.356	-31.570
1.20	0.25	90	0.05	11.558	11.133	11.193*	-5.201	12340.234
			0.10	12.637	12.221	12.251*	11.476	283.794
		100	0.05	3.251	2.666	3.583*	1.384	-1583.130
			0.10	4.029	3.443	4.242*	1.659	-330.630
		110	0.05	0.463	0.110	0.386*	75.965	41899.761
			0.10	0.596	0.192	0.507*	8.597	725.664
	1.00	90	0.05	16.477	14.630*	501.276	-1.51×10^{7}	1.78×10^{11}
			0.10	20.358	18.616*	90.851	-1.08×10^{6}	1.28×10^{11}
		100	0.05	12.835	6.815*	3.26×10^{6}	-1.61×10^{12}	-7.29×10^{18}
			0.10	12.245	10.300*	1854.881	-6.55×10^{7}	-1.69×10^{12}
		110	0.05	3.955	2.214*	594.275	9.44×10^{6}	-9.14×10^{11}
			0.10	6.038	4.232*	235.393	-6.39×10^{5}	7.25×10^{10}

Table A.3: Mean dollar error of MGEE approximations of jump-diffusion processes. The columns *MGEE2*, *MGEE3*, and *MGEE4* of the rows *Uncalibrated*, $h_2(\tilde{\sigma})$, $h_3(\tilde{\sigma})$, $h_4(\tilde{\sigma})$ are the error-averages of the 48 cases. For each case, the option approximation with the MGEE of second- (column $MGEE2$), third- (column $MGEE3$) and fourth- (column $MGEE4$) order moments are calculated. Columns AV2012 and Li2012 are Alexander and Venkatramanan (2012) and Li et al. (2010) approximations. The column 'Wiener-MC' represents the average difference between the uncalibrated 'Wiener' process and the calibrated 'Wiener' process (all cases including $\lambda = 1$ and $\lambda = 10$).

Objective Function	Wiener-MC	AV2012	Li2010	MGEE2	MGEE3	MGEE4
Uncalibrated	0.2441	0.6028	0.3105	0.1531*	0.3312	2.4641
$h_2(\tilde{\boldsymbol{\sigma}})$	-	0.8468	0.2283	0.0122*	0.0870	32.5970
$h_3(\tilde{\boldsymbol{\sigma}})$	0.0783	1.1194	0.3736	0.0115*	0.1371	18.4856
$h_4(\tilde{\boldsymbol{\sigma}})$	0.1838	1.3484	0.47868	0.0353*	0.1856	11.1462

Table A.4: Mean dollar error of MGEE approximations of jump-diffusion processes. The columns *MGEE2*, *MGEE3* and *MGEE4* of the rows *Uncalibrated*, $h_2(\tilde{\sigma})$, $h_3(\tilde{\sigma})$, $h_4(\tilde{\sigma})$ are the error-averages of the 48 cases. For each case, the option approximation with the MGEE of second- (column $MGEE2$), third- (column $MGEE3$) and fourth- (column $MGEE4$) order moments are calculated. Columns AV2012 and Li2012 are Alexander and Venkatramanan (2012) and Li et al. (2010) approximations. The column 'Wiener-MC' represents the average difference between the uncalibrated 'Wiener' process and the calibrated 'Wiener' process (only cases with $\lambda = 1$).

Objective Function	Wiener-MC	AV2012	Li2010	MGEE2	MGEE3	MGEE4
Uncalibrated	0.1402	0.7479	0.2156	0.0506	0.0426*	0.1374
$h_2(\tilde{\boldsymbol{\sigma}})$	-	1.1364	0.2073	0.0133*	0.0157	0.0303
$h_3(\tilde{\boldsymbol{\sigma}})$	0.1028	1.5575	0.4113	0.0103*	0.0159	0.0699
$h_4(\tilde{\boldsymbol{\sigma}})$	0.2607	1.9107	0.5734	0.0418	0.0279*	0.1021

Table A.5: Number of best approximations of jump-diffusion processes for different calibration methods. The columns *MGEE2*, *MGEE3*, and *MGEE4* of the rows *Uncalibrated*, $h_2(\tilde{\sigma})$, $h_3(\tilde{\sigma})$, $h_4(\tilde{\sigma})$ are the number of cases with best approximation from the 48 different cases. For each case, the option approximation with the MGEE of second- (column $MGEE2$), third- (column $MGEE3$) and fourth- (column $MGEE4$) order moments are calculated. Columns AV2012 and Li2012 are Alexander and Venkatramanan (2012) and Li et al. (2010) approximations. The column 'Wiener-MC' represents the case where there is no improvement with a MGEE of second-, third- or fourth-order.

Objective Function	Wiener-MC	AV2012	Li2010	MGEE2	MGEE3	MGEE4
Uncalibrated	0	6	0	13	19*	10
$h_2(\tilde{\boldsymbol{\sigma}})$	-	0	3	27*	13	5
$h_3(\tilde{\boldsymbol{\sigma}})$	4	0	3	24*	11	6
$h_4(\tilde{\boldsymbol{\sigma}})$	3	0	3	17	19*	6
Total	7	6	9	81	62	27

Table A.6: Percentage improvement when moments of higher-order are included in the MGEE approximations of jump-diffusion processes for different calibration methods ($\lambda = 1$ and $\lambda = 10$). The columns *MGEE2*, *MGEE3*, and *MGEE4* of the rows *Uncalibrated*, $h_2(\tilde{\sigma})$, $h_3(\tilde{\sigma})$, $h_4(\tilde{\sigma})$ are the percentage improvement in the error-averages of the 48 cases (all cases including $\lambda = 1$ and $\lambda = 10$).

Objective Function	MGEE2	MGEE3	MGEE4
Uncalibrated	0.0910*	-0.0871	-2.2200
$h_2(\tilde{\boldsymbol{\sigma}}) = \lVert M_{l_1,l_2} \rVert_2$	-0.0122*	-0.0870	-32.5970
$h_3(\tilde{\boldsymbol{\sigma}}) = \lVert M_{l_1,l_2} \rVert_2 + \lVert M_{l_1,l_2,l_3} \rVert_2$	0.0668*	-0.0588	-18.4073
$h_4(\tilde{\boldsymbol{\sigma}}) = \lVert M_{l_1,l_2} \rVert_2 + \lVert M_{l_1,l_2,l_3} \rVert_2 + \lVert M_{l_1,l_2,l_3,l_4} \rVert_2$	0.1485*	-0.0018	-10.9624

Table A.7: Percentage improvement when moments of higher-order are included in the MGEE approximations of jump-diffusion processes for different calibration methods (only cases with $\lambda = 1$). The columns *MGEE2*, *MGEE3*, and *MGEE4* of the rows *Uncalibrated*, $h_2(\tilde{\sigma})$, $h_3(\tilde{\sigma})$, $h_4(\tilde{\sigma})$ are the percentage improvement in the error-averages of the 48 cases (only cases with $\lambda = 1$).

Objective Function	MGEE2	MGEE3	MGEE4
Uncalibrated	0.0896	0.0976*	0.0028
$h_2(\tilde{\boldsymbol{\sigma}}) = \|M_{l_1,l_2}\|_2$	-0.0133*	-0.0157	-0.0303
$h_3(\tilde{\boldsymbol{\sigma}}) = \|M_{l_1,l_2}\|_2 + \|M_{l_1,l_2,l_3}\|_2$	0.0925*	0.0869	0.0329
$h_4(\tilde{\boldsymbol{\sigma}}) = \|M_{l_1,l_2}\|_2 + \|M_{l_1,l_2,l_3}\|_2 + \|M_{l_1,l_2,l_3,l_4}\|_2$	0.2189	0.2328*	0.1586

Table A.8: Percentage of the Monte Carlo simulated paths that are evaluated with the MGEE density and generate positive value. The columns *Uncalibrated*, ($h_2(\tilde{\sigma})$,$MGEE2$), ($h_3(\tilde{\sigma})$,$MGEE3$) and ($h_4(\tilde{\sigma})$,$MGEE4$) correspond to the percentage of the 20,000,000 simulation paths that have positive values when they are evaluated under the MGEE density.

λ	ν	t	r	*Uncalibrated*	($h_2(\tilde{\boldsymbol{\sigma}})$,$MGEE2$)	($h_3(\tilde{\boldsymbol{\sigma}})$,$MGEE3$)	($h_4(\tilde{\boldsymbol{\sigma}})$,$MGEE4$)
1	0.20	0.25	0.05	0.6008	1.0000	0.9589	0.7439
			0.10	0.5999	1.0000	0.9561	0.9299
		1.00	0.05	0.5526	1.0000	0.9649	0.5591
			0.10	0.5505	1.0000	0.9666	0.7609
	0.05	0.25	0.05	0.9672	1.0000	0.9996	0.9765
			0.10	0.9951	1.0000	1.0000	0.9860
		1.00	0.05	0.9964	1.0000	1.0000	0.9968
			0.10	0.9976	1.0000	0.9977	0.7141
10	0.20	0.25	0.05	0.9369	1.0000	0.9580	0.9153
			0.10	0.9379	1.0000	0.9569	0.9152
		1.00	0.05	0.8863	1.0000	0.9857	0.9091
			0.10	0.8865	1.0000	0.9584	0.8889
	0.05	0.25	0.05	0.8496	1.0000	0.9983	0.6052
			0.10	0.8724	1.0000	0.9998	0.7572
		1.00	0.05	0.6597	1.0000	0.9967	0.9939
			0.10	0.6642	1.0000	0.9992	0.9936

Table A.9: Algorithm performance when moments of higher-order are included in the MGEE approximations of jump-diffusion processes for different calibration methods. The columns *MGEE2*, *MGEE3*, and *MGEE4* of the rows *Uncalibrated*, $h_2(\tilde{\boldsymbol{\sigma}})$, $h_3(\tilde{\boldsymbol{\sigma}})$, $h_4(\tilde{\boldsymbol{\sigma}})$ are the average running time of the option pricing and calibration algorithms for the 48 cases (there is a description of the cases in Table A.1). The column 'MC' is the average running time of the Monte Carlo algorithm with 20,000,000 simulations.

Objective Function	MC	MGEE2	MGEE3	MGEE4
Uncalibrated	581	172	591	3760
$h_2(\tilde{\boldsymbol{\sigma}}) = \|M_{l_1,l_2}\|_2$	581	177	603	3750
$h_3(\tilde{\boldsymbol{\sigma}}) = \|M_{l_1,l_2}\|_2 + \|M_{l_1,l_2,l_3}\|_2$	581	183	631	3925
$h_4(\tilde{\boldsymbol{\sigma}}) = \|M_{l_1,l_2}\|_2 + \|M_{l_1,l_2,l_3}\|_2 + \|M_{l_1,l_2,l_3,l_4}\|_2$	581	185	652	4210

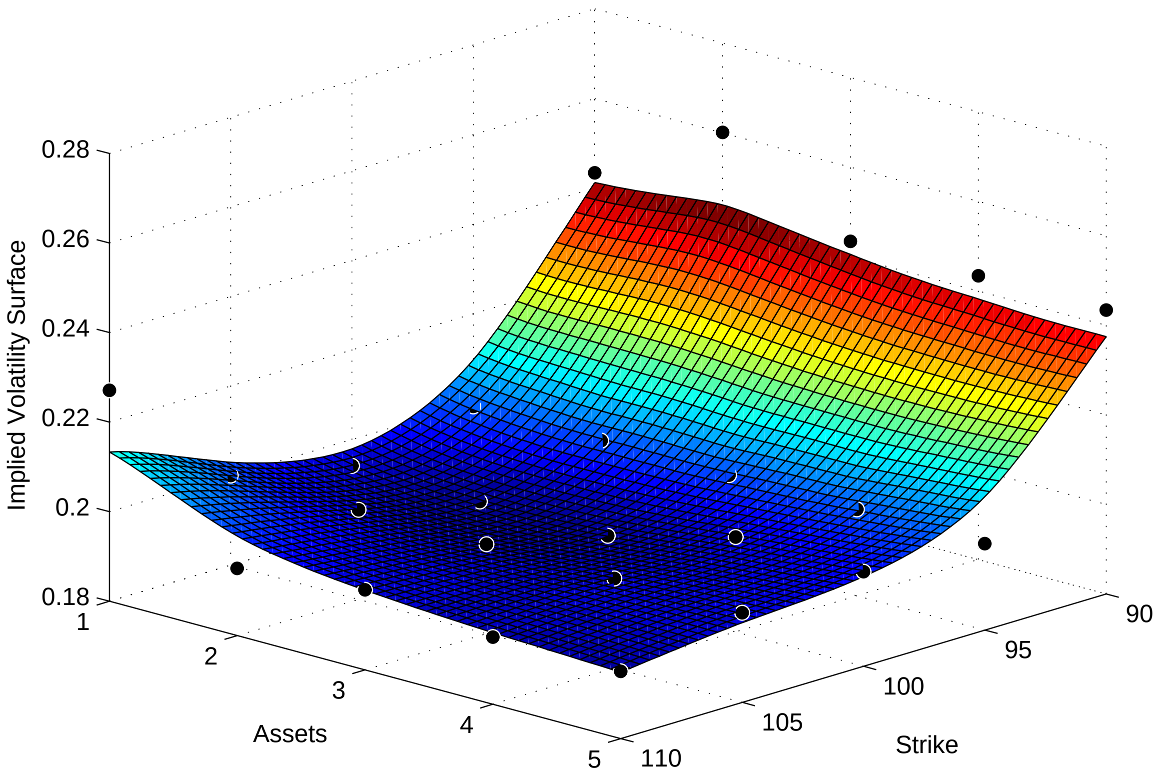

Figure A.1: Implied volatility surface of a basket option priced with a MGEE fitting a 5-variate q-Gaussian process with $q = 1.10$.

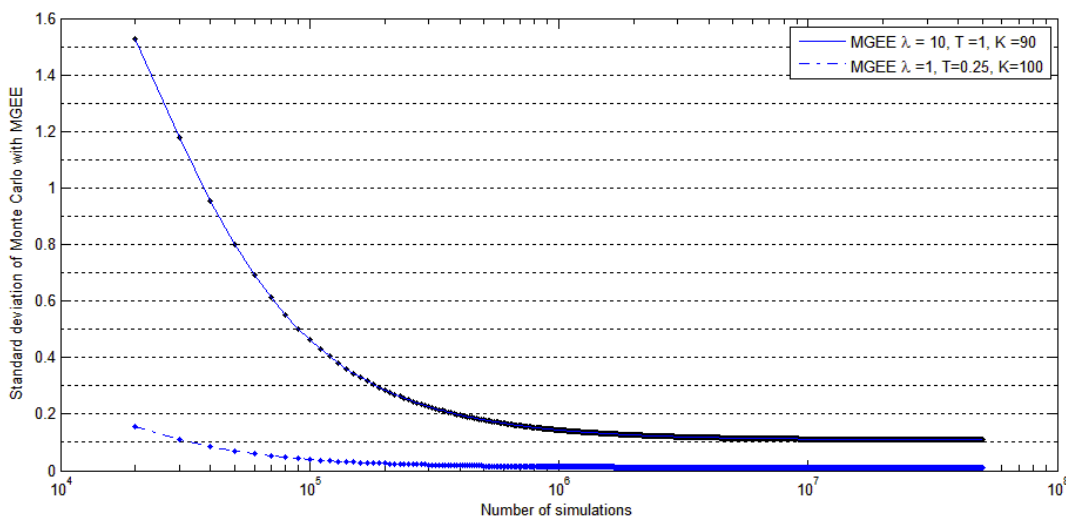

Figure A.2: Standard deviation of Monte Carlo integration algorithm for different number of simulated paths.