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Multivariate truncated moments☆

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ABSTRACT

We derive formulae for the higher order tail moments of the lower truncated multivariate standard normal (MVSN), Student's t, lognormal and a finite-mixture of multivariate normal distributions (FMVN). For the MVSN we propose a recursive formula for moments of arbitrary order as a generalization of previous research. For the Student's t-distribution, the recursive formula is an extension of the normal case and when the degrees of freedom $\nu \to \infty$ the tail moments converge to the normal case. For the lognormal, we propose a general result for distributions in the positive domain. Potential applications include robust statistics, reliability theory, survival analysis and extreme value theory. As an application of our results we calculate the exceedance skewness and kurtosis and we propose a new definition of multivariate skewness and kurtosis using tensors with the moments in their components. The tensor skewness and kurtosis captures more information about the shape of distributions than previous definitions.

1. Introduction

The first results on truncated distributions were developed based on the multivariate standard normal (MVSN). There are two main results, [6] and [33]. Birnbaum and Meyer proposed a general method and expressions for finding the central moments of a lower truncated MVSN distribution. Using this method, they calculate the first and second moments explicitly. The formulae are written in terms of the univariate standard normal cumulative distribution function. Tallis calculated the moment generating function (MGF) of a lower truncated MVSN distribution. With the MGF, it is possible to derive all moments, but Tallis produced an explicit derivation only for the first and second moments. Subsequent research extended Tallis' work: Finney [9] derived an expansion series to express the density of an arbitrary truncated bivariate distribution in terms of the cumulants of a standard normal distribution using the Cornish and Fisher [8] approximation. Applications of truncated moments can be found in several areas of physical sciences, such as econometrics: Lee [20], Amemiya [1] used Tallis' results to develop extensions of the censored data model, a regression model with truncated data due to Tobin [34]. As Cohen [7] mentions, most applications of truncated distributions originate as a result of sampling over a truncated interval of the population. Lee [21] derived a recursive formula to calculate the moments of a doubly-truncated MVSN distribution. However, due to the formula's complexity, it has not been possible to derive explicit formulae for the moments. Gupta and Tracy [11] extended Lee's results to derive inequalities for the absolute value moments of a doubly-truncated arbitrary multivariate distribution. Manjunath and Wilhelm [25] extended Tallis' results to doubly-truncated non-standard MVSN distributions. They used a Leppard and Tallis [22] algorithm to calculate the mean and covariance of the doubly-truncated multivariate non-standard normal distribution using Tallis' MGF.

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Results on the bivariate standard normal were produced after the multivariate case, and thus it could be inferred that Birnbaum and Meyer's results were not fully spread over the literature. Rosenbaum [31] calculated explicit formulae for the first and second moments of a lower truncated bivariate standard normal (BVSN). Rao et al. [30] calculated the sampling correlation of a bivariate normal, with lower truncation of only one variable. He also derived moments up to the fifth order when only one variable is truncated. Ang and Chen [2] extended Rosenbaum results for the doubly truncated bivariate standard normal case, and used these formulae to test for asymmetries of correlation in different financial market regimes. Arnold et al. [3] calculated the marginal of a bivariate normal distribution with double truncation over one variable as in [30].

In this paper, we expand the literature on the calculation of truncated moments. It is the first time formulae have been derived for the third and fourth order central moments of a number of lower truncated multivariate distributions. The first major contribution is a formula to calculate moments of an arbitrary order of the lower truncated MVSN distribution. This result is a generalization of the research on truncated moments of Birnbaum and Meyer, Tallis, Finney and Lee.

Although truncated moments can be calculated with raw integration, these integrals require complex numerical algorithms with no convergence criteria. All previous formulae found in the literature are expressed in terms of the multivariate cumulative distribution function, which enable us to use a suite of algorithms developed for these specific integrals with convergence criteria. To compute this function, we need to calculate the integral of the multivariate normal density function. Genz and Bretz [10] collected several methods to calculate this integral. Plackett [29] presented a method that reduces integrals of the sixth order to single integrals. Horrace [14] presented some inequality results useful for convergence of computation algorithms. Our research presents the convergence criteria of the formulae.

Results on the MVSN are extended to the non-standard case for a lower truncated FMVN distribution. We derive the first four order moments of the lower truncated FMVN distribution. Previous results on third and fourth order moments of the non-truncated case of the FMVN were developed in [12]. They used the results of [4,23,13] for calculating the moments. They derived all cross moments using linear algebra notation equivalent to tensor notation. Using this notation the first moment is a vector, the second moment is a matrix, the third moment is a matrix that represents a tensor of third order and the fourth order moment is a matrix that represents a tensor of fourth order. Our results can be used with tensor calculus for defining matrices similar to Haas and Balestra matrices.

Another contribution of this research is the calculation of truncated moments of arbitrary order for the truncated multivariate Student's t-distribution. This distribution is useful for applications where models have distributions with heavy tails. Moments of arbitrary order for lower truncated distributions have been calculated for the bivariate standard Student's t-distribution by Nadarajah [28]. The first four central moments of the doubly-truncated univariate standard Student's t-distribution are derived in [16]. His calculations are based on the fact that a Student's t-distribution can be represented as the product of two distributions: the multivariate normal and the inverse of a univariate Gamma distribution. Using our results on moments over the lower truncated standard normal we extend the results of Kim to the multivariate case.

A general result for calculating moments of lower truncated multivariate distributions with positive domain is presented. Using this result, moments of arbitrary order of the lower truncated multivariate lognormal (MVL) distribution are derived.

The structure of this paper is as follows: Section 2 contains the definitions and the theory of truncated moments. Section 3 presents results and calculations of moments of third, fourth and n-th order for the lower truncated MVSN distribution. Section 4 derives results on the first four order moments for the lower truncated FMVN case. Section 5 presents the results for truncated moments of arbitrary order of the lower truncated multivariate Student's t. In Section 6, a result to calculate tail moments of positive distributions and the moments of the lower truncated MVL distribution is presented. Section 7 presents a review of the multivariate skewness and kurtosis measures with new definitions using tensors and we calculate the skewness and kurtosis of the lower truncated distributions defined in Section 2.

2. Moments and cumulants

Define X as a random variable of dimension n whose components are $X_1, \ldots, X_n$. Let X have a absolutely continuous distribution function F_X. We assume that F_X is differentiable and that the joint density function f_X exists. Let α_s be nonnegative integers, a_s be truncation points, for $s = \{1, 2, \ldots, n\}$. The p-order lower truncated tail moment function of X is defined by,

$$m_p(\mathbf{x}; \alpha_s; a_s) = E[X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} | a_1 < X_1, a_2 < X_2, \ldots, a_n < X_n], \tag{1}$$

where $\alpha_s \geq 0$ and $\sum_{s=1}^{n} \alpha_s = p$. Another equivalent definition for moments of order p is,

$$m_{p,I}(\mathbf{x}; a_s) = E[X_{i_1} X_{i_2} \cdots X_{i_p} | a_1 < X_1, a_2 < X_2, \ldots, a_n < X_n], \tag{2}$$

with $I = \{i_1, \ldots, i_j, \ldots, i_p\}$, $i_j \in \{1, 2, \ldots, n\}$. If $i_1 = i_2 = \cdots = i_p$, they are called non-central moments, otherwise they are called cross moments. For example, if $i_1 = \cdots = i_p = 1$, then

$$m_{p,\{i_1=1,i_2=1,\ldots,i_p=1\}}(\mathbf{x}; a_s) = m_{p,\{1,\ldots,1\}}(\mathbf{x}; a_s) = E[X_1^p | a_s < X_s] = m_p(\mathbf{x}; \alpha_1 = p, \alpha_2 = 0, \ldots, \alpha_n = 0; a_s).$$

These moments can be computed with the integral:

$$m_p(\mathbf{x}; \alpha_s, a_s) = \int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n} f_X}{F_X(a_s)} dx_1 \cdots dx_n,$$

or equivalently with the integral:

$$m_{p,I}(\mathbf{x}; a_s) = \int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} \frac{x_{i_1} \cdots x_{i_p} f_X}{F_X(a_s)} dx_1 \cdots dx_n.$$

Similar functions can be defined by changing the tail of truncation, such as the upper truncated: $a'_s \geq x_s$; however, we develop results only for the lower truncated moment function as by symmetry the upper truncated moments can be calculated using these results. We refer to this lower truncated tail moment just as the tail moment. A natural extension is the tail moments for double truncation $a_s \leq X_s \leq a'_s$; the results for these cases are very similar with duplicate number of terms in the final formulae.

3. Multivariate normal case

In the present section, we derive a formula for moments of p-th order of the lower truncated MVSN distribution. Tallis [33] derived the MGF and expressions for the first and second moments. Although Birnbaum and Meyer [6] found expressions for the first and second moments in the multivariate case and suggested that their method is useful to find moments beyond the second, it is not a straightforward method to derive these formulae. Therefore we follow Tallis' approach. Using his notation, we differentiate the MGF and find the partial derivatives.

Let X have a multivariate normal distribution with density:

$$\phi_n(x_1, \ldots, x_n; \mathbf{R}) = \phi_n(x_s; \mathbf{R}) = (2\pi)^{-n/2} |\mathbf{R}|^{-1/2} \exp\left(-\frac{1}{2}\mathbf{x}'\mathbf{R}^{-1}\mathbf{x}\right), \tag{3}$$

with $s = \{1, 2, \ldots, n\}$, where $\mathbf{R}$ is a correlation matrix defined as

$$\mathbf{R} = \begin{pmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,n} \\ \rho_{2,1} & 1 & \cdots & \rho_{2,n} \\ \vdots & & \ddots & \vdots \\ \rho_{n,1} & \rho_{n,2} & \cdots & 1 \end{pmatrix}. \tag{4}$$

Let a_s be again truncation points. The lower truncated MVSN distribution is defined as (3) with $a_s \leq X_s$. For notational purposes, we refer to X using vector notation $\mathbf{x}$ or its component notation x_s as in [33].

Define the abbreviated integral operator as

$$\int_{a_1}^{\infty} \cdots \int_{a_n}^{\infty} (\cdot) dx_1 \cdots dx_n = \overset{(n)}{\int_{a_s}^{\infty}} (\cdot) dx_s, \tag{5}$$

so the distribution function is

$$\Phi_n(x_s; \mathbf{R}) = \overset{(n)}{\int_{x_s}^{\infty}} \phi_n(z_s; \mathbf{R}) dz_s. \tag{6}$$

Let L be the total probability of truncated density function ϕ, $L = \Phi_n(a_s; \mathbf{R})$. The MGF of $\mathbf{x}$ is:

$$E\left[\exp(\mathbf{tx})\right] = G(\mathbf{t}, a_s) = L^{-1}(2\pi)^{-n/2} |\mathbf{R}|^{-1/2} \exp(T) \overset{(n)}{\int_{a_s}^{\infty}} \exp\left(-\frac{1}{2}(\mathbf{x} - \zeta)'\mathbf{R}^{-1}(\mathbf{x} - \zeta)\right) dx_s, \tag{7}$$

where $\zeta = \mathbf{Rt}$ and $T = \frac{1}{2}\mathbf{t}'\mathbf{Rt}$, $\mathbf{t}$ and $\mathbf{x} - \zeta$ are column vectors. If we define $b_s = a_s - \zeta_s$ then (7) becomes:

$$E\left[\exp(\mathbf{tx})\right] = L^{-1} \exp(T) \Phi_n(b_s; \mathbf{R}). \tag{8}$$

To obtain arbitrary order moments a change of variable is applied, and then partial derivatives of (8) with respect to $\mathbf{t}$ must be derived and evaluated at $\mathbf{t} = \mathbf{0}$.

Before deriving the moments, we define the notation of conditional distributions with the purpose of simplifying the final formula. We extend the notation used by Stuart et al. [32] and Tallis [33] to be able to provide general results.

3.1. Partition notation

Define the partition over the vector $\mathbf{x} = (\mathbf{x}_{h_1 \cdots h_p \cdot s}, \mathbf{x}_{h_1 \cdots h_p})$, $h_1, \ldots, h_p \in \{1, \ldots, n\}$, $0 \leq p \leq n$, i.e. the subvector $\mathbf{x}_{h_1 \cdots h_p} = (X_{h_1}, \ldots, X_{h_p})$ and the subvector[1] $\mathbf{x}_{h_1 \cdots h_p \cdot s}$ will have as components $\mathbf{x} \setminus \mathbf{x}_{h_1 \cdots h_p}$. We may partition $\mathbf{R}$ as four submatrices:

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_{h_1 \cdots h_p \cdot s} & \mathbf{R}_{h_1 \cdots h_p, 12} \\ \mathbf{R}_{h_1 \cdots h_p, 21} & \mathbf{R}_{h_1 \cdots h_p} \end{pmatrix}, \tag{9}$$

[1] The notation $h_1 \cdots h_p \cdot s$ means all indices of vector s but $\{h_1, \ldots, h_p\}$. This notation is used in [15].

where $\mathbf{R}_{h_1\cdots h_p\cdot s}$ is the correlation matrix of $\mathbf{x}_{h_1\cdots h_p\cdot s}$, $\mathbf{R}_{h_1\cdots h_p}$ is the correlation matrix of $\mathbf{x}_{h_1\cdots h_p}$ and $\mathbf{R}_{h_1\cdots h_p,12}$ is a matrix with the correlation of the set $h_1,\ldots,h_p$ against the set $h_1,\ldots,h_p\cdot s$. Therefore, $\mathbf{x}_{h_1\cdots h_p}$ is distributed $N(\mathbf{0},\mathbf{R}_{h_1\cdots h_p})$. Denote $f(\mathbf{x})$ to be the density function of $\mathbf{x}$ and $f_{h_1\cdots h_p}(\mathbf{x}_{h_1\cdots h_p})$ the marginal density of $\mathbf{x}_{h_1\cdots h_p}$. Then $f(x_s, x_{h_1}=a_{h_1},\ldots,x_{h_p}=a_{h_p}) = f_{h_1\cdots h_p}(\mathbf{a}_{h_1\cdots h_p})f(\mathbf{x}_{h_1\cdots h_p\cdot s}|\mathbf{a}_{h_1\cdots h_p})$ for $s\in\{1,\ldots,n\}$, $s\neq h_1\neq\cdots\neq h_p$ with $\mathbf{a}_{h_1,\ldots,h_p}=(a_{h_1},\ldots,a_{h_p})$. As a result, we have the following expression:

$$\phi_n(x_{h_1\cdots h_p\cdot s}, x_{h_1}=a_{h_1},\ldots,x_{h_p}=a_{h_p};\mathbf{R}) = \phi_p(a_{h_1},\ldots,a_{h_p};\mathbf{R}_{h_1\cdots h_p})\phi_{n-p}(x_{h_1\cdots h_p\cdot s};\mathbf{C}_{h_1\cdots h_p\cdot s}) \tag{10}$$

where $\mathbf{C}_{h_1\cdots h_p\cdot s}$ is the covariance matrix of $f(\mathbf{x}_{h_1\cdots h_p\cdot s}|\mathbf{x}_{h_1\cdots h_p})$. Then,

$$\overset{(n-p)}{\int_{a_s}^{\infty}} \phi_n(x_{h_1\cdots h_p\cdot s}, x_{h_1}=a_{h_1},\ldots,x_{h_p}=a_{h_p};\mathbf{R})d_{x_{h_1\cdots h_p}} = \phi_p(a_{h_1},\ldots,a_{h_p};\mathbf{R}_{h_1\cdots h_p})\Phi_{n-p}(a_{h_1\cdots h_p\cdot s};\mathbf{C}_{h_1\cdots h_p\cdot s}), \tag{11}$$

where $a_{h_1\cdots h_p\cdot s}$ is the set of complement thresholds with a linear transformation. To find the value of $\mathbf{C}_{h_1\cdots h_p\cdot s}$, we notice from (10) and (11) that $f(\mathbf{x}_{h_1\cdots h_p\cdot s}|\mathbf{a}_{h_1\cdots h_p})$ is distributed $N(\boldsymbol{\mu}_{h_1\cdots h_p\cdot s},\mathbf{C}_{h_1\cdots h_p\cdot s})$ with $\boldsymbol{\mu}_{h_1\cdots h_p\cdot s}=\mathbf{R}_{h_1\cdots h_p,12}\mathbf{R}^{-1}_{h_1\cdots h_p}\mathbf{a}_{h_1\cdots h_p}$. The partial correlation matrix can be calculated as in [32]:

$$\mathbf{C}_{h_1\cdots h_p\cdot s} = \mathbf{R}_{h_1\cdots h_p\cdot s} - \mathbf{R}_{h_1\cdots h_p,12}\mathbf{R}^{-1}_{h_1\cdots h_p}\mathbf{R}_{h_1\cdots h_p,21}. \tag{12}$$

To find the value of vector $\mathbf{a}_{h_1\cdots h_p\cdot s}$, we denote by $a_{h_1\cdots h_p\cdot s,k}$ its k-th component and $\mu_{h_1\cdots h_p\cdot s,w}$ the component w-th of vector $\boldsymbol{\mu}_{h_1\cdots h_p\cdot s}$. The set $a_{h_1\cdots h_p\cdot s}$ is defined by:

$$a_{h_1\cdots h_p\cdot s,k} = a_k - \mu_{h_1\cdots h_p\cdot s,k}, \tag{13}$$

where $k\in\{k_1,\ldots,k_{n-p}\}$. We use the following notation for expression (11):

$$F_{h_1,\ldots,h_p}(a_{h_1},\ldots,a_{h_p}) = \phi_p(a_{h_1},\ldots,a_{h_p};\mathbf{R}_{h_1\cdots h_p})\Phi_{n-p}(a_{h_1\cdots h_p\cdot s};\mathbf{C}_{h_1\cdots h_p\cdot s}). \tag{14}$$

Finally expressions are needed to calculate the partial derivative $\frac{\partial\left(\overset{(n)}{\int_{a_s}^{a_s^*}}\phi_n(\cdot)\right)}{\partial t_i}$. Denote by $c_{h_1\cdots h_p\cdot s}(i,j)$ the components i,j of the matrix $\mathbf{C}_{h_1\cdots h_p\cdot s}$. A derivation of components $c_{h_1\cdots h_p.s}(i,j)$ is provided in Appendix A.1.

In (7) the threshold points a_s were changed in a linear transformation by b_s that are dependent on t_i, $i=1,\ldots,n$. Then we notice that $\frac{\partial\phi_p(b_{h_1},\ldots,b_{h_p};\mathbf{R}_{h_1\cdots h_p})}{\partial t_i} = \phi_p(b_{h_1},\ldots,b_{h_p};\mathbf{R}_{h_1\cdots h_p})U_i(b_{h_1},\ldots,b_{h_p})$, where:

$$U_i(b_{h_1},\ldots,b_{h_p}) = \frac{\partial\left(-\frac{1}{2}\mathbf{b}'_{h_1\cdots h_p}\mathbf{R}^{-1}_{h_1\cdots h_p}\mathbf{b}_{h_1\cdots h_p}\right)}{\partial t_i}, \tag{15}$$

where $\mathbf{b}_{h_1\cdots h_p}$ is defined similar to $\mathbf{a}_{h_1\cdots h_p}$. A derivation of $U_i(b_{h_1},\ldots,b_{h_p})$ is given in Appendix A.2.

Lemma 3.1. *Let X be a vector with a lower truncated MVSN distribution with correlation matrix $\mathbf{R}$. Let b_s, $s=\{1,2,\ldots,n\}$ be linear transformed truncation points over X as (8). The partial derivative with respect to t_k is then given as:*

$$\frac{\partial}{\partial t_k}\left(F_{h_1,\ldots,h_p}(b_{h_1},\ldots,b_{h_p})\right) = U_i(b_{h_1},\ldots,b_{h_p})F_{h_1,\ldots,h_p}(b_{h_1},\ldots,b_{h_p}) + \sum_{\substack{h_{p+1}=1\\ h_{p+1}\neq\cdots\neq h_1}}^{n} c_{h_1\cdots h_p\cdot s}(k,v)F_{h_1,\ldots,h_{p+1}}(b_{h_1},\ldots,b_{h_{p+1}}). \tag{16}$$

Proof. The result follows immediately from Definitions (12), (11) and (15). □

To simplify the notation, we denote:

$$\sum_{\substack{h_{p+1}=1\\ h_{p+1}\neq\cdots\neq h_1}}^{n} = \sum_{h_{p+1}\neq\cdots\neq h_1},$$

in what follows.

Proposition 3.2. *Let X be a vector with a lower truncated MVSN random variable with correlation matrix $\mathbf{R}$. Let a_s, $s=1,2,\ldots,n$ be truncation points over X and $i_1,i_2,i_3\in\{1,\ldots,n\}$. The third order moments of X are:*

$$
\begin{aligned}
m_{3,\{i_1,i_2,i_3\}}(\mathbf{x}, a_s) &= E\left[X_{i_1}X_{i_2}X_{i_3}|a_s \le X_s, s \in \{1,2,\dots,n\}\right] = \left.\frac{\partial^3 G(\mathbf{t}, a_s)}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}}\right|_{\mathbf{t=0}} \\
&= L^{-1}\left\{\sum_{h=1}^{n} F_{h_1}(a_{h_1})\left(\rho_{i_3,h_1}\rho_{i_1,i_2} + \rho_{i_2,h_1}\rho_{i_1,i_3} + \rho_{i_1,h_1}\rho_{i_2,i_3}\right) - \sum_{h_1=1}^{n}\rho_{i_1,h_1}\rho_{i_2,h_1}\rho_{i_3,h_1}F_{h_1}(a_{h_1})\right. \\
&\quad + \sum_{h_1=1}^{n}\rho_{i_1,h_1}U_{i_2}(a_{h_1})\left(U_{i_3}(a_{h_1})F_{h_1}(a_{h_1}) + \sum_{h_2\neq h_1} c_{h_1\cdot s}(i_3,h_2)F_{h_1,h_2}(a_{h_1},a_{h_2})\right) \\
&\quad + \sum_{h_1=1}^{n}\rho_{i_1,h_1}\sum_{h_2\neq h_1} c_{h_1\cdot s}(i_2,h_2)\left(U_{i_3}(a_{h_1},a_{h_2})F_{h_1,h_2}(a_{h_1},a_{h_2})\right. \\
&\quad \left.\left. + \sum_{h_3\neq h_1\neq h_2} c_{h_1h_2\cdot s}(i_3,h_3)F_{h_1,h_2,h_3}(b_{h_1},b_{h_2},b_{h_3})\right)\right\}.
\end{aligned}
$$

Proof. See Appendix A.3. □

3.2. Numerical efficiency of Proposition 3.2

We have derived the third order moment of the lower truncated MVSN. To check the formula, we developed code in MATLAB using numerical integration with quadrature integral methods. This integral method has a convergence criterion and in Table E.1 of Appendix E, we can see the convergence of the integral to the value of Proposition 3.2's formula. We can see that the numerical algorithm is exponential time consuming when precision is demanded, while the formula has a constant running time. Although there exist several numerical approximations to estimate the moments of a distribution, the use of this formula presents the advantage of having a convergence criterion that arises from the research developed over MVSN rectangular integration. For more information of efficiency we suggest to review [10]. This checking process is repeated for the subsequent formulae.

Now we calculate the fourth order moment. To simplify notation we denote the derivative of the p-variate marginal $F_{h_1,\dots,h_p}(a_{h_1},\dots,a_{h_p})$ as:

$$
Q_i(a_{h_1},\dots,a_{h_p}) = \frac{\partial}{\partial t_i}(F_{h_1,\dots,h_p}(a_{h_1},\dots,a_{h_p})). \tag{17}
$$

Proposition 3.3. *Let X be a vector with a lower truncated MVSN with correlation matrix $\mathbf{R}$. Let a_s, $s = 1,2,\dots,n$ be truncation points over X and $i_1,i_2,i_3,i_4 \in \{1,\dots,n\}$. Fourth order moments of X are:*

$$
\begin{aligned}
m_{4,\{i_1,i_2,i_3,i_4\}}(\mathbf{x}, a_s) &= E[X_{i_1}X_{i_2}X_{i_3}X_{i_4}] = \left.\frac{\partial^4 G(\mathbf{t}, a_s)}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}\partial t_{i_4}}\right|_{\mathbf{t=0}} \\
&= L^{-1}\left\{\sum_{\substack{j_1,j_2,k_1,k_2\in\{i_1,i_2,i_3,i_4\}\\ j_1\neq j_2\neq k_1\neq k_2}}\rho_{j_1,j_2}\rho_{k_1,k_2}\Phi_n(a_s;\mathbf{R}) + \sum_{h_1=1}^{n}(\rho_{i_3,i_4}\rho_{i_1,h_1}Q_j(a_{h_1})\right. \\
&\quad + \rho_{i_2,i_4}\rho_{i_1,h_1}Q_{i_3}(a_{h_1}) + \rho_{i_2,i_3}\rho_{i_1,h_1}Q_{i_4}(a_{h_1}) + \rho_{i_1,i_2}\rho_{i_3,h_1}Q_{i_4}(a_{h_1}) + \rho_{i_1,i_3}\rho_{i_2,h_1}Q_{i_4}(a_{h_1}) \\
&\quad + \rho_{i_1,i_4}\rho_{i_2,h_1}Q_{i_4}(a_{h_1})) + \sum_{h_1=1}^{n}\rho_{i_1,h_1}\left\{-\rho_{i_2,h_1}\rho_{i_3,h_1}Q_{i_4}(a_{h_1}) - \rho_{i_2,h_1}\rho_{i_4,h_1}Q_{i_3}(a_{h_1})\right. \\
&\quad + U_{i_2}(a_{h_1})\left(-\rho_{i_3,h_1}\rho_{i_4,h_1}F_{h_1}(a_{h_1}) + U_{i_3}(a_{h_1})Q_{i_4}(a_{h_1}) + \sum_{h_2\neq h_1}c_{h_1\cdot s}(i_3,h_2)Q_{i_4}(a_{h_1},a_{h_2})\right) \\
&\quad + \sum_{h_2\neq h_1}c_{h_1\cdot s}(i_2,h_2)\left\{\left(\frac{\rho_{i_4,h_1}\left(\rho_{h_1,i_3} - \rho_{h_1,h_2}\rho_{h_2,i_3}\right) + \rho_{h_2,i_4}\left(\rho_{h_2,i_3} - \rho_{h_1,h_2}\rho_{h_1,i_3}\right)}{1-\rho_{h_1,h_2}^2}\right)\right. \\
&\quad \times F_{h_1,h_2}(a_{h_1},a_{h_2}) + U_{i_3}(a_{h_1},a_{h_2})Q_{i_4}(a_{h_1},a_{h_2}) \\
&\quad \left.\left.\left. + \sum_{h_3\neq h_1\neq h_2}c_{h_1h_2\cdot s}(i_3,h_3)Q_{i_4}(a_{h_1},a_{h_2},a_{h_3})\right\}\right\}\right\}.
\end{aligned}
$$

Proof. See Appendix A.4. □

Using the same procedure we now derive a general procedure to calculate moments of arbitrary order.

Corollary 3.4. *Let X be a vector with a lower truncated MVSN with correlation matrix* $\mathbf{R}$*. Let* $a_s, s = 1, 2, \ldots, n$ *be truncation points over X. Denote the indices of the partial derivatives by* $I = \{i_1, \ldots, i_p\}$*. Define indices* $h_1, \ldots, h_p \in I$ *such that* $h_1 \neq \cdots \neq h_p$*. The p-th order moment of X is:*

$$m_{p,\{i_1,\ldots,i_p\}}(\mathbf{x}; a_s) = L^{-1}\left(\left.\frac{\partial^p \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\cdots\partial t_{i_p}}\right|_{\mathbf{t=0}} + \sum_{h_1,h_2}\rho_{h_1,h_2}\left.\frac{\partial^{p-2}\Phi_n(b_s;\mathbf{R})}{\partial t_{h_3}\cdots\partial t_{h_p}}\right|_{\mathbf{t=0}} + \sum_{h_1,h_2,h_3,h_4}\rho_{h_1,h_2}\rho_{h_3,h_4}\left.\frac{\partial^{p-4}\Phi_n(b_s;\mathbf{R})}{\partial t_{h_5}\cdots\partial t_{h_p}}\right|_{\mathbf{t=0}}\cdots + \sum_{h_1,\ldots,h_p}\rho_{h_1,h_2}\cdots\rho_{h_{p-1},h_p}\Phi_n(b_s;\mathbf{R})\right),$$

for even p and

$$m_{p,\{i_1,\ldots,i_p\}}(\mathbf{x}; a_s) = L^{-1}\left(\left.\frac{\partial^p \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\cdots\partial t_{i_p}}\right|_{\mathbf{t=0}} + \sum_{h_1,h_2}\rho_{h_1,h_2}\left.\frac{\partial^{p-2}\Phi_n(b_s;\mathbf{R})}{\partial t_{h_3}\cdots\partial t_{h_p}}\right|_{\mathbf{t=0}} + \sum_{h_1,h_2,h_3,h_4}\rho_{h_1,h_2}\rho_{h_3,h_4}\left.\frac{\partial^{p-4}\Phi_n(b_s;\mathbf{R})}{\partial t_{h_3}\cdots\partial t_{h_p}}\right|_{\mathbf{t=0}}\cdots + \sum_{h_1,\ldots,h_{p-1}}\rho_{h_1,h_2}\cdots\rho_{h_{p-2},h_{p-1}}\frac{\partial\Phi_n(b_s;\mathbf{R})}{\partial t_{h_p}}\right),$$

for odd p where:

$$\left.\frac{\partial^p \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\cdots\partial t_{i_p}}\right|_{\mathbf{t=0}} = \sum_{h_1=1}^{n}\rho_{i_1,h_1}\frac{\partial^{p-2}Q_{i_2}(b_{h_1})}{\partial t_{i_3}\cdots t_{i_p}}$$

$$\begin{aligned}\frac{\partial^p Q_l(a_{h_1}\cdots a_{h_q})}{\partial t_{i_1}\cdots\partial t_{i_p}} &= \frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(\frac{\partial U_l(a_{h_1},\ldots,a_{h_q})}{\partial t_{i_1}}F_{h_1,\ldots,h_q}(a_{h_1},\ldots,a_{h_q})\right)\\ &+ \frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(U_l(a_{h_1},\ldots,a_{h_q})Q_{i_1}(a_{h_1},\ldots,a_{h_q})\right)\\ &+ \sum_{h_{q+1}\neq h_1\cdots\neq h_q} c_{(h_1\cdots h_q)\cdot s}(k,h_{q+1})\frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(Q_{i_1}(a_{h_1},\ldots,a_{h_{q+1}})\right),\end{aligned}$$

for $q = \{1, \ldots, n-1\}$.

Proof. See Appendix A.5. □

With the intention of comparing the symmetry in both tails of the distributions, Ang and Chen [2] define a function called the exceedance correlation. A generalization for moments is defined as exceedance moments.

Definition 3.1. Let X be a random vector with known density distribution and a_s a vector of thresholds and ς a value such that $a_s = \varsigma, s = \{1, \ldots, n\}$. The exceedance moments of X are:

$$\bar{m}_{p,I}(\mathbf{x},\varsigma) = \begin{cases} m_{p,I}(\mathbf{x}, a_s) \equiv m_{p,I}(\mathbf{x},\varsigma), & \text{if } \varsigma \le 0,\\ m_{p,I}(-\mathbf{x}, -a_s) \equiv m_{p,I}(-\mathbf{x},-\varsigma), & \text{if } \varsigma > 0.\end{cases} \quad (18)$$

The first six exceedance moments of a BVN with $\rho = 0.8$ are plotted in Fig. A.1 of Appendix A.11.

Example 1. Define a lower truncated bivariate normal (BVN) with $\mu = (0, 0)$. In Appendix B, Figs. B.1 and B.3 are plots of the exceedance moment $\bar{m}_{3,\{i_1,i_1,i_1\}} \equiv \bar{m}_{30}$ and Figs. B.2 and B.4 plot the exceedance moment $\bar{m}_{2,\{i_1,i_1,i_2\}} \equiv \bar{m}_{21}$ for different unconditional correlation values. For this case, the distribution is a standard BVN. We can observe that for positive unconditional correlations the exceedance moments $\bar{m}_{30}$ are convex and for negative correlations they are concave. Exceedance moments $\bar{m}_{21}$ are convex in both cases. In Figs. B.5 and B.6, we note that the exceedance correlation is increasing in the tails when the unconditional correlation is negative and this is important for similar research with the class of assets having negative correlations (Bond prices vs. Equities).

Example 2. Define a lower truncated BVN with $\mu = (0, 0)$ and standard deviations $\sigma_1 = 1, \sigma_2 = 0.5$. In Appendix B, Figs. B.7 and B.8 are plots of the exceedance moment $\bar{m}_{30}$ and $\bar{m}_{03}$ for different unconditional correlations. We note that standard deviations change the rate of decreasing $\bar{m}_{30}$ in the tails. For example, $\bar{m}_{30}$ with unconditional correlation $\rho = -0.5$ is convex while $\bar{m}_{03}$ with $\rho = -0.5$ is concave.

Example 3. In Figs. B.9–B.14 of Appendix B, we plot the exceedance moments of fourth order $\bar{m}_{40}$, $\bar{m}_{31}$ and $\bar{m}_{22}$ for the same standard BVN of Example 1. We observe convexity throughout except in Fig. B.10.

Moments of the doubly-truncated case have been calculated for the first and the second order of a bivariate standard normal by Ang and Chen [2] and later in the MVSN by Manjunath and Wilhelm [25]. They extended Tallis' results regarding lower truncated moments. We demonstrate the procedure for extending our results to calculate third order moments for a doubly-truncated distribution and we use this procedure as an example for the extension to the n-order moment of a doubly-truncated distribution. Let X have a MVSN distribution. Define truncation points $a_s \le X_s \le a^*$, $s = \{1, \dots, n\}$. The MGF in (8) becomes:

$$E\left[\exp(\mathbf{tx})\right] = L^{-1}(2\pi)^{-n/2}|\mathbf{R}|^{-1/2}\exp(T)\int_{b_s}^{b_s^*}{}^{(n)} \exp\left(-\frac{1}{2}\mathbf{x}'\mathbf{R}^{-1}\mathbf{x}\right)dx_s, \tag{19}$$

with $b_s = a_s - \zeta_s$ and $\zeta = \mathbf{Rt}$. Then the marginal defined in (11) and used for the derivation of moments will be changed by,

$$\left(\phi_p(a_{h_1}, \dots, a_{h_p}; \mathbf{R}_{h_1\cdots h_p}) - \phi_p(a^*_{h_1}, \dots, a^*_{h_p}; \mathbf{R}_{h_1\cdots h_p})\right)\int_{a_s}^{a_s^*}{}^{(n-p)} \phi_{n-p}(x_{h_1\cdots h_p\cdot s}; \mathbf{C}_{h_1\cdots h_p\cdot s})dx_{h_1\cdots h_p\cdot s}$$
$$= F_{h_1,\dots,h_p}(a_{h_1}, \dots, a_{h_p}) - F_{h_1,\dots,h_p}(a^*_{h_1}, \dots, a^*_{h_p}).$$

Terms $F_{h_q}(a_{h_q}), q = \{1, \dots, p\}$ will be substituted by $F_{h_q}(a_{h_q}) - F_{h_q}(a^*_{h_q})$. The second change is of the derivative $U_i(b_{h_1}, \dots, b_{h_p})$ defined in (15). All terms b_{h_q} have to be changed by $b_{h_q} - b^*_{h_q}$ where $b^*_{h_q} = a^*_{h_q} - \zeta_{h_q}$. Define $a_{i_1,*} = a_{i_q} - a^*_{i_q}$ Finally at Lemma 3.1 two terms will appear on the right hand side for each $F_{h_1,\dots,h_p}$. As a result, we have the third moment of a doubly-truncated MVSN distribution.

Proposition 3.5. *Let X be a doubly-truncated MVSN variable with correlation matrix $\mathbf{R}$. Let $a_s, a_s^* s = 1, 2, \dots, n$ be truncation points over X and $i_1, i_2, i_3 \in \{1, \dots, n\}$. Third order moments of X are:*

$$\begin{aligned}
m_{3,\{i_1,i_2,i_3\}}(\mathbf{x}, a_s, a_s^*) &= E\left[X_{i_1}X_{i_2}X_{i_3}|a_s \le X_s \le a_s^*\right]\\
&= L^{-1}\left\{\sum_{h_1=1}^{n}(F_{h_1}(a_{h_1}) - F_h(a^*_{h_1}))\left(\rho_{i_3,h_1}\rho_{i_1,i_2} + \rho_{i_2,h_1}\rho_{i_1,i_3} + \rho_{i_1,h_1}\rho_{i_2,i_3}\right)\right.\\
&\quad - \sum_{h_1=1}^{n}\rho_{i_1,h_1}\rho_{i_2,h_1}\rho_{i_3,h_1}(F_{h_1}(a_{h_1}) - F_{h_1}(a^*_{h_1}))\\
&\quad + \sum_{h_1=1}^{n}\rho_{i_1,h_1}U_{i_2}(a_{h_1,*})\left(U_{i_3}(a_{h_1,*})(F_{h_1}(a_{h_1}) - F_{h_1}(a^*_{h_1}))\right.\\
&\quad \left. + \sum_{h_2\ne h_1}c_{h_1\cdot s}(i_3, h_2)\sum_{j_1\in\{a_{h_1},a^*_{h_1}\}, j_2\in\{a_{h_3},a^*_{h_3}\}}(-1)^r F_{h_1,h_3}(j_1, j_2)\right)\\
&\quad + \sum_{h_1=1}^{n}\rho_{i_1,h_1}\sum_{h_2\ne h_1}c_{h_1\cdot s}(i_2, h_2)\left(U_{i_3}(a_{h_1,*}, a_{h_2,*})\sum_{j_1\in\{a_{h_1},a^*_{h_1}\}, j_2\in\{a_{h_2},a^*_{h_2}\}}(-1)^r F_{h_1,h_3}(j_1, j_2)\right.\\
&\quad \left.\left. + \sum_{h_3\ne h_1\ne h_2}c_{h_1h_2\cdot s}(i_3, h_3)\sum_{\substack{j_1\in\{a_{h_1},a^*_{h_1}\}, j_2\in\{a_{h_2},a^*_{h_2}\}\\ j_3\in\{a_{h_3},a^*_{h_3}\}}}(-1)^r F_{h_1,h_3}(j_1, j_2, j_3)\right)\right\},
\end{aligned}$$

with r the number of superior extreme indices $\left(a^*_{h_1}, a^*_{h_2}, a^*_{h_3}\right)$ inside the sum.

Proof. The result follows immediately from Proposition 3.2 and changes to the upper limit of integration described above. □

Moments of n-th order can be derived by applying this set of substitutions iteratively. Extensions to the non-standard case are given in Section 4 for the general case of FMVN distributions.

4. Multivariate finite mixture of normal distributions case

Let X have a multivariate finite-mixture of normal distributions (FMVN) with k-components. The pdf of X is defined as:

$$f(x_1,\dots,x_n)=\sum_{j=1}^{k}\omega_j\phi_n(x_s;\boldsymbol{\mu}_j,\mathbf{V}_j)=(2\pi)^{-n/2}\sum_{j=1}^{k}\omega_j|\mathbf{V}_j|^{-1/2}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu}_j\right)'\mathbf{V}_j^{-1}\left(\mathbf{x}-\boldsymbol{\mu}_j\right)\right), \tag{20}$$

where $\omega_j, j=1,\dots,k$ are the mixing weights, $\sum_{j=1}^{k}\omega_j=1$ and $\boldsymbol{\mu}_j,\mathbf{V}_j$ are the mean vector and the covariance matrix of the component density $\phi_n(x_s;\boldsymbol{\mu}_j,\mathbf{V}_j)$.

$$\mathbf{V}_j=\begin{pmatrix}\sigma_{j;1,1}^2 & \cdots & \sigma_{j;1,n}\\ \vdots & \ddots & \vdots\\ \sigma_{j;n,1} & \cdots & \sigma_{j;n,n}^2\end{pmatrix}. \tag{21}$$

Define truncation points $a_s\le X_s$, $s=\{1,\dots,n\}$. The distribution of X truncated on a_s will be defined as the *lower truncated FMVN*. The cdf of X is defined as:

$$\begin{aligned}F(x_1,\dots,x_n)&=\sum_{j=1}^{k}\omega_j\Phi_n(x_s;\boldsymbol{\mu}_j,\mathbf{V}_j)\\&=(2\pi)^{-n/2}\sum_{j=1}^{k}\omega_j|\mathbf{V}_j|^{-1/2}\overset{(n)}{\int_{x_s}^{\infty}}\exp\left(-\frac{1}{2}\left(\mathbf{z}-\boldsymbol{\mu}_j\right)'\mathbf{V}_j^{-1}\left(\mathbf{z}-\boldsymbol{\mu}_j\right)\right)dz_s.\end{aligned} \tag{22}$$

To derive the moments, we use the MGF as in Section 2 for the MVSN case. Define L the total probability, $L=\sum_{j=1}^{k}\omega_j\Phi_n(a_s;\mathbf{R}_j)$. The MGF of X is formulated as:

$$\begin{aligned}E[\exp(\mathbf{tx})]&=G(\mathbf{t},a_s,\boldsymbol{\omega})\\&=L^{-1}(2\pi)^{-n/2}\sum_{j=1}^{k}\omega_j|\mathbf{V}_j|^{-1/2}\overset{(n)}{\int_{a_s}^{\infty}}\exp\left(-\frac{1}{2}\left\{\left(\mathbf{x}-\boldsymbol{\mu}_j\right)'\mathbf{V}_j^{-1}\left(\mathbf{x}-\boldsymbol{\mu}_j\right)-2\mathbf{tx}\right\}\right)dx_s.\end{aligned}$$

Add and subtract $\mathbf{t}\boldsymbol{\mu}_j$ to each exponential term and define $\zeta_j=\mathbf{V}_j\mathbf{t}$ and $T_j=\mathbf{t}\boldsymbol{\mu}_j+\frac{1}{2}\mathbf{t}'\mathbf{V}_j\mathbf{t}$. The last expression becomes:

$$\begin{aligned}G(\mathbf{t},a_s,\boldsymbol{\omega})&=L^{-1}(2\pi)^{-n/2}\sum_{j=1}^{k}\omega_j|\mathbf{V}_j|^{-1/2}\exp(\mathbf{t}\boldsymbol{\mu}_j)\overset{(n)}{\int_{a_s}^{\infty}}\exp\left(-\frac{1}{2}\left\{\left(\mathbf{x}-\boldsymbol{\mu}_j\right)'\mathbf{V}_j^{-1}\left(\mathbf{x}-\boldsymbol{\mu}_j\right)-2\mathbf{t}\left(\mathbf{x}-\boldsymbol{\mu}_j\right)\right\}\right)dx_s\\&=L^{-1}(2\pi)^{-n/2}\sum_{j=1}^{k}\omega_j|\mathbf{V}_j|^{-1/2}\exp(T_j)\overset{(n)}{\int_{a_s}^{\infty}}\exp\left(-\frac{1}{2}\left\{\left(\mathbf{x}-\boldsymbol{\mu}_j-\zeta_j\right)'\mathbf{V}_j^{-1}\left(\mathbf{x}-\boldsymbol{\mu}_j-\zeta_j\right)\right\}\right)dx_s.\end{aligned}$$

Using the decomposition $\mathbf{V}_j=\mathbf{D}^{1/2}\mathbf{R}_j\mathbf{D}^{1/2}$ with $\mathbf{D}$ being a diagonal matrix and $d_{i,i}=\sigma_i^2$, we can calculate $\mathbf{R}_j=\mathbf{D}^{-1}\mathbf{V}_j\mathbf{D}^{-1}$. Applying the change of variable $\mathbf{y}=(\mathbf{x}-\boldsymbol{\mu}_j-\zeta)\mathbf{D}^{-1/2}$ and new limits of integration $b_{j,s}=(a_s-\mu_{j,s}-\zeta_j)/\sigma_{j;s,s}$ with $\mu_{j,s}$ the component s of the vector $\boldsymbol{\mu}_j$, we finally yield:

$$G(\mathbf{t},a_s,\boldsymbol{\omega})=L^{-1}\sum_{j=1}^{k}\omega_j\exp(T_j)\Phi_n(b_{j,s};\mathbf{R}_j). \tag{23}$$

Evaluating at $\mathbf{t}=\mathbf{0}$, the limits of integration $b_{j,s}$ transform into $\xi_{j,s}=(a_s-\mu_{j,s})/\sigma_{j;s,s}$. Using the same notation as the MVSN case, we denote the p-variate marginal:

$$F_{h_1,\dots,h_p}(\xi_{j,h_1},\dots,\xi_{j,h_p})=\phi_p(\xi_{j,h_1}\cdots\xi_{j,h_p})\Phi_{n-p}(\xi_{j,h_1\cdots h_p\cdot s};\mathbf{C}_{j,h_1\cdots h_p\cdot s}), \tag{24}$$

and the partial derivative,

$$
\begin{aligned}
Q_k(\xi_{j,h_1},\dots,\xi_{j,h_p}) &= \frac{\partial}{\partial t_k}\left(F_{h_1,\dots,h_p}(\xi_{j,h_1},\dots,\xi_{j,h_p})\right)\\
&= U_k(\xi_{j,h_1}\cdots\xi_{j,h_p})F_{h_1,\dots,h_p}(\xi_{j,h_1},\dots,\xi_{j,h_p})\\
&\quad+\sum_{h_{p+1}\neq h_p\neq\cdots\neq h_1} c_{j,h_1\cdots h_{p+1}\cdot s}(k,h_{p+1})F_{h_1,\dots,h_{p+1}}(\xi_{j,h_1},\dots,\xi_{j,h_{p+1}}).
\end{aligned}
\tag{25}
$$

Using the results of Section 3, we derive the first four moments.

Proposition 4.1. *Let X be a lower truncated FMVN with pdf as (20) with $\omega_j, j=1,\dots,k$ the mixing weights, $\sum_{j=1}^{k}\omega_j=1$ and $\boldsymbol{\mu}_j,\mathbf{V}_j$ the mean vector and the covariance matrix of the component density $\phi_n(x_s;\boldsymbol{\mu}_j,\mathbf{V}_j)$. Let $a_s, s=1,2,\dots,n$ be the truncation points over X. The first four moments of X are:*

$$
m_{1,\{i_1\}}(\mathbf{x},a_s,\boldsymbol{\omega}) = L^{-1}\sum_{j=1}^{k}\omega_j\left(\mu_{j,i_1}\Phi_n(\xi_{j,s};\mathbf{R}_j)+\sum_{h_1=1}^{n}\sigma_{j;h_1,i_1}F_{h_1}(\xi_{j,h_1})\right),
$$

$$
\begin{aligned}
m_{2,\{i_1,i_2\}}(\mathbf{x},a_s,\boldsymbol{\omega}) &= L^{-1}\sum_{j=1}^{k}\omega_j\Bigg(\left(\mu_{j,i_1}\mu_{j,i_2}+\sigma_{j;i_1,i_2}\right)\Phi_n(\xi_{j,s};\mathbf{R}_j)\\
&\quad+\sum_{h_1=1}^{n}\left(\mu_{j,i_1}\sigma_{j;h_1,i_2}+\mu_{j,i_2}\sigma_{j;h_1,i_1}\right)F_{h_1}(\xi_{j,h_1})+\sum_{h_1=1}^{n}\sigma_{j;h_1,i_1}Q_{i_2}(\xi_{j,h_1})\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
m_{3,\{i_1,i_2,i_3\}}(\mathbf{x},a_s,\boldsymbol{\omega}) &= L^{-1}\sum_{j=1}^{k}\omega_j\Bigg\{\left(\mu_{j,i_1}\mu_{j,i_2}\mu_{j,i_3}+\mu_{j,i_1}\sigma_{j;i_2,i_3}+\mu_{j,i_2}\sigma_{j;i_1,i_3}+\mu_{j,i_3}\sigma_{j;i_1,i_2}\right)\Phi_n(\xi_{j,s};\mathbf{R}_j)\\
&\quad\times\sum_{h_1=1}^{n}\mu_{j,i_3}\sigma_{j;h_1,i_1}Q_{i_2}(\xi_{j,h_1})+\sum_{h_1=1}^{n}\mu_{j,i_2}\sigma_{j;h_1,i_1}Q_{i_3}(\xi_{j,h_1})+\sum_{h_1=1}^{n}\mu_{j,i_1}\sigma_{j;h_1,i_2}Q_{i_3}(\xi_{j,h_1})\\
&\quad+\sum_{h_1=1}^{n}F_{h_1}(\xi_{j,h_1})\left(\left(\mu_{j,i_1}\mu_{j,i_2}+\sigma_{j;i_1,i_2}\right)\sigma_{j;h_1,i_3}+\left(\mu_{j,i_1}\mu_{j,i_3}+\sigma_{j;i_1,i_3}\right)\sigma_{j;h_1,i_2}\right.\\
&\quad\left.+\left(\mu_{j,i_2}\mu_{j,i_3}+\sigma_{j;i_2,i_3}\right)\sigma_{j;h_1,i_1}\right)+\sum_{h_1=1}^{n}\sigma_{j;h_1,i_1}\left(\frac{\partial U_{i_2}(\xi_{j,h_1})}{\partial t_{i_3}}F_{h_1}(\xi_{j,h_1})\right.\\
&\quad\left.\left.+U_{i_2}(\xi_{j,h_1})Q_{i_3}(\xi_{j,h_1})+\sum_{h_2\neq h_1}^{n}c_{h_1\cdot s}(i_2,h_2)Q_{i_3}(\xi_{j,h_1},\xi_{j,h_2})\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
m_{4,\{i_1,i_2,i_3,i_4\}}(\mathbf{x},a_s,\boldsymbol{\omega}) &= L^{-1}\sum_{j=1}^{k}\omega_j\left\{\sum_{h_1=1}^{n}\left(\left(\mu_{j,i_3}\mu_{j,i_4}+\sigma_{i_3,i_4}\right)\sigma_{j;h_1,i_1}Q_{i_2}(\xi_{j,h_1})\right.\right.\\
&\quad+\left(\mu_{j,i_2}\mu_{j,i_4}+\sigma_{i_2,i_4}\right)\sigma_{j;h_1,i_1}Q_{i_3}(\xi_{j,h_1})+\left(\mu_{j,i_2}\mu_{j,i_3}+\sigma_{i_2,i_3}\right)\sigma_{j;h_1,i_1}Q_{i_4}(\xi_{j,h_1})\\
&\quad+\left(\mu_{j,i_1}\mu_{j,i_2}+\sigma_{i_1,i_2}\right)\sigma_{j;h_1,i_3}Q_{i_4}(\xi_{j,h_1})+\left(\mu_{j,i_1}\mu_{j,i_3}+\sigma_{i_1,i_3}\right)\sigma_{j;h_1,i_2}Q_{i_4}(\xi_{j,h_1})\\
&\quad\left.+\left(\mu_{j,i_1}\mu_{j,i_4}+\sigma_{i_1,i_4}\right)\sigma_{j;h_1,i_2}Q_{i_3}(\xi_{j,h_1})\right)+\sum_{h_1=1}^{n}\left\{\sum_{\substack{k_1,k_2\in\{i_1,i_2,i_3,i_4\}\\ k_3,k_4\in\{i_1,i_2,i_3,i_4\}\setminus\{k_1,k_2\}\\ k_1\neq k_2\neq k_3\neq k_4}}\left\{F_{h_1}(\xi_{j,h_1})\right.\right.\\
&\quad\times\left(\mu_{j,k_1}\mu_{j,k_2}\mu_{j,k_3}+\mu_{j,k_1}\sigma_{j;k_2,k_3}+\mu_{j,k_2}\sigma_{j;k_1,k_3}+\mu_{j,k_3}\sigma_{j;k_1,k_2}\right)\left(\sigma_{j;h1,k_4}\right)\\
&\quad+\mu_{j,k_1}\sigma_{j;h_1,k_2}\left(\frac{\partial U_{k_3}(\xi_{j,h_1})}{\partial t_{k_4}}F_{h_1}(\xi_{j,h_1})+U_{k_3}(\xi_{j,h_1})Q_{k_4}(\xi_{j,h_1})\right.\\
&\quad\left.+\sum_{h_2\neq h_1}^{n}c_{h_1\cdot s}(k_3,k_4)Q_{k_3}(\xi_{j,h_1},\xi_{j,h_2})\right)+\sigma_{j;h_1,k_1}\left(\frac{\partial U_{k_2}(\xi_{j,h_1})}{\partial t_{k_3}}Q_{k_4}(\xi_{j,h_1})\right.
\end{aligned}
$$

$$+ \frac{\partial U_{k_2}(\xi_{j,h_1})}{\partial t_{k_4}} Q_{k_3}(\xi_{j,h_1}) + U_{k_2}(\xi_{j,h_1}) \frac{\partial Q_{k_3}(\xi_{j,h_1})}{\partial t_{k_4}}$$

$$+ \sum_{h_2 \neq h_1} c_{h_1 \cdot s}(k_2, h_2) \frac{\partial Q_{k_3}(\xi_{j,h_1})}{\partial t_{k_4}} \Bigg) \Bigg\} + \Bigg(\mu_{j,i_1}\mu_{j,i_2}\mu_{j,i_3}\mu_{j,i_4} + \sigma_{j;i_1,i_2}\sigma_{j;i_3,i_4}$$

$$+ \sigma_{j;i_1,i_3}\sigma_{j;i_2,i_4} + \sigma_{j;i_1,i_4}\sigma_{j;i_2,i_3} + \sum_{\substack{k_1,k_2 \in \{i_1,i_2,i_3,i_4\} \\ k_3,k_4 \in \{i_1,i_2,i_3,i_4\} \setminus \{k_1,k_2\} \\ k_1 \neq k_2 \neq k_3 \neq k_4}} \mu_{j,k_1}\mu_{j,k_2}\sigma_{k_3,k_4} \Bigg) \Phi_n(\xi_{j,s}) \Bigg\} \Bigg\} .$$

Proof. See Appendix A.6. □

5. Multivariate Student's *t* case

In this section, we derive the third and fourth central moments of a lower truncated multivariate standard Student's t-distribution. Let X have a lower truncated multivariate Student's t-distribution with $\nu > 0$ degrees of freedom, mean vector $\boldsymbol{\mu}$ and correlation matrix $\mathbf{R}$. The lower truncated density is defined as:

$$f(x_1, \ldots, x_n; \nu; \boldsymbol{\mu}; \mathbf{R}) = \frac{\Gamma((\nu+n)/2)}{(\pi\nu)^{\nu/2}\Gamma(\nu/2)|\mathbf{R}|^{1/2}} \left(1 + \frac{1}{\nu}(\mathbf{x}-\boldsymbol{\mu})'\mathbf{R}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)^{-(\nu+n)/2}, \tag{26}$$

for $a_1 \le X_1, \ldots, a_n \le X_n$ and 0 otherwise. Γ is the gamma function, and $\pi = \Gamma(1/2)$. If the mean vector is zero, then we refer to this distribution as a standard Student's t.

Let $Z = (Z_1, \ldots, Z_n)$ be a random vector having a MVSN distribution with correlation matrix $\mathbf{R}$, and η a univariate Gamma distribution with mean $\alpha = \nu/2$ and variance $\beta = 2/\nu$ with pdf:

$$f_\eta(x; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp(-x/\beta). \tag{27}$$

The multivariate standard Student's t-distribution with ν degrees of freedom can be expressed as $X = \eta^{-1/2}Z$ [19]. We use this fact to develop our results.

Lemma 5.1. *Let Z have a standard normal distribution with pdf (3). Let η have a Gamma distribution with pdf (27). Define the lower truncation points for Z:*

$$\eta^{1/2}a_i \le Z_i, \quad i = 1, \ldots, n, \tag{28}$$

and let $X = (X_1, \ldots, X_n) = \eta^{-1/2}Z$. Then X has a lower truncated multivariate standard Student's t-distribution with ν degrees of freedom, $a_i \le X_i$ and

$$E_\eta\left[\eta^{-i/2}\phi_n\left(\eta^{1/2}\mathbf{x}; \mathbf{0}, \mathbf{R}\right)\right] = \frac{\Gamma((\nu-i)/2)\nu^{\nu/2}}{2^{(i+n)/2}\Gamma(\nu/2)\Gamma(1/2)^n|\mathbf{R}|^{1/2}} \left(\mathbf{x}'\mathbf{R}^{-1}\mathbf{x} + \nu\right)^{-(\nu-i)/2}, \tag{29}$$

where E_η is the expected value conditional on the distribution of η.

Proof. See Appendix A.7. □

Proposition 5.2. *Let X have a lower truncated multivariate standard Student's t-distribution with truncation points $a_s \le X_s$ and $i_1, i_2, i_3, i_4 \in \{1, \ldots, n\}$. Let $L = \Phi_n(a_s; \mathbf{R})$. The first four order moments of $\mathbf{x}$ are:*

$$m_{1,\{i_1\}}(\mathbf{x}, a_s) = L^{-1} \sum_{h_1=1}^{n} \rho_{i_1,h_1}\gamma_{1,h_1}\Phi_{n-1}(a_{h_1 \cdot s}; \mathbf{C}_{h_1 \cdot s}),$$

$$m_{2,\{i_1,i_2\}}(\mathbf{x}, a_s) = L^{-1} \Bigg(\sum_{h_1=1}^{n} \rho_{i_1,h_1} \Bigg(\rho_{i_2,h_1} a_{h_1} \gamma_{1,h_1} \Phi_{n-1}(a_{h_1 \cdot s}; \mathbf{C}_{h_1 \cdot s})$$

$$+ \sum_{h_2 \neq h_1} (\rho_{i_2,h_2} - \rho_{h_1,h_2}\rho_{i_2,h_1}) \gamma_{2,h_1 h_2} \Phi_{n-2}(a_{h_1 h_2 \cdot s}; \mathbf{C}_{h_1 h_2 \cdot s}) \Bigg) \Bigg) + \rho_{i_1,i_2},$$

$$
\begin{aligned}
m_{3,\{i_1,i_2,i_3\}}(\mathbf{x}, a_s) &= L^{-1}\Bigg\{\sum_{h_1=1}^{n}\gamma_{1,h_1}\Phi_{n-1}(a_{h_1\cdot s};\mathbf{C}_{h_1\cdot s})\left(\rho_{i_3,h_1}\rho_{i_1,i_2}+\rho_{i_2,h_1}\rho_{i_1,i_3}+\rho_{i_1,h_1}\rho_{i_2,i_3}\right)\\
&\quad-\sum_{h_1=1}^{n}\rho_{i_1,h_1}\rho_{i_2,h_1}\rho_{i_3,h_1}\gamma_{1,h_1}\Phi_{n-1}(a_{h_1\cdot s};\mathbf{C}_{h_1\cdot s})\\
&\quad+\sum_{h_1=1}^{n}\rho_{i_1,h_1}U_{i_2}(a_{h_1})\Bigg(U_{i_3}(a_{h_1})\gamma_{1,h_1}\Phi_{n-1}(a_{h_1\cdot s};\mathbf{C}_{h_1\cdot s})\\
&\quad+\sum_{h_3\neq h_1}c_{h_1\cdot s}(i_3,h_3)\gamma_{2,h_1h_2}\Phi_{n-2}(a_{h_1h_3\cdot s};\mathbf{C}_{h_1h_3\cdot s})\Bigg)\\
&\quad+\sum_{h_1=1}^{n}\rho_{i_1,h_1}\sum_{h_2\neq h_1}c_{h_1\cdot s}(i_2,h_2)\Bigg(U_{i_3}(a_{h_1},a_{h_2})\gamma_{2,h_1h_2}\Phi_{n-2}(a_{h_1h_2\cdot s};\mathbf{C}_{h_1h_2\cdot s})\\
&\quad+\sum_{h_3\neq h_1\neq h_2}c_{h_1h_2\cdot s}(i_3,h_3)\gamma_{3,h_1h_2h_3}\Phi_{n-3}(a_{h_1h_2h_3\cdot s};\mathbf{C}_{h_1h_2h_3\cdot s})\Bigg)\Bigg\},\\
m_{4,\{i_1,i_2,i_3,i_4\}}(\mathbf{x}, a_s) &= L^{-1}\Bigg\{\sum_{\substack{k_1,k_2,k_3,k_4\in\{i_1,i_2,i_3,i_4\}\\ k_1\neq k_2\neq k_3\neq k_4}}\rho_{k_1,k_2}\rho_{k_3,k_4}\Phi_n(a_s;\mathbf{R})\\
&\quad+\sum_{h_1=1}^{n}\left(\rho_{i_3,i_4}\rho_{i_1,h_1}Q_{i_2,\gamma}(a_{h_1})+\rho_{i_2,i_4}\rho_{i_1,h_1}Q_{i_3,\gamma}(a_{h_1})+\rho_{i_2,i_3}\rho_{i_1,h_1}Q_{i_4,\gamma}(a_{h_1})\right.\\
&\quad\left.+\ \rho_{i_1,i_2}\rho_{i_3,h_1}Q_{i_4,\gamma}(a_{h_1})+\rho_{i_1,i_3}\rho_{i_2,h_1}Q_{i_4,\gamma}(a_{h_1})+\rho_{i_1,i_4}\rho_{i_2,h_1}Q_{i_4,\gamma}(a_{h_1})\right)\\
&\quad+\sum_{h_1=1}^{n}\rho_{i_1,h_1}\Bigg\{-\rho_{i_2,h_1}\rho_{i_3,h_1}Q_{i_4,\gamma}(a_{h_1})-\rho_{i_2,h_1}\rho_{i_4,h_1}Q_{i_3,\gamma}(a_{h_1})\\
&\quad+U_{i_2}(a_{h_1})\Bigg(-\rho_{i_3,h_1}\rho_{i_4,h_1}\gamma_{1,h_1}\Phi_{n-1}(a_{h_1\cdot s};\mathbf{C}_{h_1\cdot s})+U_{i_3}(a_{h_1})Q_{i_4,\gamma}(a_{h_1})\\
&\quad+\sum_{h_2\neq h_1}c_{h_1\cdot s}(i_2,h_2)Q_{i_4,\gamma}(a_{h_1},a_{h_2})\Bigg)\\
&\quad+\sum_{h_2\neq h_1}c_{h_1\cdot s}(i_2,h_2)\Bigg\{\left(\frac{\rho_{i_4,h_1}\left(\rho_{h_1,i_3}-\rho_{h_1,h_2}\rho_{h_2,i_3}\right)+\rho_{h_2,i_4}\left(\rho_{h_2,i_3}-\rho_{h_1,h_2}\rho_{h_1,i_3}\right)}{1-\rho^2_{h_1,h_2}}\right)\\
&\quad\times\gamma_{2,h_1h_2}\Phi_{n-2}(a_{h_1h_3\cdot s};\mathbf{C}_{h_1h_3\cdot s})\\
&\quad+\ U_{i_3}(a_{h_1},a_{h_2})Q_{i_4,\gamma}(a_{h_1},a_{h_2})+\sum_{h_3\neq h_1\neq h_2}c_{h_1h_2\cdot s}(i_3,h_3)Q_{i_4,\gamma}(a_{h_1},a_{h_2},a_{h_3})\Bigg\}\Bigg\}\Bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
&Q_{i_4,\gamma}(a_{h_1},\ldots,a_{h_q}) = U_{i_4}(a_{h_1},\ldots,a_{h_q})\gamma_{q,h_1,\ldots,h_q}\Phi_{n-q}\left(a_{(h_1\cdots h_q)\cdot s};\mathbf{R}_{(h_1\cdots h_q)\cdot s}\right)\\
&\qquad+\sum_{h_{q+1}\neq h_1\cdots\neq h_q}c_{(h_1\cdots h_q)\cdot s}(i_4,h_{q+1})\gamma_{q+1,\{h_1,\ldots,h_{q+1}\}}\Phi_{n-q-1}\left(a_{(h_1\cdots h_{q+1})\cdot s};\mathbf{C}_{(h_1\cdots h_{q+1})\cdot s}\right),\\
&\gamma_{q,h_1\cdots h_q} = \left(\frac{\Gamma((\nu-q)/2)\nu^{\nu/2}}{2^{(n+q)/2}\Gamma(\nu/2)\Gamma(1/2)^n|\mathbf{R}_{h_1\cdots h_q}|^{1/2}}\left(\mathbf{a}'_{h_1\cdots h_q}\mathbf{R}^{-1}_{h_1\cdots h_q}\mathbf{a}_{h_1\cdots h_q}+\nu\right)^{-(\nu-q)/2}\right).
\end{aligned}
$$

Proof. See Appendix A.8. □

Corollary 5.3. *Let X have a lower truncated multivariate standard Student's t-distribution with truncation points $a_i \le X_i$. The p-th order moments of $\mathbf{x}$ are:*

$$m_{p,I}(\mathbf{x}, a_s) = L^{-1}\left(\sum_{j_1=1}^{n}\rho_{i_1,j_1}\frac{\partial^{p-2}Q_{i_2,\gamma}(a_h)}{\partial t_{i_3}\cdots t_{i_p}} + \sum_{j_1,j_2\in\{i_1,\ldots,i_p\}}\rho_{j_1,j_2}\sum_{l_1=1}^{n}\rho_{k_1,l_1}\frac{\partial^{p-4}Q_{k_2,\gamma}(a_h)}{\partial t_{k_3}\cdots t_{k_{p-2}}}\right.$$
$$\left. + \sum_{j_1,j_2,j_3,j_4\in\{i_1,\ldots,i_p\}}\rho_{j_1,j_2}\rho_{j_3,j_4}\sum_{l_1=1}^{n}\rho_{k_1,l_1}\frac{\partial^{p-6}Q_{k_2,\gamma}(a_h)}{\partial t_{k_3}\cdots t_{k_{p-4}}} + \cdots + \rho_{j_1,j_2}\cdots\rho_{j_{p-1},j_p}\Phi_n(a_s;\mathbf{R})\right),$$

for odd p and

$$= L^{-1}\left(\sum_{j_1=1}^{n}\rho_{i_1,j_1}\frac{\partial^{p-2}Q_{i_2,\gamma}(a_h)}{\partial t_{i_3}\cdots t_{i_p}} + \sum_{j_1,j_2\in\{i_1,\ldots,i_p\}}\rho_{j_1,j_2}\sum_{l_1=1}^{n}\rho_{k_1,l_1}\frac{\partial^{p-4}Q_{k_2,\gamma}(a_h)}{\partial t_{k_3}\cdots t_{k_{p-2}}}\right.$$
$$\left. + \sum_{j_1,j_2,j_3,j_4\in\{i_1,\ldots,i_p\}}\rho_{j_1,j_2}\rho_{j_3,j_4}\sum_{l_1=1}^{n}\rho_{k_1,l_1}\frac{\partial^{p-6}Q_{k_2,\gamma}(a_h)}{\partial t_{k_3}\cdots t_{k_{p-4}}} + \cdots + \sum_{j_1,\ldots,j_p}\rho_{j_1,j_2}\cdots\rho_{j_{p-1},j_p}\sum_{l_1=1}^{n}\gamma_{1,l_1}\Phi_{n-1}(\eta^{-1/2}a_s;\mathbf{R})\right),$$

for even p where

$$\frac{\partial^p Q_{l,\gamma}(a_{h_1}\cdots a_{h_q})}{\partial t_{i_1}\cdots\partial t_{i_p}} = \frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(\frac{\partial U_l(a_{h_1},\ldots,a_{h_q})}{\partial t_{i_1}}\gamma_{q,h_1,\ldots,h_q}(a_{h_1},\ldots,a_{h_q};\mathbf{R}_{h_1\cdots h_{q+1}})\right.$$
$$\left.\times\ \Phi_{n-q}\left(a_{(h_1\cdots h_q)\cdot s};\mathbf{R}_{(h_1\cdots h_q)\cdot s}\right)\right) + \frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(U_l(a_{h_1},\ldots,a_{h_q})Q_{i_1,\gamma}(a_{h_1},\ldots,a_{h_q})\right)$$
$$+ \sum_{h_{q+1}\neq h_1\cdots\neq h_q} c_{(h_1\cdots h_q)\cdot s}(k,h_{q+1})\frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(Q_{i_1,\gamma}(a_{h_1},\ldots,a_{h_{q+1}})\right),$$

for $q = \{1,\ldots,n-1\}$.

Proof. See Appendix A.9. □

6. Multivariate lognormal case

In this section we provide a new result for distributions defined over the positive domain. Let X be a random vector. Define lower truncation points a_s, $s = \{1,\ldots,n\}$ such that $a_s \leq X_s$. The pdf of X is defined as f_X and the cdf F_X. The p-th order product tail moment $m_p(\mathbf{x};\alpha_s;a_s)$ is defined as in (2) with $\alpha = \{\alpha_1,\alpha_2,\ldots,\alpha_n\}$. This moment can be computed with the integral:

$$m_p(\mathbf{x};\alpha_s;a_s) = \frac{\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}f_X dx_1\cdots dx_n}{F_X(a_s)}. \tag{30}$$

Now we define the distribution of the incomplete cross moments with joint pdf h:

$$h_{X,\boldsymbol{\alpha}} = \frac{x_1^{\alpha_1}\cdots x_n^{\alpha_n}f_X}{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}f_X dx_1\cdots dx_n}, \tag{31}$$

for $x_s > 0$. If we denote by $H_{X,\boldsymbol{\alpha}}(a_s)$ the joint cdf of the incomplete cross moments with $X_s > 0$, $s = \{1,\ldots,n\}$, we have that:

$$m_p(\mathbf{x};\alpha_s;a_s) = H_{X,\boldsymbol{\alpha}}(a_s). \tag{32}$$

If a multivariate distribution is closed under incomplete cross moments, the computation of the previous density is directly given by:

$$m_p(\mathbf{x};\alpha_s;a_s) = H_{X,\boldsymbol{\alpha}}(a_s) = (H_{X,\boldsymbol{\alpha}}(\mathbf{0}))^{-1}F_{Y,\boldsymbol{\alpha}}(a_s). \tag{33}$$

To calculate truncated moments for the non-standard case, denote by $\mathbf{S}$ the covariance matrix of X, and $\boldsymbol{\mu}$ the mean vector of X.

Proposition 6.1. *Let X be a random vector with lower truncated multivariate standard lognormal (MVL) distribution with correlation matrix $\mathbf{R}$. Define truncation points a_s such that $0 < a_s \leq X_s$. The moments of order p of distribution X truncated at a_s are:*

$$m_p(\mathbf{x};\alpha_s;a_s) = L^{-1}\exp\left(\frac{1}{2}\boldsymbol{\alpha}'\mathbf{R}\boldsymbol{\alpha}\right)\Phi_n\left(b_s;\mathbf{R}\right),$$

where $b_s = \log(a_s) - \sum_i \rho_{s,i}\alpha_i$ *and* $L = \Phi_n(\xi_s; \mathbf{R})$ *with* $\xi_s = \log(a_s)$*. If* X *has a lower truncated MVL distribution with mean vector* $\boldsymbol{\mu}$ *and covariance matrix* $\mathbf{V}$ *the moments of order* p *of* X *are:*

$$m_p(\mathbf{x}; \alpha_s; a_s) = L^{-1} \exp\left(\frac{1}{2}\boldsymbol{\alpha}'\mathbf{V}\boldsymbol{\alpha} + \boldsymbol{\mu}'\boldsymbol{\alpha}\right) \Phi_n\left(b_s; \mathbf{R}\right),$$

where $b_s = (\log(a_s) - \mu_s - \sum_i \sigma_{s,i}\alpha_i)/\sigma_i$ *and* $L = \Phi_n(\xi_s; \mathbf{R})$ *with* $\xi_s = (\log(a_s) - \mu_s)/\sigma_i$*.*

Proof. See Appendix A.10. □

Example 4. Define three distributions: a standard BVN with $\mu = (0, 0)$ and $\rho = 0.8$, a bivariate Student's t with $\nu = 4$ degrees of freedom and the same correlation, and a lognormal with the same correlation. In Appendix C, Figs. C.1–C.3 show the contour plot of these distributions. Figs. C.4–C.8 show the third and fourth order exceedance moments of these distributions. As the lognormal is defined only in the positive domain, we plot the positive tail. It is evident that the Student's t-distribution exceedance moments increase more rapidly than the normal and lognormal. The lognormal exceedance moments are higher than those of the Student's t up to $\varsigma \approx 3$ because the lognormal distribution has a mean of $E(X) = \exp(\mu) = (1, 1)$.

7. Application: skewness and kurtosis of lower truncated multivariate distributions

In a multivariate setting, the measures of skewness and kurtosis developed by Mardia [26] are the standard measures used in statistics.

Mardia [26] *skewness and kurtosis definitions:* Let $X = (X_1, X_2, \dots, X_n)$ be a multivariate random vector. Let $\mu = (\mu_1, \dots, \mu_n)$ denote the mean vector of X and $\mathbf{V}$ the covariance matrix. Denote by $Y = (X - \mu)\mathbf{V}^{-1/2}$ the standardized vector. Let Z be a random vector with the same distribution but independent of Y. Mardia's skewness measure of X is:

$$\beta_{1,n} = E\left[(YZ)^3\right]. \tag{34}$$

A fundamental property of skewness measures is that they are invariant under non-singular transformations. Mardia's kurtosis measure is:

$$\beta_{1,n} = E\left[(Y'Y)^2\right]. \tag{35}$$

This measure is also invariant under non-singular transformations. Mardia [26] shows an application where he tested the normality from two artificially generated samples: one generated from a symmetric distribution and the other from a skewed one, and the results confirm the applicability of these measures to recognize the deviations from the normal distribution in a sample. However, Mardia's measures have the problem that they report zero values for distributions of different shapes like the class of elliptical symmetric distributions.[2]

Klar [17] mentions that this measure can be asymptotically distribution-free with the class of elliptical symmetric distributions but only when a projection over a direction is made with the consequence of not capturing the asymmetry properly for certain distributions.

Klar [17] found that Mardia's skewness definition is not asymptotically distribution-free within the class of elliptical distributions and Mori et al.'s definition is not well balanced. Klar developed a robust measure of skewness using some properties of the skewness measure of [26,27]. He proposed a new measure of multivariate skewness and kurtosis whose limit laws are asymptotically distribution-free under elliptical symmetry.

A more recent attempt to define multivariate skewness and kurtosis has been made by Kollo [18]. We follow his notation.

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ an $mr \times ns$ partitioned matrix with blocks $\mathbf{B}_{i,j}$ of $m \times n$ similar to (9). The star product of $\mathbf{A}$, $\mathbf{B}$ is an $r \times s$ matrix:

$$\mathbf{A} \star \mathbf{B} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}\mathbf{B}_{ij}. \tag{36}$$

Kollo [18] *definitions:* Let X, Y be multivariate random vectors defined as in Mardia's definition of skewness. The tensor skewness is defined as the n-vector:

$$\mathbf{b}_n(X) = \mathbf{1}_{n\times n} \star E(Y \otimes Y \otimes Y) \tag{37}$$

where $\otimes$ is the Kronecker product of two matrices. $E(Y \otimes Y' \otimes Y)$ is the third moment of Y in matrix notation. The kurtosis is defined as the $n \times n$ matrix:

$$\mathbf{B}_{n,n}(X) = \mathbf{1}_{p\times p} \star E(Y \otimes Y' \otimes Y \otimes Y'). \tag{38}$$

2 Malkovich and Afifi [24] proposed a definition of multivariate skewness based on projections of the random variable onto a line. The algorithm selects the square of the value that maximizes reportedly in the projection. Unfortunately, this measure has the same problem as Mardia's measure, as Baringhaus and Henze [5] remarks. Mori et al. [27] developed a new skewness measure to fix the problem of Mardia's measure.

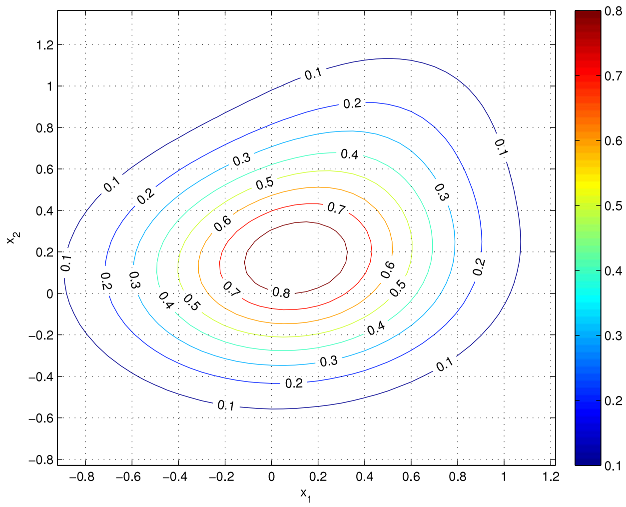

Fig. 1. Contour of a mixture of two BVNs.

Kollo analyzed and contrasted the basic properties and characteristics of these measures against the measures of [26,27]. These measures have the advantage that they are more descriptive of asymmetry and fat-tailedness of distributions.

Kollo's skewness and kurtosis measures include all cross moments and therefore are more informative about the shape of a distribution than Mardia's measures. However, in his definition cross moments are averaged and it could be the case that one cross moment dominates all components of the skewness vector or kurtosis matrix (Appendix D, Figs. D.1–D.4). For this reason, we propose a definition of the tensor skewness and tensor kurtosis with cross moments in their components. We use these measures to calculate the skewness and kurtosis of the truncated distributions of Sections 3–6.

For example, Fig. 1 is the contour of a mixture of two BVNs. Mardia's skewness of this distribution is approximately $\beta_1 = 0.0608685$. This number shows that the distribution is slightly skewed to the positive side but it does not give an idea of which of the component variables is more skewed. In contrast the moments $m_{30} = 0.0918868$ and $m_{03} = 0.1524966$ give the information that the second variable contributes more than the first to the total asymmetry.

We calculate the skewness and kurtosis of the lower truncated distributions developed in Sections 3–6. First, we use Mardia's definitions as they are standard in the literature.

Proposition 7.1. *Let X be a lower truncated multivariate random vector. Denote by μ the mean vector of X and $\mathbf{V}$ the covariance matrix. Denote by $Y = (X - \mu)\mathbf{V}^{-1/2}$ the standardized vector. Let $a_s \le X_s$, $s = 1, 2, \ldots, n$ be truncation points over X. Mardia's skewness of X is:*

$$\beta_1(\mathbf{x}, a_s) = \sum_{i_1=1}^{n} \left(m_{3,\{i_1,i_1,i_1\}}(\mathbf{y}, a_s)\right)^2 + 3 \sum_{i_1 \ne i_2 = 1}^{n} \left(m_{3,\{i_1,i_1,i_2\}}(\mathbf{y}, a_s)\right)^2 + 6 \sum_{i_1 \ne i_2 \ne i_3 = 1}^{n} \left(m_{3,\{i_1,i_2,i_3\}}(\mathbf{y}, a_s)\right)^2, \tag{39}$$

where $m_{3,\{i_1,i_1,i_1\}}(\mathbf{y}, a_s)$ is the third order moment of Y. Mardia's kurtosis of X is:

$$\beta_2(\mathbf{x}, a_s) = \sum_{i_1=1}^{n} \left(m_{4,\{i_1,i_1,i_1,i_1\}}(\mathbf{y}, a_s)\right)^2 + \sum_{i_1 \ne i_2 = 1}^{n} \left(m_{4,\{i_1,i_1,i_2,i_2\}}(\mathbf{y}, a_s)\right)^2, \tag{40}$$

where $m_{4,\{i_1,i_2,i_3,i_4\}}(\mathbf{y}, a_s)$ is the fourth order moment of Y.

Proof. In [17], Mardia's skewness measure is presented as:

$$\beta_1(\mathbf{x}, a_s) = \sum_{i_1=1}^{n} \left(E[Y_{i_1}^3]\right)^2 + 3 \sum_{i_1 \ne i_2 = 1}^{n} \left(E[Y_{i_1}^2 Y_{i_2}]\right)^2 + 6 \sum_{i_1 \ne i_2 \ne i_3 = 1}^{n} \left(E[Y_{i_1} Y_{i_2} Y_{i_3}]\right)^2,$$

and Mardia's kurtosis measure as,

$$\beta_2(\mathbf{x}, a_s) = \sum_{i_1=1}^{n} \left(E[Y_{i_1}^4]\right)^2 + \sum_{i_1 \ne i_2 = 1}^{n} \left(E[Y_{i_1}^2 Y_{i_2}^2]\right)^2.$$

Substituting our definition for truncated moments inside the last expressions, the result follows. □

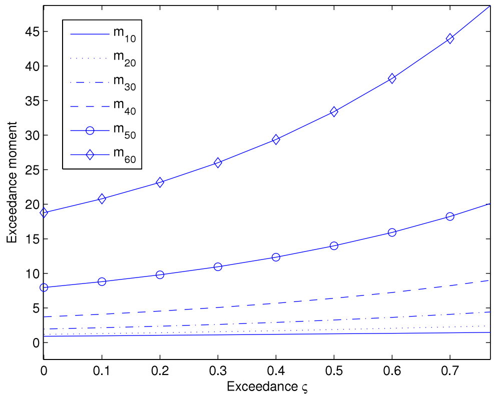

Fig. A.1. Plot of truncated moments of a BVN with $\rho = 0.8$.

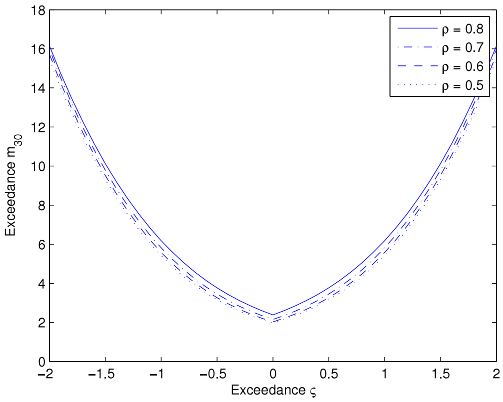

Fig. B.1. Plot of m_{30} for BVN with $\rho > 0$.

As mentioned in [18], Mardia's skewness measure is inaccurate for measuring asymmetry because its value is equal to zero for the elliptical symmetric class of distributions and not only for normal distributions. This implies that distributions with asymmetries in the tails can have zero as a skewness value. Kollo's skewness measure is in agreement with Mardia's measure and the new kurtosis measure will include all fourth order moments. They will provide a broader view of the concepts of asymmetry and heavy-tailedness for many applications such as financial risk management.

Proposition 7.2. *Let X be a lower truncated multivariate random vector. Denote by μ the mean vector of X and $\mathbf{V}$ the covariance matrix. Denote by $Y = (X - \mu)\mathbf{V}^{-1/2}$ the standardized vector. Let $a_s \leq X_s$, $s = 1, 2, \ldots, n$ be truncation points over X. Kollo's skewness of X is the vector:*

$$\mathbf{b}_n(\mathbf{x}, a_s) = \left(\sum_{i_1,i_2=1}^{n} m_{3,\{i_1,i_2,1\}}(\mathbf{y}, a_s), \ldots, \sum_{i_1,i_2=1}^{n} m_{3,\{i_1,i_2,n\}}(\mathbf{y}, a_s) \right), \tag{41}$$

where $m_{3,\{i_1,i_1,i_3\}}(\mathbf{y}, a_s)$ is the third order moment of Y. Kollo's kurtosis of X is:

$$\mathbf{B}_{n\times n}(\mathbf{x}, a_s) = \begin{pmatrix} \sum_{i_1,i_2=1}^{n} m_{4,\{i_1,i_2,1,1\}}(\mathbf{y}, a_s) & \cdots & \sum_{i_1,i_2=1}^{n} m_{4,\{i_1,i_2,1,n\}}(\mathbf{y}, a_s) \\ \vdots & \ddots & \vdots \\ \sum_{i_1,i_2=1}^{n} m_{4,\{i_1,i_2,n,1\}}(\mathbf{y}, a_s) & \cdots & \sum_{i_1,i_2=1}^{n} m_{4,\{i_1,i_2,n,n\}}(\mathbf{y}, a_s) \end{pmatrix}, \tag{42}$$

where $m_{4,\{i_1,i_2,i_3,i_4\}}(\mathbf{y}, a_s)$ is the fourth order moment of Y.

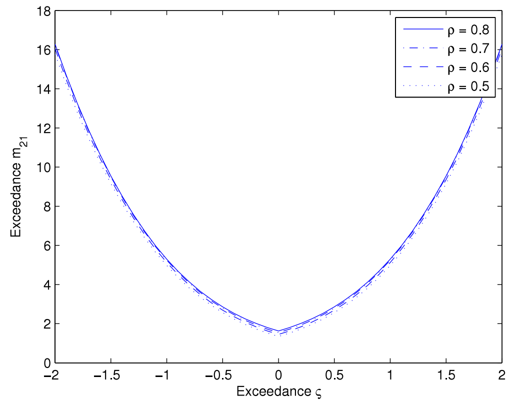

Fig. B.2. Plot of m_{21} for BVN with $\rho > 0$.

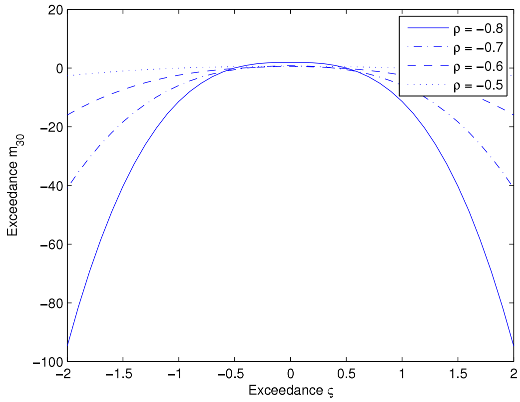

Fig. B.3. Plot of m_{30} for BVN with $\rho < 0$.

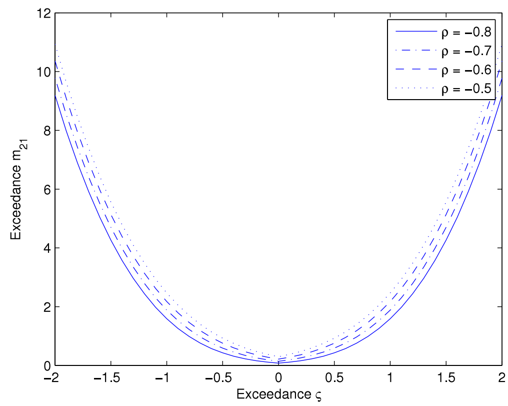

Fig. B.4. Plot of m_{21} for BVN with $\rho < 0$.

Proof. Kollo [18] states another expression to define skewness by its components:

$$\mathbf{b}_n(\mathbf{x}, a_s) = E\left[\sum_{i_1,i_2=1}^{n} \left(Y_{i_1}Y_{i_2}\right) Y\right],$$

and kurtosis:

$$\mathbf{B}_{n\times n}(\mathbf{x}, a_s) = E\left[\sum_{i_1,i_2=1}^{n} \left(Y_{i_1}Y_{i_2}\right) YY'\right].$$

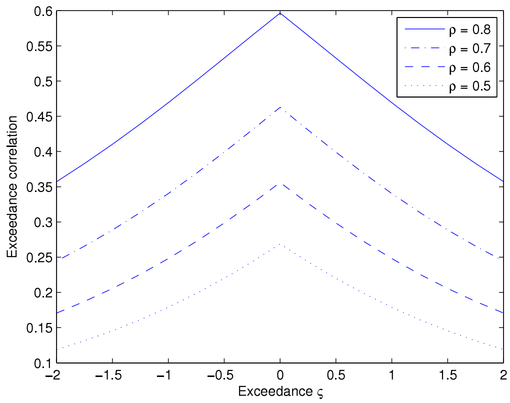

Fig. B.5. Plot of exceedance correlation for BVN with $\rho > 0$.

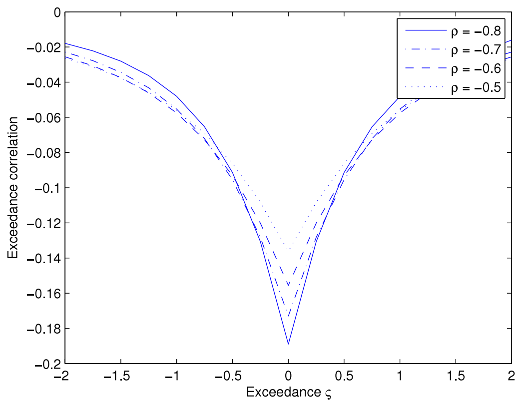

Fig. B.6. Plot of exceedance correlation for BVN with $\rho < 0$.

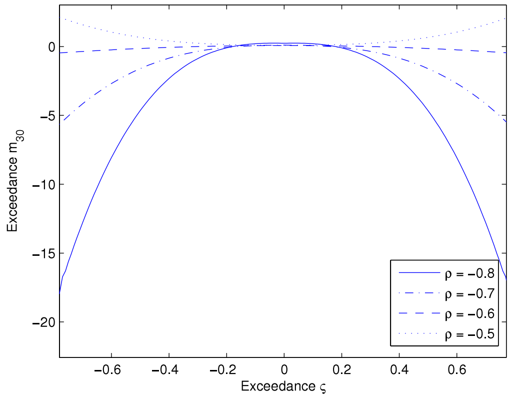

Fig. B.7. Plot of m_{30} for BVN with $\rho < 0$.

Applying linearity of the expectation operator and using the definitions of truncated moments from Sections 3–6, the result follows. □

The advantage of Kollo's measures having more information about the shape of the distribution in the tail comes with the disadvantage of having more numbers for defining asymmetry and tail-fatness. In many applications a possible motivation for a more complex skewness measure is that each component represents the asymmetry of each variable. However, the components of this measure are not equivalent to the third order moment co-skewness and systematic co-skewness measures that have economic importance in asset pricing theory.

For this reason, we introduce new skewness and kurtosis measures using tensors with the cross moments as the components of the tensors. In this way, our skewness measures will be equivalent to the co-skewness and co-kurtosis, reconciling the statistical definition of higher moments with the economic motivation.

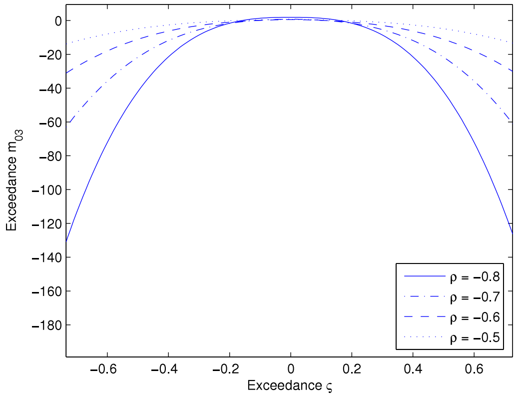

Fig. B.8. Plot of m_{03} for BVN with $\rho < 0$.

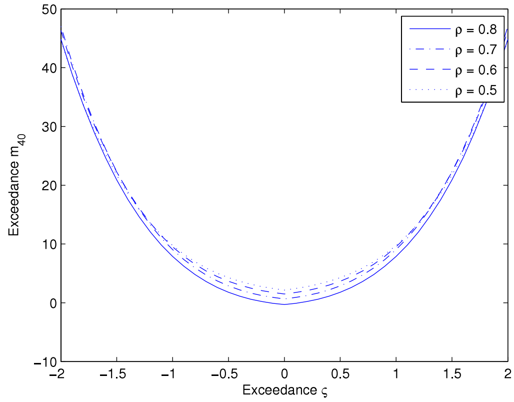

Fig. B.9. Plot of m_{40} for BVN with $\rho > 0$.

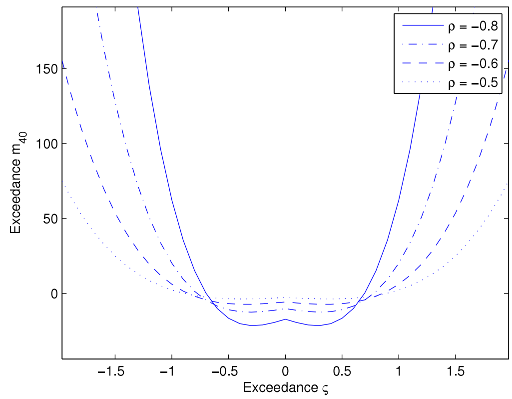

Fig. B.10. Plot of m_{40} for BVN with $\rho < 0$.

Definition 7.1. Let X be a random vector of dimension n with mean vector μ and covariance matrix $\mathbf{V}$. Denote by $Y = (X - \mu)\mathbf{V}^{-1/2}$ the standardized vector. The skewness of X is defined as the third order tensor:

$$\hat{b}_{i_1,i_2,i_3} = E[Y_{i_1} Y_{i_2} Y_{i_3}]. \tag{43}$$

The kurtosis of X is the fourth order tensor:

$$\hat{b}_{i_1,i_2,i_3,i_4} = E[Y_{i_1} Y_{i_2} Y_{i_3} Y_{i_4}]. \tag{44}$$

Proposition 7.3. *Let X be a lower truncated multivariate random vector. Denote by μ the mean vector of X and $\mathbf{V}$ the covariance matrix. Denote by $Y = (X - \mu)\mathbf{V}^{-1/2}$ the standardized vector. Let $a_s \le X_s, s = 1, 2, \ldots, n$ be truncation points over X. The* *tensor skewness of X is:*

$$\hat{b}_{i_1,i_2,i_3} = \hat{m}_{3,\{i_1,i_2,i_3\}}(\mathbf{y}, a_s), \tag{45}$$

and the kurtosis skewness is:

$$\hat{b}_{i_1,i_2,i_3,i_4} = \hat{m}_{4,\{i_1,i_2,i_3,i_4\}}(\mathbf{y}, a_s). \tag{46}$$

Proof. The result follows immediately from Definition 7.1. □

We extend definitions of exceedance moments to exceedance skewness and kurtosis.

Definition 7.2. Let X be a random vector with known density distribution, a_s a vector of thresholds and ς a value such that $a_s = \varsigma$, $s = \{1, \ldots, n\}$. The exceedance skewness and kurtosis of X are as follows.

Mardia's:

$$\bar{\beta}_1(\mathbf{x}, \varsigma) = \begin{cases} \beta_1(\mathbf{x}, a_s) \equiv \beta_1(\mathbf{x}, \varsigma), & \text{if } \varsigma \le 0, \\ \beta_1(-\mathbf{x}, -a_s) \equiv \beta_1(-\mathbf{x}, -\varsigma), & \text{if } \varsigma > 0. \end{cases}$$

$$\bar{\beta}_2(\mathbf{x}, \varsigma) = \begin{cases} \beta_2(\mathbf{x}, a_s) \equiv \beta_2(\mathbf{x}, \varsigma), & \text{if } \varsigma \le 0, \\ \beta_2(-\mathbf{x}, -a_s) \equiv \beta_2(-\mathbf{x}, -\varsigma), & \text{if } \varsigma > 0. \end{cases}$$

Kollo's:

$$\bar{\mathbf{b}}_n(\mathbf{x}, \varsigma) = \begin{cases} \mathbf{b}_n(\mathbf{x}, a_s) \equiv \mathbf{b}_n(\mathbf{x}, \varsigma), & \text{if } \varsigma \le 0, \\ \mathbf{b}_n(-\mathbf{x}, -a_s) \equiv \mathbf{b}_n(-\mathbf{x}, -\varsigma), & \text{if } \varsigma > 0. \end{cases}$$

$$\bar{\mathbf{B}}_{n\times n}(\mathbf{x}, \varsigma) = \begin{cases} \mathbf{B}_{n\times n}(\mathbf{x}, a_s) \equiv \mathbf{B}_{n\times n}(\mathbf{x}, \varsigma), & \text{if } \varsigma \le 0, \\ \mathbf{B}_{n\times n}(-\mathbf{x}, -a_s) \equiv \mathbf{B}_{n\times n}(-\mathbf{x}, -\varsigma), & \text{if } \varsigma > 0. \end{cases}$$

Example 5. In Appendix C, Figs. C.4–C.8 are plots of Mardia's and Kollo's exceedance skewness and kurtosis using the distributions of Example 4. The plots are on a log-scale and we observe the same relationships between the distributions of the exceedance moments of third and fourth order.

An important conclusion with Mardia and Kollo skewness and kurtosis measures is that it is possible to extract the principal asymmetry and heavy-tailedness properties of the distributions; however, when more information about the distributions is needed, the use of all moments as in tensor skewness and tensor kurtosis is fundamental.

Appendix A. Mathematical derivations

A.1. Derivation of $\mathbf{C}_{h_1\cdots h_n\cdot s}$

The values of the components of matrices $\mathbf{C}_{h_1\cdot s}$, $\mathbf{C}_{h_1h_2\cdot s}$ and $\mathbf{C}_{h_1h_2h_3\cdot s}$ are:

$$c_{h_1\cdot s}(k, q) = \left(\rho_{k,q} - \rho_{q,h_1}\rho_{k,h_1}\right), \tag{47}$$

$$c_{h_1h_2\cdot s}(k, u) = \left(\rho_{k,u} - \sum_{i\in\{h_1,h_2\}} \rho_{k,i}\rho_{u,i} - \rho_{k,u}\rho^2_{h_1,h_2} + \sum_{\substack{i,j\in\{h_1,h_2\}\\ i\ne j}} \rho_{h_1,h_2}\rho_{k,i}\rho_{u,j}\right) \Big/ (1-\rho^2_{h_1,h_2}), \tag{48}$$

$$\begin{aligned} c_{h_1h_2h_3\cdot s}(k, v) = {} & \left(\frac{1}{D_{h_1h_2h_3}}\right)\left(-\rho_{k,v} + \sum_{i\in\{h_1,h_2,h_3\}} \rho_{k,i}\rho_{v,i} + \sum_{\substack{i,j\in\{h_1,h_2,h_3\}\\ i\ne j}} \rho_{k,v}\rho^2_{i,j} - \sum_{\substack{i,j\in\{h_1,h_2,h_3\}\\ i\ne j}} \rho_{k,i}\rho_{v,j}\rho_{i,j}\right. \\ & \left. - \sum_{\substack{i,j,l\in\{h_1,h_2,h_3\}\\ i\ne j\ne l}} \rho_{k,i}\rho_{v,i}\rho_{j,l} + \sum_{\substack{i,j,l\in\{h_1,h_2,h_3\}\\ i\ne j\ne l}} \rho_{k,i}\rho_{v,j}\rho_{i,l}\rho_{j,l} - 2\rho_{k,v}\rho_{h_1,h_2}\rho_{h_1,h_3}\rho_{h_2,h_3}\right), \end{aligned} \tag{49}$$

where

$$D_{h_1h_2h_3} = -2\rho_{h_1,h_2}\rho_{h_2,h_3}\rho_{h_1,h_3} - 1 + \sum_{\substack{j_1,j_2\in\{h_1,h_2,h_3\}\\ j_1\ne j_2}} \rho^2_{j_1,j_2}, \tag{50}$$

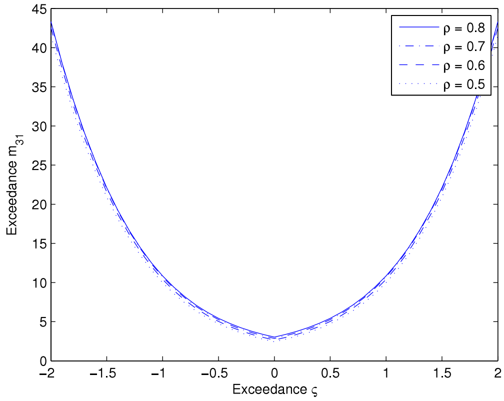

Fig. B.11. Plot of m_{31} for BVN with $\rho > 0$.

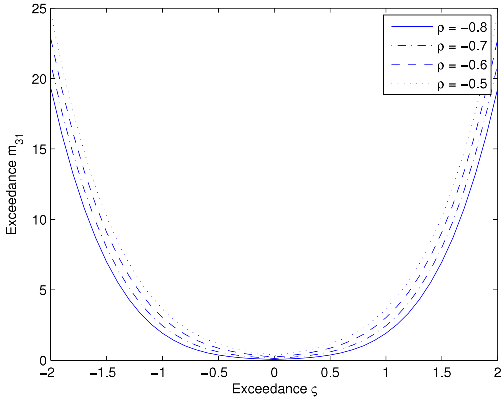

Fig. B.12. Plot of m_{31} for BVN with $\rho < 0$.

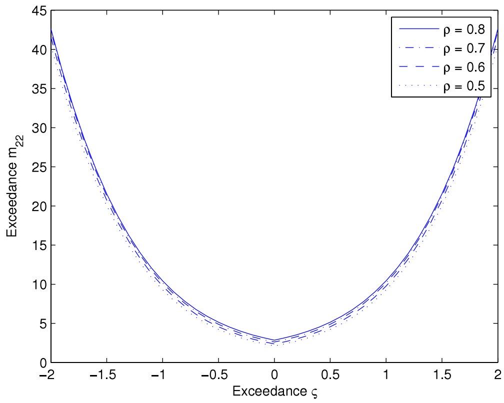

Fig. B.13. Plot of m_{22} for BVN with $\rho > 0$.

and $\rho_{i,j}$ are the unconditional correlation coefficients defined in (4). For the general case, we calculate components using the expression:

$$\mathbf{C}_{h_1\cdots h_p\cdot s} = \mathbf{R}_{h_1\cdots h_p\cdot s} - \mathbf{R}_{h_1\cdots h_p,12}\mathbf{R}^{-1}_{h_1\cdots h_p}\mathbf{R}_{h_1\cdots h_p,21}.$$

A.2. Derivation of $U_k(b_1, \ldots, b_{h_n})$

In the case of the partial derivative $U_i(\cdot)$, after some calculations we have:

$$U_i(b_{h_1}) = \rho_{i,h_1} b_{h_1},$$

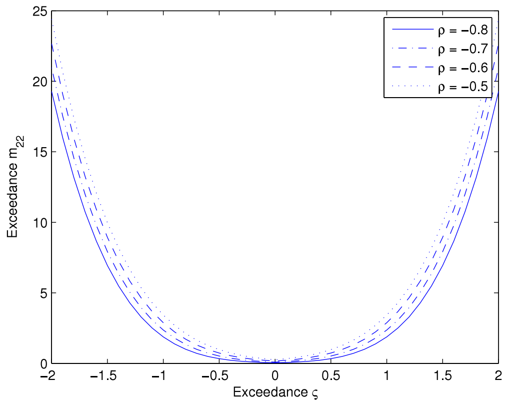

Fig. B.14. Plot of m_{22} for BVN with $\rho < 0$.

$$U_i(b_{h_1}, b_{h_2}) = \left(\frac{b_{h_2}\left(\rho_{h_1,i} - \rho_{h_1,h_2}\rho_{h_2,i}\right) + b_{h_1}\left(\rho_{h_2,i} - \rho_{h_1,h_2}\rho_{h_1,i}\right)}{1 - \rho^2_{h_1,h_2}} \right),$$

$$U_i(b_{h_1}, b_{h_2}, b_{h_3}) = \left(\frac{1}{D_{h_1h_2h_3}} \right) \sum_{\substack{j_1,j_2,j_3 \in \{h_1,h_2,h_3\} \\ j_1 \neq j_2 \neq j_3}} b_{j_1} \left(-\rho_{j_1,i} + \rho^2_{j_2,j_3}\rho_{j_1,i} + \sum_{\substack{k_1,k_2 \in \{j_2,j_3\} \\ k_1 \neq k_2}} \rho_{j_1,k_1}\rho_{k_1,k_2} - \sum_{\substack{k_1,k_2 \in \{j_2,j_3\} \\ k_1 \neq k_2}} \rho_{j_1,k_1}\rho_{k_1,k_2}\rho_{k_2,i} \right).$$

For the general case, we calculate components using the expression:

$$U_i(b_{h_1}, \ldots, b_{h_p}) = \frac{\partial \left(-\frac{1}{2}\mathbf{b}'_{h_1\cdots h_p}\mathbf{R}^{-1}_{h_1\cdots h_p}\mathbf{b}_{h_1\cdots h_p} \right)}{\partial t_i}.$$

A.3. Proof of Proposition 3.2

Proof. With change of variable $Y_s = X_s - \zeta_s$, the MGF (7) becomes [33]:

$$G(\mathbf{t}, a_s) = L^{-1}\exp(T)\Phi_n(b_s; \mathbf{R}), \tag{51}$$

where $b_s = a_s - \sum_{h=1}^{n} \rho_{i,j}t_h$. To obtain the moments, we find partial derivatives of (7) over $\mathbf{t}$, and evaluate at $\mathbf{t} = \mathbf{0}$. The third moment will be the partial derivative of (51) evaluated at $\mathbf{t} = 0$:

$$\begin{aligned}\frac{\partial^3 G(\mathbf{t}, a_s)}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}} = L^{-1}\Bigg(&\frac{\partial e^T}{\partial t_{i_3}}\frac{\partial^2\Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}} + e^T\frac{\partial^3\Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}} + \frac{\partial e^T}{\partial t_{i_2}}\frac{\partial^2\Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_3}} + \frac{\partial^2 e^T}{\partial t_{i_2}\partial t_{i_3}}\frac{\partial\Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}} \\ &+ \frac{\partial^2 e^T}{\partial t_{i_1}\partial t_{i_2}}\frac{\partial\Phi_n(b_s;\mathbf{R})}{\partial t_{i_3}} + \frac{\partial^3 e^T}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}}\Phi_n(b_s;\mathbf{R}) + \frac{\partial^2 e^T}{\partial t_{i_1}\partial t_{i_3}}\frac{\partial\Phi_n(b_s;\mathbf{R})}{\partial t_{i_2}} + \frac{\partial e^T}{\partial t_{i_1}}\frac{\partial^2\Phi_n(b_s;\mathbf{R})}{\partial t_{i_2}\partial t_{i_3}}\Bigg).\end{aligned} \tag{52}$$

The first, third, sixth and eighth terms of (52) are zero for $\mathbf{t} = \mathbf{0}$. In the fourth term of (52), the partial derivative $\frac{\partial^2 e^T}{\partial t_{i_1}\partial t_{i_2}}$ becomes $\rho_{i,j}$ and:

$$\frac{\partial\Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}} = \sum_{h_1=1}^{n}\rho_{i,h_1}\int_{b_1}^{\infty}\cdots\int_{b_{h_1-1}}^{\infty}\int_{b_{h_1+1}}^{\infty}\cdots\int_{b_n}^{\infty}\phi_n(x_s, x_{h_1} = b_{h_1};\mathbf{R})dx_s. \tag{53}$$

The fifth and seventh terms of (52) are equal to the fourth term, exchanging i, j and k, respectively. We define the univariate marginal $F_h(b_h)$ as:

$$F_{h_1}(b_{h_1}) = \int_{b_1}^{\infty}\cdots\int_{b_{h_1-1}}^{\infty}\int_{b_{h_1+1}}^{\infty}\cdots\int_{b_n}^{\infty}\phi_n(x_s, x_{h_1} = b_{h_1};\mathbf{R})dx_s \tag{54}$$

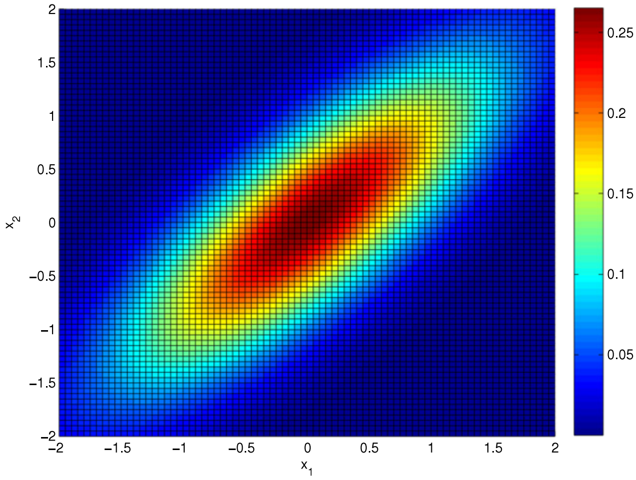

Fig. C.1. Contour of BVN with $\rho = 0.8$.

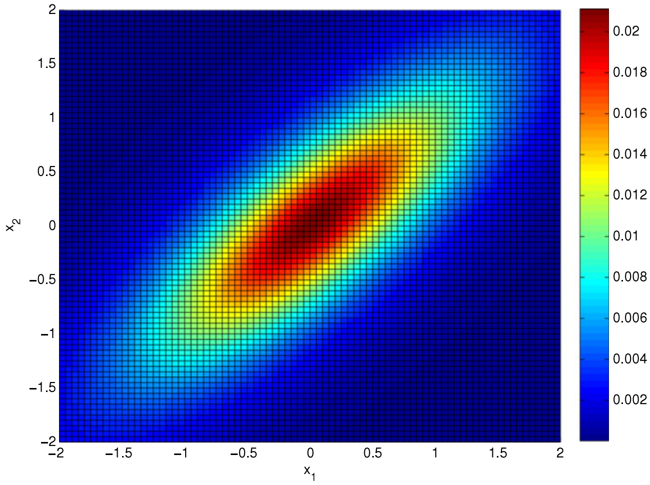

Fig. C.2. Contour of bivariate Student's t with $\nu = 4$, $\rho = 0.8$.

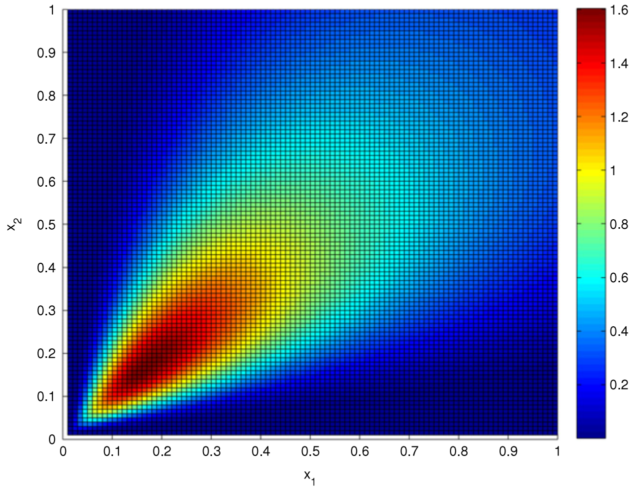

Fig. C.3. Contour of bivariate lognormal with $\rho = 0.8$.

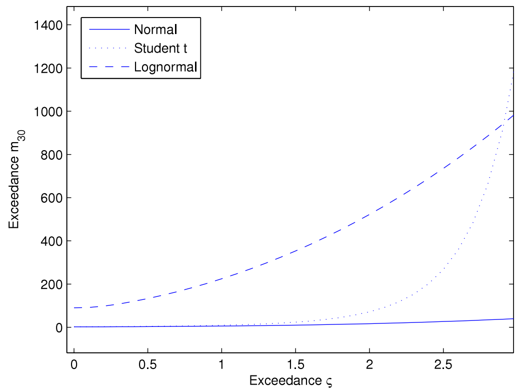

Fig. C.4. Plot of m_{30} with $\rho = 0.8$.

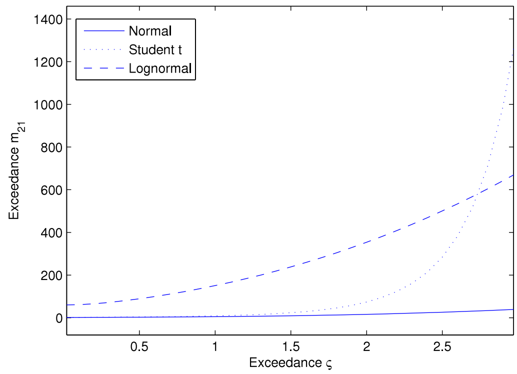

Fig. C.5. Plot of m_{21} with $\rho = 0.8$.

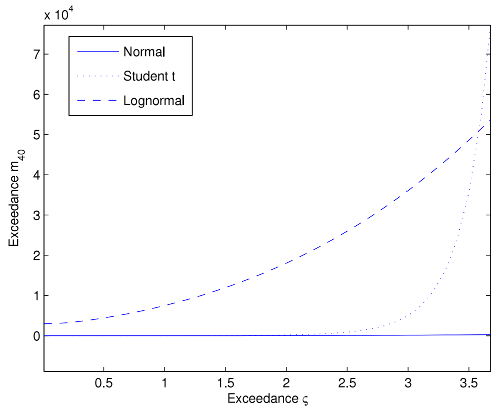

Fig. C.6. Plot of m_{40} with $\rho = 0.8$.

and the bivariate marginal $F_{h_1,h_2}(a_{h_1}, a_{h_2})$ as:

$$F_{h_1,h_2}(b_{h_1}, b_{h_2}) = \int_{b_1}^{\infty} \cdots \int_{b_{h_1-1}}^{\infty} \int_{b_{h_1+1}}^{\infty} \cdots \int_{b_{h_2-1}}^{\infty} \int_{b_{h_2+1}}^{\infty} \cdots \int_{b_n}^{\infty} \phi_n(x_s, x_{h_1} = b_{h_1}, x_{h_2} = b_{h_2}; \mathbf{R}) dx_s.$$

Using this notation, the first partial derivative can be written as:

$$\frac{\partial \Phi_n(b_s; \mathbf{R})}{\partial t_{i_1}} = \sum_{h_1=1}^{n} \rho_{i,h_1} F_{h_1}(b_{h_1}), \tag{55}$$

and the second partial derivative as [25]:

$$\frac{\partial^2 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}} = \sum_{h_1=1}^{n} \rho_{i,h_1} \frac{\partial F_{h_1}(b_{h_1})}{\partial t_{i_2}} = \sum_{h_1=1}^{n} \rho_{i,h_1} \left(U_{i_2}(b_{h_1}) F_{h_1}(b_{h_1}) + \sum_{h_2\neq h_1} c_{h_1\cdot s}(j,h_2) F_{h_1,h_2}(b_{h_1},b_{h_2}) \right). \tag{56}$$

We derive a formula for $\frac{\partial^3 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}}$:

$$\begin{aligned}\frac{\partial^3 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}} &= \frac{\partial}{\partial t_{i_3}} \left(\frac{\partial^2 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}} \right) \\ &= \sum_{h_1=1}^{n} \rho_{i,h_1} \left(\frac{\partial U_{i_2}(b_{h_1})}{\partial t_{i_3}} F_{h_1}(b_{h_1}) + U_{i_2}(b_{h_1}) \frac{\partial F_{h_1}(b_{h_1})}{\partial t_{i_3}} + \sum_{h_2\neq h_1} c_{h_1\cdot s}(j,h_2) \frac{\partial F_{h_1,h_2}(b_{h_1},b_{h_2})}{\partial t_{i_3}} \right).\end{aligned} \tag{57}$$

The partial derivative $\frac{\partial U_{i_2}(b_h)}{\partial t_{i_3}} = -\rho_{j,h}\rho_{k,h}$. The partial derivative of the bivariate marginal is derived as:

$$\frac{\partial F_{h_1,h_2}(b_{h_1},b_{h_2})}{\partial t_{i_3}} = U_{i_3}(b_{h_1},b_{h_2}) F_{h_1,h_2}(b_{h_1},b_{h_2}) + \sum_{h_3\neq h_1\neq h_2} c_{h_1h_2\cdot s}(k,h_3) F_{h_1,h_2,h_3}(b_{h_1},b_{h_2},b_{h_3}). \tag{58}$$

Then (57) becomes,

$$\begin{aligned}\frac{\partial^3 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}} &= \sum_{h_1=1}^{n} \rho_{i,h_1} \Bigg\{ -\rho_{j,h_1}\rho_{k,h_1} F_{h_1}(b_{h_1}) + U_{i_2}(b_{h_1}) \\ &\quad \times \left(U_{i_3}(b_{h_1}) F_{h_1}(b_{h_1}) + \sum_{h_2\neq h_1} c_{h_1\cdot s}(k,h_2) F_{h_1,h_2}(b_{h_1},b_{h_2}) \right) + \sum_{h_2\neq h_1} c_{h_1\cdot s}(j,h_2) \\ &\quad \times \left(U_{i_3}(b_{h_1},b_{h_2}) F_{h_1,h_2}(b_{h_1},b_{h_2}) + \sum_{h_3\neq h_1\neq h_2} c_{h_1h_2\cdot s}(k,h_3) F_{h_1,h_2,h_3}(b_{h_1},b_{h_2},b_{h_3}) \right) \Bigg\}.\end{aligned} \tag{59}$$

Combining (47), (48), (15) and (55) and evaluating at $\mathbf{t} = \mathbf{0}$, b_s becomes a_s and the result follows. □

A.4. Proof of Proposition 3.3

Proof. Following the procedure of Proposition 3.2, we found the fourth order partial derivatives of the MGF defined at (51) and then we evaluate for $\mathbf{t} = \mathbf{0}$:

$$\begin{aligned}\frac{\partial^4 G(\mathbf{t},a_s)}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}\partial t_{i_4}} &= L^{-1} \Bigg\{ \frac{\partial^2 e^T}{\partial t_{i_3}\partial t_{i_4}} \frac{\partial^2 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}} + \frac{\partial e^T}{\partial t_{i_3}} \frac{\partial^3 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_4}} + \frac{\partial e^T}{\partial t_{i_4}} \frac{\partial^3 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}} \\ &\quad + e^T \frac{\partial^4 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}\partial t_{i_4}} + \frac{\partial^2 e^T}{\partial t_{i_2}\partial t_{i_4}} \frac{\partial^2 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_3}} + \frac{\partial e^T}{\partial t_{i_2}} \frac{\partial^3 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_3}\partial t_{i_4}} \\ &\quad + \frac{\partial^3 e^T}{\partial t_{i_2} t_{i_3}\partial t_{i_4}} \frac{\partial \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}} + \frac{\partial^2 e^T}{\partial t_{i_2} t_{i_3}} \frac{\partial^2 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_4}} + \frac{\partial^3 e^T}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_4}} \frac{\partial \Phi_n(b_s;\mathbf{R})}{\partial t_{i_3}} \\ &\quad + \frac{\partial^2 e^T}{\partial t_{i_1} t_{i_2}} \frac{\partial^2 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_3}\partial t_{i_4}} + \frac{\partial^4 e^T}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}\partial t_{i_4}} \Phi_n(b_s;\mathbf{R}) + \frac{\partial^3 e^T}{\partial t_{i_1} t_{i_2} t_{i_3}} \frac{\partial \Phi_n(b_s;\mathbf{R})}{\partial t_{i_4}} \\ &\quad + \frac{\partial^3 e^T}{\partial t_{i_1}\partial t_{i_3}\partial t_{i_4}} \frac{\partial \Phi_n(b_s;\mathbf{R})}{\partial t_{i_2}} + \frac{\partial^2 e^T}{\partial t_{i_1} t_{i_3}} \frac{\partial^2 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_2}\partial t_{i_4}} + \frac{\partial^2 e^T}{\partial t_{i_1}\partial t_{i_4}} \frac{\partial^2 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_2} t_{i_3}} + \frac{\partial e^T}{\partial t_{i_1}} \frac{\partial^3 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_2}\partial t_{i_3}\partial t_{i_4}} \Bigg\}.\end{aligned} \tag{60}$$

The second, third, sixth, seventh, ninth, twelfth, thirteenth and last terms of (60) are zero for $\mathbf{t} = \mathbf{0}$. Using the proof of Proposition 3.2 and Definition (17), the first term is:

$$\frac{\partial^2 e^T}{\partial t_{i_3}\partial t_{i_4}} \frac{\partial^2 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}} = \rho_{k,l} \sum_{h_1=1}^{n} \rho_{i,h_1} Q_{i_2}(b_{h_1}). \tag{61}$$

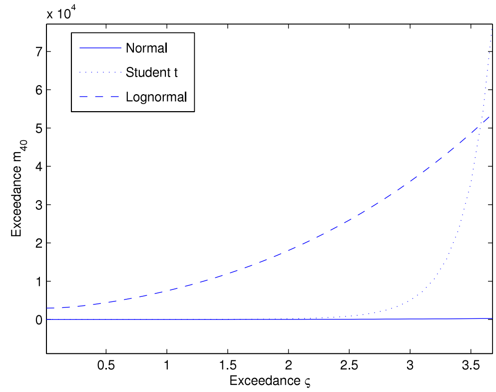

Fig. C.7. Plot of m_{31} with $\rho = 0.8$.

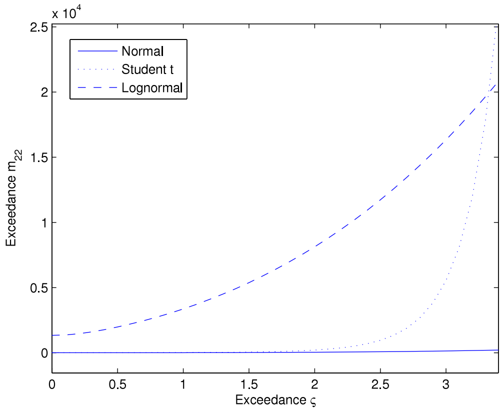

Fig. C.8. Plot of m_{22} with $\rho = 0.8$.

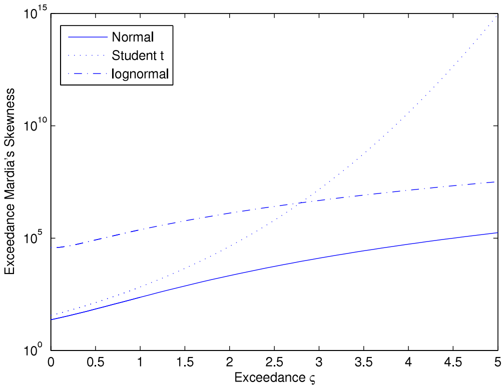

Fig. D.1. Plot of Mardia's skewness with $\rho = 0.8$.

The fifth, eighth, tenth, fourteenth, and fifteenth terms of (60) are similar to (61), exchanging i, j, k, l, respectively. The eleventh term of (60) for $\mathbf{t} = \mathbf{0}$ becomes,

$$\frac{\partial^4 e^T}{\partial t_{i_1} \partial t_{i_2} \partial t_{i_3} \partial t_{i_4}} \Phi_n(b_s; \mathbf{R}) = \sum_{\substack{v_1, v_2 \in \{i,j,k,l\} \\ v_3, v_4 \in \{i,j,k,l\} \setminus \{v_1, v_2\} \\ v_1 \neq v_2 \neq v_3 \neq v_4}} \rho_{v_1, v_2} \rho_{v_3, v_4} \Phi_n(b_s; \mathbf{R}) = \sum_{\substack{v_1, v_2 \in \{i,j,k,l\} \\ v_3, v_3 \in \{i,j,k,l\} \setminus \{v_1, v_2\} \\ v_1 \neq v_2 \neq v_3 \neq v_4}} \rho_{v_1, v_2} \rho_{v_3, v_4} L. \tag{62}$$

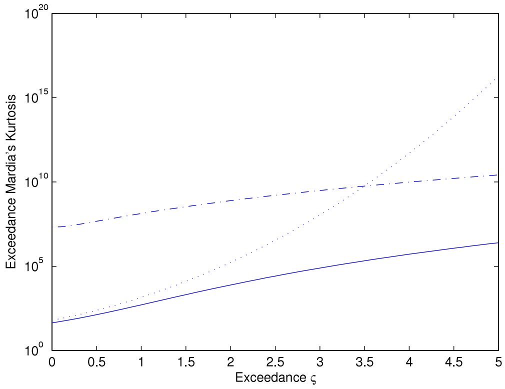

Fig. D.2. Plot of Mardia's kurtosis with $\rho = 0.8$.

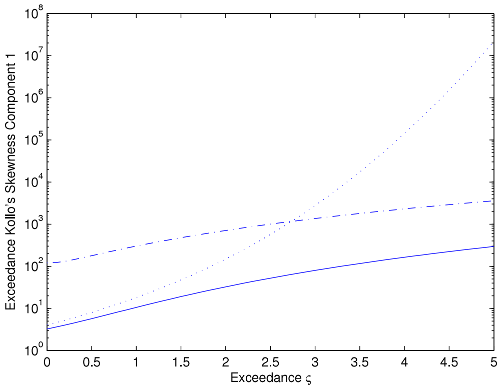

Fig. D.3. Plot of Kollo's skewness first vector with $\rho = 0.8$.

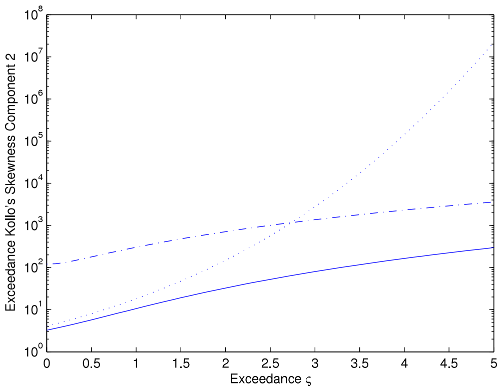

Fig. D.4. Plot of Kollo's skewness second vector with $\rho = 0.8$.

The fourth term of (60) is:

$$
\begin{aligned}
\frac{\partial^4 \Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}\partial t_{i_4}} &= \sum_{h_1=1}^{n} \rho_{i,h_1} \Bigg\{ -\rho_{j,h_1}\rho_{k,h_1} \frac{\partial F_{h_1}(b_{h_1})}{\partial t_{i_4}} + \frac{\partial U_{i_2}(b_{h_1})}{\partial t_{i_4}} \Bigg(U_{i_3}(b_{h_1}) F_{h_1}(b_{h_1}) \\
&\quad + \sum_{h_2 \neq h_1} c_{h_1 \cdot s}(k,h_2) F_h(b_{h_1}, b_{h_2}) \Bigg) + U_{i_2}(b_h) \Bigg(\frac{\partial U_{i_3}(b_{h_1})}{\partial t_{i_4}} F_{h_1}(b_{h_1}) + U_{i_3}(b_{h_1}) \frac{\partial F_{h_1}(b_{h_1})}{\partial t_{i_4}} \\
&\quad + \sum_{h_2 \neq h_1} c_{h_1 \cdot s}(k,h_2) \frac{\partial F_{h_1,h_2}(b_{h_1}, b_{h_2})}{\partial t_{i_4}} \Bigg)
\end{aligned}
$$

$$+\sum_{h_2\neq h_1} c_{h_1\cdot s}(j,h_2)\left(\left(\frac{\partial U_{i_3}(b_{h_1},b_{h_2})}{\partial t_{i_4}}F_{h_1,h_2}(b_{h_1},b_{h_2})+U_{i_3}(b_{h_1},b_{h_1})\frac{\partial F_{h_1,h_2}(b_{h_1},b_{h_2})}{\partial t_{i_4}}\right)\right.$$
$$\left.\left.+\sum_{h_3\neq h_1\neq h_2} c_{h_1h_2\cdot s}(k,h_3)\frac{\partial F_{h_1,h_2,h_3}(b_{h_1},b_{h_2},b_{h_3})}{\partial t_{i_4}}\right)\right\}.$$

But $\frac{\partial F_{h_1}(b_{h_1})}{\partial t_{i_4}} = Q_{i_4}(b_{h_1})$, $\frac{\partial F_{h_1,h_2}(b_{h_1},b_{h_2})}{\partial t_{i_4}} = Q_{i_4}(b_{h_1},b_{h_2})$ and $\frac{\partial F_{h_1,h_2,h_3}(b_{h_1},b_{h_2},b_{h_3})}{\partial t_{i_4}} = Q_{i_4}(b_{h_1},b_{h_2},b_{h_3})$. Using the results of Lemma 3.2 and Definition (17), we have formulae for $Q_{i_4}(b_{h_1})$, $Q_{i_2}(b_{h_1})$, $Q_{i_4}(b_{h_1},b_{h_2})$ and $Q_{i_4}(b_{h_1},b_{h_2},b_{h_3})$.

Combining (61) and (62) with (49) and (15), using the definition of the marginals in (10) and evaluating at $\mathbf{t}=\mathbf{0}$, $b_s=a_s$ the result follows. □

A.5. Proof of Corollary 3.4

Proof. Following the same procedure as in Propositions 3.2 and 3.3, we derive a general term for the MGF of the lower truncated MVSN. Denote the simplified notation $m_{p,\{i_1,\ldots,i_p\}}(\mathbf{x},a_s)=m_p$. We derive the MGF for a fifth time and we notice that the terms of the first five moments have this pattern:

$$m_1L = e^T\frac{\partial\Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}},$$
$$m_2L = e^T\frac{\partial^2\Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}}+\frac{\partial^2 e^T}{\partial t_{i_1}\partial t_{i_2}}\Phi_n(b_s;\mathbf{R}),$$
$$m_3L = e^T\frac{\partial^3\Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}}+\sum_{\substack{h_1,h_2,h_3\in\{i_1,i_2,i_3\}\\ h_3\neq h_1\neq h_2}}\frac{\partial^2 e^T}{\partial t_{h_1}\partial t_{h_2}}\frac{\partial\Phi_n(b_s;\mathbf{R})}{\partial t_{h_3}},$$
$$m_4L = e^T\frac{\partial^4\Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}\partial t_{i_4}}+\sum_{\substack{h_1,h_2,h_3,h_4\in\{i_1,i_2,i_3,i_4\}\\ h_4\neq h_3\neq h_2\neq h_1}}\frac{\partial^2 e^T}{\partial t_{h_1}\partial t_{h_2}}\frac{\partial^2\Phi_n(b_s;\mathbf{R})}{\partial t_{h_3}\partial t_{h_4}}+\frac{\partial^4 e^T}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}\partial t_{i_4}}\Phi_n(b_s;\mathbf{R}),$$
$$m_5L = e^T\frac{\partial^5\Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}\partial t_{i_4}\partial t_{i_5}}+\sum_{\substack{h_1,h_2,h_3,h_4\in\{i_1,i_2,i_3,i_4,i_5\}\\ h_4\neq h_3\neq h_2\neq h_1}}\frac{\partial^2 e^T}{\partial t_{h_1}\partial t_{h_2}}\frac{\partial^3\Phi_n(b_s;\mathbf{R})}{\partial t_{h_3}\partial t_{h_4}}$$
$$+\sum_{\substack{h_1,h_2,h_3,h_4,h_5\in\{i_1,i_2,i_3,i_4,i_5\}\\ h_5\neq h_4\neq h_3\neq h_2\neq h_1}}\frac{\partial^4 e^T}{\partial t_{h_1}\partial t_{h_2}\partial t_{h_3}\partial t_{h_4}}\frac{\partial\Phi_n(b_s;\mathbf{R})}{\partial t_{h_5}}.$$

Define sum indices $h_1,\ldots,h_p$ such that $\sum_{h_1,\ldots,h_p}\equiv\sum_{h_1,\ldots,h_p\in\{i_1,\ldots,i_p\},h_1\neq\cdots\neq h_p}$ and $i_1,\ldots,i_p\in\{1,\ldots,n\}$. Then a general expression for the first p-partial derivatives of the MGF will be:

$$\frac{\partial^p G(\mathbf{t},a_s)}{\partial t_{i_1}\cdots\partial t_{i_p}} = L^{-1}\left(e^T\frac{\partial^p\Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\cdots\partial t_{i_p}}+\sum_{h_1,h_2}\frac{\partial^2 e^T}{\partial t_{h_1}\partial t_{h_2}}\frac{\partial^{p-2}\Phi_n(b_s;\mathbf{R})}{\partial t_{h_3}\cdots\partial t_{h_p}}\right.$$
$$\left.+\sum_{h_1,h_2,h_3,h_4}\frac{\partial^4 e^T}{\partial t_{h_1}\partial t_{h_2}\partial t_{h_3}\partial t_{h_4}}\frac{\partial^{p-4}\Phi_n(b_s;\mathbf{R})}{\partial t_{h_5}\cdots\partial t_{h_p}}+\cdots+\frac{\partial^p e^T}{\partial t_{i_1}\cdots\partial t_{i_p}}\Phi_n(b_s;\mathbf{R})\right),$$

and evaluating at $\mathbf{t}=\mathbf{0}$, we have the moments:

$$m_p = \left.\frac{\partial^p G(\mathbf{t},b_s)}{\partial t_{i_1}\cdots\partial t_{i_p}}\right|_{\mathbf{t}=\mathbf{0}} = L^{-1}\left(\left.\frac{\partial^p\Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\cdots\partial t_{i_p}}\right|_{\mathbf{t}=\mathbf{0}}+\sum_{h_1,h_2}\rho_{h_1,h_2}\left.\frac{\partial^{p-2}\Phi_n(b_s;\mathbf{R})}{\partial t_{h_3}\cdots\partial t_{h_p}}\right|_{\mathbf{t}=\mathbf{0}}\right.$$
$$\left.+\sum_{h_1,h_2,h_3,h_4\in\{i_1,\ldots,i_p\}}\rho_{h_1,h_2}\rho_{h_3,h_4}\left.\frac{\partial^{p-4}\Phi_n(b_s;\mathbf{R})}{\partial t_{h_5}\cdots\partial t_{h_p}}\right|_{\mathbf{t}=\mathbf{0}}+\cdots+\sum_{h_1,\ldots,h_p}\rho_{h_1,h_2}\cdots\rho_{j_{h-1},h_p}\Phi_n(b_s;\mathbf{R})\right).$$

Then we notice, using Lemma 3.1,

$$\left.\frac{\partial^p\Phi_n(b_s;\mathbf{R})}{\partial t_{i_1}\cdots\partial t_{i_p}}\right|_{\mathbf{t}=\mathbf{0}} = \sum_{h_1=1}^{n}\rho_{i_1,h_1}\frac{\partial^{p-2}Q_{i_2}(b_{h_1})}{\partial t_{i_3}\cdots t_{i_p}},$$

where the partial derivative of $Q_{i_{i_4}}$ is derived using the formula:

$$\begin{aligned}\frac{\partial^p Q_l(a_{h_1}\cdots a_{h_q})}{\partial t_{i_1}\cdots\partial t_{i_p}} &= \frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(\frac{\partial U_l(a_{h_1},\ldots,a_{h_q})}{\partial t_{i_1}}F_{h_1,\ldots,h_q}(a_{h_1},\ldots,a_{h_q})\right)\\ &+\frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(U_l(a_{h_1},\ldots,a_{h_q})Q_{i_1}(a_{h_1},\ldots,a_{h_q})\right)\\ &+\sum_{h_{q+1}\neq h_1\cdots\neq h_q} c_{(h_1\cdots h_q)\cdot s}(k,h_{q+1})\frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(Q_{i_1}(a_{h_1},\ldots,a_{h_{q+1}})\right).\end{aligned}$$

This recursive formula provides an algorithm to calculate moments of arbitrary order. □

A.6. Proof of Proposition 4.1

Proof. First and second order moments are derived using the results of Proposition 3.2 replacing $\mathbf{R}$, $\mathbf{T}$, b_s, and a_s by $\mathbf{R}_j$, $\mathbf{T}_j$, $b_{j,s}$ and $\xi_{j,s}$ by $\xi_{j,s}=(a_s-\mu_{j,s})/\sigma_{j;s,s}$. We apply the same procedure for the proof of Proposition 3.2:

$$\begin{aligned}m_{1,\{i_1\}}(\mathbf{x},a_s,\boldsymbol{\omega}) &= \left.\frac{\partial G(\mathbf{t},a_s,\boldsymbol{\omega})}{\partial t_{i_1}}\right|_{\mathbf{t}=\mathbf{0}}\\ &= L^{-1}\sum_{j=1}^{k}\omega_j\left.\left(\frac{\partial \exp(T_j)}{\partial t_{i_1}}\Phi_n(b_{j,s};\mathbf{R}_j)+\exp(T_j)\frac{\partial\Phi_n(b_{j,s};\mathbf{R}_j)}{\partial t_{i_1}}\right)\right|_{\mathbf{t}=\mathbf{0}}\\ &= L^{-1}\sum_{j=1}^{k}\omega_j\left(\mu_{j,i_1}\Phi_n(\xi_{j,s};\mathbf{R}_j)+\sum_{h_1=1}^{n}\sigma_{j;h_1,i_1}\phi_1(\xi_{j,h_1})\Phi_{n-1}(\xi_{h_1\cdot s};\mathbf{C}_{h_1\cdot s})\right),\end{aligned}$$

$$\begin{aligned}m_{2,\{i_1,i_2\}}(\mathbf{x},a_s,\boldsymbol{\omega}) &= \left.\frac{\partial^2 G(\mathbf{t},a_s,\boldsymbol{\omega})}{\partial t_{i_1}\partial t_{i_2}}\right|_{\mathbf{t}=\mathbf{0}}\\ &= L^{-1}\sum_{j=1}^{k}\omega_j\left(\frac{\partial^2\exp(T_j)}{\partial t_{i_1}\partial t_{i_2}}\Phi_n(b_{j,s};\mathbf{R}_j)+\frac{\partial\exp(T_j)}{\partial t_{i_1}}\frac{\partial\Phi_n(b_{j,s};\mathbf{R}_j)}{\partial t_{i_2}}\right.\\ &\left.\left.+\frac{\partial\exp(T_j)}{\partial t_{i_2}}\frac{\partial\Phi_n(b_{j,s};\mathbf{R}_j)}{\partial t_{i_1}}+\exp(T_j)\frac{\partial^2\Phi_n(b_{j,s};\mathbf{R}_j)}{\partial t_{i_1}\partial t_{i_2}}\right)\right|_{\mathbf{t}=\mathbf{0}}\\ &= L^{-1}\sum_{j=1}^{k}\omega_j\left(\left(\mu_{j,i_1}\mu_{j,i_2}+\sigma_{i_1,i_2}\right)\Phi_n(\xi_{j,s};\mathbf{R}_j)\right.\\ &\left.+\sum_{h_1=1}^{n}\left(\mu_{j,i_1}\sigma_{j;h_1,i_2}+\mu_{j,i_2}\sigma_{j;h_1,i_1}\right)\phi_1(\xi_{j,h_1})\Phi_{n-1}(\xi_{h_1\cdot s};\mathbf{C}_{h_1\cdot s})+\sum_{h_1=1}^{n}\sigma_{j;h_1,i_1}Q_{i_2}(\xi_{j,h_1})\right).\end{aligned}$$

The third moments are derived using the results of Proposition 3.2. We have that:

$$\frac{\partial^3\exp(T_j)}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}}=\mu_{j,i_1}\mu_{j,i_2}\mu_{j,i_3}+\sigma_{i_2,i_3}\mu_{j,i_1}+\sigma_{i_1,i_3}\mu_{j,i_3}+\sigma_{i_1,i_2}\mu_{j,i_3}.$$

Applying a similar technique of MGF partial derivative calculation as in (52) and then using the proof of Proposition 3.2, the result follows. For moments of fourth order, we apply the procedure to the proof of Proposition 3.3. We have:

$$\begin{aligned}\frac{\partial^4\exp(T_j)}{\partial t_{i_1}\partial t_{i_2}\partial t_{i_3}\partial t_{i_4}} &= \mu_{j,i_1}\mu_{j,i_2}\mu_{j,i_3}\mu_{j,i_4}+\sigma_{j;i_1,i_2}\sigma_{j;i_3,i_4}+\sigma_{j;i_1,i_3}\sigma_{j;i_2,i_4}+\sigma_{j;i_1,i_4}\sigma_{j;i_2,i_3}\\ &+\sum_{\substack{k_1,k_2\in\{i_1,i_2,i_3,i_4\}\\ k_3,k_4\in\{i_1,i_2,i_3,i_4\}\setminus\{k_1,k_2\}\\ k_1\neq k_2\neq k_3\neq k_4}}\mu_{j,k_1}\mu_{j,k_2}\sigma_{k_3,k_4},\end{aligned}$$

using the appropriate changes in variables and the result is derived. □

A.7. Proof of Lemma 5.1

Proof. We apply the change of variable $W_i = \eta^{-1/2} Z_i$. Then the distribution of X conditional on η is:

$$f_{\eta^{-1/2}Z|\eta} = (2\pi)^{-n/2} |\mathbf{R}|^{-1/2} \exp\left(-\frac{1}{2\eta^{-1}} \mathbf{w}'\mathbf{R}^{-1}\mathbf{w}\right) \eta^{1/2}, \tag{63}$$

for $a_i \le W_i$, and 0 otherwise. But (63) is the pdf of $N(\mathbf{0}, \eta^{-1}\mathbf{R})$. We have that $f_{\eta^{-1/2}Z} = f_{\eta^{-1/2}Z|\eta} f_\eta$ with f_η equal to (27) with parameters $\alpha = \nu/2$, $\beta = 2/\nu$. Hence,

$$\begin{aligned} f_{\eta^{-1/2}Z} &= (2\pi)^{-n/2} |\mathbf{R}|^{-1/2} \exp\left(-\frac{1}{2\eta^{-1}} \mathbf{w}'\mathbf{R}^{-1}\mathbf{w}\right) \eta^{1/2} \frac{1}{(2/\nu)^{\nu/2} \Gamma(\nu/2)} \eta^{\nu/2-1} \exp\left(-\frac{\eta}{2/\nu}\right) \\ &= \frac{\Gamma((\nu+n)/2)}{(\Gamma(1/2)\nu)^{\nu/2} \Gamma(\nu/2) |\mathbf{R}|^{1/2}} \left(1 + \frac{1}{\nu} \mathbf{w}'\mathbf{R}^{-1}\mathbf{w}\right)^{-(\nu+n)/2}, \end{aligned}$$

for $a_i \le W_i$, which is the density function of a multivariate standard Student's t-distribution. Now we calculate the expectation on η using the definition:

$$\begin{aligned} E_\eta\left[\eta^{-i/2} \phi_n\left(\eta^{1/2}\mathbf{x}; \mathbf{0}, \mathbf{R}\right)\right] &= \int_{\eta=0}^{\infty} \eta^{-i/2} \left\{(2\pi)^{-n/2} |\mathbf{R}|^{-1/2} \exp\left(-\frac{\eta}{2} \mathbf{x}'\mathbf{R}^{-1}\mathbf{x}\right)\right\} \left\{\frac{1}{(2/\nu)^{\nu/2}} \frac{\eta^{\nu/2-1} \exp\left(-\frac{\nu}{2}\eta\right)}{\Gamma(\nu/2)}\right\} d\eta \\ &= \frac{\nu^{\nu/2}}{2^{\nu/2+n/2} \Gamma(1/2)^n \Gamma(\nu/2) |\mathbf{R}|^{1/2}} \int_{\eta=0}^{\infty} \eta^{-i/2+\nu/2-1} \exp\left(-\frac{1}{2} \mathbf{x}'\mathbf{R}^{-1}\mathbf{x}\right) d\eta. \end{aligned} \tag{64}$$

Then we apply the following change of variable $w = \frac{\eta}{2}\left(\mathbf{x}'\mathbf{R}^{-1}\mathbf{x} + \nu\right)$, and $d\eta = \frac{2}{\mathbf{x}'\mathbf{R}^{-1}\mathbf{x}+\nu} dw$ and (64) becomes:

$$\frac{\nu^{\nu/2}}{2^{\nu/2+n/2} \Gamma(1/2)^n \Gamma(\nu/2) |\mathbf{R}|^{1/2}} \int_{\eta=0}^{\infty} \left(\frac{2w}{\mathbf{x}'\mathbf{R}^{-1}\mathbf{x}+\nu}\right)^{-i/2+\nu/2-1} \exp(-w) \left(\frac{2}{\mathbf{x}'\mathbf{R}^{-1}\mathbf{x}+\nu}\right) dw. \tag{65}$$

Using the definition of the $\Gamma(\cdot)$ function in (65), the result follows. □

A.8. Proof of Proposition 5.2

Proof. Let Z have a lower truncated standard normal distribution with pdf (3), with truncation points $\eta^{-1/2} a_s \le Z_s$, $s = \{1, \ldots, n\}$, and η has a Gamma distribution with pdf as (27). Using Lemma 5.1, we can express the distribution of X as $f_X = f_{\eta^{-1/2}Z|\eta}$. Define the total probability $L = \Phi_n(\eta^{-1/2} a_s; \mathbf{R})$. We calculate the first moment:

$$\begin{aligned} m_{1,\{i_1\}}(\mathbf{x}; a_s) &= E[X_{i_1} | a_s \le X_s] \\ &= E_\eta[E[\eta^{-1/2} Z_{i_1} | \eta, a_s \le X_s]] \\ &= E_\eta[E[\eta^{-1/2} Z_{i_1} | \eta^{-1/2} a_s \le X_s]]. \end{aligned}$$

Using results of Section 3 for a MVSN distribution, we can calculate the inner expected value. Before we adjust the limits of integration from a_s to $\eta^{1/2} a_s$, therefore $\zeta_s = \eta^{1/2}\mathbf{R}\mathbf{t}$ and consequently $\frac{\partial^p e^T}{\partial t_{i_1} \cdots \partial t_{i_p}} = \eta^{p/2} e^T \frac{\partial^p T}{\partial t_{i_1} \cdots \partial t_{i_p}}$. Then the first moment is,

$$m_{1,\{i_1\}}(\mathbf{x}; a_s) = E_\eta\left[\eta^{-1/2}\left(L^{-1} \sum_{h_1=1}^{n} \rho_{i_1,h_1} \phi_1(\eta^{1/2} a_{h_1}; 1)\right)\right] \Phi_{n-1}(a_{h_1 \cdot s}; \mathbf{C}_{h_1 \cdot s}).$$

Then using Lemma 5.1,

$$\begin{aligned} m_{1,\{i_1\}}(\mathbf{x}; a_s) &= L^{-1} \sum_{h_1=1}^{n} \rho_{i_1,h_1} E_\eta\left[\eta^{-1/2} \phi_1(\eta^{1/2} a_{h_1}; 1)\right] \Phi_{n-1}(a_{h_1 \cdot s}; \mathbf{C}_{h_1 \cdot s}) \\ &= L^{-1} \sum_{h_1=1}^{n} \rho_{i_1,h_1} \gamma_{1,h_1} \Phi_{n-1}(a_{h_1 \cdot s}; \mathbf{C}_{h_1 \cdot s}), \end{aligned}$$

where

$$\gamma_{1,h_1} = \left(\frac{\Gamma((\nu-1)/2)\nu^{\nu/2}}{2^{(n+1)/2} \Gamma(\nu/2) \Gamma(1/2)^n} \left(a_{h_1}^2 + \nu\right)^{-(\nu-1)/2}\right).$$

The second moment is,

$$\begin{aligned}
m_{2,\{i_1,i_2\}}(\mathbf{x}; a_s) &= E[X_{i_1}X_{i_2}|a_s \le X_s]\\
&= E[X_{i_1}X_{i_2}|\eta, a_s \le X_s]\\
&= E_\eta[E[\eta^{-1}Z_{i_1}Z_{i_2}|\eta, a_s \le X_s]]\\
&= E_\eta\Bigg[\eta^{-1}L^{-1}\sum_{h_1=1}^{n}\rho_{i_1,h_1}\Bigg(\rho_{i_2,h_1}\eta^{1/2}a_{h_1}\phi_1(\eta^{1/2}a_{h_1};1)\Phi_{n-1}(\eta^{1/2}a_{h_1\cdot s};\mathbf{C}_{h_1\cdot s})\\
&\quad + \sum_{h_2\ne h_1}(\rho_{i_2,h_2}-\rho_{h_1,h_2}\rho_{i_2,h_1})\phi_2(\eta^{1/2}a_{h_1},\eta^{1/2}a_{h_2};\mathbf{R}_{h_1h_2})\Phi_{n-2}(\eta^{1/2}a_{h_1h_2\cdot s};\mathbf{C}_{h_1h_2\cdot s})\Bigg)+\rho_{i_1,i_2}\Bigg]\\
&= L^{-1}\Bigg(\sum_{h_1=1}^{n}\rho_{i_1,h_1}\Bigg(\rho_{i_2,h_1}a_{h_1}E_\eta\left[\eta^{-1/2}\phi_1(\eta^{-1/2}a_{h_1};1)\right]\Phi_{n-1}(a_{h_1\cdot s};\mathbf{C}_{h_1\cdot s})\\
&\quad + \sum_{h_2\ne h_1}(\rho_{i_2,h_2}-\rho_{h_1,h_2}\rho_{i_2,h_1})E_\eta\left[\eta^{-1}\phi_2(\eta^{-1/2}a_{h_1},\eta^{-1/2}a_{h_2};\mathbf{R}_{h_1h_2})\right]\Phi_{n-2}(a_{h_1h_2\cdot s};\mathbf{C}_{h_1h_2\cdot s})\Bigg)\Bigg)+\rho_{i_1,i_2}\\
&= L^{-1}\Bigg(\sum_{h_1=1}^{n}\rho_{i_1,h_1}\Bigg(\rho_{i_2,h_1}a_{h_1}\gamma_{1,h_1}\Phi_{n-1}(a_{h_1\cdot s};\mathbf{C}_{h_1\cdot s})\\
&\quad + \sum_{h_2\ne h_1}(\rho_{i_2,h_2}-\rho_{h_1,h_2}\rho_{i_2,h_1})\gamma_{2,h_1h_2}\Phi_{n-2}(a_{h_1h_2\cdot s};\mathbf{C}_{h_1h_2\cdot s})\Bigg)\Bigg)+\rho_{i_1,i_2},
\end{aligned}$$

where

$$\gamma_{2,h_1h_2} = \left(\frac{\Gamma((\nu-2)/2)\nu^{\nu/2}}{2^{(n+2)/2}\Gamma(\nu/2)\Gamma(1/2)^n|\mathbf{R}_{h_1h_2}|^{1/2}}\left(\mathbf{a}'_{h_1h_2}\mathbf{R}^{-1}_{h_1h_2}\mathbf{a}_{h_1h_2}+\nu\right)^{-(\nu-2)/2}\right).$$

Before calculating the third order moments, we calculate the value of $U_{i_3}(\eta^{1/2}a_{h_1}) = \eta^{1/2}U_{i_3}(a_{h_1})$ and $\frac{\partial U_{i_3}(\eta^{1/2}a_{h_1})}{\partial t_i} = \eta\frac{\partial U_{i_3}(a_{h_1})}{\partial t_i}$ then,

$$\begin{aligned}
m_{3,\{i_1,i_2,i_3\}}(\mathbf{x}; a_s) &= E[X_{i_1}X_{i_2}X_{i_3}|a_s \le X_s]\\
&= E[X_{i_1}X_{i_2}X_{i_3}|\eta, a_s \le X_s]\\
&= E_\eta[E[\eta^{-3/2}Z_{i_1}Z_{i_2}Z_{i_3}|\eta, a_s \le X_s]]\\
&= E_\eta\Bigg[L^{-1}\Bigg(\eta^{-1/2}\sum_{h_1=1}^{n}F_{h_1}(\eta^{1/2}a_{h_1})\left(\rho_{i_3,h_1}\rho_{i_1,i_2}+\rho_{i_2,h_1}\rho_{i_1,i_3}+\rho_{i_1,h_1}\rho_{i_2,i_3}\right)\\
&\quad -\eta^{-1/2}\sum_{h_1=1}^{n}\rho_{i_1,h_1}\rho_{i_2,h_1}\rho_{i_3,h_1}F_{h_1}(\eta^{1/2}a_{h_1})\\
&\quad +\sum_{h_1=1}^{n}\rho_{i_1,h_1}U_{i_2}(a_{h_1})\Bigg(U_{i_3}(a_{h_1})\eta^{-1/2}F_{h_1}(\eta^{1/2}a_{h_1})+\sum_{h_2\ne h_1}c_{h_1\cdot s}(i_3,h_2)\eta^{-1}F_{h_1,h_2}(\eta^{1/2}a_{h_1},\eta^{1/2}a_{h_2})\Bigg)\\
&\quad +\sum_{h_1=1}^{n}\rho_{i_1,h_1}\sum_{h_2\ne h_1}c_{h_1\cdot s}(i_2,h_2)\Bigg(U_{i_3}(a_{h_1},a_{h_2})\eta^{-1}F_{h_1,h_2}(\eta^{1/2}a_{h_1},\eta^{1/2}a_{h_2})\\
&\quad +\sum_{h_3\ne h_1\ne h_2}c_{h_1h_2\cdot s}(i_3,h_3)\eta^{-3/2}F_{h_1,h_2,h_3}(\eta^{1/2}a_{h_1},\eta^{1/2}a_{h_2},\eta^{1/2}a_{h_3})\Bigg)\Bigg)\Bigg]\\
&= L^{-1}\Bigg(\sum_{h_1=1}^{n}\gamma_{1,h_1}\Phi_{n-1}(a_{h_1\cdot s};\mathbf{C}_{h_1\cdot s})\left(\rho_{i_3,h_1}\rho_{i_1,i_2}+\rho_{i_2,h_1}\rho_{i_1,i_4}+\rho_{i_1,h_1}\rho_{i_2,i_3}\right)\\
&\quad -\sum_{h_1=1}^{n}\rho_{i_1,h_1}\rho_{i_2,h_1}\rho_{i_3,h_1}\gamma_{1,h_1}\Phi_{n-1}(a_{h_1\cdot s};\mathbf{C}_{h_1\cdot s})
\end{aligned}$$

$$+\sum_{h_1=1}^{n}\rho_{i_1,h_1}U_{i_2}(a_{h_1})\left(U_{i_3}(a_{h_1})\gamma_{1,h_1}\Phi_{n-1}(a_{h_1\cdot s};\mathbf{C}_{h_1\cdot s})+\sum_{h_2\neq h_1}c_{h_1\cdot s}(i_3,h_2)\gamma_{2,h_1h_2}\Phi_{n-2}(a_{h_1h_2\cdot s};\mathbf{C}_{h_1h_2\cdot s})\right)$$
$$+\sum_{h_1=1}^{n}\rho_{i_1,h_1}\sum_{h_2\neq h_1}c_{h_1\cdot s}(i_2,h_2)\left(U_{i_3}(a_{h_1},a_{h_2})\gamma_{2,h_1h_2}\Phi_{n-2}(a_{h_1h_2\cdot s};\mathbf{C}_{h_1h_2\cdot s})\right.$$
$$\left.\left.+\sum_{h_3\neq h_1\neq h_2}c_{h_1h_2\cdot s}(i_3,h_3)\gamma_{3,h_1h_2h_3}\Phi_{n-3}(a_{h_1h_2h_3\cdot s};\mathbf{C}_{h_1h_2h_3\cdot s})\right)\right),$$

where

$$\gamma_{3,h_1h_2h_3}=\left(\frac{\Gamma((\nu-3)/2)\nu^{\nu/2}}{2^{(n+3)/2}\Gamma(\nu/2)\Gamma(1/2)^n|\mathbf{R}_{h_1h_2h_3}|^{1/2}}\left(\mathbf{a}'_{h_1h_2h_3}\mathbf{R}^{-1}_{h_1h_2h_3}\mathbf{a}_{h_1h_2h_3}+\nu\right)^{-(\nu-3)/2}\right).$$

And the fourth moments are,

$$\begin{aligned}
m_{4,\{i_1,i_2,i_3,i_4\}}(\mathbf{x};a_s) &= E[X_{i_1}X_{i_2}X_{i_3}X_{i_4}|a_s\le X_s]\\
&= E[X_{i_1}X_{i_2}X_{i_3}X_{i_4}|\eta^{-1/2}a_s\le X_s]\\
&= E_\eta[E[\eta^{-2}Z_{i_1}Z_{i_2}Z_{i_3}Z_{i_4}|\eta^{-1/2}a_s\le X_s]]\\
&= E_\eta\Bigg[L^{-1}\Bigg\{\sum_{\substack{v_1,v_2,v_3,v_4\in\{i_1,i_2,i_3,i_4\}\\ v_1\neq v_2\neq v_3\neq v_4}}\rho_{v_1,v_2}\rho_{v_3,v_4}\Phi_n(\eta^{1/2}a_s;\mathbf{R})\\
&\quad+\sum_{h_1=1}^{n}\Bigg\{\rho_{i_3,i_4}\rho_{i_1,h_1}Q_{i_2,\eta}(a_{h_1})+\rho_{i_2,i_4}\rho_{i_1,h_1}Q_{i_3,\eta}(a_{h_1})+\rho_{i_2,i_3}\rho_{i_1,h_1}Q_{i_4,\eta}(a_{h_1})\\
&\quad+\rho_{i_1,i_2}\rho_{i_3,h_1}Q_{i_4,\eta}(a_{h_1})+\rho_{i_1,i_3}\rho_{i_2,h_1}Q_{i_4,\eta}(a_{h_1})+\rho_{i_1,i_4}\rho_{i_2,h_1}Q_{i_4,\eta}(a_{h_1})\\
&\quad+\rho_{i_1,h_1}\Bigg\{-\rho_{i_2,h_1}\rho_{i_3,h_1}Q_{i_4,\eta}(a_{h_1})-\rho_{i_2,h_1}\rho_{i_4,h_1}Q_{i_3,\eta}(a_{h_1})\\
&\quad+U_{i_2}(a_{h_1})\Bigg(-\rho_{i_3,h_1}\rho_{i_4,h_1}\eta^{-1/2}F_{h_1}(\eta^{1/2}a_{h_1})+U_{i_3}(a_{h_1})Q_{i_4,\eta}(a_{h_1})\\
&\quad+\sum_{h_2\neq h_1}c_{h_1\cdot s}(i_2,h_2)Q_{i_4,\eta}(a_{h_1},a_{h_2})\Bigg)\\
&\quad+\sum_{h_2\neq h_1}c_{h_1\cdot s}(i_2,h_2)\Bigg\{\left(\frac{\rho_{i_4,h_1}\left(\rho_{h_1,i_3}-\rho_{h_1,h_2}\rho_{h_2,i_3}\right)+\rho_{h_2,i_4}\left(\rho_{h_2,i_3}-\rho_{h_1,h_2}\rho_{h_1,i_3}\right)}{1-\rho^2_{h_1,h_2}}\right)\\
&\quad\times\eta^{-1}F_{h_1,h_2}(\eta^{1/2}a_{h_1},\eta^{1/2}a_{h_2})\\
&\quad+\;U_{i_3}(a_{h_1},a_{h_2})Q_{i_4,\eta}(a_{h_1},a_{h_2})+\sum_{h_3\neq h_1\neq h_2}c_{h_1h_2\cdot s}(i_3,h_3)Q_{i_4,\eta}(a_{h_1},a_{h_2},a_{h_3})\Bigg\}\Bigg\}\Bigg\}\Bigg\}\Bigg]\\
&= L^{-1}\Bigg\{\sum_{\substack{v_1,v_2,v_3,v_4\in\{i_1,i_2,i_3,i_4\}\\ v_1\neq v_2\neq v_3\neq v_4}}\rho_{v_1,v_2}\rho_{v_3,v_4}\Phi_n(a_s;\mathbf{R})\\
&\quad+\sum_{h_1=1}^{n}\rho_{i_3,i_4}\rho_{i_1,h_1}Q_{i_2,\gamma}(a_{h_1})+\rho_{i_2,i_4}\rho_{i_1,h_1}Q_{i_3,\gamma}(a_{h_1})\\
&\quad+\rho_{i_2,i_3}\rho_{i_1,h_1}Q_{i_4,\gamma}(a_{h_1})+\rho_{i_1,i_2}\rho_{i_3,h_1}Q_{i_4,\gamma}(a_{h_1})+\rho_{i_1,i_3}\rho_{i_2,h_1}Q_{i_4,\gamma}(a_{h_1})\\
&\quad+\rho_{i_1,i_4}\rho_{i_2,h_1}Q_{i_4,\gamma}(a_{h_1})+\sum_{h_1=1}^{n}\rho_{i_1,h_1}\Bigg\{-\rho_{i_2,h_1}\rho_{i_3,h_1}Q_{i_4,\gamma}(a_{h_1})-\rho_{i_2,h_1}\rho_{i_4,h_1}Q_{i_3,\gamma}(a_{h_1})
\end{aligned}$$

$$+ U_{i_2}(a_{h_1}) \Bigg(- \rho_{i_3,h_1}\rho_{i_4,h_1}\gamma_{1,h_1}\Phi_{n-1}(a_{h_1\cdot s}; \mathbf{C}_{h_1\cdot s}) + U_{i_3}(a_{h_1})Q_{i_4,\gamma}(a_{h_1})$$

$$+ \sum_{h_2\neq h_1} c_{h_1\cdot s}(i_2,h_2)Q_{i_4,\gamma}(a_{h_1},a_{h_2})\Bigg)$$

$$+ \sum_{h_2\neq h_1} c_{h_1\cdot s}(i_2,h_2)\Bigg\{\left(\frac{\rho_{i_4,h_1}\left(\rho_{h_1,i_3}-\rho_{h_1,h_2}\rho_{h_2,i_3}\right)+\rho_{h_2,i_4}\left(\rho_{h_2,i_3}-\rho_{h_1,h_2}\rho_{h_1,i_3}\right)}{1-\rho^2_{h_1,h_2}}\right)$$

$$\times \gamma_{2,h_1h_2}\Phi_{n-2}(a_{h_1h_2\cdot s}; \mathbf{C}_{h_1h_2\cdot s}) + U_{i_3}(a_{h_1},a_{h_2})Q_{i_4,\gamma}(a_{h_1},a_{h_2})$$

$$+ \sum_{h_3\neq h_1\neq h_2} c_{h_1h_2\cdot s}(i_3,h_3)Q_{i_4,\gamma}(a_{h_1},a_{h_2},a_{h_3})\Bigg\}\Bigg\}\Bigg\},$$

where

$$Q_{i_4,\eta}(a_{h_1},\ldots,a_{h_q}) = U_{i_4}(a_{h_1},\ldots,a_{h_q})\eta^{-q/2}\phi_q(a_{h_1},\ldots,a_{h_q};\mathbf{R}_{h_1,\ldots,h_q})\Phi_{n-q}\left(a_{(h_1\cdots h_q)\cdot s};\mathbf{R}_{(h_1\cdots h_q)\cdot s}\right)$$
$$+ \sum_{h_{q+1}\neq h_1\cdots\neq h_q} c_{(h_1\cdots h_q)\cdot s}(i_3,h_{q+1})\eta^{-(q+1)/2}\phi_{q+1}(a_{h_1},\ldots,a_{h_{q+1}};\mathbf{R}_{h_1,\ldots,h_{q+1}})\Phi_{n-q-1}\left(a_{(h_1\cdots h_{q+1})\cdot s};\mathbf{C}_{(h_1\cdots h_{q+1})\cdot s}\right),$$

$$Q_{i_4,\gamma}(a_{h_1},\ldots,a_{h_q}) = U_{i_4}(a_{h_1},\ldots,a_{h_q})\gamma_{q,h_1,\ldots,h_q}\Phi_{n-q}\left(a_{(h_1\cdots h_q)\cdot s};\mathbf{R}_{(h_1\cdots h_q)\cdot s}\right)$$
$$+ \sum_{h_{q+1}\neq h_1\cdots\neq h_q} c_{(h_1\cdots h_q)\cdot s}(i_3,h_{q+1})\gamma_{q+1,h_1,\ldots,h_{q+1}}\Phi_{n-q-1}\left(a_{(h_1\cdots h_{q+1})\cdot s};\mathbf{C}_{(h_1\cdots h_{q+1})\cdot s}\right),$$

$$\gamma_{q,h_1\cdots h_q} = \left(\frac{\Gamma((\nu-q)/2)\nu^{\nu/2}}{2^{(n+q)/2}\Gamma(\nu/2)\Gamma(1/2)^n|\mathbf{R}_{h_1\cdots h_q}|^{1/2}}\left(\mathbf{a}'_{h_1\cdots h_q}\mathbf{R}^{-1}_{h_1\cdots h_q}\mathbf{a}_{h_1\cdots h_q}+\nu\right)^{-(\nu-q)/2}\right). \quad \square$$

A.9. Proof of Corollary 5.3

Let Z have a lower truncated standard normal distribution with pdf (3), truncation points $\eta^{-1/2}a_s \le Z_s$, $s = \{1,\ldots,n\}$, and η having a Gamma distribution with pdf as (27). Using Lemma 5.1, we can express the distribution of X as $f_X = f_{\eta^{-1/2}Z|\eta}$. Define the total probability $L = \Phi_n(\eta^{-1/2}a_s;\mathbf{R})$. Then combining the result on p-th order moments of a lower truncated MVSN distribution and Lemma 5.1, the p-th moments of X are,

$$m_{p,I}(\mathbf{x},a_s) = E[X_{i_1}\cdots X_{i_p}|a_s\le X_s]$$

$$= E[X_{i_1}\cdots X_{i_p}|\eta^{-1/2}a_s\le X_s]$$

$$= E_\eta[E[\eta^{-p/2}Z_{i_1}\cdots Z_{i_p}|\eta^{-1/2}a_s\le X_s]]$$

$$= E_\eta\Bigg[\eta^{-p/2}L^{-1}\Bigg(\frac{\partial^p\Phi_n(b_{s,\eta};\mathbf{R})}{\partial t_{i_1}\cdots\partial t_{i_p}}\Bigg|_{\mathbf{t}=\mathbf{0}} + \sum_{j_1,j_2\in\{i_1,\ldots,i_p\}}\rho_{j_1,j_2}\eta^1\frac{\partial^{p-2}\Phi_n(b_{s,\eta};\mathbf{R})}{\partial t_{k_1}\cdots\partial t_{k_{p-2}}}\Bigg|_{\mathbf{t}=\mathbf{0}}$$

$$+ \sum_{j_1,j_2,j_3,j_4\in\{i_1,\ldots,i_p\}}\rho_{j_1,j_2}\rho_{j_3,j_4}\eta^2\frac{\partial^{p-4}\Phi_n(b_{s,\eta};\mathbf{R})}{\partial t_{k_1}\cdots\partial t_{k_{p-4}}}\Bigg|_{\mathbf{t}=\mathbf{0}}$$

$$+ \cdots + \rho_{j_1,j_2}\cdots\rho_{j_{p-1},j_p}\eta^{p/2}\Phi_n(\eta^{-1/2}b_{s,\eta};\mathbf{R})\Bigg)\Bigg]$$

$$= L^{-1}\Bigg(\sum_{j_1=1}^n\rho_{i_1,j_1}\frac{\partial^{p-2}Q_{i_2,\gamma}(a_h)}{\partial t_{i_3}\cdots t_{i_p}} + \sum_{j_1,j_2\in\{i_1,\ldots,i_p\}}\rho_{j_1,j_2}\sum_{l_1=1}^n\rho_{k_1,l_1}\frac{\partial^{p-4}Q_{k_2,\gamma}(a_h)}{\partial t_{k_3}\cdots t_{k_{p-2}}}$$

$$+ \sum_{j_1,j_2,j_3,j_4\in\{i_1,\ldots,i_p\}}\rho_{j_1,j_2}\rho_{j_3,j_4}\sum_{l_1=1}^n\rho_{k_1,l_1}\frac{\partial^{p-6}Q_{k_2,\gamma}(a_h)}{\partial t_{k_3}\cdots t_{k_{p-4}}} + \cdots + \rho_{j_1,j_2}\cdots\rho_{j_{p-1},j_p}\Phi_n(a_s;\mathbf{R})\Bigg),$$

for odd p and

$$= L^{-1}\left(\sum_{j_1=1}^{n}\rho_{i_1,j_1}\frac{\partial^{p-2}Q_{i_2,\gamma}(a_h)}{\partial t_{i_3}\cdots t_{i_p}} + \sum_{j_1,j_2\in\{i_1,\ldots,i_p\}}\rho_{j_1,j_2}\sum_{l_1=1}^{n}\rho_{k_1,l_1}\frac{\partial^{p-4}Q_{k_2,\gamma}(a_h)}{\partial t_{k_3}\cdots t_{k_{p-2}}} + \sum_{j_1,j_2,j_3,j_4\in\{i_1,\ldots,i_p\}}\rho_{j_1,j_2}\rho_{j_3,j_4}\sum_{l_1=1}^{n}\rho_{k_1,l_1}\frac{\partial^{p-6}Q_{k_2,\gamma}(a_h)}{\partial t_{k_3}\cdots t_{k_{p-4}}} + \cdots + \sum_{j_1,\ldots,j_p}\rho_{j_1,j_2}\cdots\rho_{j_{p-1},j_p}\sum_{l_1=1}^{n}\gamma_{1,l_1}\Phi_{n-1}(\eta^{-1/2}a_s;\mathbf{R})\right),$$

for even p where

$$\begin{aligned}\frac{\partial^p Q_{l,\eta}(a_{h_1}\cdots a_{h_q})}{\partial t_{i_1}\cdots\partial t_{i_p}} &= \frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(\frac{\partial U_l(a_{h_1},\ldots,a_{h_q})}{\partial t_{i_1}}\eta^{-q/2}\phi_q(a_{h_1},\ldots,a_{h_q};\mathbf{R}_{h_1\cdots h_{q+1}})\right.\\ &\quad\left.\times\ \Phi_{n-q}\left(a_{(h_1\cdots h_q)\cdot s};\mathbf{R}_{(h_1\cdots h_q)\cdot s}\right)\right) + \frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(U_l(a_{h_1},\ldots,a_{h_q})Q_{i_1,\gamma}(a_{h_1},\ldots,a_{h_q})\right)\\ &\quad+\sum_{h_{q+1}\neq h_1\cdots\neq h_q}c_{(h_1\cdots h_q)\cdot s}(k,h_{q+1})\frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(Q_{i_1,\gamma}(a_{h_1},\ldots,a_{h_{q+1}})\right),\\ \frac{\partial^p Q_{l,\gamma}(a_{h_1}\cdots a_{h_q})}{\partial t_{i_1}\cdots\partial t_{i_p}} &= \frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(\frac{\partial U_l(a_{h_1},\ldots,a_{h_q})}{\partial t_{i_1}}\gamma_{q,h_1,\ldots,h_q}(a_{h_1},\ldots,a_{h_q};\mathbf{R}_{h_1\cdots h_{q+1}})\right.\\ &\quad\left.\times\ \Phi_{n-q}\left(a_{(h_1\cdots h_q)\cdot s};\mathbf{R}_{(h_1\cdots h_q)\cdot s}\right)\right) + \frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(U_l(a_{h_1},\ldots,a_{h_q})Q_{i_1,\gamma}(a_{h_1},\ldots,a_{h_q})\right)\\ &\quad+\sum_{h_{q+1}\neq h_1\cdots\neq h_q}c_{(h_1\cdots h_q)\cdot s}(k,h_{q+1})\frac{\partial^{p-1}}{\partial t_{i_2}\cdots\partial t_{i_p}}\left(Q_{i_1,\gamma}(a_{h_1},\ldots,a_{h_{q+1}})\right).\end{aligned}$$

A.10. Proof of Proposition 6.1

To simplify the notation, we define $\log(\mathbf{x})$ as the function that returns the vector $(\log(x_1),\ldots,\log(x_n))$. The pdf of X is defined as:

$$f(x_1,\ldots,x_n;\mathbf{R}) = f(x_s;\mathbf{R}) = (2\pi)^{-n/2}|\mathbf{R}|^{-1/2}\left(\prod_{i=1}^{n}x_i^{-1}\right)\exp\left(-\frac{1}{2}\log(\mathbf{x})'\mathbf{R}^{-1}\log(\mathbf{x})\right), \tag{66}$$

where $x_s > 0$. The joint cdf of X is defined as:

$$F(x_1,\ldots,x_n;\mathbf{R}) = F(x_s;\mathbf{R}) = \int_{a_s}^{\infty(n)} f(x_s;\mathbf{R})dx_s \tag{67}$$

$$= \Phi_n(\log(\mathbf{x});\mathbf{R}), \tag{68}$$

where $\Phi_n(\mathbf{x};\mathbf{R})$ is the cdf of the MVSN defined in (6).

Denote the total probability by $L = F(\mathbf{0};\mathbf{R})$, and $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$. Using (31), the distribution of the incomplete cross moments of X is:

$$g_{X,\boldsymbol{\alpha}} = \frac{x_1^{\alpha_1}\cdots x_n^{\alpha_n}f(x_s;\mathbf{R})}{{}^{(n)}\int_0^{\infty}f(x_s;\mathbf{R})dx_s} = \frac{x_1^{\alpha_1}\cdots x_n^{\alpha_n}f(x_s;\mathbf{R})}{F(\mathbf{0};\mathbf{R})} = L^{-1}\left(x_1^{\alpha_1}\cdots x_n^{\alpha_n}f(x_s;\mathbf{R})\right). \tag{69}$$

If we calculate the joint cdf of (69), we will have the moments of order p of X:

$$\begin{aligned}G_{X,\boldsymbol{\alpha}}(a_s) &= m_{p,\boldsymbol{\alpha}}(\mathbf{x},a_s) = L^{-1}\left(\int_{a_s}^{\infty(n)} x_1^{\alpha_1}\cdots x_n^{\alpha_n}f(x_s;\mathbf{R})dx_s\right)\\ &= L^{-1}\left(\int_{a_s}^{\infty(n)}(2\pi)^{-n/2}|\mathbf{R}|^{-1/2}\left(\prod_{i=1}^{n}x_i^{\alpha_i-1}\right)\exp\left(-\frac{1}{2}\log(\mathbf{x})'\mathbf{R}^{-1}\log(\mathbf{x})\right)dx_s\right).\end{aligned}$$

If we apply the change of variable $\mathbf{x} = \exp(\mathbf{t})$, then we have,

$$m_{p,\boldsymbol{\alpha}}(\mathbf{x},a_s) = L^{-1}\left(\int_{\log(a_s)}^{\infty(n)}(2\pi)^{-n/2}|\mathbf{R}|^{-1/2}\exp\left(\mathbf{t}'(\boldsymbol{\alpha}-\mathbf{1})\right)\exp\left(-\frac{1}{2}\mathbf{t}'\mathbf{R}^{-1}\mathbf{t}\right)\exp\left(\mathbf{t}'\mathbf{1}\right)dt_s\right). \tag{70}$$

Table E.1
Values of the integral approximation of m_{30} and the value given by the exact formula.

Precision	Integral approximation of m_{30}	Time (s)
1e−4	1.230270608467717	0.23
1e−5	1.229793521590841	0.27
1e−6	1.229761999969332	0.67
1e−7	1.229791237708743	1.48
1e−8	1.229791596774500	3.20
1e−9	1.229791639693225	8.25
1e−10	1.229791641608072	21.14
1e−11	1.229791640698883	47.73
1e−12	1.229791640531571	127.25
1e−13	1.229791640511376	342.26
1e−14	1.229791640510763	762.14
Value of $m_{3,\{1,1,1\}}(\mathbf{x}, a_s)$	1.229791640510507	0.35

The last expression can be transformed as,

$$m_{p,\boldsymbol{\alpha}}(\mathbf{x}, a_s) = L^{-1}\left(\int_{\log(a_s)}^{\infty (n)} (2\pi)^{-n/2}|\mathbf{R}|^{-1/2} \exp\left(\frac{1}{2}\boldsymbol{\alpha}'\mathbf{R}\boldsymbol{\alpha} - \frac{1}{2}(\mathbf{t}-\boldsymbol{\varsigma})'\mathbf{R}^{-1}(\mathbf{t}-\boldsymbol{\varsigma})\right) dt_s\right),$$

for $s = \{1, \dots, n\}$ with $\boldsymbol{\varsigma} = \mathbf{R}\boldsymbol{\alpha}$. Define $\mathbf{z} = \mathbf{t} - \varsigma$ and the last expression becomes,

$$\begin{aligned} m_{p,\boldsymbol{\alpha}}(\mathbf{x}, a_s) &= L^{-1}\left(\int_{\log(a_s)-\sum_i \rho_{s,i}\alpha_i}^{\infty (n)} (2\pi)^{-n/2}|\mathbf{R}|^{-1/2} \exp\left(\frac{1}{2}\boldsymbol{\alpha}'\mathbf{R}\boldsymbol{\alpha}\right) \exp\left(-\frac{1}{2}\mathbf{z}'\mathbf{R}^{-1}\mathbf{z}\right) dz_s\right) \\ &= L^{-1} \exp\left(\frac{1}{2}\boldsymbol{\alpha}'\mathbf{R}\boldsymbol{\alpha}\right) \Phi_n(b_s; \mathbf{R}), \end{aligned} \tag{71}$$

where $b_s = \log(a_s) - \sum_i \rho_{s,i}\alpha_i$.

Consider now the non-standard case. Denote the covariance matrix of the distribution as $\mathbf{V}$. Using the decomposition $\mathbf{V} = \mathbf{D}'\mathbf{R}\mathbf{D}$, as in the previous section we can calculate the correlation matrix $\mathbf{R}$ as $\mathbf{R} = \mathbf{D}^{-1}\mathbf{V}\mathbf{D}^{-1}$. Then, using the same arguments as for standard case, it can be demonstrated for the non-standard case that:

$$m_{p,\boldsymbol{\alpha}}(\mathbf{x}, a_s) = L^{-1} \exp\left(\frac{1}{2}\boldsymbol{\alpha}'\mathbf{V}\boldsymbol{\alpha} + \boldsymbol{\mu}'\boldsymbol{\alpha}\right) \Phi_n(b_s; \mathbf{R}),$$

where $b_s = (\log(a_s) - \mu_i - \sum_i \sigma_{s,i}\alpha_i)/\sigma_i$.

A.11. *Exceedance moments of BVN for first to sixth order*

See Fig. A.1.

Appendix B. Third and fourth order exceedance moments BVN

See Figs. B.1–B.14.

Appendix C. Third and fourth order exceedance moments normal, Student's *t* and lognormal

See Figs. C.1–C.8.

Appendix D. Mardia's and Kollo's exceedance skewness and kurtosis

See Figs. D.1–D.4.

Appendix E. Tables

See Table E.1.

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