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# Renormalisation flow and geodesics on the moduli space of four dimensional $N = 2$ supersymmetric Yang–Mills theory

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## Abstract

It is shown that the  $\beta$ -functions for four dimensional  $N = 2$  supersymmetric Yang–Mills theory without matter give integral curves on the moduli space some of which are geodesics of the natural metric on the moduli space. In particular the flow lines which cross-over from the weak coupling limit (asymptotically free theory) to the singular points, representing the strong coupling limit, are geodesics. A possible connection with irreversibility is discussed. © 1998 Elsevier Science B.V.

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The purpose of this letter is to investigate the geometrical and topological structure of the renormalisation flow in an exactly solved four dimensional field theory  $N = 2$  supersymmetric  $SU(2)$  Yang–Mills theory without matter [1]. The  $\beta$ -function for the complex coupling,  $\tau = \frac{\theta_{\text{eff}}}{2\pi} + \frac{4\pi i}{g_{\text{eff}}^2}$ , of this model was calculated in [2] and the resulting flow diagram is shown in Fig. 1. This diagram has special points: the weak coupling limit  $\tau = i\infty$ , the strong coupling limit at  $\tau = 0, \pm 1$  where there are singularities due to extra massless degrees of freedom associated with composite objects (monopoles and dyons), and the points at  $\tau = (\pm 1 + i)/2$  where the Higgs field  $\phi$  satisfies  $\text{Tr}\langle\phi^2\rangle = 0$ . The points at

$\tau = \pm 1$  and  $\tau = 0$  are physically equivalent under the symmetry  $\tau \rightarrow \tau + 1$ , as are the two points  $\tau = (\pm 1 + i)/2$ . At these last two points  $\langle\phi\rangle \neq 0$  even though  $\text{Tr}\langle\phi^2\rangle = 0$ , so that  $SU(2)$  symmetry is not restored in the quantum regime, but these points are the quantum mechanical vestige of the place where the symmetry would be restored in the classical theory. It will be shown that the flow lines which cross over between all these special points are geodesics of the Seiberg–Witten metric, but none of the other flow lines is a geodesic. These are geodesics with a “frictional” force and the geodesic nature of the cross-over appears to be related to irreversibility of the renormalisation flow.

$N = 2$  supersymmetric Yang–Mills theory has features in common with both the standard model of electro-weak theory (symmetry breaking and mass generation by the Higgs mechanism) and with quan-

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tum chromo-dynamics (asymptotic freedom and confinement) [1]. There are also many differences, of course, but as a model for testing physical ideas in four dimensions Seiberg and Witten's low energy effective action gives us an unprecedented arena for experimentation. In common with QCD,  $N=2$  supersymmetric Yang–Mills theory exhibits dimensional transmutation in that there exists a natural length scale,  $\Lambda$ , at which the interactions become strong in the sense that the dimensionless gauge coupling,  $g_{\text{eff}}(\Lambda)$ , becomes large. It is convenient, as above, to combine  $g_{\text{eff}}(\Lambda)$  with an effective topological parameter,  $\theta_{\text{eff}}(\Lambda)$ , into the complex quantity

$$\tau = \frac{\theta_{\text{eff}}}{2\pi} + \frac{4\pi i}{g_{\text{eff}}^2}.$$

The value of  $\tau$  depends on Higgs expectation value,

$$\langle \phi \rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix},$$

which is not fixed in this theory but can take on a range of values corresponding to different vacua. Using the gauge invariant quantity  $u = \text{Tr} \langle \phi^2 \rangle$  to parameterise the symmetry breaking, Seiberg and Witten determined the zero momentum functional form of the dimensionless quantity  $\tau$  as a function of  $u/\Lambda^2 \equiv \tilde{u}$ ,

$$\tau = \frac{iK'}{K} \quad (1)$$

where  $K'(k^2) = K(k'^2)$  and  $K(k^2)$  are standard elliptic integrals [3], with  $k'^2 = 1 - k^2$  and  $k^2 = \frac{2}{\tilde{u}+1}$  (the explicit form (1) can be found in the review [4]). It is then straightforward to obtain the  $\beta$ -functions of the theory, including non-perturbative effects [2],

$$\begin{aligned} \beta(\tau) &:= \Lambda \frac{\partial \tau}{\partial \Lambda} \Big|_u = -2u \frac{\partial \tau}{\partial u} \Big|_\Lambda \\ &= \frac{1}{i\pi} \left( \frac{\vartheta_3^4 + \vartheta_4^4}{\vartheta_3^4 \vartheta_4^4} \right). \end{aligned} \quad (2)$$

The  $\vartheta$ -functions used here are, in the conventions of Whittaker and Watson [3],

$$\vartheta_2 = 2q^{1/4} \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n})^2 \quad (3)$$

$$\vartheta_3 = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1})^2$$

$$\vartheta_4 = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1})^2$$

with  $q = e^{\pi i \tau}$ .

These  $\beta$ -functions represent a vector flow,  $\boldsymbol{\beta} = \beta \frac{\partial}{\partial \tau} + \bar{\beta} \frac{\partial}{\partial \bar{\tau}}$ , on a one (complex) dimensional manifold, parameterised by  $\tau$ , which has the topology of a sphere with three holes [1]. The flow is most easily visualised by noting that it is radial in the  $\tilde{u}$ -plane (by definition) and then transforming to the  $\tau$ -plane. The  $\beta$ -function is singular at  $\tau = \pm 1, 0$  and vanishes at  $\tau = \frac{1}{2}(\pm 1 + i)$ . This last point corresponds to  $u = 0$ , which is the point at which one would have  $a = 0$  classically, restoring the gauge symmetry to the full  $SU(2)$ , but quantum mechanically  $|a|$  has a lower bound ( $\approx 0.7628$ ) so  $a = 0$  can never be reached in the quantum regime. Nevertheless the point  $u = 0$  corresponds to the minimum value of  $|a|$  in the quantum theory<sup>2</sup>.

Note that this vector flow represents a real physical flow that follows from changing  $\Lambda$  at fixed  $u$  (or  $u$  at fixed  $\Lambda$ ). It is thus similar in spirit to the Callan–Symanzik flow of Q.E.D., which one obtains by changing the physical electron mass. Here physical masses are also being varied, since  $u$  (and thus  $a$ ) is changing at fixed  $\Lambda$ .

The question posed here is, how does the above flow relate to the geometry of the punctured sphere? Seiberg and Witten also described a metric on the punctured sphere which is compatible with the  $\Gamma(2)$  action on the upper half  $\tau$ -plane. In terms of the Higgs expectation value,  $a$ ,

$$ds_{\text{SW}}^2 = (\text{Im}\tau) da d\bar{a}. \quad (4)$$

Using  $\tau$  co-ordinates this is

$$ds_{\text{SW}}^2 = \pi^2 (\text{Im}\tau) \left| \frac{\vartheta_3^4 \vartheta_4^4}{\vartheta_2^2} \right|^2 d\tau d\bar{\tau}, \quad (5)$$

or in  $u$ -co-ordinates

$$ds_{\text{SW}}^2 = \frac{1}{\pi^2} \frac{(K'\bar{K} + \bar{K}'K)}{\sqrt{1+u}\sqrt{1+\bar{u}}} du d\bar{u}. \quad (6)$$

<sup>2</sup> I am grateful to Denjoe O'Connor for useful discussions on this point.

The relationship between the Seiberg-Witten metric above and the Poincaré metric on the upper half-plane,

$$ds_p^2 = \frac{1}{(\text{Im}\tau)^2} d\tau d\bar{\tau}, \tag{7}$$

was discussed in [5].

Physically the Seiberg–Witten metric seems to be the more relevant – it gives rise to a positive Ricci curvature which diverges at  $\tau = \pm 1$  and 0, reflecting the singularities at these points,

$$R = \frac{1}{\pi^2 (\text{Im}\tau)^3} \left| \frac{\partial_2^2}{\partial_3^4 \partial_4^4} \right|^2, \tag{8}$$

and  $R$  tends to zero in the weak coupling limit,  $\tau \rightarrow i\infty$ . The boundaries of the fundamental region (Fig. 1) are clearly geodesics of the Poincaré metric and it is natural to ask if this property also holds for the Seiberg–Witten metric. A detailed calculation shows that it does! Indeed the only flow lines of the figure that are geodesics of the Seiberg–Witten metric are the bold lines which cross-over between the special points  $\tau = \pm 1, \tau = 0, \tau = i\infty$  and  $\tau = \frac{1}{2}(\pm 1 + i)$ .

To prove that the lines  $\tau = is, \tau = 1 + is, \tau = \frac{1}{2} + is$  (where  $s$  is a real parameter) are geodesics, consider the geodesic equation for a vector field  $\xi = \xi^\tau \frac{\partial}{\partial \tau} + \xi^{\bar{\tau}} \frac{\partial}{\partial \bar{\tau}}$ ,

$$\nabla_\xi \xi = \lambda \xi. \tag{9}$$

Explicitly, for the Levi–Civita connection associated with a Kähler metric,

$$\begin{aligned} \xi^\tau \frac{\partial \xi^\tau}{\partial \tau} + \xi^{\bar{\tau}} \frac{\partial \xi^\tau}{\partial \bar{\tau}} + \Gamma_{\tau\tau}^\tau \xi^\tau \xi^\tau &= \lambda \xi^\tau \\ \xi^\tau \frac{\partial \xi^{\bar{\tau}}}{\partial \tau} + \xi^{\bar{\tau}} \frac{\partial \xi^{\bar{\tau}}}{\partial \bar{\tau}} + \Gamma_{\bar{\tau}\bar{\tau}}^{\bar{\tau}} \xi^{\bar{\tau}} \xi^{\bar{\tau}} &= \lambda \xi^{\bar{\tau}}. \end{aligned} \tag{10}$$

For a holomorphic vector field,  $\xi^\tau = \xi(\tau)$  and  $\xi^{\bar{\tau}} = \bar{\xi}(\bar{\tau})$ , with  $\bar{\xi}$  the complex conjugate of  $\xi$ , these equations reduce to

$$\begin{aligned} \xi \frac{\partial \xi}{\partial \tau} + T_{\tau\tau}^\tau \xi^2 &= \lambda \xi \\ \bar{\xi} \frac{\partial \bar{\xi}}{\partial \bar{\tau}} + T_{\bar{\tau}\bar{\tau}}^{\bar{\tau}} \bar{\xi}^2 &= \lambda \bar{\xi}, \end{aligned} \tag{11}$$

with the condition that  $\lambda = \frac{\partial \xi}{\partial \tau} + T_{\tau\tau}^\tau \xi = \frac{\partial \bar{\xi}}{\partial \bar{\tau}} + T_{\bar{\tau}\bar{\tau}}^{\bar{\tau}} \bar{\xi}$  be real. For the  $\beta$ -function above and the connection for the Seiberg–Witten metric this is easily checked using the representation (3) for the  $\vartheta$ -functions. Since  $q = e^{-\pi s}$  along  $\tau = is, q = -e^{-\pi s}$  along  $\tau = 1 + is$  and  $q = ie^{-\pi s/2}$  along  $\tau = \frac{1}{2} + is$ , the reality conditions on  $\lambda$  are readily verified using (2) and (5). For the circular arch  $\tau = \frac{1}{2}(1 + e^{i\phi})$  corresponding to  $-1 < u < 1$ , a proof can be constructed using the  $u$ -coordinate, in which one only need prove that

$$\Gamma_{uu}^u = \frac{\{\bar{K}'(\partial_u K) + \bar{K}(\partial_u K')\}}{K'\bar{K} + \bar{K}'K} - \frac{1}{2(u+1)} \tag{12}$$

is real along the real line segment  $-1 < u < 1$ . This follows by noting that  $k^2 = \frac{u-1}{u+1}$  is real and negative for  $-1 < u < 1$ , thus  $K'$  is real on this segment,  $K$  however is not real, but one can show using (see Ref. [3])

$$\frac{dK}{d(k^2)} = \frac{E - k^2 K}{2k^2 k'^2}, \quad \frac{dK'}{d(k^2)} = \frac{k^2 K' - E'}{2k^2 k'^2},$$

$$K'E + E'K - K'K = \pi/2, \tag{13}$$

and the fact that  $K'$  is real, that  $T_{uu}^u$  is in fact real along  $-1 < u < 1$ , and thus this line segment is a geodesic. Hence the bold lines in Fig. 1 are geodesics in the Seiberg-Witten metric. That none of the other flows is geodesic follows from the fact that they are all repulsed from the singular points  $\tau = \pm 1, 0$ . Since these points have infinite positive curvature any

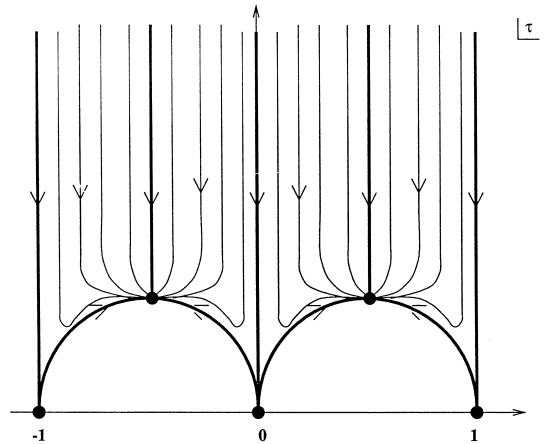


Fig. 1. Renormalisation flow for  $SU(2) N=2$  supersymmetric Yang–Mills. The thick lines bounding the fundamental domain are geodesic.

geodesic would necessarily be attracted to and focused into these points, not repulsed by them.

One can actually prove a slightly stronger result. The two form

$$d\beta = \partial_{[a}(G_{b]c} \beta^c) dx^a \wedge dx^b \quad (14)$$

does not vanish in general (the flow is not gradient), but it does vanish along the thick lines of the figure. On a Kähler manifold this is equivalent to saying that  $\partial_{\bar{\tau}}(G_{\tau\bar{\tau}} \beta^{\tau})$  is real on these lines. Combined with the observation that the  $\beta$ -functions are conformally Killing for the Seiberg–Witten metric it can be shown that this in fact implies that these lines are geodesics.

It seems possible that these observations may be related to irreversibility of the flow. Gradient flow, with a positive definite metric [6], automatically implies a c-theorem [7]. If the flow is gradient then there exists a differentiable function,  $U(g)$ , such that  $\beta^a = G^{ab} \partial_b U(g)$  and so

$$\frac{dU(g)}{d \ln \Lambda} = \beta^a \partial_a U(g) = \beta^a \beta^b G_{ab} \geq 0 \quad (15)$$

for a positive definite metric. The function  $U(g)$  is therefore monotonic along the flow trajectories. The flow here is not gradient with the Seiberg–Witten metric in general, but the two form  $d\beta$  does vanish along the lines of cross-over, which then implies that these lines are geodesics. Geodesic cross-over, when the metric is obtained as the second derivative of the effective potential, has been noted in some other models (specifically Gaussian models in  $2 \leq d \leq 4$  and the  $1-d$  Ising model [8], and the  $O(N)$  model in  $3-d$  for large  $N$  [9]) where it can be related to a maximisation of the relative entropy. Thus it seems plausible that geodesic flow is related to an increase in relative entropy along the lines of crossover. Note that geodesic flow does not imply reversibility – the

geodesic flow described here is more closely analogous to motion under friction. The “frictional” force along the trajectories is represented by the function  $\lambda$  in Eq. (11). In the weak coupling regime  $\lambda \rightarrow -1$  while at the singular points  $\lambda \rightarrow -\infty$ , indicating that the frictional force becomes infinite at these points. It has been checked numerically that  $\lambda$  is always negative along the thick lines of Fig. 1 – in line with the concept of irreversibility in the infra-red (small  $u$ ) direction.

It would obviously be of interest to explore these matters in the case of supersymmetric theories with matter and for more general gauge groups, but the absence of such concrete formulae as Eq. (1), would probably render the pedestrian analysis presented here inadequate to the task – one would need more sophisticated mathematical techniques.

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