

Geodesic Renormalisation Group Flow

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ABSTRACT

It is shown that the renormalisation group flow in coupling constant space can be interpreted in terms of a dynamical equation for the couplings analogous to viscous fluid flow under the action of a potential. For free scalar field theory the flow is geodesic in two dimensions, while for $D \neq 2$ it is only geodesic in certain limits, e.g. for vanishing external source. For the 1-D Ising model the renormalisation flow is geodesic when the external magnetic field vanishes.

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Introduction

The machinery of the renormalisation group has evolved steadily over the 40 years since its inception and recently the notion of a geometry on the space of couplings has received some attention [1] [2], (for a review in a thermodynamic context see [3]). The aim of this paper is to use these geometrical notions to interpret the β -functions as a dynamical flow and search for a dynamical equation governing the evolution of the couplings of a quantum field theory under changes of scale. It will be argued that, with a specific choice of metric on the space of couplings, the flow of the couplings is analogous to fluid flow in a curved space, under the influence of a potential, including friction. Under certain circumstances the flow is geodesic, with the constraint that the kinetic energy equals the potential energy.

For example for free massive field theories coupled to a constant source, the mass m^2 and the source j parameterise a two dimensional space and it will be shown that the renormalisation group flow is geodesic along the j -axis ($m^2 = 0$) and along the m^2 -axis ($j = 0$), but not otherwise, (except in two dimensions which is special in that all trajectories are geodesics).

The concept of geodesic flow requires introducing a metric on the space of couplings [2] [4] [5]. The significance of a metric was highlighted by Zamolodchikov who utilised the concept in the proof of the c-theorem in two dimensional field theory [4]. A related metric in D -dimensions was considered by O'Connor and Stephens in [2] and this is the metric which will be studied here.

The construction used by O'Connor and Stephens was to consider the partition function Z and free energy W to be differentiable functions on the n -dimensional space of couplings, parameterised by $g^a, a = 1, \dots, n$.

Then from the identity

$$1 = \int D\varphi e^{-S[\varphi]+W} \quad (1)$$

follows the formula

$$dW = \langle dS \rangle \quad (2)$$

where $dS = (\int \Phi_a(x) d^D x) dg^a$ is an operator value one - form. O'Connor and Stephen's metric is

$$\langle (dS - dW) \otimes (dS - dW) \rangle. \quad (3)$$

In order to be able to pass to the infinite volume limit densities will be used here. Defining

$$\tilde{\Phi}_a(x) = \Phi_a(x) - \langle \Phi_a(x) \rangle, \quad (4)$$

we shall take

$$G_{ab} = \int \langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(0) \rangle d^D x \quad (5)$$

to be the metric.

In special co-ordinates in which the action is linear in the couplings, G_{ab} can easily be derived from the free energy density w , defined via $W = \int w d^D x$, from

$$G_{ab} = -\partial_a \partial_b w. \quad (6)$$

Note however that equation (6) is not covariant under general co-ordinate transformations, whereas equation (5) is. In general there may be singularities in $\langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(y) \rangle$ as $|x - y| \rightarrow 0$ which are strong enough to render the integral in (5) divergent and it will be assumed that these can be regularised. It will also be assumed that $\langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(y) \rangle$ falls off fast enough for large $|x - y|$ for (5) to be finite. The formalism will now be illustrated with two examples - free field theory and then the 1-D Ising model.

Free Field Theory

Consider free field theory in D Euclidean dimensions, with action

$$S(\varphi; j, m^2) = \int d^D x \left\{ \frac{1}{2} \varphi (-\square^2 + m^2) \varphi + j \varphi \right\}. \quad (7)$$

There are two parameters in this theory, j (which shall be taken to be independent of position for simplicity) and m^2 . The partition function is

$$Z(j, m^2) = \int D\varphi e^{-S(\varphi; j, m^2)}. \quad (8)$$

Performing the Gaussian functional integral gives the free energy density, using dimensional continuation with dimension D ,

$$w(j, m^2) = - \left(\frac{m^2}{4\pi} \right)^{D/2} \frac{\Gamma(2 - \frac{D}{2})}{D(\frac{D}{2} - 1)} - \frac{j^2}{2m^2}. \quad (9)$$

Since the action is linear in j and m^2 the metric as given by (6) is easily calculated giving

$$ds^2 = -(\partial_a \partial_b w) dx^a dx^b = \frac{1}{m^2} dj^2 + \left[\frac{\Gamma(2 - \frac{D}{2})}{2(4\pi)^{\frac{D}{2}}} (m^2)^{\frac{D}{2} - 2} + \frac{j^2}{m^6} \right] (dm^2)^2 - \frac{2j}{m^4} dj dm^2. \quad (10)$$

As pointed out in [2], it is convenient to use $\phi = \langle \varphi \rangle = -j/m^2$, rather than j itself as a co-ordinate, because this diagonalises the metric. This leads to

$$ds^2 = m^2 d\phi^2 + \frac{1}{2} \frac{1}{(4\pi)^{D/2}} (m^2)^{\frac{D}{2} - 2} \Gamma\left(2 - \frac{D}{2}\right) (dm^2)^2. \quad (11)$$

To expose the geometry more clearly a further change of co-ordinates is useful.

Let

$$\begin{aligned}\theta &= 2\sqrt{\pi} \left\{ \frac{D}{4} \sqrt{\frac{2}{\Gamma(2 - \frac{D}{2})}} \right\}^{\frac{2}{D}} \phi \\ r &= \frac{4}{D} \sqrt{\frac{\Gamma(2 - \frac{D}{2})}{2}} \left(\frac{m^2}{4\pi} \right)^{\frac{D}{4}}\end{aligned}\quad (12)$$

then

$$ds^2 = dr^2 + r^{\frac{4}{D}} d\theta^2. \quad (13)$$

This co-ordinate transformation is singular for $D = 4$, but perfectly regular for $0 < D < 4$. It is clear from (13) that $D = 2$ gives the flat metric in $r - \theta$ space, but all other dimensions lead to a curved metrics. The curvature scalar is

$$\mathcal{R} = -\frac{2(2-D)}{D^2 r^2} = -\frac{(2-D)}{4} \frac{1}{\Gamma(2 - \frac{D}{2})} \left(\frac{m^2}{4\pi} \right)^{-\frac{D}{2}} \quad (14)$$

which is equivalent to the large volume limit of [2].

Note that $\mathcal{R} = 0$ in $D = 4$ for finite m^2 , though it is positive for finite r , due to the singular nature of the co-ordinate transformation (12) in four dimensions. For $D = 0$, $\mathcal{R} = -(1/2)$ for finite m^2 , giving the Lobachevski plane for the Gaussian distribution in ordinary statistics, [6]. It is, of course, m^2 which is the physical parameter.

We shall now turn to a discussion of the renormalisation flow in this geometry. Since the model is a free field theory, the couplings simply scale according to their canonical dimensions, and β -functions can be defined by using dimensionless parameters

$$m^2 \rightarrow m^2/\kappa^2, \quad \phi \rightarrow \phi/\kappa^{\frac{D}{2}-1},$$

where κ is a fiducial scale (renormalisation point), giving

$$\begin{aligned}\beta^{m^2} &= \kappa \frac{dm^2}{d\kappa} = -2m^2 \\ \beta^\phi &= \kappa \frac{d\phi}{d\kappa} = -\left(\frac{D}{2} - 1\right)\phi\end{aligned}\quad (15)$$

or

$$\beta^r = -\frac{D}{2}r \quad \beta^\theta = -\left(\frac{D}{2} - 1\right)\theta \quad (16)$$

in the $(r - \theta)$ co-ordinate system. The vector field $\vec{\beta} = -(\frac{D}{2} - 1)\theta \frac{\partial}{\partial \theta} - \frac{D}{2}r \frac{\partial}{\partial r}$ is plotted in figures 1, 2 and 3 for dimensions 1, 2 and 3 respectively. The two dimensional case is plotted in polar co-ordinates in figure 2 since it is clearly natural to make θ periodic

in this instance, with period 2π . This is reflected in two dimensional conformal field theory where the field $\vartheta = \sqrt{2\pi}\varphi$ is not a primary field, but the operator $:e^{ie\vartheta}:$ is primary, with correlator

$$\langle :e^{ie\vartheta(x)} :: e^{ie\vartheta(y)} : \rangle \approx \frac{1}{|x-y|^{e^2}}, \quad (17)$$

[7]. It appears, therefore, that the operator $z = \frac{m}{\sqrt{2\pi}}e^{i\vartheta}$ is the most natural, not only from the point of view of conformal field theory but also from the geometrical point of view taken here. There is no obvious reason for making the fields periodic in the other dimensions.

It seems natural to enquire if the vector flows plotted in the figures are related in any way to the metric in equation (13). Clearly the integral curves of (16) in $D = 2$ are radial and thus geodesics of the metric (13). This is not true for $D \neq 2$. If a curve $x^a(t)$ is parameterised by t and has tangent vector $\zeta^a = \frac{dx^a}{dt}$ the condition that $x^a(t)$ be a geodesic is given by the geodesic equation

$$\zeta^b \nabla_b \zeta^a = \lambda \zeta^a \quad (18)$$

where the right hand side, $\lambda \neq 0$, allows for the possibility that t might not be affine [8].

If $\vec{\beta}$ in (16) is used in the geodesic equation (18) with the metric (13) one finds the following conditions on $r(t)$ and $\theta(t)$

$$\begin{aligned} \left(\frac{D}{2}\right)^3 r - \left(\frac{D}{2} - 1\right)^2 r^{\frac{4-D}{D}} \theta^2 &= -\left(\frac{D}{2}\right)^2 \lambda r \\ \left(\frac{D^2}{4} - 1\right)\theta &= -\left(\frac{D}{2} - 1\right)\lambda\theta. \end{aligned} \quad (19)$$

If $D \neq 2$, the only solutions, for $0 < D < 4$, are either $r = 0$ (i.e. $m^2 = 0$) with $\lambda = -\left(\frac{D}{2} + 1\right)$ or $\theta = 0$ with $\lambda = -\frac{D}{2}$. Thus the bold lines in figures 1 and 3 are geodesics. The other curves fail to satisfy the geodesic condition (18) for $D \neq 2$, and can be interpreted as being repulsed from the ϕ -axis by a “force”. This conclusion also holds for $D = 4$ where the only renormalisation group flows which are geodesic are $m^2 = 0$ with $\lambda = -3$ or $\phi = 0$ with $\lambda = -2$.

The 1-D Ising Model

Clearly it is of interest to extend the above analysis beyond free field theory to the case of a non-trivial interacting theory. Consider therefore one of the simplest interacting theories - the one dimensional Ising model.

The one dimensional Ising model on a periodic lattice of N sites is described in [9]. The partition function is

$$Z_N = \sum_{\{\sigma\}} \exp \left[K \sum_{j=1}^N \sigma_j \sigma_{j+1} + h \sum_{j=1}^N \sigma_j \right] \quad (20)$$

where $K = \frac{J}{kT}$ and $h = \frac{H}{kT}$, with J the spin coupling and H the external magnetic field, (periodic boundary conditions require $\sigma_{N+1} \equiv \sigma_1$). $Z_N(K, h)$ can be conveniently expressed in terms of the transfer matrix

$$V = \begin{pmatrix} V_{++} & V_{+-} \\ V_{-+} & V_{--} \end{pmatrix} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix} \quad (21)$$

as $Z_N = \text{Tr} V^N$.

Diagonalising V gives the eigenvalues

$$\lambda_{\pm} = e^K \left\{ \cosh h \pm \sqrt{\sinh^2 h + e^{-4K}} \right\}. \quad (22)$$

Thus

$$Z_N = \lambda_+^N \left[1 + \left(\frac{\lambda_-}{\lambda_+} \right)^N \right] \quad (23)$$

and in the limit of large N

$$(1/N) \ln Z_N = K + \ln \left(\cosh h + \sqrt{\sinh^2 h + e^{-4K}} \right),$$

which is essentially the negative of the free energy per unit volume.

Using equation (6) the line element is easily obtained

$$\begin{aligned} ds^2 &= \frac{e^{-4K}}{(\sinh^2 h + e^{-4K})^{3/2}} \\ &\times \left[\frac{4e^{-4K} \cosh h + 8 \sinh^2 h (\cosh h + \sqrt{\sinh^2 h + e^{-4K}})}{(\cosh h + \sqrt{\sinh^2 h + e^{-4K}})^2} dK^2 \right. \\ &\quad \left. + 4 \sinh h dK dh + \cosh h dh^2 \right]. \end{aligned} \quad (24)$$

It is convenient to change to a set of co-ordinates in which the metric is diagonal. To this end define $\rho = e^{2K} \sinh h$ and use K and ρ as co-ordinates. The metric is then

$$ds^2 = \frac{1}{\sqrt{(1+\rho^2)(e^{4K} + \rho^2)}} \left[\frac{4 e^{4K} dK^2}{(\sqrt{1+\rho^2} + \sqrt{e^{4K} + \rho^2})^2} + \frac{d\rho^2}{(1+\rho^2)} \right]. \quad (25)$$

The Ricci scalar is easily calculated and is found to be

$$\mathcal{R} = \frac{1}{2} \left[1 + \sqrt{\frac{e^{4K} + \rho^2}{1 + \rho^2}} \right] \quad (26)$$

or, in terms of the physical co-ordinates K and h ,

$$\mathcal{R} = \left(\frac{1}{2} \right) \left(1 + \frac{e^{2K} \cosh h}{\sqrt{e^{4K} \sinh^2 h + 1}} \right). \quad (27)$$

This expression was first obtained by Janyszek and Mrugała, [10], and the function is depicted in Fig. 4. Note that the curvature diverges at the critical point $T = 0$, ($K \rightarrow \infty$).

To investigate the behaviour of the renormalisation flow in this lattice model the β -functions are replaced by a discrete recursive map, [11]. In its simplest form this map is obtained by asking: can one find new couplings K' and h' such that

$$Z_{\frac{N}{2}}(K', h') = A^N Z_N(K, h) ? \quad (28)$$

(A is a normalisation factor.)

Equation (28) is easily satisfied by demanding

$$\begin{pmatrix} e^{K'+h'} & e^{-K'} \\ e^{-K'} & e^{K'-h'} \end{pmatrix} = A^2 \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}^2 \quad (29)$$

giving the recursive formulae:

$$e^{2h'} = e^{2h} \frac{\cosh(2K + h)}{\cosh(2K - h)} \quad (30)$$

$$e^{4K'} = \frac{\cosh(4K) + \cosh(2h)}{2 \cosh^2(h)}. \quad (31)$$

The normalisation factor A is unimportant for the present analysis.

Note that the combination $e^{2K'} \sinh(h') = e^{2K} \sinh(h)$ is a renormalisation transformation invariant. Strictly speaking this transformation is not always invertible and so it is incorrect to refer to it as a renormalisation group transformation. The renormalisation flow is depicted in Fig. 5.

Since the renormalisation transformation is a discrete map here, the β -functions cannot be defined in terms of continuous a vector field. However, it still makes sense to ask if any of the flow curves in Fig. 5 are geodesics of the metric (24). The answer is that only the special curve running between the two fixed points $K = 0$ and $K = \infty$

is a geodesic, i.e. the seperatrix $h = 0$, the other flow lines are “repulsed” from the K -axis and never cross it, but crowd together asymptotically in the manner of a caustic in geometrical optics or fluid mechanics. The geodesic distance between the two fixed points is $\pi/2$. This behaviour is suggestive of a dynamical interpretation of renormalisation flow.

General Considerations

One is led to enquire into the nature of the “force” and to investigate the possibility of a dynamical description of the renormalisation group flow in terms of an equation for the β -functions. It will be shown that the β -functions obey a dynamical equation analogous to the viscous flow of a fluid in a curved space under the influence of a potential, with a constraint which dictates that the kinetic energy equals the potential energy.

The argument hinges on the renormalisation group equation for the two point correlators of the theory

$$\kappa \frac{\partial}{\partial \kappa} \langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(y) \rangle + \beta^c \partial_c \langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(y) \rangle + \partial_a \beta^c \langle \tilde{\Phi}_c(x) \tilde{\Phi}_b(y) \rangle + \partial_b \beta^c \langle \tilde{\Phi}_a(x) \tilde{\Phi}_c(y) \rangle = 0. \quad (32)$$

If the couplings are scaled to be dimensionless, then all $\tilde{\Phi}_a(x)$ have canonical mass dimension D , thus the usual scaling argument gives

$$\begin{aligned} \left(x^\mu \frac{\partial}{\partial x^\mu} + y^\mu \frac{\partial}{\partial y^\mu} \right) \langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(y) \rangle + 2D \langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(y) \rangle \\ = -\beta^c \partial_c \langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(y) \rangle - \partial_a \beta^c \langle \tilde{\Phi}_c(x) \tilde{\Phi}_b(y) \rangle \\ - \partial_b \beta^c \langle \tilde{\Phi}_a(x) \tilde{\Phi}_c(y) \rangle. \end{aligned} \quad (33)$$

Integrating over all y and using translational invariance leads to

$$\beta^c \partial_c G_{ab} + (\partial_a \beta^c) G_{cb} + (\partial_b \beta^c) G_{ca} = -D G_{ab}. \quad (34)$$

This equation states that $\vec{\beta}$ is a conformal Killing vector for the metric G_{ab} and it is straightforward to check this property using (13) and (16). That the renormalisation group equation can be written in terms of a Lie derivative was noted by Lässig in [12] and explored in more detail in [13]. A subtlety arises in $D = 4$ in that the integral over y diverges and $G_{m^2 m^2}$ is infinite, due to the short distance singularity as $y \rightarrow x$. However, if $4 - D = \epsilon$ is kept positive and the limit $\epsilon \rightarrow 0$ is only taken after the scalar curvature (14) is calculated, it appears that this is a co-ordinate singularity rather than a pathology in the geometry.

If the Lie derivative on the left hand side of equation (34) is written in terms of the Levi-Civita connection for the metric G_{ab}

$$\beta^c \partial_c G_{ab} + (\partial_a \beta^c) G_{cb} + (\partial_b \beta^c) G_{ac} = (\nabla_a \beta^c) G_{cb} + (\nabla_b \beta^c) G_{ac} \quad (35)$$

and then equation (34) is contracted with β^a , one finds

$$\beta^b \nabla_b \beta^a = -G^{ab} \nabla_b U - D\beta^a \quad (36)$$

where the ‘‘potential’’ is given by $U = \frac{1}{2} \beta^b \beta^c G_{bc}$. The right hand side can be interpreted as a force, and the potential, U , pushes $\vec{\beta}$ away from geodesic flow unless $\nabla_b U \propto \beta_b$. The second term on the right hand side of (36) plays the role of an isotropic frictional force.

This dynamical picture is compatible with the analysis of geodesic flow for free field theories since in this case $U = \frac{1}{2} \left\{ \frac{D^2 r^2}{4} + \left(\frac{D}{2} - 1\right)^2 \theta^2 r^{4/D} \right\}$ giving

$$G^{ab} \partial_b U = \begin{pmatrix} \left\{ \left(\frac{D}{2}\right)^2 r + \frac{2}{D} \left(\frac{D}{2} - 1\right)^2 \theta^2 r^{\frac{4}{D}-1} \right\} \\ \left(\frac{D}{2} - 1\right)^2 \theta \end{pmatrix} \quad (37)$$

and

$$\begin{aligned} G^{ab} \partial_b U &= -\frac{D}{2} \beta^a && \text{for } \theta = 0 \\ G^{ab} \partial_b U &= -\left(\frac{D}{2} - 1\right) \beta^a && \text{for } r = 0, \quad D < 4 \end{aligned} \quad (38)$$

so both these cases result in geodesic flow from equation (36). Note the special case of two dimensions where $U = (1/2)r^2$ is the harmonic oscillator potential.

The derivation of equation (36) relied only on the renormalisation group equation (32) and the fact that the composite operators, $\tilde{\Phi}_a$, have canonical dimension D , and so is completely general, even for interacting field theories. However, the question of singularities in $\langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(y) \rangle$ as $|x - y| \rightarrow 0$ must be addressed. If $\langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(y) \rangle \sim \frac{1}{|x-y|^k}$ with $k < D$, then $G_{ab} = \int d^D x \langle \tilde{\Phi}_a(x) \tilde{\Phi}_b(0) \rangle$ is finite (assuming the correlator $\tilde{\Phi}_a(x) \tilde{\Phi}_b(0) \rightarrow 0$ fast enough for large $|x|$). But if $k \geq D$, then G_{ab} will be singular and must be regularised. Singular metric components do not necessarily indicate singular geometry and the curvature may be finite as the regulator is removed, as the example of free field theory in $D = 4$ shows.

In general, however, it may prove necessary to subtract counter-terms from the two point correlators. This has the effect of modifying equation (34), so that it reads

$$\beta^c \partial_c G_{ab} + (\partial_a \beta^c) G_{cb} + (\partial_b \beta^c) G_{ac} = -D G_{ab} - \chi_{ab} \quad (39)$$

where χ_{ab} is a new tensor introduced by the subtraction procedure (see for example [4]). The dynamical equation (36) is then modified by the introduction of a non-isotropic ‘‘friction’’ and becomes

$$\beta^b \nabla_b \beta^a = -G^{ab} \partial_b U - D\beta^a - \chi^a_b \beta^b. \quad (40)$$

The condition for geodesic flow is now $G^{ab}\partial_b U \propto \beta^a$ and $\chi^a{}_b \beta^b \propto \beta^a$. Equation (40) holds even for renormalisation schemes in which the β functions have explicit κ dependence.

In conclusion, it has been demonstrated that the metric (5) can give geodesic renormalisation group flow in free field theories, under certain circumstances. This has been interpreted in terms of the dynamical equation (36), analogous to fluid flow with friction in a curved space, under the influence of a potential with the constraint $U = \frac{1}{2} |\beta|^2$, similar to the virial theorem for a collection of harmonic oscillators in statistical equilibrium. For a general theory the extra non-isotropic frictional term in (40) may be necessary. Geodesic renormalisation flow has also been demonstrated in the one dimensional Ising model for the crossover between the two fixed points at $T = \infty$ and $T = 0$ with vanishing magnetic field.

Clearly it would be of particular interest to examine these ideas in a non-trivial interacting field theory, where it is to be expected that the more general dynamical equation (40) will be relevant and work is in progress on this.

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Fig. 1: RG flow for free field theory in 1-D
The bold lines are geodesics

Fig. 2: RG flow for free field theory in 2-D
All RG trajectories are geodesics

Fig. 3: RG flow for free field theory in 3-D
The bold lines are geodesics

Fig. 4: Ricci curvature for the 1-D Ising model
The line of the central ridge is a geodesic

Fig. 5: The renormalisation flow for the 1-D Ising model
The K -axis ($h = 0$) is a geodesic