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Derivative for Discrete Choquet Integrals

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Abstract. In this paper we study necessary and sufficient conditions for the existence of the derivative for fuzzy measures when we are considering the Choquet integral. Results apply to discrete domains. The main result is based on the definition we introduce of compatible permutation for two pairs of measures (μ, ν).

As an application of the main result, we present the conditions for possibility measures.

1 Introduction

Choquet integral [2] permits to integrate a function with respect to a fuzzy measure. Fuzzy measures [3,12], also known as capacities and non-additive measures, generalize standard measures replacing the additivity condition by a monotonicity one. Then, when a fuzzy measure is additive, Choquet integral reduces to the Lebesgue integral.

The Radon-Nikodym derivative is a very important concept related to the Lebesgue integral. The Radon-Nikodym theorem establishes that we can express one additive measure with respect to another one under some conditions. In particular, the condition of absolute continuity between two measures plays a pivotal role.

In addition to its intrinsic mathematical interest, the Radon-Nikodym derivative is useful in practical applications. More particularly, it has been used to define distances and divergences between pairs of measures. In particular, f-divergences, which are defined in terms of Radon-Nikodym derivatives, are extensively used in statistics and information theory. Recall that the Hellinger distance, the Kullback-Leibler divergence, the Rényi distance and the variation distance are all examples of f-divergences. They are used to compare probability distributions, and the Kullback-Leibler divergence can also be used to define the entropy.

Because of its theoretical and applied interest, the problem of defining Radon-Nikodym-like derivatives for fuzzy measures is a relevant question. In the continuous case, Graf, Mesiar and Sipos, Nguyen, Rébillé, and Sugeno [5–9] have studied the existence and computation of a Radon-Nikodym like derivative for

© Springer Nature Switzerland AG 2019
V. Torra et al. (Eds.): MDAI 2019, LNAI 11676, pp. 138–147, 2019.
https://doi.org/10.1007/978-3-030-26773-5_13

non-additive measures in the context of Choquet integrals. We have considered the problem ourselves in the context of defining f-divergence for fuzzy measures. We have considered both discrete and continuous case. See e.g. [11,13] (for f-divergence and Hellinger distances) and [10] (for the definition of the entropy). This derivative has also been used in [1] to define an alternative expression for f-divergence.

In this paper we consider the problem of existence of the derivative when the reference set is finite. More particularly, we consider the problem of finding necessary and sufficient conditions on the existence of the derivative.

The structure of the paper is as follows. In Sect. 2 we review the concepts that are needed in the paper. In Sect. 3 we present the main results. In Sect. 4, as the application of the main result, we present the conditions for possibility measures.

2 Preliminaries

Let us consider the universal set $X := \{x_1, x_2. \cdots, x_n\}$. Let us review the definitions of fuzzy measure and Choquet integral.

Definition 1. *A set function μ such that $\mu(\emptyset) = 0$ and that is monotonic with respect to the set inclusion (i.e., $\mu(A) \leq \mu(B)$ when $A \subset B$) is called a fuzzy measure, non-additive measure, capacity or monotonic game.*

It is often also required that μ satisfies $\mu(X) = 1$. We do not require this condition in this paper.

Definition 2. *A fuzzy measure μ on $(X, 2^X)$ is called a possibility measure, if $\mu(A \cup B) = \mu(A) \vee \mu(B)$ for $A, B \in 2^X$. Here $\vee$ is understood as the maximum.*

Definition 3. *Let μ be a fuzzy measure and f be a function $f : X \to [0, \infty)$. The Choquet integral of the function f with respect to the fuzzy measure μ is defined by*

$$(C) \int f d\mu = \int_0^\infty \mu(\{x | f(x) \geq \alpha\}) d\alpha \quad (1)$$

Let $A \subset X$. The Choquet integral of the function f over A with respect to the fuzzy measure μ is defined by

$$(C) \int_A f d\mu = \int_0^\infty \mu(\{x | f(x) \geq \alpha\} \cap A) d\alpha \quad (2)$$

When μ is additive, this expression corresponds to the classical Lebesgue integral. Using Eq. 2 we can consider defining measures in terms of other measures. That is, we can define a measure μ from another measure ν as follows

$$\mu(A) = (C) \int_A f d\nu = \int_0^\infty \nu(\{x | f(x) \geq \alpha\} \cap A) d\alpha \quad (3)$$

Given μ and ν in Eq. 3, we can consider the problem of finding the function f. When the measures are additive, this corresponds to the Radon-Nikodym derivative, as the Choquet integral reduces to the Lebesgue integral.

3 Condition for the Existence of a Derivative

While for additive fuzzy measures the Radon-Nikodym derivative exist when the measures are absolutely continuous, this is not the case for fuzzy measures. Because of that, it is rellevant to study when the derivative exist. We give some conditions for its existence in this section. From now on, we will consider measures ν such that $\nu(\{x_i\}) \neq 0$ for all x_i.

Definition 4. *Let μ, ν be fuzzy measures on $(X, 2^X)$. We say that μ and ν are compatible if there exists a permutation σ on $\{1, 2, \ldots, n\}$ such that*

$$\frac{\mu(\{x_{\sigma(1)}\})}{\nu(\{x_{\sigma(1)}\})} \geq \frac{\mu(\{x_{\sigma(2)}\})}{\nu(\{x_{\sigma(2)}\})} \geq \cdots \geq \frac{\mu(\{x_{\sigma(n)}\})}{\nu(\{x_{\sigma(n)}\})}.$$

A permutation σ satisfying this condition is said to be a compatible permutation for (μ, ν).

From the definition of compatible permutation for a pair of measures (μ, ν), it is easy to prove the following proposition.

Proposition 1. *Let σ be a compatible permutation for (μ, ν). Then, we have*

$$\begin{vmatrix} \mu(\{x_{\sigma(k)}\}) & \nu(\{x_{\sigma(k)}\}) \\ \mu(\{x_{\sigma(k+1)}\}) & \nu(\{x_{\sigma(k+1)}\}) \end{vmatrix} \geq 0$$

for $k = 1, 2, \ldots, n-1$.

We will now give the main theorem of this paper. Let us now consider the following. Let $x_k \in X$ and $f(x_k) := \frac{\mu(\{x_k\})}{\nu(\{x_k\})}$ for $k = 1, 2, \cdots, n$.

Since

$$(C)\int_{\{x_k\}} f d\nu = f(x_k)\nu(\{x_k\}),$$

we have

$$\mu(\{x_k\}) = (C)\int_{\{x_k\}} f d\nu.$$

Let $A_2 := \{x_{i_1}, x_{i_2}\}$. Suppose that

$$\mu(A_2) = (C)\int_{A_2} f d\nu.$$

with $f(x_{\sigma(i_1)}) \geq f(x_{\sigma(i_2)})$

Since

$$\begin{aligned}(C)\int_{A_2} f d\nu &= f(x_{\sigma(i_2)})[\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}\}) - \nu(\{x_{\sigma(i_1)}\})] + f(x_{\sigma(i_1)})\nu(\{x_{\sigma(i_1)}\}) \\ &= \frac{\mu(\{x_{\sigma(i_2)}\})}{\nu(\{x_{\sigma(i_2)}\})}[\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}\}) - \nu(\{x_{\sigma(i_1)}\})] + \frac{\mu(\{x_{\sigma(i_1)}\})}{\nu(\{x_{\sigma(i_1)}\})}\nu(\{x_{\sigma(i_1)}\}) \\ &= \frac{\mu(\{x_{\sigma(i_2)}\})}{\nu(\{x_{\sigma(i_2)}\})}[\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}\}) - \nu(\{x_{\sigma(i_1)}\})] + \mu(\{x_{\sigma(i_1)}\}),\end{aligned}$$

we have

$$\mu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}\}) - \mu(\{x_{\sigma(i_1)}\}) = \frac{\mu(\{x_{\sigma(i_2)}\})}{\nu(\{x_{\sigma(i_2)}\})}[\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}\}) - \nu(\{x_{\sigma(i_1)}\})],$$

that is,

$$\nu(\{x_{\sigma(i_2)}\})[\mu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}\}) - \mu(\{x_{\sigma(i_1)}\})] = \mu(\{x_{\sigma(i_2)}\})[\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}\}) - \nu(\{x_{\sigma(i_1)}\})] \quad (4)$$

Let $A_3 := \{x_{i_1}, x_{i_2}, x_{i_3}\}$. Suppose that Eq. 4 and

$$\mu(A_3) = (C)\int_{A_3} f d\nu.$$

with $f(x_{\sigma(i_1)}) \geq f(x_{\sigma(i_2)}) \geq f(x_{\sigma(i_3)})$.

Then, we have

$$\begin{aligned}
(C)\int_{A_3} f d\nu &= f(x_{\sigma(i_3)})[\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, x_{\sigma(i_3)}\}) - \nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, \})] \\
&\quad + f(x_{\sigma(i_2)})[\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}\}) - \nu(\{x_{\sigma(i_1)}\})] + f(x_{\sigma(i_1)})\nu(\{x_{\sigma(i_1)}\}) \\
&= \frac{\mu(\{x_{\sigma(i_3)}\})}{\nu(\{x_{\sigma(i_3)}\})}[\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, x_{\sigma(i_3)}\}) - \nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, \})] \\
&\quad + \frac{\mu(\{x_{\sigma(i_2)}\})}{\nu(\{x_{\sigma(i_2)}\})}[\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}\}) - \nu(\{x_{\sigma(i_1)}\})] + \frac{\mu(\{x_{\sigma(i_1)}\})}{\nu(\{x_{\sigma(i_1)}\})}\nu(\{x_{\sigma(i_1)}\}) \\
&= \frac{\mu(\{x_{\sigma(i_3)}\})}{\nu(\{x_{\sigma(i_3)}\})}[\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, x_{\sigma(i_3)}\}) - \nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, \})] \\
&\quad + \mu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}\}) - \mu(\{x_{\sigma(i_1)}\}) + \mu(\{x_{\sigma(i_1)}\}) \\
&= \frac{\mu(\{x_{\sigma(i_3)}\})}{\nu(\{x_{\sigma(i_3)}\})}[\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, x_{\sigma(i_3)}\}) - \nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, \})] + \mu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}\})
\end{aligned}$$

Then we have

$$\begin{aligned}
&\nu(\{x_{\sigma(i_3)}\})[\mu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, x_{\sigma(i_3)}\}) - \mu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}\})] \\
&\qquad = \mu(\{x_{\sigma(i_3)}\})[\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, x_{\sigma(i_3)}\}) - \nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}\})]
\end{aligned}$$

Let $A_k := \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$. Then, by induction, if

$$\mu(A_k) = (C)\int_{A_k} f d\nu,$$

then we have

$$\begin{aligned}
&\nu(\{x_{\sigma(i_k)}\})[\mu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, \ldots, x_{\sigma(i_k)}\}) - \mu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)} \ldots, x_{\sigma(i_{k-1})}\})] \\
&\qquad = \mu(\{x_{\sigma(i_3)}\})[\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, \ldots, x_{\sigma(i_k)}\}) - \nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)} \ldots, x_{\sigma(i_{k-1})}\})].
\end{aligned}$$

Therefore we have the next theorem

Theorem 1. *Let μ, ν be fuzzy measures on $(X, 2^X)$ and σ be a compatible permutation for (μ, ν). Then, there exists a function f on X such that*

$$\mu(A_k) = (C)\int_{A_k} f d\nu$$

for all $A_k = \{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\} \subset X$ if and only if

$$\begin{vmatrix} 1 & \mu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)} \ldots, x_{\sigma(i_{k-1})}\}) & \nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)} \ldots, x_{\sigma(i_{k-1})}\}) \\ 1 & \mu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, \ldots, x_{\sigma(i_k)}\}) & \nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, \ldots, x_{\sigma(i_k)}\}) \\ 0 & \mu(\{x_{\sigma(i_k)}\}) & \nu(\{x_{\sigma(i_k)}\}) \end{vmatrix} = 0 \quad (5)$$

for all $k = 2, \ldots, n$.

Here, $x_{\sigma(i_k)}$ will be the element with smallest $\mu(\{x_{\sigma(i_k)}\})/\nu(\{x_{\sigma(i_k)}\})$ in the set $x_{\sigma(i_1)}, x_{\sigma(i_2)}, \ldots, x_{\sigma(i_k)}$.

We illustrate this theorem with an example. We give two measures on a reference set of three elements that are compatible. We show that the determinants of Theorem 1 are zero and thus, there exists a derivative of one measure with respect to the other one.

Example 1. Let $X := \{x_1, x_2, x_3\}$ and let μ and ν two non-additive measures defined as in Table 1.

Table 1. Two measures μ and ν that are compatible.

A	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_2, x_3\}$	$\{x_1, x_3\}$	$\{x_1, x_2, x_3\}$
$\mu(A)$	0.2	0.3	0.4	0.5	0.6	0.55	0.8
$\nu(A)$	0.1	0.3	0.8	0.4	0.9	0.8	1

We can observe that

$$\frac{\mu(\{x_1\})}{\nu(\{x_1\})} > \frac{\mu(\{x_2\})}{\nu(\{x_2\})} > \frac{\mu(\{x_3\})}{\nu(\{x_3\})}.$$

Let us now check that the determinants are zero for all $A_k \subseteq X$. We need to consider only $k = 2$ and $k = 3$ as there are only 3 elements in X.

Let us begin with $k = 2$, and we need to consider the sets $\{x_1, x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_3\}$. Then, for the first set we obtain the following determinant that is equal to zero:

$$\begin{vmatrix} 1 & \mu(\{x_1\}) & \nu(\{x_1\}) \\ 1 & \mu(\{x_1, x_2\}) & \nu(\{x_1, x_2\}) \\ 0 & \mu(\{x_2\}) & \nu(\{x_2\}) \end{vmatrix} = \begin{vmatrix} 1 & 0.2 & 0.1 \\ 1 & 0.5 & 0.4 \\ 0 & 0.3 & 0.3 \end{vmatrix} = 0.$$

For the second set we obtain the following determinant that is also equal to zero:

$$\begin{vmatrix} 1 & \mu(\{x_1\}) & \nu(\{x_1\}) \\ 1 & \mu(\{x_1,x_3\}) & \nu(\{x_1,x_3\}) \\ 0 & \mu(\{x_3\}) & \nu(\{x_3\}) \end{vmatrix} = \begin{vmatrix} 1 & 0.2 & 0.1 \\ 1 & 0.55 & 0.8 \\ 0 & 0.4 & 0.8 \end{vmatrix} = 0.$$

Similarly, for the third set we obtain the following determinant that is also equal to zero:

$$\begin{vmatrix} 1 & \mu(\{x_2\}) & \nu(\{x_2\}) \\ 1 & \mu(\{x_2,x_3\}) & \nu(\{x_2,x_3\}) \\ 0 & \mu(\{x_3\}) & \nu(\{x_3\}) \end{vmatrix} = \begin{vmatrix} 1 & 0.3 & 0.3 \\ 1 & 0.6 & 0.9 \\ 0 & 0.4 & 0.8 \end{vmatrix} = 0.$$

Then, for $k = 3$ we need to consider the only set with 3 elements. That is, $\{x_1, x_2, x_3\}$. In this case we have the following determinant that is also equal to zero.

$$\begin{vmatrix} 1 & \mu(\{x_1,x_2\}) & \nu(\{x_1,x_2\}) \\ 1 & \mu(\{x_1,x_2,x_3\}) & \nu(\{x_1,x_2,x_3\}) \\ 0 & \mu(\{x_3\}) & \nu(\{x_3\}) \end{vmatrix} = \begin{vmatrix} 1 & 0.5 & 0.4 \\ 1 & 0.8 & 1 \\ 0 & 0.4 & 0.8 \end{vmatrix} = 0.$$

Therefore, Theorem 1 implies that defining

$$f(x_1) = \frac{\mu(\{x_1\})}{\nu(\{x_1\})}, f(x_2) = \frac{\mu(\{x_2\})}{\nu(\{x_2\})}, f(x_3) = \frac{\mu(\{x_3\})}{\nu(\{x_3\})},$$

or, more specifically, with

$$f(x_1) = 2.0, f(x_2) = 1.0 \text{ and } f(x_3) = 0.5$$

we have

$$\mu(A) = (C)\int_A f d\nu,$$

for all A. This last equation can be checked with straightforward computation.

4 Possibility Measures

Let us consider two possibility measures μ and ν. We will reconsider for this type of measures Theorem 1 and make the condition for the existence of the derivative simpler.

Definition 5. *Let μ and ν be compatible fuzzy measures on $(X, 2^x)$ and σ be a compatible permutation on (μ, ν). Then, μ (resp. ν) is said to be weakly monotone decreasing for σ if*

$$\mu(\{x_{\sigma(1)}\}) \geq \mu(\{x_{\sigma(2)}\}) \geq \cdots \geq \mu(\{x_{\sigma(n)}\})$$

(resp. $\nu(\{x_{\sigma(1)}\}) \geq \nu(\{x_{\sigma(2)}\}) \geq \cdots \geq \nu(\{x_{\sigma(n)}\})$).

Suppose that μ and ν are weakly monotone increasing for σ
Since

$$\mu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)} \ldots, x_{\sigma(i_{k-1})}\}) = \mu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, \ldots, x_{\sigma(i_k)}\}) = \mu(\{x_{\sigma(i_1)})$$

and

$$\nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)} \ldots, x_{\sigma(i_{k-1})}\}) = \nu(\{x_{\sigma(i_1)}, x_{\sigma(i_2)}, \ldots, x_{\sigma(i_k)}\}) = \nu(\{x_{\sigma(i_1)}),$$

for any $k = 2, \ldots, n$ we have that we can prove that the following equality holds

$$\begin{vmatrix} 1 & \mu(\{x_{\sigma(i_1)}, \ldots, x_{\sigma(i_{k-1})}\}) & \nu(\{x_{\sigma(i_1)}, \ldots, x_{\sigma(i_{k-1})}\}) \\ 1 & \mu(\{x_{\sigma(i_1)}, \ldots, x_{\sigma(i_k)}\}) & \nu(\{x_{\sigma(i_1)}, \ldots, x_{\sigma(i_k)}\}) \\ 0 & \mu(\{x_{\sigma(i_k)}\}) & \nu(\{x_{\sigma(i_k)}\}) \end{vmatrix}$$

and

$$= \begin{vmatrix} 1 & \mu(\{x_{\sigma(i_1)}\}) & \nu(\{x_{\sigma(i_1)}\}) \\ 1 & \mu(\{x_{\sigma(i_1)}\}) & \nu(\{x_{\sigma(i_1)}\}) \\ 0 & \mu(\{x_{\sigma(i_k)}\}) & \nu(\{x_{\sigma(i_k)}\}) \end{vmatrix} = 0$$

for all $k = 2, \ldots, n$.

Therefore, as this implies that Eq. 5 holds for all $k = 2, \ldots, n$, applying Theorem 1, we have the next theorem.

Theorem 2. *Let μ and ν be compatible possibility measures on $(X, 2^x)$ and σ be a compatible permutation on (μ, ν).*

If μ and ν are weakly monotone increasing for σ, there exists a function f on X such that

$$\mu(A_k) = (C) \int_{A_k} f d\nu$$

for all $A_k = \{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\} \subset X$.

Let us consider a special case, let ν be a 0-1 possibility measure such that $\nu(A) = 1$, if $A \neq \emptyset$, $\nu(A) = 0$,if $A \neq \emptyset$.

Then for every possibility measure μ, μ and ν are compatible.

Let σ be a compatible permutation for (μ, ν). Then we have

$$\frac{\mu(\{x_{\sigma(1)}\})}{1} \geq \frac{\mu(\{x_{\sigma(2)}\})}{1} \geq \cdots \geq \frac{\mu(\{x_{\sigma(n)}\})}{1}.$$

μ and ν are both weakly monotone decreasing.

Therefore we have the next corollary.

Corollary 1. *Let ν be a 0-1 possibility measure such that $\nu(A) = 1$, if $A \neq \emptyset$, $\nu(A) = 0$,if $A \neq \emptyset$. For every possibility measure μ, there exists a function f on X such that*

$$\mu(A_k) = (C) \int_{A_k} f d\nu$$

for all $A_k = \{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\} \subset X$.

If μ and ν have some strict condition, we have the converse of Theorem 2.

Definition 6. *Let μ, ν be fuzzy measures on $(X, 2^X)$. We say that μ and ν are strict compatible if there exists a permutation σ on $\{1, 2, \ldots, n\}$ such that*

$$\frac{\mu(\{x_{\sigma(1)}\})}{\nu(\{x_{\sigma(1)}\})} > \frac{\mu(\{x_{\sigma(2)}\})}{\nu(\{x_{\sigma(2)}\})} > \cdots > \frac{\mu(\{x_{\sigma(n)}\})}{\nu(\{x_{\sigma(n)}\})}.$$

Suppose that possibility measures μ and ν are strict compatible and μ is not weakly monotone increasing. Then, there exist $l, m (1 \leq l < m \leq n)$ such that $\mu(\{x_{\sigma(i_l)}\}) < \mu(\{x_{\sigma(i_m)}\})$.

Let $A_m = \{x_{\sigma(i_l)}, x_{\sigma(i_m)}\}$, and let us define $\alpha_1, \alpha_2, \beta_1, \beta_2$ and D as follows:

$$\mu(\{x_{\sigma(i_l)}\}) = \alpha_1, \mu(\{x_{\sigma(i_m)}\}) = \alpha_2, \nu(\{x_{\sigma(i_l)}\}) = \beta_1, \nu(\{x_{\sigma(i_m)}\}) = \beta_2$$

and

$$D = \begin{vmatrix} 1 & \mu(\{x_{\sigma(i_l)}\}) & \nu(\{x_{\sigma(i_l)}\}) \\ 1 & \mu(\{x_{\sigma(i_l)}, x_{\sigma(i_m)}\}) & \nu(\{x_{\sigma(i_l)}, x_{\sigma(i_m)}\}) \\ 0 & \mu(\{x_{\sigma(i_m)}\}) & \nu(\{x_{\sigma(i_m)}\}) \end{vmatrix}.$$

Observe that from these definitions it follows $\alpha_1 < \alpha_2$. Then, we have for D the following:

$$D = \begin{vmatrix} 1 & \alpha_1 & \beta_1 \\ 1 & \alpha_2 & \beta_1 \vee \beta_2 \\ 0 & \alpha_2 & \beta_2 \end{vmatrix} = \beta_2(\alpha_2 - \alpha_1) - \alpha_2(\beta_1 \vee \beta_2 - \beta_1)$$

Then, if $\beta_1 \geq \beta_2$, we have

$$D = \beta_2(\alpha_2 - \alpha_1) - \alpha_2(\beta_1 - \beta_1) = \beta_2(\alpha_2 - \alpha_1) > 0,$$

and if $\beta_1 < \beta_2$ we have

$$D = \beta_2(\alpha_2 - \alpha_1) - \alpha_2(\beta_2 - \beta_1) = \alpha_2\beta_1 - \alpha_1\beta_2.$$

Since (μ, ν) is strict and as σ is a compatible permutation for (μ, ν) (i.e., $\alpha_1/\alpha_2 \leq \beta_1/\beta_2$), we have that

$$D = \beta_1\beta_2(\frac{\alpha_2}{\beta_2} - \frac{\alpha_1}{\beta_1}) < 0.$$

In any case, we have $D \neq 0$.

Therefore we have the next proposition.

Proposition 2. *Let μ, ν be fuzzy measures on $(X, 2^X)$ which are strict compatible, and let σ be a compatible permutation for (μ, ν).*

Suppose that there exists a function f on X such that

$$\mu(A_k) = (C)\int_{A_k} f d\nu$$

for all $A_k = \{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\} \subset X$, $k = 2, \ldots, n$. Then μ and ν are both weakly monotone decreasing.

Table 2. Possibility measures μ and ν defined by the measures on the singletons.

A	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_4\}$
$\mu(A)$	0.9	0.8	0.6	0.4
$\nu(A)$	0.8	0.8	0.7	0.5

Example 2. Let $X := \{x_1, x_2, x_3, x_4\}$ and possibility measures defined as in Table 2.

Then (μ, ν) are strictly compatible, and μ and ν are weakly monotone.

From Proposition 2, it follows that there exists a function f on X such that

$$\mu(A_k) = (C)\int_{A_k} f d\nu$$

for all $A_k = \{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\} \subset X$, $k = 2, \ldots, n$.

5 Conclusion

In this paper we have studied the problem of existence of Radon-Nikodym-like derivatives for fuzzy measures. We have proven a theorem on the necessary and sufficient conditions based on the definition of compatible permutation for pairs of measures. We have introduced this definition. We have also shown how these results apply to possibility measures.

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