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COSET GEOMETRIES WITH TRIALITIES AND THEIR REDUCED INCIDENCE GRAPHS

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Abstract. In this article we explore combinatorial trialities of incidence geometries. We give a construction that uses coset geometries to construct examples of incidence geometries with trialities and prescribed automorphism group. We define the reduced incidence graph of the geometry to be the oriented graph obtained as the quotient of the geometry under the triality. Our chosen examples exhibit interesting features relating the automorphism group of the geometry and the automorphism group of the reduced incidence graphs.

1. Introduction

The projective space of dimension n over a field F is an incidence geometry $PG(n, F)$. It has n types of elements; the projective subspaces: the points, the lines, the planes, and so on. The elements are related by incidence, defined by inclusion.

A collinearity of $PG(n, F)$ is an automorphism preserving incidence and type. According to the Fundamental theorem of projective geometry, every collinearity is composed by a homography and a field automorphism. A duality of $PG(n, F)$ is an automorphism preserving incidence that maps elements of type k to elements of type $n - k - 1$. Dualities are also called correlations or reciprocities. Geometric dualities, that is, dualities in projective spaces, correspond to sesquilinear forms. Therefore the classification of the sesquilinear forms also give a classification of the geometric dualities. A polarity is a duality δ that is an involution, that is, $\delta^2 = \mathrm{Id}$. A duality can always be expressed as a composition of a polarity and a collinearity.

A projective configuration of points and lines is self-dual if it is preserved by a projective duality. More generally, a combinatorial configuration is self-dual if there is an incidence-preserving bijection between the points and the lines. Self-dual configurations were for example studied by Coxeter for their connection with interesting graphs [**4**].

The incidence graph of a combinatorial configuration is a bipartite graph with the points in one vertex set and the lines in the other, and an edge between a point and a line if they are incident. Artzy observed that the incidence graph of a

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self-dual configuration contained redundant information, and introduced a reduced incidence graph [**1**]. He also used the reduced incidence graph for finding planar realizations of self-dual combinatorial configurations over finite fields. Polarity graphs of finite projective planes and generalized quadrangles are examples of reduced incidence graphs. Polarity graphs have recently been used for finding vertex-transitive graphs of diameter two approaching the Moore bound [**2**].

Triality is a more obscure phenomenon than is duality, for several reasons. The history of triality goes back to Cartan [**3**]. Tits classified the geometric trialities with at least one absolute point [**9**]. Classic geometric triality occurs in a 6-dimensional hyperbolic quadric in $PG(7, \mathbb{F})$, as a cyclic permutation of an incidence geometry consisting of three types: points, 3-solids of type I and 3-solids of type II. Another way to see it is as a cyclic (non-linear) permutation between the three irreducible representations of Spin(8). This triality occurs as a consequence of the fact that the Dynkin diagram of D_4 has symmetry group S_3. The symmetries of the diagram give rise to outer automorphisms of the symmetry group of the projective quadric. The dualities in projective geometry also come from symmetries of the Dynkin diagram, but in that case the diagram is A_n and the symmetry is an involution.

Just as dualities and polarities can occur in incidence geometries combinatorially, without specifying a realization in projective geometry, so can trialities. We show how to construct incidence geometries with a given automorphism group G and a triality which is induced by an outer automorphism of G. Our examples use outer automorphisms induced by field automorphisms. They were also studied by Tits [**9**], but the geometries that we obtain appear to be new. Also, we use a generalization of Artzy's reduced incidence graph to represent the geometries, and discover an interesting relation between this graph and the field extension.

2. Constructing coset geometries with trialities

An incidence pregeometry is a quadruple $\Gamma = (X, *, t, I)$ where X is a set of elements, I is a set of types, t is a surjective type function $t\colon X \twoheadrightarrow I$ and $*$ is a binary relation on the set X, such that elements of the same type are not related. The cardinality $|I|$ is called the rank of Γ. The relation $*$ is called the incidence relation of Γ. As any symmetric binary relation, it can be represented using a symmetric matrix or an undirected graph. The graph representing the incidence relation is called the incidence graph of the incidence geometry. It is an n-partite graph, with the elements of each type in each partition.

A flag of a pregeometry is a set of pairwise incident elements. The rank of a flag is the number of types in the flag. Since all elements in a flag have different type, this coincides with the number of elements in the flag. The type of a flag F is the set $t(F)$. A chamber is a flag of type I. A pregeometry is a geometry (incidence geometry) if every flag is contained in a chamber.

The residue of a flag F in a geometry Γ is the set of elements in Γ that are not in F but that are incident with all elements of F. The residue of a flag in a geometry is a geometry. The rank of a residue of a flag is the number of types in the residue. Since Γ is a geometry, every flag is contained in a chamber, therefore the rank of the residue of a flag of rank k in a geometry of rank n is $n-k$.

A geometry is connected if its incidence graph is connected. It is residually connected if all residues of rank at least two are connected, as a geometry. Clearly any residually connected geometry of rank at least two is connected, being the connected residue of the empty flag.

If Γ is an incidence geometry of rank at least r, then an r-ality of Γ is an incidence-preserving automorphism of order r of Γ, permuting r types cyclically.

A coset pregeometry is an incidence pregeometry Γ constructed from a group G and a set of subgroups $\{H_1, \ldots, H_n\}$ of G (called maximal parabolic subgroups), so that

- the type set is $I = \{1, \ldots, n\}$,
- the elements are the right cosets of H_i,
- the type function maps every coset $H_i g$ to i and
- two elements are incident if their intersection is not empty.

A coset pregeometry that is a geometry is called a coset geometry. We are interested in geometries with trialities, but in principle, a pregeometry can also have a triality. Lemma 1 suggests a method for finding coset geometries with trialities.

Lemma 1. *A coset geometry* $\Gamma = (G, \{H_1, H_2, H_3\})$ *has a triality if* G *has an automorphism* σ *such that* $\sigma(H_1) = H_2$, $\sigma(H_2) = H_3$ *and* $\sigma(H_3) = H_1$.

Proof. Let G be a group and H_1, H_2, H_3 three subgroups of G. Assume that σ is as above stated, then σ maps cosets of H_i to cosets of $H_{i+1 \pmod 3}$. Also, if x and y are two cosets, then $x \cap y = \emptyset$ if and only if $\sigma(x) \cap \sigma(y) = \emptyset$. Therefore σ induces an incidence-preserving automorphism τ of Γ, permuting the types H_1, H_2 and H_3 cyclically, a triality. $\square$

Method 1. Given a group G, choose a subgroup $H_1 < G$ and a group automorphism σ of G as in Lemma 1. If the coset pregeometry $\Gamma = \big(G, \{H_1, \sigma(H_1), \sigma^2(H_1)\}\big)$ is a geometry, then it is a geometry with a triality.

A similar approach can be used to construct coset geometries with r-alities. This will be explored in later work.

3. The reduced incidence graph of a geometry with an r-ality

The reduced incidence graph $\mathcal{G}_\delta^R$ of an incidence geometry Γ of rank 2 with types $\{1, 2\}$ and a duality δ, is the graph constructed from the incidence graph $\mathcal{G}$ of Γ in the following way:

- Vertices of $\mathcal{G}_\delta^R$ are ordered pairs of vertices of $\mathcal{G}$ of the form $[x] = (x, \delta(x))$, where $t(x) = 1$.
- There is an edge from $[x]$ to $[y]$ if $x * \delta(y)$. If $x * \delta(x)$, then there is a half-edge at $[x]$.

The reduced incidence graph of a rank two geometry was first defined by Artzy in 1956 [**1**]. His definition was different from ours, in that it used loops instead of half-edges. Later the concept was interpreted in terms of voltage graphs by Pisanski.

Every edge of the reduced incidence graph represents two incidences: $x * \delta(y)$ and $\delta(x) * \delta^2(y)$. A half-edge represents only one incidence, the incidence between x and $\delta(x)$.

Lemma 2 ([**1**])**.** *The reduced incidence graph of an incidence geometry Γ of rank 2 with a duality δ is a lossless representation of Γ. However, the same geometry can have many different reduced incidence graphs, while the incidence graph is always uniquely determined.*

We now extend the notion of reduced incidence graph to an incidence geometry Γ of rank r with an incidence-preserving automorphism of Γ permuting r types cyclically (an r-ality).

Definition 3. The reduced incidence graph $\mathcal{G}_\sigma^R$ of a pregeometry Γ of rank r, with incidence graph $\mathcal{G}$, and an incidence-preserving automorphism σ permuting the r types $I = \{1, \ldots, r\}$ cyclically, is the labeled oriented graph defined as follows:

- Vertices are the ordered tuples of vertices from the incidence graph $\mathcal{G}$ of the form $[x] = (x, \sigma(x), \ldots, \sigma^{r-1}(x))$, with $t(x) = 1$.
- There is an oriented edge labeled σ^i from $[x]$ to $[y]$, with $x \neq y$, if $x * \sigma^i(y)$ for some $i \in \{1, \ldots, r-1\}$. If $x * \sigma^i(x)$ for some $i \in \{1, \ldots, r-1\}$, then there is a half-edge (a dart) at the vertex $[x]$, labeled σ^i.

The edges in the incidence graph are given by the incidence of Γ, and we see that a reduced incidence graph has $1/r$ the number of vertices and $1/r$ the amount of edges compared to the incidence graph. More precisely, the following lemma is easily proven.

Lemma 4. *The incidence graph $\mathcal{G}$ is a graph cover of degree r of the reduced incidence graph $\mathcal{G}_\sigma^R$. If the r-ality σ has order r, that is, if $\sigma^r = \mathrm{Id}$, then $\mathcal{G}_\sigma^R$ is the quotient graph under the action of σ on $\mathcal{G}$.*

For a background on graph covers, graph quotients and voltage graphs, see for example [**7**]. Note that the latter statement of Lemma 4 is not true for a duality δ which is not a polarity, because then there will be some orbit of an edge that has cardinality different from $r = 2$.

An r-ality σ for $r = 2$ is a duality. In that case, all edges have the same label: σ. The labels therefore do not carry any information and can be removed. Also, the orientations on the edges can be removed. When σ is a polarity, this is clear, since $a * \sigma(b)$ implies $\sigma(a) * \sigma^2(b)$, which is the same as $\sigma(a) * b$. When σ is not a polarity, orientation can still be removed, because $\mathcal{G}_\sigma^R$ can be considered to represent the polarity obtained from σ by setting $\sigma(x) := \sigma^{-1}(x)$ for all x of type 2. By removing the labels and orientations, Artzy's original definition of a reduced incidence graph of a rank two geometry is recovered.

If $r = 3$, then σ is a triality. The labels are σ and σ^2, and if $\sigma^3 = \mathrm{Id}$, then $a * \sigma(b)$ implies $\sigma^2(a) * b$. So edge orientation suffices to recover $\mathcal{G}$ from $\mathcal{G}_\sigma^R$. If

$\sigma^3 \neq \mathrm{Id}$, $\mathcal{G}_\sigma^R$ represents the triality obtained from σ by setting $\sigma^3 = (\sigma^2 \circ \sigma)^{-1}$. Therefore, for trialities, $\mathcal{G}_\sigma^R$ is an oriented graph with half-edges (no labels).

An absolute point of a duality δ is a point p such that $p \in \delta(p)$. Similarly, we define an absolute element of an r-ality σ to be an element x such that x and $\sigma(x)$ are incident. By definition, absolute points of σ correspond to half-edges in $\mathcal{G}_\sigma^R$. The study of absolute points of dualities and trialities has lead to many interesting results in incidence geometry, including the polar spaces and the generalized polygons [**6, 9**].

4. Examples

4.1. The Suzuki groups

The Suzuki groups can be constructed as matrix groups in four dimensions over a finite field $\mathbb{F}_q$ with $q = 2^{2n+1}$. Several constructions exist, for example [**8, 11**]. Tits gave a construction as automorphism groups of a non-classical ovoid in $PG(3, \mathbb{F})$[**10**] now commonly called the Suzuki-Tits ovoid. The order of $Sz(q)$ is $q^2(q^2+1)(q-1)$. The outer automorphism group is the cyclic group C_{2n+1}, induced by the field automorphisms acting on the coefficients of the matrices. Therefore, there will be outer automorphisms of order 3 if and only if $2n + 1$ is a multiple of 3. There are maximal subgroups of order $q^2(q-1)$, $2(q-1)$ and $4(q \pm \sqrt{2q} + 1)$ as well as subfield subgroups $Sz(q')$ where $q' = 2^m$ for every divisor m of $2n+1$ [**11**].

We want to construct a residually connected geometry with a triality. Therefore we want to pick a maximal subgroup H with the property that the cosets of $\sigma(H)$ and $\sigma^2(H)$ that have non-empty intersection with H should form a connected rank two coset geometry, where σ is the outer automorphism of the group of order 3.

For $n = 1$, so that $q = 8$, the construction gives a residually connected coset geometry Γ of rank 3 with 2080 elements of each type and a triality τ induced by the field automorphism. The type-preserving automorphism group of the geometry is $Sz(8)$, as expected. The incidence graph is 3-partite and has $3 \cdot 2080 = 6240$ vertices. The reduced incidence graph $\mathcal{G}_\tau^R$ is an oriented graph on 2080 vertices. The automorphism group of this oriented graph is $Sz(2)$, a group of order 20. The triality τ has 20 absolute points, giving 20 half-edges in $\mathcal{G}_\tau^R$. The 20 vertices with half-edges form a single orbit under the action of $Sz(2)$. The triples of elements in Γ represented by these 20 vertices form a subgeometry, on which $Sz(2)$ acts.

4.2. The projective special linear group of the line

Just as the Suzuki groups, the projective special linear group $PSL(2, q)$ has outer automorphisms induced by the field automorphisms of the field extension. By choosing q to be the cube of a power of a prime, we ensure that the automorphism group of $PSL(2, q)$ contains an automorphism σ of order 3.

For $q \in \{8, 27, 64, 125, 512\}$ we constructed a coset geometry that is thin, residually connected and flag-transitive, with a triality and type-preserving automorphism group $PSL(2, q)$ using Method 1. The automorphism in the case $q = 8$ is inner while in the other cases, it is outer. Following the pattern of the previous experiment, we calculated the reduced incidence graph of these geometries, using the triality. In all our examples, except for $q = 8$, the automorphism group of the reduced incidence graph of the coset geometry constructed using $q = p^n$, p a power of a prime, is $PSL(2, p)$.

4.3. An example from the smallest Mathieu group

Let G be the permutation representation of degree 12 of the Mathieu group M_{11} acting on the 12 cosets of one of its $PSL(2, 11)$ subgroups. Write the points $\{1, \ldots, 12\}$. Let G_i be the stabilizer of point I. Since G is 3-primitive (hence 3-transitive), the coset geometry $\Gamma(G, \{G_1, G_2, G_3\})$ is firm, residually connected and flag-transitive. Moreover, the type preserving automorphism group of Γ is S_{12} and the full automorphism group (including type-permuting automorphisms) is $S_{12} \times S_3$, meaning that this geometry has trialities. However, the automorphism group of G is M_{11}. The automorphism of G giving the triality is inner for M_{11}.

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