



BOUNDARY SMOOTHNESS PROPERTIES OF $Lip\alpha$ ANALYTIC FUNCTIONS

By

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Abstract. Let U be an open set and $b \in \text{bdy}(U)$. Let $0 < \alpha < 1$. Let $A(U)$ denote the space of $Lip\alpha$ functions that are analytic on U , and $a(U)$ the subspace $Lip\alpha \cap A(U)$. The space $a(U \cup \{b\})$, consisting of the functions that are analytic near b , is dense in $a(U)$. Let k be a natural number. We say that $a(U)$ admits a k -th order continuous point derivation (cpd) at b if the functional $f \mapsto f^{(k)}(b)$ is continuous on $a(U \cup \{b\})$, with respect to the $Lip\alpha$ norm.

Theorem $a(U)$ admits a k -th order cpd at b if and only if

$$\sum_{n=1}^{\infty} 2^{(k+1)n} M_*^{1+\alpha}(A_n(b) \sim U) < +\infty.$$

Here M_*^β denotes β -dimensional lower Hausdorff content, and $A_n(b)$ denotes the annulus

$$\{z \in \mathbb{C} : |z - b| \in [2^{-n-1}, 2^{-n}]\}.$$

There is a weak-star topology on $A(U)$, and the space $A(U \cup \{b\})$ is weak-star dense in $A(U)$. We say that $A(U)$ admits a k -th order weak-star cpd at b if the functional $f \mapsto f^{(k)}(b)$ is weak-star continuous on $A(U \cup \{b\})$.

Theorem $A(U)$ admits a k -th order weak-star cpd at b if and only if

$$\sum_{n=1}^{\infty} 2^{(k+1)n} M^{1+\alpha}(A_n(b) \sim U) < +\infty.$$

This time, M^β denotes ordinary β -dimensional Hausdorff content.

1. Introduction

Let $0 < \alpha < 1$. For $E \subset \mathbb{C}$ and $f : E \rightarrow \mathbb{C}$ let

$$\|f\|'_{Lip\alpha(E)} = \sup \left\{ \frac{|f(z) - f(w)|}{|z - w|^\alpha} : z \neq w \right\}.$$

We call $\|f\|'_{Lip\alpha(E)}$ the $Lip\alpha(E)$ seminorm of f . We denote

$$Lip\alpha(E) = \{f \in \mathbb{C}^E : \|f\|'_{Lip\alpha} < +\infty\}.$$

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This is a Banach space when endowed with the norm

$$\|f\|_{\text{Lip}\alpha} = |f(b)| + \|f'\|_{\text{Lip}\alpha},$$

where b is any fixed point of E . We abbreviate $\text{Lip}\alpha(\mathbb{C})$ to $\text{Lip}\alpha$. The subspace $\text{lip}\alpha \subset \text{Lip}\alpha$ consists of those $f \in \text{Lip}\alpha$ such that

$$\lim_{\delta \downarrow 0} \sup_{0 < |z-w| < \delta} \frac{|f(z) - f(w)|}{|z - w|^\alpha} = 0.$$

For open sets $U \subset \mathbb{C}$ we denote

$$A(U) = \{f \in \text{Lip}\alpha : \bar{\partial}f = 0 \text{ on } U\},$$

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Here $\bar{\partial}f$ denotes the distributional $\bar{\partial}$ -derivative

$$\bar{\partial}f = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$

In view of Weyl's Lemma, " $\bar{\partial}f = 0$ on U " is a way of saying that the restriction $f|_U$ is an analytic function.

This paper is about the extent to which the functions belonging to $A(U)$ or $a(U)$ may be better-behaved at points of $\text{bdy}U$ than are typical elements of $\text{Lip}\alpha$ or $\text{lip}\alpha$. Specifically, we consider the question of the existence of bounded point derivations. We will explain this concept shortly. First, we review some classical facts.

Suppose b is an isolated point of $\text{bdy}U$. Then, since the elements $f \in A(U)$ are bounded and analytic on a deleted neighbourhood of b , it follows that they extend analytically across b , and since they are continuous, they are already analytic on $U \cup \{b\}$.

Similarly, if a line segment I forms a relatively-open subset of $\mathbb{C} \sim U$, then each function $f \in A(U)$ extends analytically across I .

These facts may be rephrased in terms of the concept of $\bar{\partial}$ - $\text{Lip}\alpha$ -null set:

A compact $K \subset \mathbb{C}$ is said to be $\bar{\partial}$ - $\text{Lip}\alpha$ -null if

$$A(U \sim K) = A(U)$$

whenever $U \subset \mathbb{C}$ is open.

Singletons and line segments are $\bar{\partial}$ - $\text{Lip}\alpha$ -null.

Not every compact set K having no interior is $\bar{\delta}$ -Lip-null. For instance, if K has positive area, then the function

$$f(z) = \frac{-1}{\pi} \int_K \frac{d\xi d\eta}{z - (\xi + i\eta)}$$

(the Cauchy transform of area restricted to K) belongs to each $\text{Lip}\alpha$ ($\alpha < 1$), and is analytic on $\mathbb{C} \sim K$, and nonconstant. Dolženko characterised the $\bar{\delta}$ -Lip α -null compact sets in terms of Hausdorff contents.

A *measure function* is a monotone nondecreasing function $h : [0, +\infty) \rightarrow [0, +\infty)$. The *Hausdorff content* M_h associated to a measure function h is defined by

$$M_h(E) = \inf_{\mathcal{S}} \sum_{B \in \mathcal{S}} h(\text{diam} B),$$

whenever $E \subset \mathbb{C}$, where \mathcal{S} runs over all countable coverings of E by balls (or, equivalently, open balls, or closed balls, or arbitrary sets). When $h(r) = r^\beta$, we denote $M_h = M^\beta$.

Dolženko's result is that a compact set K is $\bar{\delta}$ -Lip α -null if and only if $M^{1+\alpha}(K) = 0$ [3].

A similar result holds for $\text{lip}\alpha$. The *lower β -dimensional Hausdorff content* of E is

$$M_*^\beta(E) = \sup_h M_h(E)$$

where h runs over all measure functions such that $h(r) \leq r^\beta$ and $r^{-\beta}h(r) \rightarrow 0$ as $r \downarrow 0$. We say that K is $\bar{\delta}$ -lip α -null if

$$a(U \sim K) = a(U), \quad \forall U \text{ open.}$$

The result [11] is that K is $\bar{\delta}$ -lip α -null if and only if $M_*^{1+\alpha}(K) = 0$.

For example, if C is the usual middle-thirds Cantor set, then the square Cantor set $C \times C$ has

$$M^{1+\alpha}(C \times C) = 0 \Leftrightarrow \alpha > \log_3 4/3,$$

$$M_*^{1+\alpha}(C \times C) = 0 \Leftrightarrow \alpha \geq \log_3 4/3,$$

so $C \times C$ is $\bar{\delta}$ -Lip α -null if and only if $\alpha > \log_3 4/3$ and $\bar{\delta}$ -lip α -null if and only if $\alpha \geq \log_3 4/3$.

Obviously, if all functions $f \in A(U)$ extend analytically across a boundary point a , then they are as smooth as can be. But it may happen that limited smoothness occurs even at points which are not of this type.

For an arbitrary set $E \subset \mathbb{C}$, let

$$A(E) = \bigcup \{A(U) : U \text{ open}, E \subset U\},$$

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The spaces $A(E)$, $a(E)$ are closed subspaces of $\text{Lip}\alpha$ in case E is open.

Lemma 1.1 *Let $b \in \mathbb{C}$ and $U \subset \mathbb{C}$ be open. Then $a(U \cup \{b\})$ is dense in $a(U)$.*

Accepting this for the moment, we note that for $k \in \mathbb{N}$ the maps

$$\delta_b^k : f \mapsto f^{(k)}(b)$$

are well-defined linear functionals on $A(U \cup \{b\})$.

Definition We say that $a(U)$ admits a k -th order continuous point derivation (cpd) at b if δ_b^k extends to a continuous linear functional on $a(U)$.

Equivalently, $a(U)$ admits a k -th order cpd at b if and only if there exists a constant $\kappa > 0$ such that

$$|f^{(k)}(b)| \leq \kappa \|f\|_{\text{Lip}\alpha}$$

whenever $f \in a(U \cup \{b\})$.

We denote

$$A_n(b) = \{z \in \mathbb{C} : 2^{-n} < |z - b| \leq 2^{-n-1}\}.$$

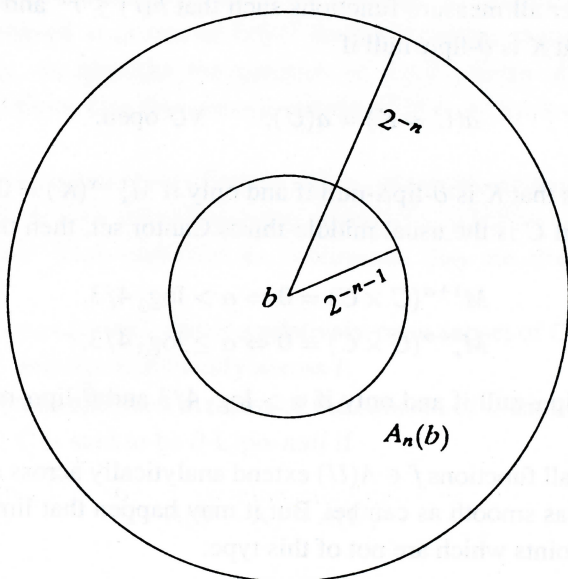


Fig. 1.

Our first main result is:

Theorem 1.2 Let $0 < \alpha < 1$, $U \subset \mathbb{C}$ be open, $b \in \mathbb{C}$, and $k \in \mathbb{N}$. Then $A(U)$ admits a k -th order cpd at b if and only if

$$\sum_{n=1}^{+\infty} 2^{(k+1)n} M_*^{1+\alpha}(A_n(b) \sim U) < +\infty.$$

The second main result is a similar theorem for $A(U)$. It involves *weak-star* continuous cpd's.

Let b be a point of the interior of some (large) closed disk D . The restriction space

$$\text{Lip}\alpha(D) = \{f|D : f \in \text{Lip}\alpha\}$$

is a Banach space (indeed, a Banach algebra) with the quotient norm. Similarly, for $\text{lip}\alpha(D)$. De Leeuw showed that

$$\text{lip}\alpha(D)^{**}$$

is isometrically isomorphic to $\text{Lip}\alpha(D)$. Thus $\text{Lip}\alpha(D)$ acquires a weak-star topology, as the dual of $\text{lip}\alpha(D)^*$. When we refer to weak-star topological concepts in the sequel, we intend that the topology be of this kind, for some suitably large D .

Lemma 1.3 Let $b \in \mathbb{C}$ and $U \subset \mathbb{C}$ be open. Then $A(U \cup \{b\})$ is weak-star dense in $A(U)$.

We say that $A(U)$ admits a k -th order weak-star cpd at b if δ_b^k extends to a weak-star continuous linear functional on $\text{Lip}\alpha(D)$. Whether this happens or not does not depend on the choice of D (with $b \in \text{int}(D)$). The condition may be expressed in terms of the De Leeuw representation (see 2.7 below): $A(U)$ admits a k -th order weak-star continuous point derivation at b if and only if there exists a finite-total-variation Borel-regular measure ρ on $\mathbb{C} \times \mathbb{C}$, having no mass on the diagonal, such that

$$f^{(k)}(b) = \int_{\mathbb{C} \times \mathbb{C}} \frac{(f(z) - f(w)) d\rho(z, w)}{|z - w|^\alpha}$$

whenever $f \in A(U \cup \{b\})$.

Theorem 1.4 Let $0 < \alpha < 1$, $U \subset \mathbb{C}$ be open, $b \in \mathbb{C}$, and $k \in \mathbb{N}$. Then $A(U)$ admits a k -th order weak-star continuous point derivation at b if and only if

$$\sum_{n=1}^{+\infty} 2^{(k+1)n} M^{1+\alpha}(A_n(b) \sim U) < +\infty.$$

These results are in a line of development which goes back to Wiener's work [16] on the problem of regular boundary points for the Dirichlet problem. Series involving capacities of intersections with annuli are called Wiener series. Wiener series have been used a lot, to characterise various kinds of thinness, and in connection with analytic functions, for instance in Melnikov's characterisation of peak points for the algebra $R(X)$ [7], and in [6], [5], [8–10,12], [15]. Like most Wiener series characterizations, the present results are quite concrete, and allow us to determine whether or not a cpd exists at any boundary point of a sufficiently-precisely described set U .

We prove the results in Section 2, and give a few illustrative examples in Section 3.

2. Proof of Results

2.1. We begin by recalling extension theorems for Lipschitz functions. Each real-valued Lip1 function on a subset of a metric space may be extended to the whole space, so that it remains Lip1 with the same constant. This is Kirszbraun's theorem [4]. Applying this to \mathbb{C} , with the metric

$$\rho(x, y) = |x - y|^\alpha$$

(remember that $0 < \alpha < 1$), we see that each real-valued Lip α function on a subset of \mathbb{C} has an extension having the same Lip α seminorm. Applying this to the real and imaginary parts, we see that each complex-valued Lip α function f on a subset X of \mathbb{C} has a Lip α extension \tilde{f} to \mathbb{C} having at most twice the seminorm:

$$\|\tilde{f}\|_{\text{Lip}\alpha(\mathbb{C})} \leq 2 \cdot \|f\|_{\text{Lip}\alpha(X)}.$$

There is a similar result for lip α functions. Given a lip α function on a subset $E \subset \mathbb{C}$, we may choose a nondecreasing concave function $\omega(r)$ such that ω dominates the modulus of continuity of f and has $r^{-\alpha}\omega(r) \rightarrow 0$ as $r \downarrow 0$. Moreover, we can arrange that $r^{-\alpha}\omega(r)$ is bounded by the Lip α seminorm of f . Applying Kirszbraun's theorem to \mathbb{C} with the metric

$$\rho(x, y) = \omega(|x - y|),$$

we find that f has an extension in lip α that has at most double the Lip α seminorm of f .

2.2. Next, we recall **Frostman's Lemma**. According to this, if a Borel set $E \subset \mathbb{R}^d$ has positive M_h content, for some measure function h , then it is possible to

find a positive measure ν , with compact support contained in E , such that the total mass of ν exceeds $M_h(E)$, and such that ν has growth $h(r)$, in the sense that

$$\nu \mathbb{B}(b, r) \leq \kappa h(r)$$

whenever $b \in \mathbb{C}$ and $r > 0$. Here, the constant κ may be taken to depend only on the dimension d . See [1].

If $M_*^\beta(E) > 0$, then the measure ν may be chosen to have compact support in E with

$$\nu(\mathbb{C}) \geq M_*^\beta(E),$$

$$\nu(\mathbb{B}(b, r)) \leq \kappa \cdot r^\beta, \quad \forall b \in \mathbb{C} \text{ and } r > 0,$$

and

$$\frac{\sup_b \nu(\mathbb{B}(b, r))}{r^\beta} \rightarrow 0 \quad \text{as } r \downarrow 0.$$

2.3. The Lip_α seminorm of a product is controlled by the Lip_α seminorm and the sup norm of the factors:

$$\|fg\|'_{\text{Lip}_\alpha} \leq \|f\|'_{\text{Lip}_\alpha} \cdot \|g\|_{L^\infty} + \|g\|'_{\text{Lip}_\alpha} \cdot \|f\|_{L^\infty}.$$

Also, the Lip_α seminorm of a differentiable function on a bounded set is controlled by the sup norm of the derivative:

$$\|g\|'_{\text{Lip}(\alpha, X)} \leq \|Dg\|_{L^\infty} \cdot \text{diam}(X)^{1-\alpha}.$$

Putting these facts together, we obtain the following estimate for the Lip_α seminorm of $f(z)/(z - b)^k$ on annuli about the point b :

Lemma 2.3 *Let f belong to Lip_α and let $b \in \mathbb{C}$. There is a constant κ_k , independent of $n \in \mathbb{N}$, such that*

$$\left\| z \mapsto \frac{f(z)}{(z - b)^k} \right\|'_{\text{Lip}_\alpha(A_n(b))} \leq \kappa \cdot 2^{kn} \cdot \|f\|_{\text{Lip}_\alpha(A_n(b))}.$$

(Note that f may be assumed to have a zero on A_n , without loss in generality.)

2.4. We recall the Decay Estimates for Lipschitz holomorphic functions.

Decay Lemma *Let $0 < \alpha < 1$, let $K \subset \mathbb{C}$ be compact and let $f \in \text{Lip}_\alpha(\mathbb{C})$ be analytic off K and vanish at ∞ . Then there are constants $\kappa > 0$ and $\kappa_k > 0$, depending on α but not on K or f , such that the following estimates hold for $z \notin K$:*

$$(1) \quad \|f\|_{L^\infty} \leq \kappa \cdot \|f\|'_{\text{Lip}_\alpha} \cdot M^{1+\alpha}(K)^{\frac{\alpha}{1+\alpha}},$$

$$(2) \quad |f(z)| \leq \frac{\kappa \cdot \|f\|'_{\text{Lip}_\alpha} \cdot M^{1+\alpha}(K)}{\text{dist}(z, K)},$$

and if $k \in \mathbb{N}$ then

$$(3) \quad |f^{(k)}(z)| \leq \frac{\kappa_k \cdot \|f\|'_{\text{Lip}\alpha} \cdot M^{1+\alpha}(K)}{\text{dist}(z, K)^{k+1}}.$$

See [11, section 12] and [13, section 7] (The argument for (3) is a routine extension of the argument for (1) that is given in the former paper.)

2.5. We recall the T-invariance properties of $\text{Lip}\alpha$.

The Vitushkin localisation operator, T_ϕ , is defined by

$$(T_\phi)f = C(\phi \cdot \bar{\partial}f),$$

for all distributions f and all test functions ϕ , where C denotes the Cauchy transform:

$$Cg = \frac{-1}{\pi z} * g,$$

for all distributions g having compact support. In view of the fact that C inverts $\bar{\partial}$ on the compactly-supported distributions, we see that $T_\phi f$ is analytic off the support of ϕ and off the support of $\bar{\partial}f$. The spaces $\text{Lip}\alpha$ and $\text{lip}\alpha$ (like all translation-symmetric concrete spaces) are mapped into themselves by the action of T_ϕ [14]. But they also have the additional property of *nice* T-invariance. This means that they are mapped equicontinuously by the sequence of operators $\{T_{\phi_n}\}$ whenever $\{\phi_n\}$ is a *standard pincher*. To be precise:

Lemma *Let ϕ_n ($n = 1, 2, 3, \dots$) be a C^∞ function having compact support, and be such that*

- (1) $\text{spt}\phi_n$ is a subset of the ball $\mathbb{B}(0, 1/n)$,
 - (2) $\sup_n \|\phi_n\|_{L^\infty} \leq \kappa_1$, where $\kappa_1 < +\infty$ is a constant independent of n .
 - (3) $\|\nabla\phi_n\|_{L^\infty} \leq \kappa_2 \cdot n$, where $\kappa_2 < +\infty$ is a constant independent of n .
- Let $0 < \alpha < 1$. Then there is a constant $\kappa(\kappa_1, \kappa_2, \alpha) > 0$ such that

$$\|T_{\phi_n} f\|_{\text{Lip}\alpha(\mathbb{C})} \leq \kappa \|f\|_{\text{Lip}\alpha(\mathbb{B}(0, 1/n))}, \quad \forall n.$$

This follows from Lemma (4.1) in [13] (see (2.2) of that paper for the definition of the quantity $N(\phi)$ there referred to).

2.6. We now prove a Quantitative Cauchy Theorem for Lipschitz functions.

Theorem (a) *Let Γ be a piecewise analytic curve bounding a region $\Omega \in \mathbb{C}$, and suppose that Γ is free of outward-pointing cusps. Let $0 < \alpha < 1$. Then there exists a constant $\kappa(\Gamma, \alpha) > 0$ such that*

$$\left| \int_\Gamma f(z) dz \right| \leq \kappa \cdot M^{1+\alpha}(\Omega \cap S)$$

whenever $\|f\|'_{\alpha, \mathbb{C}} \leq 1$, S is closed, and f is analytic on $\Omega \sim S$.

(b) The constant $\kappa(\Gamma, \alpha)$ depends only on the equivalence class of Γ under the action of the conformal group of \mathbb{C} . In other words, it is the same for any curve obtained from Γ by rigid motion and dilation.

(c) A similar result holds with $\text{Lip}\alpha$ replaced by $\text{lip}\alpha$ and $M^{1+\alpha}$ replaced by $M_*^{1+\alpha}$.

Versions of this theorem have appeared previously. For the case when Γ is a circle, it was essentially proven by Dolzhenko (cf. [Garnett 1972, p.65, Lemma 2.2]). As we shall see, the general case is not far from the circle case.

Proof. 1. It is enough to prove the result for the case when Γ is a simple closed Jordan curve, i.e. $\text{bdy}\Gamma$ is connected. Once this case is proven, the general case follows on cutting up Ω into a finite number of pieces and using the subadditivity of $M^{1+\alpha}$.

2. For the same reason, it is enough to prove the case in which Γ consists of three analytic arcs, making a curvilinear triangle in which none of the vertices is an outward-pointing cusp. We may also assume that S is a subset of $\Omega \cup \Gamma$.

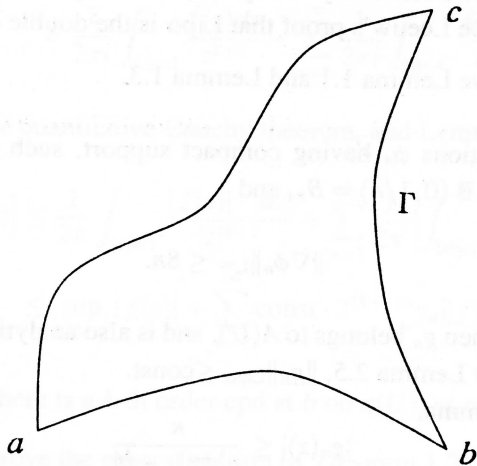


Fig. 2.

3. Using a mixture of rectilinear triangles and curvilinear triangles partly bounded by pieces of Γ , we can cover S by a finite number of patches P_n such that (1) each P_n is bounded by a piecewise-analytic curvilinear triangle,

$$(2) \quad M^{1+\alpha}(S) \geq \text{const} \cdot \sum_n (\text{diam} P_n)^{1+\alpha},$$

and (3) $\text{perimeter}(P_n) \leq \text{const} \cdot \text{diam}(P_n)$. This depends upon the fact that there are no outward cusps in Γ . The constants depend on the shape of Γ but do not change if Γ is rescaled, translated, or rotated.

4. Then, using Cauchy's Theorem, and denoting by c_n any chosen point of P_n , we get

$$\begin{aligned} \left| \int_{\Gamma} f(z) dz \right| &\leq \sum_n \left| \int_{\text{bdy} P_n} (f(z) - f(c_n)) dz \right| \\ &\leq \text{const} \sum_n \text{diam}(P_n) \cdot \text{diam}(P_n)^\alpha \\ &\leq \kappa \cdot M^{1+\alpha}(S). \end{aligned}$$

□

2.7. We recall the De Leeuw representation of $\text{lip}\alpha^*$ [2]. If $T \in \text{lip}\alpha(B)^*$, then it can be represented as follows. There is a Borel-regular measure ρ on $B \times B$, having no mass on the diagonal, having finite total variation, and such that

$$Tf = \int \frac{(f(z) - f(w)) d\rho(z, w)}{|z - w|^\alpha}$$

whenever $f \in \text{lip}\alpha(B)$. The proof is a simple application of the Hahn-Banach theorem and the F. Riesz representation theorem. Using this representation the double-dual action of a function $f \in \text{Lip}\alpha(B)$ on T is given by the same formula. (In essence, that is De Leeuw's proof that $\text{Lip}\alpha$ is the double dual of $\text{lip}\alpha$.)

2.8. We now prove Lemma 1.1 and Lemma 1.3.

Fix $f \in A(U)$.

Choose C^∞ functions ϕ_n having compact support, such that $\phi_n = 1$ near b , $0 \leq \phi_n \leq 1$, $\text{spt}\phi_n \subset \mathbb{B}(0, 1/n) = B_n$, and

$$\|\nabla\phi_n\|_{L^\infty} \leq 8n.$$

Form $g_n = T_{\phi_n} f$. Then g_n belongs to $A(U)$, and is also analytic on the complement of B_n . Moreover, by Lemma 2.5, $\|g_n\|_{\text{Lip}\alpha} \leq \text{const}$.

By the Decay Lemma,

$$|g_n(z)| \leq \frac{\kappa}{n^{1+\alpha}|z - b|}$$

whenever $|z - b| > 1/n$, and thus by a simple argument

$$|g_n(z)| \leq \kappa/n^\alpha, \quad \forall z \in \mathbb{C}.$$

Since $\|g_n\|_{\text{Lip}\alpha}$ is bounded, we deduce from the De Leeuw representation and the Lebesgue dominated convergence theorem that $g_n \rightarrow 0$ weak-star in $\text{Lip}\alpha$. Thus $f - g_n \rightarrow f$ weak-star, and $f - g_n \in A(U \cup \{b\})$. This proves Lemma 1.3.

If $f \in a(U)$, we may use the fact that $T_\phi 1 = 0$ to get the estimate

$$\|T_{\phi_n} f\|_{\text{Lip}\alpha} \leq \text{const} \cdot \|f - f(b)\|_{\text{Lip}\alpha(B_n)} \rightarrow 0,$$

and so conclude that in that case $g_n \rightarrow 0$ in $\text{Lip}\alpha$ norm. Thus we obtain Lemma 1.1. \square

2.9. Now we prove the first part of Theorem 1.2: If the series converges, then there is a k -th order cpd on $a(U)$ at the point b .

We may suppose that $b = 0$.

Let $\gamma_n = M_*^{1+\alpha}(A_n(b) \sim U)$. Suppose that

$$\sum_n 2^{(k+1)n} \gamma_n < +\infty.$$

Let $f \in A(U \cup \{0\})$. Choose N such that f is analytic on $\mathbb{B}(0, 2^{-N})$. Then by the Cauchy integral formula,

$$\begin{aligned} f^{(k)}(0) &= \frac{1}{2\pi i} \int_{|z|=2^{-N}} \frac{f(z) dz}{z^{k+1}} \\ &= \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z) dz}{z^{k+1}} - \sum_{n=1}^{N-1} \frac{1}{2\pi i} \int_{\text{bdy}A_n} \frac{f(z) dz}{z^{k+1}} \\ &= \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z) dz}{z^{k+1}} - \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{\text{bdy}A_n} \frac{f(z) dz}{z^{k+1}}, \end{aligned}$$

and hence, using the quantitative Cauchy theorem, and Lemma 2.3, we get

$$\begin{aligned} |f^{(k)}(0)| &\leq \frac{1}{2\pi} \int_{|z|=1} \frac{|f(z)| \cdot |dz|}{|z|^{k+1}} + \sum_{n=1}^{\infty} \frac{1}{2\pi} \left| \int_{\text{bdy}A_n} \frac{f(z) dz}{z^{k+1}} \right| \\ &\leq \sup_{\mathbb{B}(0,1)} |f(z)| + \sum_{n=1}^{\infty} \text{const} \cdot 2^{(k+1)n} \gamma_n \|f\|_{\text{Lip}\alpha}. \end{aligned}$$

This shows that there is a k -th order cpd at b on $a(U)$, as required. \square

2.10. Next, we prove the other direction of Theorem 1.2.

Let γ_n be as in the last subsection, and suppose that

$$\sum_n 2^{(k+1)n} \gamma_n = +\infty.$$

Choose $\epsilon_n \downarrow 0$ such that

$$\sum_{n=1}^{\infty} 2^{(k+1)n} \epsilon_n \gamma_n = +\infty,$$

and $2^{(k+1)n} \epsilon_n \gamma_n \leq 1$ for all n .

Applying Frostman's Lemma, we obtain, for each n , a positive measure ν_n on $A_n \sim U$ such that

$$\nu_n \mathbb{B}(z, r) \leq \epsilon_n \cdot r^{1+\alpha}, \quad \forall z \in \mathbb{C}, \forall r > 0,$$

$$\sup_z \frac{\nu_n \mathbb{B}(z, r)}{r^{1+\alpha}} \rightarrow 0 \quad \text{as } r \downarrow 0,$$

and

$$\|\nu_n\| = \kappa \epsilon_n \gamma_n.$$

Let

$$f_n(z) = \int \left(\frac{\zeta}{|\zeta|} \right)^{k+1} \frac{d\nu_n(\zeta)}{\zeta - z}.$$

Then $\|f_n\|_{\text{Lip}\alpha} \leq \kappa \epsilon_n$, $f_n \in a(U)$, f_n is analytic off A_n , and

$$f_n^{(k)}(0) = k! \int \frac{d\nu_n(\zeta)}{|\zeta|^{k+1}}$$

lies between fixed constant multiples of $2^{(k+1)n} \epsilon_n \gamma_n$.

For each n , choose $p \geq n$ such that

$$\sum_n^p 2^{(k+1)m} \epsilon_m \gamma_m \in [1, 2].$$

Let

$$g_n(z) = \sum_{m=n}^p f_m(z).$$

From the Decay Lemma, we get

$$|f_m(z)| \leq \text{const} \cdot \epsilon_m \cdot 2^{-\alpha m},$$

$$|f_m(z)| \leq \frac{\text{const} \cdot \epsilon_m \gamma_m}{\text{dist}(z, A_m)},$$

$$|f'_m(z)| \leq \frac{\text{const} \cdot \epsilon_m \gamma_m}{\text{dist}(z, A_m)^2}.$$

Fix $z, w \in \mathbb{C}$ with $z \neq w$. Fix m with $n \leq m \leq p$. The following five cases cover all the possibilities:

Case 1: z or w belongs to A_{m-1} or A_m or A_{m+1} .

Case 2: $|z| > 2^{-m+1}$ and $|w| > 2^{-m+1}$, and $|z - w| < 2^{-m}$.

Case 3: $|z| < 2^{-m-2}$ and $|w| < 2^{-m-2}$.

Case 4: $|z| > 2^{-m+1}$ and $|w| > 2^{-m+1}$, and $|z - w| \geq 2^{-m}$.

Case 5: $|z| < 2^{-m-2}$ and $|w| > 2^{-m+1}$.

In Case 1, we have

$$\frac{|f_m(z) - f_m(w)|}{|z - w|^\alpha} \leq \|f_m\|_{\text{Lip}\alpha} \leq \text{const} \cdot \epsilon_n.$$

In Case 2 or Case 3, we may connect z to w using an arc Γ of length at most $\pi|z - w|$ such that

$$|\zeta - t| \geq 2^{-m-2}, \quad \forall \zeta \in \Gamma \quad \forall t \in A_m.$$

Thus integrating the decay estimate for $f_m^t(\zeta)$ gives

$$\frac{|f_m(z) - f_m(w)|}{|z - w|^\alpha} \leq \kappa \cdot 2^{2m} \epsilon_m \gamma_m \cdot 2^{(\alpha-1)n}.$$

In Case 4 or Case 5, we get

$$\begin{aligned} \frac{|f_m(z) - f_m(w)|}{|z - w|^\alpha} &\leq \frac{|f_m(z)| + |f_m(w)|}{2^{-\alpha(m+2)}} \\ &\leq \text{const} \cdot 2^{2m} \epsilon_m \gamma_m \cdot 2^{(\alpha-1)n}. \end{aligned}$$

Thus we have

$$\begin{aligned} \frac{|g_n(z) - g_n(w)|}{|z - w|^\alpha} &\leq 6\kappa \cdot \epsilon_n + \kappa \cdot 2^{(\alpha-1)n} \sum_{m=n}^p 2^{2m} \epsilon_m \gamma_m \\ &\leq \kappa \cdot (\epsilon_n + 2^{(\alpha-1)n}). \end{aligned}$$

Thus the functions g_n have $\text{Lip}\alpha$ norms tending to zero, and yet $g_n^{(k)}(0)$ is bounded away from zero, whence there is no cpd at 0.

This completes the proof of Theorem 1.2. \square

2.11. There is a weak-star continuous point derivation on $A(U)$ at b if and only if there is an estimate

$$|f^{(k)}(0)| \leq \text{const} \cdot \|f\|_{\text{Lip}\alpha},$$

valid for $f \in A(U \cup \{b\})$. Thus the argument just given needs practically no change to prove Theorem 1.4. \square

3. Examples

3.1. If $\text{bdy}U$ is smooth near b , then there is no cpd at b . More generally, if there is a sector

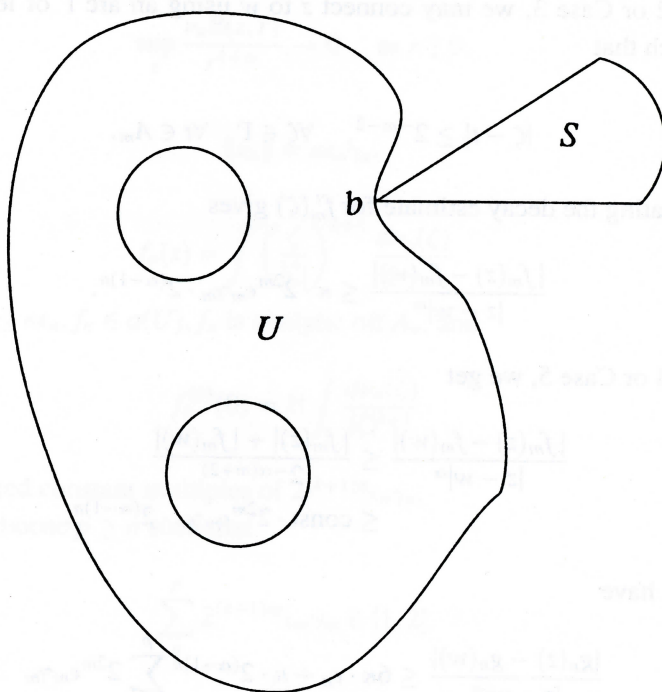


Fig. 3.

$$S = \{z \in \mathbb{C} : 0 \leq |z - b| \leq \delta, |\phi - \arg z| \leq \epsilon\}$$

with $\delta > 0$ and $\epsilon > 0$, that lies outside U , then there is no cpd at b . For, given such a sector, and $n > \log_2 \delta$, the set $A_n(b) \sim U$ will contain a disk of radius $\eta 2^{-n}$ for some $\eta > 0$, independent of n . Then

$$M_*^{1+\alpha}(A_n \sim U) \geq (\eta 2^{-n})^{1+\alpha},$$

so the series

$$\sum_n 4^n M_*^{1+\alpha}(A_n \sim U)$$

diverges.

3.2. Let $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be a monotone increasing function, and let U be the set

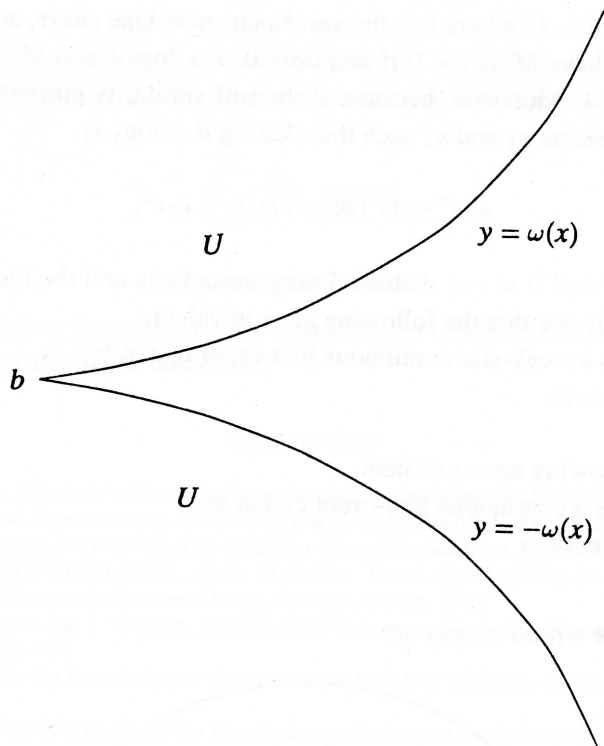


Fig. 4.

$$U = \{z = x + iy : |y| \leq \omega(x) \text{ if } x > 0\}.$$

If $\omega(r)/r \rightarrow 0$ as $r \downarrow 0$, then $\mathbb{C} \sim U$ has a cusp at 0. In this situation, there are positive constants κ_1 and κ_2 such that $A_n(0) \sim U$ contains a ball of radius $\kappa_1 \omega(2^{-n})$ and is contained in a ball of radius $\kappa_2 \omega(2^{-n})$. Thus there is a cpd of order k on $a(U)$ at 0 if and only if

$$\sum_n 2^{(k+1)n} \omega(2^{-n})^{1+\alpha} < +\infty.$$

For instance, if the region is that outside the cubic cusp

$$U = \{x + iy : |y| > x^{3/2}\},$$

then there is a first-order cpd on $a(U)$ as soon as $\alpha > 1/3$, but there is a second-order cpd for no $\alpha < 1$.

For an exponential cusp, there are cpd's of every order for every $\alpha > 0$.

For this kind of set the *same* condition characterises the existence of weak-star continuous cpd's on $A(U)$.

3.3. Let $U = \mathbb{C} \sim \Gamma$, where Γ is the von Koch snowflake curve, and let b be any point of Γ . We have $M^\beta(\Gamma) = 0$ if and only if $\beta > \log_3 4$ and $M_*^\beta(\Gamma) = 0$ if and only if $\beta \geq \log_3 4$. Moreover, because of the self-similarity properties of Γ , there are positive constants κ_1 and κ_2 such that (letting $d = \log_3 4$)

$$\kappa_1 r^d \leq M^d(\mathbb{B}(z, r) \cap \Gamma) \leq \kappa_2 r^d,$$

whenever $z \in \Gamma$ and $0 < r < \text{diam}\Gamma$. Using these facts and the theorems proved above, we readily see that the following are equivalent:

- (1) $A(U)$ admits a weak-star continuous first-order cpd at b ;
- (2) Γ is $\bar{\delta}$ -Lip α -null;
- (3) $\alpha > \log_3 4$.

Also, the following are equivalent:

- (1') $a(U)$ admits a continuous first-order cpd at b ;
- (2') Γ is $\bar{\delta}$ -lip α -null;
- (3') $\alpha \geq \log_3 4$.

3.4. Let U be a road-runner set

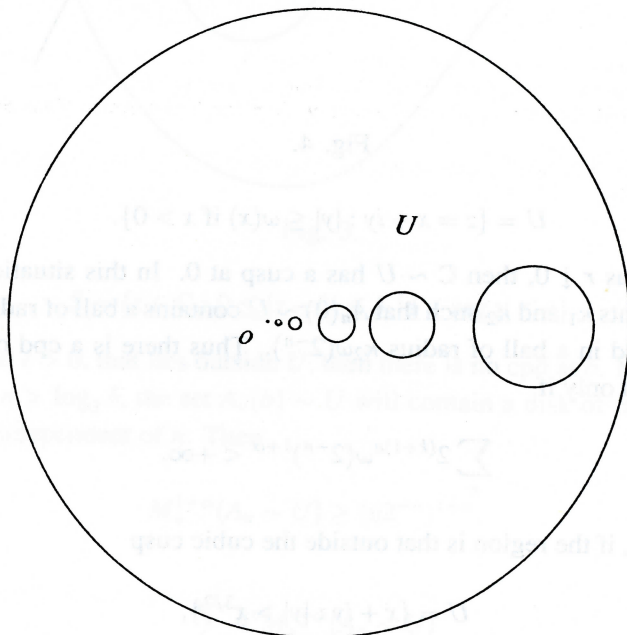


Fig. 5.

$$U = \mathbb{U}(0, 1) \sim \bigcup_{n=1}^{\infty} \mathbb{B}(a_n, r_n),$$

where $a_n > 0$, $r_n > 0$, and $0 < a_{n+1} + r_{n+1} < a_n - r_n < 1$, for each n . Then application of the above theorems shows that $a(U)$ admits a k -th order cpd at 0 if and only if $A(U)$ admits a k -th order weak-star cpd, and that both happen if and only if

$$\sum_{n=1}^{\infty} \frac{r_n}{a_n^{k+1}} < +\infty.$$

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