STABILITY OF NATURAL ENERGY FUNCTIONALS AT RIEMANNIAN SUBIMMERSIONS

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Abstract. We derive variational formulas of natural first order functionals and obtain criteria for stability in particular at Riemannian subimmersions.

1. Introduction

Any function $\Phi\colon \operatorname{Sym}^+(m) \to \mathbb{R}_0^+$ on symmetric $(m\times m)$ matrices which is invariant under conjugation by $\mathrm{O}(m)$ gives rise to a functional $E_\Phi\colon C^\infty(M^m, X^k) \to \mathbb{R}_0^+$. Here (M^m, g) and (X^k, h) denote compact Riemannian manifolds of dimensions m and k, carrying Riemannian metrics g and h, respectively. We denote by $df^* \in \Gamma \operatorname{Hom}(f^*TX, TM)$ the adjoint of the differential $df \in \Gamma(\operatorname{Hom}(TM, f^*TX))$ with respect to the Riemannian metrics on M and X. More explicitely it is defined by the condition that $h(df\, v, x) = g(v, df^* x)$ holds for any $p \in M$ and $v \in T_pM$, $x \in T_{f(p)}X$. The Φ-energy of a smooth map $f\colon M \to X$ is the integral

$$E_\Phi(f) = \int_M \Phi(df^* df) \ .$$

Essentially the Φ-energies are the natural locally computable functionals whose density depends explicitely only on the first derivatives of f and does not involve (derivatives of) the Riemannian curvature tensors (cf. [2], [10]).

Examples of such functionals are the 2-energy, $\Phi(A) = \operatorname{Tr}(A)$, the p-energies, $\Phi(A) = (\operatorname{Tr}(A))^{p/2}$ (cf. [6]), the exponential energy, $\Phi(A) = e^{\operatorname{Tr}(A)}$ (cf. [7], [5]), but also $\Phi(A) = \operatorname{Tr}(A^p)$, $\Phi(A) = \operatorname{Tr}(e^A)$, the volume and the Jacobians, $\Phi(A) = (\sigma_l(A))^q$, where $\sigma_l(A)$ denotes the l-th elementary symmetric polynomial in the eigenvalues of A. By a theorem of Glaeser, [8], all these functions Φ may be written as $\Phi(A) = \Phi^s(\operatorname{Tr}(A), \operatorname{Tr}(A)^2, \ldots, \operatorname{Tr}(A)^m)$ with some smooth function $\Phi^s\colon \mathbb{R}^m \to \mathbb{R}$. Functionals with $\Phi(A) = F(\operatorname{Tr}(A))$ have been studied by Ara in [1].

In this paper we will derive (un)stability criteria for the Φ-energies at Riemannian subimmersions. These are maps f such that $df^* df$ is an orthogonal projection of constant rank. Examples of such maps are immersions (cf. [5], [6], [1], [3]) and Riemannian submersions like G-maps $f\colon G/H \to G/K$, $H \subset K \subset G$, between compact normal homogeneous spaces.

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By a Φ-harmonic map we mean a critical point for E_Φ. Such a map f is called stably Φ-harmonic if the index form, i.e., the second variation, of E_Φ is nonnegative at f. At a Riemannian subimmersion f of rank k these properties depend on a few parameters determined by Φ: Let $\lambda_k(\Phi), \lambda'_k(\Phi), \mu_k(\Phi), \nu_k(\Phi) \in \mathbb{R}$ be the parameters for Φ defined by expanding the first and second derivatives of Φ at the orthogonal projection $\mathfrak{p}_k \colon \mathbb{R}^m \to \mathbb{R}^k$ in terms of traces (cf. (2.8) and (2.9))

$$d_{\mathfrak{p}_k}\Phi A = \lambda_k \operatorname{Tr} A + \lambda'_k \operatorname{Tr} A_{\mathcal{V}} \quad \text{and} \quad d^2_{\mathfrak{p}_k}\Phi(A) = \mu_k \operatorname{Tr}(A^2_{\mathcal{H}}) + \nu_k(\operatorname{Tr} A)^2 + \dots \ .$$

In Proposition 3.2 we show that for all Φ with $\lambda_k(\Phi) \neq 0$ the Φ-harmonic Riemannian subimmersions are those with minimal image and fibres. If $\lambda_k(\Phi) = 0$, then any Riemannian subimmersion is Φ-harmonic.

The second variation of the 2-energy E_{Tr} has always finite index at a harmonic map but usually there are few stably critical maps of the 2-energy. For instance, the identity map on the standard k-sphere S^k is unstable for E_{Tr} if $k \geq 3$, cf. [13]. More generally, a stable harmonic map $S^m \to X$ or $X \to S^k$ is constant if $m, k > 2$, cf. [16], [11]. In these assertions the sphere may be replaced by certain Riemannian symmetric spaces, see [12], [9]. In [1], [5], [6] stability criteria for isometries have been derived for functionals of the type $\int_M F(\|df\|)$, $F\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$, such as the exponential energy or the p-energy. For Riemannian submersions in this case, see [14].

For the Φ-energy at a Riemannian subimmersion we compute the leading symbol of the second variation. By Proposition 3.8 the second variation $d^2_f E_\Phi$ of the Φ-energy at a minimal Riemannian subimmersion f has finite index if $\lambda_k(\Phi)$, $\lambda_k(\Phi) + \lambda'_k(\Phi)$, $\lambda_k(\Phi) + 2\mu_k(\Phi) + 2\nu_k(\Phi)$ and $\lambda_k(\Phi) + \mu_k(\Phi)$ are all positive.

Any Riemannian submersion $f\colon M \to X$ of rank k with totally geodesic fibres is Φ-harmonic. For vector fields v along f the second variation of E_Φ at f is given in Theorem 3.10. Some results of Urakawa on the 2-energy, cf. [15], immediately carry over to these more general functionals. For instance, if the identity map on X is Φ-unstable, then the same holds for f (cf. (3.13)). Conversely, if the identity map on X is stably Φ-harmonic, we can make f stably Φ-harmonic by shrinking its fibres. For equivariant maps among normal homogeneous spaces one can explicitely compute the index of the second variation by translating the Jacobi operator into Lie theory and computing its small eigenvalues, analogous to [15]. This will be pursued elsewhere.

2. Preliminaries

2.1. Variational formulas for the Φ-Energy

We first recall the general variational formulas for the Φ-energy from [3]. We may always extend Φ to a function $\Phi\colon \mathrm{M}(m) \to \mathbb{R}$ on all real $(m \times m)$-matrices which remains invariant under conjugation by orthogonal matrices. A straightforward calculation yields the following formulas for the first and second variations. Denote by ∇^M, ∇^X the Levi-Civita connections on M, X, and by ∇ the induced connection on f^*TX or $TM^* \otimes f^*TX$. A smooth map $f\colon M \to X$ is Φ-harmonic, i.e., a critical point of the Φ-energy if

$$d_f E(v) = \frac{d}{dt} E_\Phi(f_t) = \int_M d_{df^*df}\Phi((\nabla v)^* df + df^* \nabla v) = 0 \tag{2.1}$$

for any smooth one parameter variation $f_t = F(t, \cdot)$, $F\colon (-\epsilon, \epsilon) \times M \to X$ with $v = (d/dt)\, f_t$. In [3] the tension field $\tau_\Phi(f) \in \Gamma f^*TX$ defined by $(d/dt)\, E_\Phi(f_t) = \langle \tau_\Phi(f) \mid w\rangle$ is computed. For the 2-energy, $\Phi = d\Phi = \mathrm{Tr}$, $d^2\Phi = 0$, we have

$$\tau(f) = \tau_{\mathrm{Tr}}(f) = \mathrm{Tr}\, \nabla df \ .$$

In [3, Proposition 2.3] the index form of the Φ-energy at a Φ-harmonic map $f\colon M \to X$ is computed. It is given by

$$\begin{aligned} d_f^2 E(v,w) &= \frac{d^2}{dtds} E_\Phi(f_{t,s}) = \int_M I_\Phi(f)(v,w) \ , \text{ where} \\ I_\Phi(f)(v,w) &= d_{df^*df}\Phi\left(-df^* R^X(\cdot, v)w - df^* R^X(\cdot, w)v\right) \\ &\quad + d_{df^*df}\Phi\left((\nabla v)^*(\nabla w) + (\nabla w)^*(\nabla v)\right) \\ &\quad + d^2_{df^*df}\Phi\left(df^*\nabla v + (\nabla v)^* df, df^* \nabla w + (\nabla w)^* df\right) \ . \end{aligned} \tag{2.2}$$

The vector fields $v = (d/dt)\, f_{t,s}$, $w = (d/ds)\, f_{t,s}$ along f are variation fields of a smooth 2-parameter family $f_{t,s} = F(t,s,\cdot)$, $F\colon (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \times M \to X$. By R^X we denote the Riemannian curvature tensor of X and by $df^* R^X(\cdot, w)v$ the homomorphism $\mathcal{R}_{v,w}\colon TM \to TM$ defined by $\langle \mathcal{R}_{v,w} x \mid y\rangle = \langle R^X(dfx, w)v \mid dfy\rangle$.

The second variation of E_Φ can be written in the form $d_f^2 E_\Phi(v,w) = \langle J_\Phi(f)v \mid w\rangle$ with a second order differential operator, the Jacobi operator of Φ, acting on vector fields v, w along f. Its leading symbol is calculated from (2.2).

Corollary 2.3 (see [3, Proposition 2.5])**.** *The leading symbol of the second variation of E_Φ is given by*

$$\begin{aligned} \langle \sigma(\xi) v \mid w\rangle = d^2\Phi\left(df^* v \otimes \xi + \xi \otimes df^* v, df^* w \otimes \xi + \xi \otimes df^* w\right) \\ + 2\langle v \mid w\rangle d_{df^*df}\Phi(\xi \otimes \xi) \ , \end{aligned}$$

where $\xi \in TM = TM^*$ by the Riemannian metric, $v, w \in f^*TX$. If $\sigma(\xi) > 0$ for all $\xi \neq 0$, then $J_\Phi(f)$ is elliptic and the second variation of E_Φ has finite index.

2.2. **A Bochner Formula**

In Section 3 some stability criteria of Riemannian submersions will be derived by comparing (2.2) with the following Bochner formula. With $\|A\|^2 = \mathrm{Tr}\, A^* A$ and $B = \nabla df$,

$B(v)$, $B(df^*v)$ defined by $\langle B(v)x \mid y\rangle = \langle v \mid (\nabla_x df)y\rangle$, resp. $B(df^*v) = \nabla_{df^*v}df$, a straightforward calculation gives for a closed manifold M that

$$\begin{aligned}\frac{1}{2}\int_M \|df^*\nabla v + (\nabla v)^* df\|^2 = \int_M \|df^*\nabla v\|^2 - \operatorname{Tr}\left(R^X(\cdot, df^*v)v\right) + (\operatorname{Tr}(df^*\nabla v))^2 \\ - \langle \nabla_{df^*v} v \mid \tau(f)\rangle - \operatorname{Tr}\left(df^*\nabla v B(v)\right) \\ + \langle \tau(f) \mid v\rangle \operatorname{Tr}\left(df^*\nabla v\right) + \operatorname{Tr}\left((\nabla v)^* B(df^*v)\right) \ .\end{aligned} \tag{2.4}$$

2.3. **Derivatives of Φ at a projection**

The endomorphism df^*df of a Riemannian subimmersion f is the projection onto the horizontal distribution. We therefore need the derivatives of Φ at a projection. Decomposing $\mathbb{R}^m = \mathbb{R}^k \oplus \mathbb{R}^{m-k}$ and accordingly

$$\operatorname{Sym}(m) = \operatorname{Sym}(k) \oplus \operatorname{Sym}(m-k) \oplus \operatorname{M}(k \times (m-k)) \ , \tag{2.5}$$

we determine the components of $d\Phi$ and $d^2\Phi$. Since the projection $\mathfrak{p}_k$ onto the first factor $\mathbb{R}^k$ is fixed by conjugation with $\operatorname{O}(k) \times \operatorname{O}(m-k)$, the first and second derivatives of Φ at $\mathfrak{p}_k$ can be expressed in terms of traces and involve only few parameters. In order to derive this expansion we choose a curve $X(s) \in \operatorname{Sym}(m)$ through $X(0) = \mathfrak{p}_k$ and a skew symmetric matrix

$$v = \begin{pmatrix} P & Z \\ -Z^t & Q \end{pmatrix} \in \mathfrak{so}(m)$$

with $P \in \mathfrak{so}(k)$, $Q \in \mathfrak{so}(m-k)$ and $Z \in \operatorname{M}(k \times (m-k))$. Since Φ is invariant under conjugation by $\operatorname{O}(m)$, we have

$$\left.\frac{d}{dt}\right|_{t=0} \Phi(e^{-tv}X(s)e^{tv}) = d_{X(s)}\Phi[X(s), v] = 0 \quad \text{and} \tag{2.6}$$

$$\left.\frac{d^2}{dsdt}\right|_{s=0,t=0} \Phi(e^{-tv}X(s)e^{tv}) = d^2_{\mathfrak{p}_k}\Phi(\dot{X} \otimes [\mathfrak{p}_k, v]) + d_{\mathfrak{p}_k}\Phi[\dot{X}, v] = 0 \ . \tag{2.7}$$

The only linear invariants on the first two summands $\operatorname{Sym}(k)$ and $\operatorname{Sym}(m-k)$ in (2.5) are multiples of traces. Since

$$[\mathfrak{p}_k, v] = \begin{pmatrix} 0 & Z \\ Z^t & 0 \end{pmatrix} ,$$

it follows from (2.6) that $d_{\mathfrak{p}_k}\Phi$ vanishes on the third summand $\operatorname{M}(k \times (m-k))$ in (2.5). Hence there are $\lambda_k = \lambda_k(\Phi), \lambda'_k = \lambda'_k(\Phi) \in \mathbb{R}$ such that

$$d_{\mathfrak{p}_k}\Phi A = \lambda_k \operatorname{Tr} A + \lambda'_k \operatorname{Tr} A_{\mathcal{V}} \tag{2.8}$$

for $A \in \mathrm{Sym}(m)$ and $A_{\mathcal{H}} \in \mathrm{Sym}(k)$, $A_{\mathcal{V}} \in \mathrm{Sym}(m-k)$ and $A_{\mathcal{VH}} \in \mathrm{M}(k \times (m-k))$ such that

$$A = \begin{pmatrix} A_{\mathcal{H}} & A_{\mathcal{VH}} \\ A_{\mathcal{VH}}^t & A_{\mathcal{V}} \end{pmatrix} .$$

For the second derivative we get from (2.7) with $\dot{X} = \begin{pmatrix} 0 & Y \\ Y^t & 0 \end{pmatrix}$, resp. $\dot{X} = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}$, that

$$\begin{aligned} d^2_{\mathfrak{p}_k}\Phi(\begin{pmatrix} 0 & Y \\ Y^t & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & Z \\ Z^t & 0 \end{pmatrix}) &= -d_{\mathfrak{p}_k}\Phi\left[\left(\begin{pmatrix} 0 & Y \\ Y^t & 0 \end{pmatrix}, \begin{pmatrix} P & Z \\ -Z^t & Q \end{pmatrix}\right)\right] \\ &= -2\lambda'_k \operatorname{Tr}(Y^t Z) \end{aligned}$$

and

$$\begin{aligned} d^2_{\mathfrak{p}_k}\Phi(\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \otimes \begin{pmatrix} 0 & Z \\ Z^t & 0 \end{pmatrix}) &= -d_{\mathfrak{p}_k}\Phi\left[\left(\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}, \begin{pmatrix} P & Z \\ -Z^t & Q \end{pmatrix}\right)\right] \\ &= -\lambda'_k \operatorname{Tr}(SQ) = 0 , \end{aligned}$$

since S is symmetric and Q is skew symmetric. Finally, since $\mathfrak{p}_k$ is fixed under conjugation with $\mathrm{O}(k) \times \mathrm{O}(m-k)$ and since the only quadratic $\mathrm{O}(q)$-invariants on $B \in \mathrm{Sym}(\mathbb{R}^q)$ are linear combinations of $\operatorname{Tr}(B^2)$ and $(\operatorname{Tr}(B))^2$, we get that

$$d^2_{\mathfrak{p}_k}\Phi(\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \otimes \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix})$$

is a linear combination of

$$\operatorname{Tr}(RU) , \operatorname{Tr}(SV) , \operatorname{Tr}(R)\operatorname{Tr}(U) , \operatorname{Tr}(S)\operatorname{Tr}(V) , \operatorname{Tr}(S)\operatorname{Tr}(U) + \operatorname{Tr}(R)\operatorname{Tr}(V) .$$

The second derivative may therefore be put into the form

$$\begin{aligned} d^2_{\mathfrak{p}_k}\Phi(A) &= d^2_{\mathfrak{p}_k}\Phi(A \otimes A) \\ &= \mu_k \operatorname{Tr}(A_{\mathcal{H}}^2) + \nu_k(\operatorname{Tr} A)^2 - 2\lambda'_k \operatorname{Tr}(A^t_{\mathcal{VH}} A_{\mathcal{VH}}) \\ &\quad + \mu'_k \operatorname{Tr}(A^2_{\mathcal{V}}) + \nu'_k(\operatorname{Tr} A_{\mathcal{V}})^2 \\ &\quad + \kappa_k \operatorname{Tr} A_{\mathcal{H}} \operatorname{Tr} A_{\mathcal{V}} \end{aligned} \tag{2.9}$$

with coefficients $\mu_k = \mu_k(\Phi), \mu'_k = \mu'_k(\Phi), \ldots \in \mathbb{R}$ determined by Φ. Only the coefficients λ_k, λ'_k, μ_k, ν_k will show up in the index form. We list these for some examples in Table 1. Here $\sigma_p(A)$ denotes the p-th elementary symmetric polynomial in the eigenvalues. It is determined by the relation

$$\det(1 + tA) = \sum_{p=0}^{m} \sigma_p(A) t^p .$$

$\Phi(A)$	$\operatorname{Tr} A$	$\operatorname{Tr}(A^p),\ p>1$	$(\operatorname{Tr} A)^p$	$e^{\operatorname{Tr} A}$	$\operatorname{Tr} e^A$	$\sigma_p(A)$
λ_k	1	p	pk^{p-1}	e^k	e	$\binom{k-1}{p-1}$
λ'_k	0	$-p$	0	0	$1-e$	$\binom{k-1}{p-2}$
μ_k	0	$p(p-1)$	0	0	e	$-\binom{k-2}{p-2}$
ν_k	0	0	$p(p-1)k^{p-2}$	e^k	0	$\binom{k-2}{p-2}$

Table 1. Coefficients for some functionals

3. Riemannian subimmersions

A Riemannian, or metric, subimmersion of rank k is a map $f\colon M \to X$ such that for every point $p \in M$ there exist a neighborhood U of p, a k-dimensional manifold Y, a metric submersion $\pi\colon U \to Y$ and an isometric immersion $\imath\colon Y \to X$ such that $f|_U = \imath \circ \pi$. We denote by $\mathcal{V} = \ker df$ and $\mathcal{H} = (\ker df)^\perp \subset TM$ its vertical and horizontal distributions and by $\mathcal{N} = (\operatorname{im} df)^\perp \subset f^*TX$ the pull back of the normal bundle of M in X. We denote by $X_\mathcal{V} = \mathfrak{p}_\mathcal{V} X$, $X_\mathcal{H} = \mathfrak{p}_\mathcal{H} X$ and $X^\perp = \mathfrak{p}_\perp X$ the orthogonal projections onto $\mathcal{V}$, $\mathcal{H}$ and $\mathcal{N}$, respectively. Let $T^{\ker} \in \Gamma \operatorname{Hom}(\mathcal{V}\otimes\mathcal{V},\mathcal{H})$ and $T^{\operatorname{im}} \in \Gamma \operatorname{Hom}(\mathcal{H}\otimes\mathcal{H},\mathcal{N})$ be the second fundamental forms of the fibres and the image of f, respectively. Let $A \in \Gamma \operatorname{Hom}(\mathcal{H}\otimes\mathcal{H},\mathcal{V})$ be the tensor field $A(h,h') = \mathfrak{p}_\mathcal{V} \nabla_h h'$. We will denote the various adjoints by the same letters and write

$$\begin{aligned}
\langle A(h,h') \mid u\rangle &= \langle \nabla^M_h h' \mid u\rangle = 1/2\,\langle u \mid [h,h']\rangle = -\langle h' \mid \nabla^M_h u\rangle = -\langle A(u)h \mid h'\rangle \\
&= -\langle h' \mid A(h,u)\rangle = -\langle A(h)h' \mid u\rangle = -\langle \nabla^M_{h'} h \mid u\rangle\ , \\
\langle h \mid T^{\ker}(u,u')\rangle &= \langle h \mid \nabla^M_u u'\rangle = -\langle \nabla^M_u h \mid u'\rangle = -\langle T^{\ker}(h)u \mid u'\rangle\ , \\
\langle r \mid T^{\operatorname{im}}(h,h')\rangle &= \langle r \mid T^{\operatorname{im}}(h)h'\rangle = \langle r \mid \nabla^X_h h'\rangle = -\langle \nabla^X_h r \mid h'\rangle = -\langle T^{\operatorname{im}}(r)h \mid h'\rangle
\end{aligned}$$

for vector (fields) $h,h' \in \mathcal{H}$, $u,u' \in \mathcal{V}$, $r \in \mathcal{N}$, see [4]. The second fundamental form ∇df may be expressed in terms of the second fundamental forms T^{im}, $T^{\ker}$ and A,

$$(\nabla_X df)Y = T^{\operatorname{im}}(X_\mathcal{H},Y_\mathcal{H}) - T^{\ker}(X_\mathcal{V},Y_\mathcal{V}) - A(X_\mathcal{H},Y_\mathcal{V}) - A(Y_\mathcal{H},X_\mathcal{V})\ . \tag{3.1}$$

In the sequel (3.1) will be used to simplify the expressions (2.1) and (2.2) for the first and second variation at a Riemannian subimmersion of rank k.

3.1. The tension field at Riemannian subimmersions

For the tension field, (2.1) and (2.8) yield the following

Proposition 3.2. *If f is a metric subimmersion of rank k, then we have*

$$\tau(f) = \operatorname{Tr} T^{\operatorname{im}} - \operatorname{Tr} T^{\ker} \tag{3.3}$$

and

$$\tau_\Phi(f) = \lambda_k(\Phi)\tau(f)\ . \tag{3.4}$$

In particular, all Φ *with* $\lambda_k(\Phi) \neq 0$ *have the same critical subimmersions of rank* k *as the* 2*-energy. A Riemannian subimmersion is harmonic for such* Φ *if and only if both the image and the fibres are minimal.*

Proof. The first formula (3.3) is immediate from (3.1) since the terms involving the A-tensor do not contribute to the trace $\tau(f) = \operatorname{Tr}\nabla df$.

For the second formula (3.4) we identify $T_pM \cong \mathbb{R}^m = \mathbb{R}^k \oplus \mathbb{R}^{m-k} \cong \mathcal{H}_p \oplus \mathcal{V}_p$, $\mathfrak{p}_k = \mathfrak{p}_\mathcal{H}$ and $\mathfrak{p}_{n-k} = \mathfrak{p}_\mathcal{V}$, and apply (2.8). The integrand in the first variational formula (2.1) thus becomes

$$d_{df^*df}\Phi((\nabla v)^*df + df^*\nabla v) = \lambda_k \operatorname{Tr}((\nabla v)^*df + df^*\nabla v) + \lambda_k' \operatorname{Tr}((\nabla v)^*df + df^*\nabla v)_\mathcal{V}\ .$$

Since $df \circ \mathfrak{p}_\mathcal{V} = 0$, the second term on the right hand side vanishes and we finally obtain

$$d_{df^*df}\Phi((\nabla v)^*df + df^*\nabla v) = \lambda_k \operatorname{Tr}((\nabla v)^*df + df^*\nabla v)\ .$$

But this is λ_k times the integrand in the first variational formula for the 2-energy. Therefore $\tau_\Phi(f) = \lambda_k(\Phi)\tau(f)$.

Finally if $\tau(f)$ vanishes then both $\operatorname{Tr} T^{\mathrm{im}} \in \Gamma\mathcal{N}$ and $\operatorname{Tr} T^{\ker} \in \Gamma\mathcal{H}$ vanish. □

3.2. **The index form at Riemannian subimmersions**

The first and second derivative of Φ at $\mathfrak{p}_\mathcal{H} = df^*df$ are given by (2.8) and (2.9). As before we will identify $v \in f^*TX \cong \mathcal{H} \oplus \mathcal{N} \subset TM \oplus \mathcal{N}$ and $\nabla v \in TM^* \otimes f^*TX \cong TM^* \otimes (\mathcal{H} \oplus \mathcal{N}) \subset TM^* \otimes (TM \oplus \mathcal{N})$.

In order to evaluate the index form (2.2) on a vector field $v \in \Gamma f^*TX = \Gamma(\mathcal{H} \oplus \mathcal{N})$ we need to compute (2.8) for $A_1 = df^*R^X(v, df\cdot)v + (\nabla v)^*(\nabla v)$ for the terms involving the first derivative of Φ and (2.9) with

$$A_2 = df^*\nabla v + (\nabla v)^*df = \begin{pmatrix} \mathfrak{p}_\mathcal{H}\,((\nabla v)^* + \nabla v)\,\mathfrak{p}_\mathcal{H} & \mathfrak{p}_\mathcal{H}\,\nabla v\,\mathfrak{p}_\mathcal{V} \\ \mathfrak{p}_\mathcal{V}(\nabla v)^*\,\mathfrak{p}_\mathcal{H} & 0 \end{pmatrix}$$

for the second order terms.

With $\operatorname{Tr} A^*A = \|A\|^2$ and $\operatorname{Tr} A_{1,\mathcal{V}} = \|\nabla v\|^2_\mathcal{V} = \sum_{i=k+1}^m |\nabla_{e_i} v|^2$, for a local orthonormal framing $\{e_{k+1}, \dots, e_m\}$ of $\mathcal{V}$, we infer

$$\begin{aligned} I_\Phi(f)(v,v) = 2\lambda_k \left(\|\nabla v\|^2\ - \operatorname{Tr}\langle R^X(df\cdot, v)v \mid df\cdot\rangle\right) + 2\lambda_k'\|\,\mathfrak{p}^\perp\,\nabla v\|^2_\mathcal{V} \\ + \mu_k \operatorname{Tr}\left(A^2_{2,\mathcal{H}}\right) + 4\nu_k(\operatorname{Tr}\mathfrak{p}_\mathcal{H}\,\nabla v)^2 \end{aligned} \tag{3.5}$$

Let $\operatorname{div} x = \operatorname{Tr} \nabla^M x$ denote the divergence of a vector field x on M. We will need the following relations.

$$
\begin{aligned}
\mathfrak{p}_\perp \nabla v_{\mathcal{H}} \, \mathfrak{p}_{\mathcal{V}} &= \mathfrak{p}_{\mathcal{H}} \nabla v^\perp \, \mathfrak{p}_{\mathcal{V}} = 0 \ , \\
\operatorname{Tr}\left(A^2_{2,\mathcal{H}}\right) &= \operatorname{Tr}\left((\mathfrak{p}_{\mathcal{H}} \nabla v)^* + \mathfrak{p}_{\mathcal{H}} \nabla v)^2\right) \\
&= \operatorname{Tr}\left((\mathfrak{p}_{\mathcal{H}} \nabla v_{\mathcal{H}})^* + \mathfrak{p}_{\mathcal{H}} \nabla v_{\mathcal{H}})^2\right) + 4\|T^{\mathrm{im}}(v^\perp)\|^2 \\
&\qquad + 8 \operatorname{Tr}\left((\mathfrak{p}_{\mathcal{H}} \nabla v_{\mathcal{H}} \, \mathfrak{p}_{\mathcal{H}}) T^{\mathrm{im}}(v_{\mathcal{H}})\right) \ , \\
\operatorname{Tr} \mathfrak{p}_{\mathcal{H}} \nabla v &= \operatorname{Tr} \mathfrak{p}_{\mathcal{H}} \nabla v_{\mathcal{H}} + \operatorname{Tr} \mathfrak{p}_{\mathcal{H}} \nabla v^\perp \\
&= \operatorname{div} v_{\mathcal{H}} - \operatorname{Tr} T^{\ker}(v_{\mathcal{H}}) + \operatorname{Tr} T^{\mathrm{im}}(v^\perp) \ .
\end{aligned} \tag{3.6}
$$

Inserting (3.6) into (3.5) yields

$$
\begin{aligned}
I_\Phi(f)(v,v) = 2\lambda_k &\left(\|\nabla v\|^2 \ - \operatorname{Tr}\langle R^X(df\cdot, v)v \mid df\cdot\rangle\right) + 2\lambda'_k \|\nabla^\perp v^\perp\|^2_{\mathcal{V}} \\
&+ \mu_k \left[\operatorname{Tr}\left((\mathfrak{p}_{\mathcal{H}} \nabla v_{\mathcal{H}})^* + \mathfrak{p}_{\mathcal{H}} \nabla v_{\mathcal{H}})^2\right) + 4\|T^{\mathrm{im}}(v^\perp)\|^2 \right. \\
&\qquad \left. + 8 \operatorname{Tr}\left((\mathfrak{p}_{\mathcal{H}} \nabla v_{\mathcal{H}} \, \mathfrak{p}_{\mathcal{H}}) T^{\mathrm{im}}(v_{\mathcal{H}})\right)\right] \\
&+ 4\nu_k \left(\operatorname{div} v_{\mathcal{H}} - \operatorname{Tr} T^{\ker}(v_{\mathcal{H}}) + \operatorname{Tr} T^{\mathrm{im}}(v^\perp)\right)^2 \ .
\end{aligned} \tag{3.7}
$$

This specializes to the formulae in [3] for isometric immersions. As in (2.3) we get

Proposition 3.8. *The leading symbol of the second variation of a functional E_Φ at a minimal Riemannian subimmersion of rank k is*

$$
\sigma(\xi) = 2\lambda_k|\xi|^2 + 2\lambda'_k|\xi_{\mathcal{V}}|^2 \, \mathfrak{p}_\perp + 2\mu_k \xi_{\mathcal{H}} \otimes \xi_{\mathcal{H}} + 2\mu_k |\xi_{\mathcal{H}}|^2 \, \mathfrak{p}_{\mathcal{H}} + 4\nu_k \xi_{\mathcal{H}} \otimes \xi_{\mathcal{H}} \ .
$$

In particular, the p-energy, the exponential energies and the Jacobians have elliptic second variation with positive symbol and finite index at Riemannian subimmersions, cf. Table 1.

3.3. **Riemannian submersions with totally geodesic fibres**

We will consider in more detail the case of Riemannian submersions with totally geodesic fibres. Then $T^{\mathrm{im}} = 0$, $T^{\ker} = 0$, $v = v_{\mathcal{H}} \in \mathcal{H} \subset TM$ and

$$
\operatorname{Tr}\langle R^X(v, df\cdot)v \mid df\cdot\rangle = -\operatorname{Ric}^X(v,v) = -\operatorname{Ric}^M(v,v) - 2\|A(v)\|^2 \ .
$$

The Bochner formula (2.4) becomes

$$
\int_M \frac{1}{2} \|\nabla v + (\nabla v)^*\|^2 = \int_M \|\nabla v\|^2 - \operatorname{Ric}^X(v,v) + (\operatorname{div} v)^2 - 2 \operatorname{Tr}\left(\nabla v \circ A(v)\right) \ . \tag{3.9}
$$

The left hand side is

$$
\frac{1}{2} \|\nabla v + (\nabla v)^*\|^2 = \frac{1}{2} \| \, \mathfrak{p}^*_{\mathcal{H}} \operatorname{L}_v g\|^2 + \|\nabla v\|^2_{\mathcal{V}} \ ,
$$

where $\mathfrak{p}^*_{\mathcal{H}} \operatorname{L}_v g$ denotes the horizontal component of the Lie derivative of the metric of M. From (3.7) and (3.9) we therefore get the following expressions for the second variation.

Theorem 3.10. *The index form of a Riemannain submersion $f\colon M \to X$ with totally geodesic fibres is given by*

$$
\begin{aligned}
(3.11)\int_M I_\Phi(f)(v,v) &= \int_M 2\lambda_k \left(\|\nabla v\|^2 - \mathrm{Ric}^X(v,v)\right) + \mu_k \| \mathfrak{p}_{\mathcal{H}}^* \mathrm{L}_v\, g\|^2 + 4\nu_k (\operatorname{div} v)^2 \\
&= \int_M 2\lambda_k \left(\|\nabla v\|^2_{\mathcal{V}} + 2\,\mathrm{Tr}\,(\nabla v \circ A(v))\right) \\
&\qquad + (\lambda_k + \mu_k)\| \mathfrak{p}_{\mathcal{H}}^* \mathrm{L}_v\, g\|^2 + (4\nu_k - 2\lambda_k)(\operatorname{div} v)^2 \\
(3.12)\qquad &= \int_M (2\lambda_k + 2\mu_k)\left(\|\nabla v\|^2 - \mathrm{Ric}^X(v,v)\right) - 2\mu_k \|\nabla v\|^2_{\mathcal{V}} \\
&\qquad + (2\mu_k + 4\nu_k)(\operatorname{div} v)^2 - 4\mu_k\,\mathrm{Tr}\,(\nabla v \circ A(v))\ .
\end{aligned}
$$

3.4. **Applications**

There are a number of immediate consequences of these formulas. Let $f\colon M^m \to X^k$ as before be a Riemannian submersion of rank k with totally geodesic fibres and $\Phi\colon\ \mathrm{M}(m) \to \mathbb{R}$ be $\mathrm{O}(m)$-invariant. In case $\lambda_k(\Phi), \mu_k(\Phi), \nu_k(\Phi) \geq 0$, it follows from (3.11) that f is stably Φ-harmonic if it is stably harmonic.

3.4.1. *Positive elliptic symbol.* By Proposition 3.8 the second variation of E_Φ has positive symbol at f if

$$\lambda_k(\Phi),\ \lambda_k(\Phi) + \mu_k(\Phi),\ \lambda_k(\Phi) + 2\mu_k(\Phi) + 2\nu_k(\Phi) > 0\ .$$

Hence in this case we always have finite index and nullity of the second variation of E_Φ. Some results of Urakawa, [15], for the 2-energy extend to such Φ-energies. For instance, from (3.11), the second variation of E_Φ at f restricted to basic vector fields coincides with the second variation of E_Φ at the identity map of X. Hence the index, nullity and smallest eigenvalue of the Jacobi operator $J_\Phi(f)$ can be compared to the corresponding quantities of the identity map on X by (cf. [15], Proposition 6.3)

$$
\begin{aligned}
&\operatorname{index} d^2_f E_\Phi \geq \operatorname{index} d^2_{\mathrm{id}_X} E_\Phi\ , \\
(3.13)\qquad &\operatorname{nullity} d^2_f E_\Phi \geq \operatorname{nullity} d^2_{\mathrm{id}_X} E_\Phi\ , \\
&\lambda_1(J_\Phi(f)) \leq \lambda_1(J_\Phi(\mathrm{id}_X)\ .
\end{aligned}
$$

Combining this with the instability result of [3, Theorem 3] yields

Proposition 3.14. *If Φ has $\lambda_k(k-2) > 2\mu_k + 2\nu_k k$, then any Riemannian submersion $f\colon M^m \to S^k$ with totally geodesic fibres onto the standard k-sphere is unstable for E_Φ.*

The canonical variation of the metric g on M is by definition the 1-parameter family g_t of metrics on M obtained by rescaling the metric on the fibres $f^{-1}(x)$, $x \in X$, by a factor $t \in \mathbb{R}^+$, i.e., $g_t = \mathfrak{p}_{\mathcal{H}}^*\, g + t^2\, \mathfrak{p}_{\mathcal{V}}^*\, g$. The second variation may be written as

$$d^2_f E_\Phi = Q^{\mathcal{V}} + Q^{\mathcal{H}}$$

with $Q^{\mathcal{V}}(v,w) = \langle v \mid J^{\mathcal{V}} w\rangle = 2\int_M \lambda_k \|\nabla v\|^2_{\mathcal{V}}$ and $Q^{\mathcal{H}}(v,w) = \langle v \mid J^{\mathcal{H}} w\rangle$ collecting the remaining horizontal terms in (3.11). The vertical Jacobi operator $J^{\mathcal{V}}$ coincides with that of [15] and the horizontal Jacobi operator $J^{\mathcal{H}}$ is the one of [15] plus the terms involving μ_k and ν_k. Let Q_t be the second variation with respect to the metric g_t. Then $Q_t = t^{-2}Q^{\mathcal{V}} + Q^{\mathcal{H}} = (t^{-2}-1)Q^{\mathcal{V}} + Q_1$. We decompose $L^2(f^*TX) = \ker J^{\mathcal{V}} \oplus (\ker J^{\mathcal{V}})^\perp$ and identify $\ker J^{\mathcal{V}}$ with the space of basic vector fields and $(\ker J^{\mathcal{V}})^\perp$ with those vector fields whose average along the fibres of f vanishes. A straightforward calculation shows that $Q^{\mathcal{H}}(v^0, v^\perp) = 0$ for $v^0 \in \ker J^{\mathcal{V}}$, $v^\perp \in (\ker J^{\mathcal{V}})^\perp$. Furthermore, $J^{\mathcal{V}}$ is the Laplacian along the fibres with values in the vector bundle f^*TX. Hence the restriction of $J^{\mathcal{V}}$ to $(\ker J^{\mathcal{V}})^\perp$ has a smallest eigenvalue $\lambda_1(J^{\mathcal{V}}) > 0$. Let $-C$ be a lower bound for the spectrum of J_Φ (for $t = 1$) and choose $t \leq (C+1)^{-1/2}$. Then

$$Q_t(v) = (t^{-2}-1)Q^{\mathcal{V}}(v^\perp) + Q_1(v^0) + Q_1(v^\perp) \geq Q_1(v^0)\ .$$

Thus, if the identity map on X is stably Φ-harmonic, we always have $Q_1(v^0) \geq 0$ and infer that f is stably Φ-harmonic with respect to sufficiently small t (cf. Theorem 7.3 in [15]). As an example, any Riemannian submersion with totally geodesic fibres is stably harmonic for the exponential energy if the fibres are suitable rescaled, since the identity map is always stable for the exponential energy (cf. [5]).

3.4.2. *Small Ricci curvature.* If the Ricci-curvature of X is small relative to the A-tensor, we use the estimate

$$\|\nabla v\|^2_{\mathcal{V}} + \|A(v)\|^2 \geq 2\left|\mathrm{Tr}\left(\nabla v \circ A(v)\right)\right|$$

together with (3.12). It follows for instance that if

$$\mathrm{Ric}^X(v,v) \leq -\|A(v)\|^2\ , \quad \lambda_k \geq |\mu_k|\ , \quad 2\nu_k \geq -\mu_k\ ,$$

then a Riemannian submersion $f\colon M \to X^k$ with totally geodesic fibres is stably Φ-harmonic. This applies to all the functionals in Table 1.

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