Communications in Mathematical Physics Manuscript-Nr.
(will be inserted by hand later)

Sets of Fibre Homotopy Classes and Twisted Order Parameter Spaces

Stefan Bechtluft-Sachs ⋆, Marco Hien ⋆⋆

Naturwissenschaftliche Fakultät I, Universität Regensburg, Universitätsstraße 31,
93053 Regensburg, Germany

Received: / Accepted:

Abstract. We propose a machinery to calculate the set of global defect indices of a regularly defected order taking values in arbitrary parameter bundles.

1. Introduction

In [2] we showed that the global defect index of an ordered medium is not in general determined by the local defect index. We also developped a method to calculate the set of global defect indices if the normal bundle of the defect as well as the order parameter bundle are trivial. In the present investigation we extend this to problems involving nontrivial bundles, such as nematics in a coordinate space with nontrivial tangent bundle.

To that end we first calculate the set $[\Delta, \mathcal{A}]_\Gamma$ of fibre homotopy classes of sections of $\mathcal{A}$ for an arbitrary locally trivial fibration $p : \mathcal{A} \to \Delta$. Assuming that $\Delta = C_f$ is the mapping cone of a cofibration $f : A \to X$ with an H-cogroup A we will present a method to calculate $[\Delta, \mathcal{A}]_\Delta$ roughly in terms of $[X, \mathcal{A}]_X$ and $[A, F]$.

This applies to defects in ordered media. Keeping the notation of [2] let $\Delta \subset M$ be the defect of a regularly defected section σ of a fibre bundle $E \to M$ and let N be the normal bundle of Δ. The set $\mathcal{A} := \bigcup_{x\in\Delta} \mathrm{Map}(SN_x, E_x)$ carries a natural topology which makes the obvious map $\mathcal{A} \to \Delta$ a locally trivial fibration with fibre $F = \mathrm{Map}(S^{n-1}, V)$. We denote by $\mathrm{Map}(SN, E)_\Delta \subset \mathrm{Map}(SN, E)$ the subset of fibre preserving maps and by $\Gamma(\mathcal{A})$ the set of cross sections of $\mathcal{A} \to \Delta$. The Exponential Law provides a homeomorphism $\Gamma(\mathcal{A}) \to \mathrm{Map}(SN, E)_\Delta$ hence a bijection between the set $[\Delta, \mathcal{A}]_\Gamma$ of fibre homotopy classes of sections of $\mathcal{A} \to \Delta$ and $[SN, E]_\Delta$ (see [3], chapter VII.2). Thus we can apply our machinery to

⋆ e-mail: stefan.bechtluft-sachs@mathematik.uni-regensburg.de
⋆⋆ e-mail: marco.hien@mathematik.uni-regensubrg.de

determine the set $[SN, E]_\Delta$ and the set $[SN, E]_\Delta^\varphi$ of fibre homotopy classes i.e. global defect indeces with local defect index φ.

Our approach will require to work in the category of pointed spaces first. Choose base points $x_0 \in \Delta$ and $\varphi \in F := \pi^{-1}(x_0)$. The set $[[\Delta, \mathcal{A}]]_\Gamma$ of fibre homotopy classes of sections *relative basepoint* can be computed from an exact sequence of Baues ([1]) generalizing the ordinary homotopy sequence of a cofibration. The set $[\Delta, \mathcal{A}]_\Gamma$ is then obtained by dividing out the action of $\pi_1(F)$. A subtlety arising here is the fact that the connecting map in the sequence in [1] is not simply an equivariant map with respect to the action of $\pi_1(F)$ but is itself given by a group action. Proposition 1 gives a compatibility formula for these two actions.

2. A sequence of sets of fibre homotopy classes

We briefly recall the long exact sequence of sets of fibre homotopy classes of sections from [1]. This sequence generalizes the homotopy sequence of a cofibration. For the moment we are working in the category of topological spaces with basepoint $*$. For topological spaces X and Y with basepoints we denote by $[[X, Y]]$ the set of homotopy classes relative basepoints and in an analogous manner by $[[X, \mathcal{A}]]_\Gamma$ the set of fibre homotopy classes of sections relative basepoints. Let A be an H-cogroup and $f : A \to X$ a cofibration. Let $i : X \to C_f$ denote the inclusion of X into the mapping cone $C_f := X \cup_f CA$ where CA is the cone over A. Furthermore we denote by SA the reduced suspension, by

$$\Sigma_* X := X \times S^1 / \{*\} \times S^1.$$

the pointed suspension of X and by $\mathcal{B} \to \Sigma_* X$ the pull-back of $\mathcal{A}$ via the canonical projection $\Sigma_* X \to X$. *Assuming* that there is a fixed cross section $\sigma : C_f \to \mathcal{A}$ with restriction $s := \sigma|_X$ we have the exact sequence (Theorem 2.4.1 (B) in [1])

$$[[\Sigma_* X, \mathcal{B}]]_\Gamma^s \xrightarrow{w_f^\#} [[SA, F]] \xrightarrow{\sigma^+} [[C_f, \mathcal{A}]]_\Gamma \xrightarrow{i^\#} [[X, \mathcal{A}|_X]]_\Gamma \xrightarrow{f^\#} [[A, F]] \qquad (1)$$

of pointed sets. Here $w_f^\#$ is a group homomorphism and σ^+ is defined by a group action of $[[SA, F]]$ on $[[C_f, \mathcal{A}]]_\Gamma$ which we will write as $\sigma^+([\alpha]) := [\sigma]_\Gamma + [\alpha]$.

We will have to take a closer look at the map σ^+. First recall the definition of the group action $+$ from [1], p. 59. Let

$$\mu : C_f \to C_f \vee SA$$

denote the map given by $\mu|_X := id_X$ and

$$(a, t) \mapsto \begin{cases} (a, 2t) \in C_f & \text{for} \quad 0 \le t \le \frac{1}{2} \\ (a, 2t-1) \in SA & \text{for} \quad \frac{1}{2} \le t \le 1 \end{cases} .$$

For every cross section $\sigma : C_f \to \mathcal{A}$ and every map $\alpha : SA \to F$ consider the map

$$s_\alpha : C_f \xrightarrow{\mu} C_f \vee SA \xrightarrow{\sigma \vee i \circ \alpha} \mathcal{A}$$

where $i : F \hookrightarrow \mathcal{A}$ is the inclusion. s_α is not a cross section, but there is a homotopy between $p \circ s_\alpha$ and id_{C_f} relative X and so we have a homotopy $K : C_f \times [0,1] \to \mathcal{A}$ completing the commutative diagram

$$\begin{array}{ccc} C_f \times 0 \cup X \times [0,1] & \stackrel{s_\alpha \cup \sigma|_X \circ pr}{\longrightarrow} & \mathcal{A} \\ \downarrow & \nearrow & \downarrow p \\ C_f \times [0,1] & \stackrel{\widetilde{H}}{\longrightarrow} & C_f \end{array}$$

Especially $\sigma_\alpha := K_1$ is a cross section in $\mathcal{A} \to C_f$, such that $\sigma_\alpha|_X = \sigma|_X$. Setting $[\sigma]_\Gamma + [\alpha] := [\sigma_\alpha]_\Gamma$ defines an action of the abelian group $[[SA, F]]$ on $[[C_f, \mathcal{A}]]_\Gamma$. For details see [1], Lemma 1.2.20 and Corollary 1.2.22.

3. Free homotopy classes

The exact sequence 1 permits to calculate the set $[[C_f, \mathcal{A}]]_\Gamma$ of section homotopy classes *relative basepoint*, whereas our proper interest lies in the set $[C_f, \mathcal{A}]_\Gamma$ of section homotopy classes without regarding basepoints. The latter is canonically identified with the orbit set of an action of $\pi_1(F)$ on $[[C_f, \mathcal{A}]]_\Gamma$, which we now proceed to describe.

The evaluation map on the space $\Gamma(\mathcal{A})$ of sections of $\mathcal{A}$ (carrying the compact-open topology),

$$\begin{array}{rcl} \pi : \Gamma(\mathcal{A}) & \to & F \\ s & \mapsto & s(x_0) \end{array}$$

is a fibration. Given a section s of $\mathcal{A}$ and a loop γ in F we can lift it to a path $\bar{\gamma}$ in $\Gamma(\mathcal{A})$ starting with s. The fibre homotopy class of the endpoint s_1 does not depend on the choices involved. Thus

$$[\gamma] \cdot [s]_\Gamma := [\bar{\gamma}(1)]_\Gamma.$$

is well-defined and yields the canonical bijection

$$[[\Delta, \mathcal{A}]]_\Gamma / \pi_1(F) \to [\Delta, \mathcal{A}]_\Gamma^{[\varphi]}$$

where the right hand side denotes the set of free homotopy classes of cross sections s of $p : \mathcal{A} \to \Delta$ for which $s(x_0)$ lies in the arc-component of φ.

For the map σ^+ of (1) we have the following formula:

Proposition 1.

$$[\gamma] \cdot ([\sigma]_\Gamma + [\alpha]) = [\gamma] \cdot \sigma^+([\alpha]) = [\gamma] \cdot [\sigma]_\Gamma + [\gamma] \cdot [\alpha] \tag{2}$$

Thus σ^+ is equivariant if and only if σ is a fixed point for the action of $\pi_1(F)$ on $[[C_f, \mathcal{A}]]_\Gamma$

Proof. Let $\bar{\gamma}$ denote a lifting of γ over the fibration $\mathrm{Map}(SA, F) \to F$ with $\bar{\gamma}(0) = \alpha$ and $\widetilde{\gamma}$ denote a lifting of γ over the fibration $\mathcal{A} \to F$ with $\widetilde{\gamma}(0) = \sigma$, so that

$$[\bar{\gamma}(1)] = [\gamma] \cdot [\alpha] \qquad \text{and} \qquad [\widetilde{\gamma}]_\Gamma = [\gamma] \cdot [\sigma]_\Gamma \ .$$

By the Exponential Law $\bar{\gamma}$ and $\widetilde{\gamma}$ define maps

$$\bar{G} : SA \times [0,1] \to F \qquad \text{and} \qquad \widetilde{G} : C_f \times [0,1] \to C_f \ .$$

Consider the map

$$\Sigma : C_f \times [0,1] \xrightarrow{\mu \times \mathrm{id}} (C_f \vee SA) \times [0,1] \xrightarrow{\widetilde{G} \cup \bar{G}} \mathcal{A}$$

with $\mu : C_f \to C_f \vee SA$ as before. Now $p \circ \Sigma$ is homotopic relative $X \times [0,1]$ to the projection $\mathrm{pr} : C_f \times [0,1] \to C_f$ along a homotopy

$$H : C_f \times [0,1] \times [0,1] \to C_f \ .$$

Let K denote the mapping completing the following commutative diagram

$$\begin{array}{ccc} C_f \times [0,1] \times 0 \cup X \times [0,1] \times [0,1] & \xrightarrow{\Sigma \cup \Sigma|_X \circ pr} & \mathcal{A} \\ \downarrow & K & \downarrow p \\ C_f \times [0,1] \times [0,1] & \xrightarrow{H} & C_f \end{array} \tag{3}$$

We define $K_{t,s} := K|_{C_f \times \{t\} \times \{s\}}$, $t, s \in [0,1]$. Restricting diagram (3) to $C_f \times i \times [0,1]$, $i = 0, 1$, we conclude

$$[K_{0,1}]_\Gamma = [\sigma]_\Gamma + [\alpha] \qquad \text{and} \qquad [K_{1,1}]_\Gamma = [\gamma] \cdot [\sigma]_\Gamma + [\gamma] \cdot [\alpha] \tag{4}$$

directly from the definition of the two group actions. The diagram (3) also includes the following commutative diagram

$$\begin{array}{ccc} C_f \times 0 \times 1 \cup \{*\} \times [0,1] \times 1 & \xrightarrow{K_{0,1} \cup \gamma} & \mathcal{A} \\ \downarrow & K_{\cdot,1} & \downarrow p \\ C_f \times [0,1] \times 1 & \xrightarrow{pr} & C_f \end{array}$$

from which we obtain

$$[K_{1,1}]_\Gamma = [\gamma] \cdot [K_{0,1}]_\Gamma = [\gamma] \cdot ([\sigma]_\Gamma + [\alpha]) \ .$$

Our assertion now follows combining this equation with (4). □

4. Examples and Applications

If Δ is obtained from X by attaching a single m-cell e^m, i.e. $\Delta = C_f$, where $f : S^{n-1} \to X$ is the attaching map, (1) reads

$$[[\Sigma_* X, \mathcal{B}]]^s_\Gamma \xrightarrow{w_f^\#} \pi_m(F) \xrightarrow{\sigma^+} [[\Delta, \mathcal{A}]]_\Gamma \xrightarrow{i^\#} [[X, \mathcal{A}|_X]]_\Gamma \xrightarrow{f^\#} \pi_{m-1}(F)$$

Let Δ be a connected finite cell complex of dimension N and c_m be the number of m-cells of Δ. By induction over the skeleta we immediately get an estimate:

Proposition 2.

$$\#[\Delta, \mathcal{A}]^{[\varphi]}_\Gamma \le \#[[\Delta, \mathcal{A}]]^{[\varphi]}_\Gamma \le \prod_{m=1}^{N} (\#\pi_m(F))^{c_m} \ .$$

In order to apply this to the setting of defect topology, we have to compute $\pi_m(F) = \pi_m(\mathrm{Map}(SN_{x_0}, E_{x_0}), \alpha)$. This group is involved in a long exact sequence

$$\begin{aligned} \cdots \to \pi_{m+1}(E_{x_0}) \xrightarrow{\rho_\alpha} \pi_{m+n-1}(E_{x_0}) \to \\ \to \pi_m(\mathrm{Map}(S^{n-1}, E_{x_0}), \alpha) \xrightarrow{\tau_\#} \pi_m(E_{x_0}) \to \cdots \end{aligned} \tag{5}$$

where $\rho_\alpha : \pi_{m+1}(E_{x_0}) \to \pi_{m+n-1}(E_{x_0})$, $\beta \mapsto [\alpha, \beta]$ is the Whitehead product with $\alpha \in \pi_{n-1}(E_{x_0})$, see [6]. We also have chosen basepoints $s_0 \in S^{n-1} = SN_{x_0}$, $v_0 = \alpha(s_0) \in E_{x_0}$ and defined the map τ to be the evaluation at s_0.

If $\Delta = S^m$ we get a bijection $\sigma^+ : \pi_m(F) \to [[S^m, \mathcal{A}]]_\Gamma$. The action of $\pi_1(F)$ on $[[S^m, \mathcal{A}]]_\Gamma$ pulls back to an action θ of $\pi_1(F)$ on $\pi_m(F)$ and $[[S^m, \mathcal{A}]]_\Gamma$ is in bijection with $\pi_m(F)/\theta$. We get

Proposition 3. *Either* $[S^m, \mathcal{A}]^{[\varphi]}_\Gamma$ *is empty or*

$$\#\pi_m(F)/\#\pi_1(F) \le \#[S^m, \mathcal{A}]^{[\varphi]}_\Gamma \le \#\pi_m(F) \ .$$

Together with the sequence (5) it is now possible to explicitly calculate lower and upper bounds for $\#[SN, E]^{[\varphi]}_{S^m}$ in cases where the corresponding homotopy groups of the fibre E_{x_0} are known.

4.1. Example Let $N \to S^m$ be a 2-dimensional complex vector bundle and $E = PN$ its projective bundle. The local defect index of the projection map $SN \to E$ is the Hopf map $\eta : S^3 \to S^2$. The number $\rho := \#[SN, E]^{[\eta]}_{S^m}$ of fibre homotopy classes of mappings $SN \to E$ with local defect index $[\eta] \in \pi_3(S^2)$ is bounded from below and above according to the following list (see [5] for the order of $\pi_k(S^l)$):

m	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\ldots \le \rho$	∞	∞	1	1	5	4	1	1	23	42	1	3	210	3
$\rho \le \ldots$	∞	∞	4	4	36	30	4	12	360	672	16	144	10080	120

4.2. Normal Nematics We need to consider the case where $E = PN$ is the projective bundle of the normal bundle of Δ. This reflects the fact that uniaxial particles are forced to be transverse to the defect. From Proposition 3 estimates for $\rho := \#[SN, E]^{[\pi]}_{S^m}$, where $\pi : S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$ is the usual projection, are easily deduced. For instance in the case $n := \mathrm{codim}(\Delta, M) = 10$, $\Delta = S^m$ we get:

m	2	3	4	5	6	7	8	9	10
$\ldots \le \rho$	1	6	1	1	1	60	1	∞	6
$\rho \le \ldots$	2	24	1	1	2	240	8	∞	96

4.3. Remark: In the following example $[[S^m, \mathcal{A}]]_\Gamma$ does not contain a fixed point for the action of $\pi_1(F)$, hence θ differs from the usual action for every σ. Let $SN := S^3 \to S^2$ be the Hopf map with fibre S^1 and $E := S^2 \times V \to S^2$ be the trivial bundle with fibre V. Assume that

$$\pi_1(V) = 0\ ,\ \pi_2(V) \neq 0 \text{ and } \pi_3(V) = 0\ ,$$

(e.g. $V := \mathbb{C}\mathrm{P}^\infty$). In this situation there is only one possible local index $0 \in \pi_1(V)$, so that

$$[SN, E]_{S^2} = [SN, E]^0_{S^2} \approx \pi_2(\mathrm{Map}_0(S^1, V))/\theta.$$

In the exact sequence (5) the Whitehead products vanish and we we obtain that

$$\pi_2(\mathrm{Map}_0(S^1, V)) \cong \pi_2(V) \neq 0\ .$$

The usual action of $\pi_1(\mathrm{Map}_0(S^1, V))$ on the higher homotopy groups is by automorphisms. Hence the orbit space $\pi_2(\mathrm{Map}_0(S^1, V))/\pi_1(\mathrm{Map}_0(S^1, V))$ has at least two elements. But as $E \to S^2$ is the trivial bundle we obtain

$$[SN, E]_{S^2} = [S^3, S^2 \times V]_{S^2} = [S^3, V] \approx \pi_3(V)/\pi_1(V)\ ,$$

and therefore $\#[SN, E]_{S^2} = 1$. This shows that in this case θ differs from the usual group action. In particular, the ambiguity of the global defect index depends non trivially on the isomorphism type of the bundles involved.

References

1. H. J. Baues: *Obstruction theory.* Lecture Notes in Mathematics 628, 387 S., Berlin-Heidelberg-New York: Springer-Verlag, 1977
2. S. Bechtluft-Sachs, M. Hien: *The Global Defect Index.* to appear in Communications in Mathematical Physics
3. G. E. Bredon: *Topology and geometry.* 557 S., Berlin-Heidelberg-New York: Springer-Verlag, 1993
4. V. Poénaru, G. Toulouse: *The crossing of defects in ordered media and the topology of 3-manifolds.* J. Phys. 38 (1977), 887-895
5. D. C. Ravenell: *Complex cobordism and stable homotopy groups of spheres.* 412 S., London-New York: Academic Press, 1986
6. G.W. Whitehead: *On products in homotopy groups.* Ann. of Math. 47, 460-475, 1946

This article was processed by the author using the LaTeX style file *cljour1* from Springer-Verlag.