SOBOLEV-POINCARÉ IMPLIES JOHN

S. Buckley and P. Koskela

Abstract. We establish necessary conditions for the validity of Sobolev-Poincaré type inequalities. We give a geometric characterisation for the validity of this inequality for simply connected plane domains.

Dedicated to F. W. Gehring on the occasion of his 70th birthday.

§1. Introduction

The Sobolev-Poincaré inequality

$$\left(\int_\Omega |u-u_\Omega|^{pn/(n-p)}\,dx\right)^{(n-p)/pn} \le C\left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p} \tag{1.1}$$

holds for $1 \le p < n$ whenever u is smooth, $u_\Omega = |\Omega|^{-1}\int_\Omega u\,dx$, and $\Omega \subset \mathbb{R}^n$ is bounded and satisfies the cone condition; by density of smooth functions (1.1) then holds for all functions in the Sobolev space $W^{1,p}(\Omega)$ consisting of all functions in $L^p(\Omega)$ whose distributional gradients belong to $L^p(\Omega)$. For $1 < p < n$, inequality (1.1) was proved by Sobolev [S1], [S2], and, for $p = 1$, this result is due to Gagliardo [G] and Nirenberg [N]. In fact, the above inequality for $1 < p < n$ can be deduced from the case $p = 1$ using Hölder's inequality. The case $p = 1$ has a geometric interpretation: the above inequality holds if and only if the domain Ω satisfies a relative isoperimetric inequality. This observation is due to Maz'ya [M1], [M2]; also see the paper of Federer and Fleming [FF].

Recently Bojarski [B] has verified the above Sobolev-Poincaré inequality for so-called John domains; also see [C] for related results. In [BK], the current authors establish Sobolev-Poincaré inequalities on John domains in the full range $0 < p < n$ for solutions to certain elliptic equations, as well as variant Sobolev-Poincaré inequalities for more general functions (see Section 2 below). A domain Ω is a John domain if it satisfies a twisted interior cone condition; see 3.1 below for a precise definition. Here John refers to F. John who used this condition in his work on elasticity [J]; Martio and Sarvas [MS] introduced this terminology. Bojarski's result also gives an estimate for the constant in (1.1) in terms of p, n, and the constant associated with the John condition, whereas no such estimate is possible using the constants in the cone condition.

1991 *Mathematics Subject Classification.* 46E35, 30C65.

The first author was partially supported by NSF Grant DMS–9207715. The second author was partially supported by NSF Grant DMS–9305742 and by the Academy of Finland.

All bounded domains satisfying a cone condition are John domains, and a prime example of a John domain that does not satisfy a cone condition is the familiar von Koch snowflake.

The main result of this note is the following theorem and its corollary that provide us with a partial converse to Bojarski's theorem.

Theorem 1.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is a domain of finite volume that satisfies a separation property (cf. below). Fix $1 \le p < n$. Then Ω satisfies the Sobolev-Poincaré inequality (1.1) if and only if Ω is a John domain.*

For simply connected plane domains the separation property is automatically valid and hence we obtain a complete characterisation.

Corollary 1.2. *Let Ω be a simply connected plane domain of finite area. Fix $1 \le p < 2$. Then Ω satisfies the Sobolev-Poincaré inequality (1.1) if and only if Ω is a John domain.*

We want to stress that the statement of Theorem 1.1 does not hold without some additional assumption on Ω: if E is a relatively closed subset of a ball B with vanishing $(n-1)$-dimensional measure, then the Sobolev-Poincaré inequality holds for $\Omega = B \setminus E$ for all $1 \le p < n$ (integrate by parts and use the inequality for B), whereas it is easy to select E so that Ω is not John. For example in $\mathbb{R}^2$, one could delete $E = \bigcup_{k=1}^{\infty} E_k$, where E_k consists of $k!$ equally spaced points on the circle $\{|x| = 1 - 2^{-k}\}$. For a less trivial example, one can delete the set $F = \bigcup_{x \in E} D_x$ where D_x is a disk centred at x with radius so small that the disks $2D_x$ are all disjoint; to see this one uses the fact that a Sobolev function u can be extended across circular boundaries without increasing $||\nabla u||_{L^p}$ by more than a fixed factor (see [HeK]). The above examples easily extend to higher dimensions but one can actually construct simply connected domains with the same properties. For example if E is a countable subset of the unit disk D given above then $F = E \times [0, 1)$ is removable for $D \times (-1, 1)$, but $D \setminus F$ is simply connected.

In Section 2 below we give a necessary condition for the Sobolev-Poincaré inequality without any additional assumptions. With the help of the separation property (defined in 3.2) we then deduce Theorem 1.1 in Section 3.

Corollary 1.2 follows from Theorem 1.1 since we shall prove that any domain Ω in $\mathbb{R}^n$ that is quasiconformally equivalent to a uniform domain satisfies a separation property. Thus the assertion of Theorem 1.1 holds for any such domain.

We wish to point out that each John domain satisfies a relative isoperimetric inequality. This follows by combining Bojarski's result with the geometric characterisation for $p = 1$. For a direct proof see the paper [HaK] by Hajłasz and Koskela. Thus Theorem 1.1 implies that a relative isoperimetric inequality characterises the validity of a Sobolev-Poincaré inequality for all $1 \le p < n$ for domains quasiconformally equivalent to a uniform domain. This can be viewed as a generalisation of the results of Maz'ya. We find this conclusion surprising and have not seen any results of this kind for $p > 1$.

It is easy to see using the Monotone Convergence Theorem and the fact that (1.1) holds for all balls with a constant independent of the radius of the ball that (1.1) remains true for $\Omega = \mathbb{R}^n$ provided we replace u_Ω by an appropriate constant. In Section 4 we consider the cases $0 < p < 1$ and $|\Omega| = \infty$, and produce versions of Theorems 1.1 and Corollary 1.2. As a special case of the results of Section 4 we record the following.

Corollary 1.3. *Let Ω be plane domain whose complement is contained in the unit disk and assume that Ω is a simply connected as a subset of the Riemann sphere. Fix $1 \le p < 2$. Then*

$$\left(\int_\Omega |u|^{2p/(2-p)}\,dx\right)^{(2-p)/2p} \le C\left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}$$

for each u in the Sobolev space $W^{1,p}(\Omega)$ if and only if $B(0,2)\cap\Omega$ is a John domain.

Corollary 1.3 and the discussion in Section 4 extend the main result of Chen, Williams and Zhao [CWZ].

The paper is organised as follows. Section 2 contains the proof of a necessary condition for Sobolev-Poincaré type inequalities. In Section 3 we prove Theorem 1.1 and Corollary 1.2 together with some extensions. In Section 4, we discuss related results. Finally in Section 5, we give an example to show that the generalisation of Theorem 1.1 in Section 4 is in some sense sharp.

§2. A necessary condition

In this section, we prove a necessary condition for the validity of Sobolev-Poincaré and Poincaré type inequalities (we reserve the former term for inequalities like (2.1) or (2.1′) below with $q = np/(n-p)$).

If Ω is a domain (i.e. connected open set), let $S(\Omega)$ denote the class of cubes whose edges are in the coordinate directions and whose concentric 3-dilates are contained in Ω. Suppose $M = M_\Omega$ is the local Hardy-Littlewood maximal operator on Ω, i.e. for any $f \in L^1_{\mathrm{loc}}(\Omega)$,

$$Mf(x) = \sup_{x\in Q\in S(\Omega)} \frac{1}{|Q|}\int_Q |f|,$$

For $0 < t < \infty$, we define the maximal operator M_t by $M_t f(x) = [M(f^t(x))]^{1/t}$. Clearly $M_t f \ge f$ on the Lebesgue set of f, and it is well-known that $M_t : L^p(\Omega) \to L^p(\Omega)$ for all $t < p \le \infty$.

For the rest of the section, let Ω be a domain of finite volume. If $p \ge 1$, we say Ω supports a Poincaré inequality with exponents $p < q$ if, for some constant C,

$$\left(\int_\Omega |u-u_\Omega|^q\,dx\right)^{1/q} \le C\left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p} \tag{2.1}$$

holds whenever u is smooth and $u_\Omega = |\Omega|^{-1}\int_\Omega u\,dx$. It then easily follows that this inequality holds, with the constant u_Ω replaced by zero, for all locally Lipschitz continuous functions u with average value zero on a set $E \subset \Omega$ of positive volume (C will then also depend on $|\Omega|/|E|$). If $0 < p < 1$, we say Ω supports a Poincaré inequality with exponents $p < q$ if, for some ball $B \subset\subset \Omega$, there exists a constant C such that

$$\left(\int_\Omega |u-u_B|^q\,dx\right)^{1/q} \le C\left(\int_\Omega [M_t(\nabla u)]^p\,dx\right)^{1/p} \tag{2.1′}$$

holds whenever u is locally Lipschitz, $t > 1$ is fixed, and $u_B = |B|^{-1} \int_B u\,dx$. It then follows that B can be replaced by any other ball $B' \subset\subset \Omega$ (C will then also depend on B'). By [BK], John domains support a Sobolev-Poincaré inequality for all $0 < p < n$.

We are now ready to state and prove a necessary condition for the validity of such a Poincaré inequality. Note that in the following theorem, the exponent $(n-p)(q-p)/p^2$ equals 1 for $q = np/(n-p)$, and lies in the interval $(0,1)$ for smaller values of $q > p$.

Theorem 2.1. *Suppose that Ω supports a Poincaré inequality with exponents $0 < p < q$, $p < n$. Fix a ball $B_0 \subset \Omega$ and let $w \in \Omega$. Then*

$$\operatorname{diam}(T) \le C(d + d^{(n-p)(q-p)/p^2}) \tag{2.2}$$

whenever T is a component of $\Omega \setminus B(w,d)$ that does not intersect B_0. The constant C depends only on p, n, $|\Omega|$, B_0, and on the constant in (2.1) or (2.1′).

Proof. We suppose first that $p \ge 1$. Let T be as above. Fix $r > d$ such that $T(r) = T \setminus B(w,r) \ne \emptyset$. We set

$$u(x) = \begin{cases} 0, & x \in \Omega \setminus T(d) \\ 1, & x \in T(r) \\ \dfrac{d(x, B(w,d))}{r-d}, & x \in A(d,r) \equiv T(d) \setminus T(r). \end{cases}$$

Note that u is locally Lipschitz continuous and vanishes in B_0, so we may apply the (q,p)-inequality to u with $|u - u_\Omega|$ replaced by $|u|$:

$$|T(r)|^{1/q} \le C\left(\int_\Omega |u|^q dx\right)^{1/q} \le C\left(\int_\Omega |\nabla u|^p dx\right)^{1/p} \le C\frac{|A(d,r)|^{1/p}}{r-d}. \tag{2.3}$$

The above arguments also show that we can replace d in (2.3) by any $d < \rho < r$ provided $A(\rho, r)$ is defined as $T(\rho) \setminus T(r)$.

Define $r_0 = d$ and for $j \ge 1$ pick r_j such that

$$|A(r_{j-1}, r_j)| = |(T \cap B(w, r_j)) \setminus B(w, r_{j-1})| = 2^{-j}|T|.$$

Then $|T(r_j)| = 2^{-j+1}|T|$ and hence the above inequality gives

$$\begin{aligned} \operatorname{diam}(T) &\le 2d + \sum_1^\infty 2|r_j - r_{j-1}| \\ &\le 2d + C\sum_1^\infty (2^{-j}|T|)^{1/p-1/q} \le 2d + C|T|^{1/p-1/q}. \end{aligned} \tag{2.4}$$

We now show that $|T'| \le Cd^{(n-p)q/p}$, where $T' = T(2d)$; combining this inequality with (2.4), it is then a routine matter to derive (2.2). We define a function ρ by setting $\rho = 0$ in $(\Omega \setminus T) \cup T'$, and $\rho = 1/d$ in $T \cap B(w, 2d)$. Set $u(x) = \inf_\gamma \int_\gamma \rho\, ds$, where the infimum is taken over all rectifiable curves that join x to B_0 in Ω. Then u is locally Lipschitz continuous, vanishes in B_0, $u \geq 1$ in T', and

$$\int_\Omega |\nabla u|^p dx \leq C d^{n-p}.$$

Thus, by applying the (q,p)-inequality to u, we conclude that

$$|T'|^{p/q} \leq C d^{n-p},$$

as desired.

The proof for $p < 1$ is similar, so let us simply sketch the necessary modifications needed to the above argument. In the first half of the proof, we assume $r \geq 4d$ and use the function

$$u(x) = \begin{cases} 0, & x \in \Omega \setminus T(2d) \\ 1, & x \in T(r) \\ \dfrac{d(x, B(w,2d))}{r-2d}, & x \in A(d,r) \equiv T(2d) \setminus T(r) \end{cases}$$

It is easily checked that $M_t(|\nabla u|)$ is zero outside $B(w,2r) \setminus B(w,d)$. Thus one gets that

$$|2r - d| \leq C|r-2d| \leq C \frac{|A(d,2r)|^{1/q}}{|T(2r)|^{1/p}}.$$

We deduce (2.4) by essentially the same argument as before (with an easy modification to take care of the assumption $r \geq 4d$).

For the second part of the proof, we choose the same function as for $p \geq 1$. We need only note that $M_t(|\nabla u|)$ is zero outside $B(w,4d)$. □

2.2 Remark. By appropriately modifying the proof of Theorem 2.1 one can check that each domain of finite volume satisfying (2.1) with $1 \leq p < q$ and $p < n$ has to be bounded; see also [A], [M2,p.214]. In fact, the assumption $p < n$ is unnecessary.

§3. Proof of Theorem 1.1

Definition 3.1. A bounded domain $\Omega \subset \mathbb{R}^n$ with a distinguished point $x_0 \in \Omega$ is called a *John domain* if it satisfies the following "twisted cone" condition: there exists a constant $C > 0$ such that for all $x \in \Omega$, there is a curve $\gamma : [0,l] \to \Omega$ parametrised by arclength such that $\gamma(0) = x$, $\gamma(l) = x_0$, and $d(\gamma(t), \Omega^c) \geq Ct$. We call such a curve a *John curve for x*.

In Definition 3.1, we shall always assume without loss of generality that $C \leq 1$ and refer to the largest such C as the John constant of Ω. If Ω is a John domain, any $y \in \Omega$ can act as the distinguished point (but a more "central" point will give a larger constant). Examples of John domains include all bounded Lipschitz domains and certain fractal domains.

Definition 3.2. We say that a domain $\Omega \subset \mathbb{R}^n$ with a distinguished point x_0 has a *separation property* if there is a constant C_0 such that the following holds: For each $x \in \Omega$ there is a curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) = x_0$, and such that for each t either $\gamma([0,t]) \subset B \equiv B(\gamma(t), C_0 d(\gamma(t), \Omega^c))$ or each $y \in \gamma([0,t]) \setminus B$ belongs to a different component of $\Omega \setminus \partial B$ than x_0.

Intuitively a separation property means that the domain does not have any "flat" tentacles and that there are no small pieces of boundary floating around. Notice that if one can show that the second case of the definition never occurs, one more or less has the John condition. This is the game plan. Let us next give examples of domains that satisfy a separation property.

Lemma 3.3. *Suppose that Ω is quasiconformally equivalent to a uniform domain G. Then Ω has a separation property. In particular, each simply connected plane domain has a separation property.*

In the above lemma, quasiconformal equivalence means that there is a homeomorphism f of $G \subset \mathbb{R}^n$ onto $\Omega \subset \mathbb{R}^n$ such that f belongs to the local Sobolev class $W^{1,n}_{\text{loc}}(G)$ and $|Df(x)|^n \le K J_f(x)$ for almost every $x \in G$, where $|Df|$ is the operator norm of the formal derivative Df of f, J_f is the Jacobian determinant of Df, and $K \ge 1$ is a fixed constant (referred to as the *dilatation* of f). When $K = 1$ and $n = 2$, this reduces to the class of conformal mappings.

We shall use only two basic properties of quasiconformal mappings in the proof of Lemma 3.3. First, quasiconformal mappings quasipreserve conformal capacity (i.e. they preserve it up to a multiplicative constant dependent on the dilatation), and secondly that if f is quasiconformal from G onto Ω, and $B(x,r)$ is a ball in Ω with $d(x, \Omega^c) = 2r$ (or Cr for some fixed $C > 1$), then $f^{-1}B$ is a subset of G whose diameter is comparable with $d(f^{-1}(x), G^c)$ and that contains a ball $B(f^{-1}(x), s)$ of comparable radius. For details of these and other properties of quasiconformal mappings, we refer the reader to [V1], [V2].

A domain G is uniform if there is a constant C such that for any pair x, y of points in G we can find a curve $\gamma : [0,l] \to G$ parametrised by arclength such that $\gamma(0) = x$, $\gamma(l) = y$, $l \le C|x-y|$, and $d(\gamma(t), G^c) \ge \frac{1}{C} \min\{t, l-t\}$. For information on uniform domains we refer the reader to the papers [GO], [J], [MS], [V2]. Notice that bounded Lipschitz domains are uniform and that each finitely connected plane domain is conformally equivalent to a uniform domain.

We first prove Theorem 1.1 and then proceed with the proof of Lemma 3.3.

Proof of Theorem 1.1. Fix $x \in \Omega$, and pick a curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and point $\gamma(1) = x_0$ as in Definition 3.2. We shall show that $d(\gamma(t), \Omega^c) \ge C \operatorname{diam}(\gamma([0,t]))$ for $0 < t < 1$. The careful reader will notice that this does not guarantee that γ is John curve for x, but it is well known and easy to check that this condition is sufficient to guarantee that γ can be modified to yield a John curve for x; see [MS, pp.385–386], [NV, pp.7–8].

To this end, let $0 < t < 1$ and $\delta(t) = d(\gamma(t), \Omega^c)$. If $\gamma([0,t]) \subset B(\gamma(t), C\delta(t))$, there is nothing to prove. Otherwise the separation property implies that $\partial B = \partial B(\gamma(t), C\delta(t))$ separates $\gamma([0,t]) \setminus B$ from x_0. If the component of $\Omega \setminus \partial B$ containing x_0 does not contain a ball centred at x_0 of radius $\delta(1)/2$, then B must have radius at least $\delta(1)/4$ since it intersects both $B(x_0, \delta(1)/2)$ and $\partial\Omega$. In this case, $B' \equiv B(\gamma(t), 4C\delta(t))$ contains $B_0 \equiv B(x_0, \delta(1)/4)$ and we may assume B' does not contain $\gamma([0,t])$ (since otherwise we are done). Thus either $\Omega \setminus \partial B$ or B'

contains B_0. In either case, we conclude from Theorem 2.1 that $\operatorname{diam}(\gamma([0,t])) \le Cd(\gamma(t),\Omega^c)$; notice that $(n-p)(q-p)/p^2 = 1$ for $q = pn/(n-p)$. Here the constant C depends on the constant in (1.1) and on $|\Omega|/d(x_0,\Omega^c)^n$. The claim follows. □

Proof of Lemma 3.3. Let f be a quasiconformal mapping of G onto Ω. Fix a point x_0 in Ω. For a given point $x \in \Omega$ let γ be a curve joining $f^{-1}(x)$ and $f^{-1}(x_0)$ as in the definition of a uniform domain. We may assume that $|t-s| \le C'|\gamma(t)-\gamma(s)|$ for each t, s; this is not immediate from the definition but requires an argument involving certain geodesics (see [GO,p.59]). Suppose that $f(\gamma[0,t])$ is not contained in $B = B(y, Cd(y,\Omega^c))$, $C > 1$, where $y = f(\gamma(t))$, and that $x_0 \notin B$. We want to show that ∂B separates $f(\gamma([0,t])) \setminus B$ from x_0 in Ω provided C is sufficiently large. Suppose that this is not the case. Then we find a continuum F that joins $w \in f(\gamma([0,t])) \setminus B$ to x_0 in Ω with $F \cap B = \emptyset$. Let $E = \overline{B}(y, d(y,\Omega^c)/2)$, let $\operatorname{cap}(E,F;\Omega)$ denote the conformal capacity of the pair E, F relative to Ω, and let w_{n-1} be the surface measure of the boundary of the unit ball. By well-known capacity estimates (see e.g. [V1]), we have

$$\operatorname{cap}(E,F;\Omega) \le \operatorname{cap}(E,\partial B;\Omega) \le \frac{w_{n-1}}{(\log(2C))^{n-1}}.$$

By the quasiconformality of f we conclude (simply by performing a change of variables) that

$$\operatorname{cap}(f^{-1}(E), f^{-1}(F); G) \le \frac{K w_{n-1}}{(\log(2C))^{n-1}}.$$

On the other hand, quasiconformality of f guarantees (cf. [V2]) that

$$d(f^{-1}(y), G^c)/C_0 \le \operatorname{diam}(f^{-1}(E)) \le C_0 d(f^{-1}(y), G^c) \tag{3.1}$$

Consequently,

$$d(f^{-1}(E), f^{-1}(F)) \le \min\{t, l-t\} \le C_1 \operatorname{diam}(f^{-1}(E)),$$

where C_1 depends only on K, n and the constant of uniformity of G. Moreover, $f^{-1}(F)$ joins points $x_1, x_2 \in \gamma([0,l]) \setminus f^{-1}(E)$, where $x_j = \gamma(t_j)$, $j = 1,2$, with $|t_1 - t_2| \ge Cd(f^{-1}(y), G^c)$. Thus by (3.1) and the properties of γ, $C_2 \operatorname{diam}(f^{-1}(F)) \ge \operatorname{diam}(f^{-1}(E))$. For a uniform domain G, $\operatorname{cap}(f^{-1}(E), f^{-1}(F); G) \ge \delta > 0$ for any such configuration; see for example [GM]. This holds because of the extension property for Sobolev functions due to Jones [J] and the corresponding estimate in $\mathbb{R}^n$. An upper bound for C follows and the proof is complete. □

§4. Further results

Let us first point out that Theorem 2.1 and the proof of Theorem 1.1 allow one to generalise Theorem 1.1 in two directions. First, the result is also valid for $0 < p < 1$ if we substitute an inequality of the form (2.1′) for (1.1). Secondly, one still gets a necessary condition for the validity of the Poincaré inequality if $p < q < np/(n-p)$.

Corollary 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain of finite volume and assume that Ω satisfies a separation property. Let $0 < p < n$. If Ω supports a Poincaré inequality with exponents $p < q$, then any point x in Ω can be joined to a fixed point $x_0 \in \Omega$ by a curve $\gamma = \gamma_x : [0,l] \to \Omega$ parametrised by arclength and with*

$$d(\gamma(t), \Omega^c) \ge C t^{p^2/(n-p)(q-p)} \tag{4.1}$$

for all $0 < t < l$.

In proving this corollary, one shows (as in the proof of Theorem 1.1) that $\operatorname{diam}(\gamma[0,t]) \le C(d + d^{(n-p)(q-p)/p^2})$, where $d = d(\gamma(t), \Omega^c)$. One can then absorb the d term on the right-hand side into the smaller power of d (essentially giving (4.1)) since d is bounded above by a constant times $|\Omega|^{1/n}$.

The index $p^2/(n-p)(q-p)$ is best possible if $p \ge 1$; we postpone an example relevant to this claim until the final section so as not to distract from the exposition. If $s = (n-p)(q-p)/p^2$, we shall for brevity refer to a domain satisfying (4.1) as an s-John domain.

We next discuss the case $|\Omega| = \infty$. The natural Sobolev-Poincaré inequality is then (4.2) below. There can be no inequality of this type with q different from $pn/(n-p)$; see [A] or [M2].

An unbounded domain $\Omega \subset \mathbb{R}^n$ is called an *unbounded John domain* if it satisfies the following "twisted double cone" condition: there exists a constant $C > 0$ such that for all $x, y \in \Omega$, there is a curve $\gamma = \gamma : [0, l] \to \Omega$ parametrised by arclength such that $\gamma(0) = x$, $\gamma(l) = y$, and $d(\gamma(t), \Omega^c) \ge C \min\{t, l - t\}$.

It is not hard to check that for bounded domains the above condition characterises John domains. Väisälä [V3] has shown that each unbounded John domain can be written as the union of a nested sequence of bounded C'-John domains; here C' depends only on C, n. Thus it is possible to apply dominated convergence to show that that each unbounded John domain admits the following Sobolev-Poincaré inequality for $1 \le p < n$:

$$\inf_{a \in \mathbb{R}} \left(\int_\Omega |u - a|^{pn/(n-p)}\, dx \right)^{(n-p)/pn} \le C \left(\int_\Omega |\nabla u|^p\, dx \right)^{1/p} . \tag{4.2}$$

For a detailed proof see the paper [H] by Hurri-Syrjänen. Let us indicate a proof for a version of (4.2) for $u \in W^{1,p}(\Omega)$; here $W^{1,p}(\Omega)$ is the usual Sobolev class consisting of functions in $L^p(\Omega)$ whose distributional gradients belong to $L^p(\Omega)$. Pick an increasing sequence Ω_j of bounded C'-John domains whose union coincides with Ω. Write a_j for the average of u over Ω. Now (4.2) holds for each Ω_j with a fixed constant and with $a = a_j$ by Bojarski's [B] result and

$$|a_j| \le ||u||_p |\Omega_j|^{-1/p} \le M |\Omega_j|^{-1/p}$$

since $u \in L^p(\Omega)$. By monotone convergence we conclude that

$$\begin{aligned} \int_\Omega |u|^{pn/(n-p)}\, dx &= \lim_j \int_{\Omega_j} |u|^{pn/(n-p)}\, dx \\ &\le 2^{pn/(n-p)} \lim_j \left(\int_{\Omega_j} |u - a_j|^{pn/(n-p)}\, dx + |a_j|^{pn/(n-p)} |\Omega_j| \right) \\ &\le C \left(\int_\Omega |\nabla u|^p\, dx \right)^{n/(n-p)} . \end{aligned}$$

In conclusion, (4.2) holds with $a = 0$ provided that Ω is an unbounded John domain (then $|\Omega| = \infty$) and $u \in W^{1,p}(\Omega)$.

Suppose now that the complement of a domain Ω is compact and Ω satisfies a cone condition. Then Ω is an unbounded John domain and the conclusion of the above paragraph applies. Thus the above paragraph extends the main theorem of [CWZ].

Certain versions of the results of Sections 2 and 3 can be extended to cover the situation when $|\Omega| = \infty$ (naturally the separation property must be modified in the spirit of the definition of unbounded John domains). We shall content ourselves with stating the following theorem whose proof we leave to the reader. Hint: Prove first that (4.2) ensures that for any $w \in \Omega$, $\Omega \setminus B(w, d)$ can have at most one component whose diameter exceeds $C'd$.

Theorem 4.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is a domain of infinite volume and assume that Ω is quasiconformally equivalent to a uniform domain. Fix $1 \leq p < n$. Then Ω satisfies the Sobolev-Poincaré inequality (4.2) if and only if Ω is an unbounded John domain.*

Combining Corollary 1.2 and Theorem 4.2 we conclude that we have completely characterised the geometry of simply connected plane domains that admit a Sobolev-Poincaré inequality.

Let us conclude this section with a couple of easy corollaries concerning uniform domains.

We say that $W^{1,p}(\Omega)$ imbeds into $C^{0,\alpha}(\overline{\Omega})$ if

$$||u||_{\alpha,\Omega} = \sup_{x\in\Omega} |u(x)| + \sup_{x,y\in\Omega, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq C||u||_{W^{1,p}(\Omega)}$$

where

$$||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)}.$$

Above, one naturally has to require that $p > n$ and one has to identify u with its continuous refinement. We also need a variant of this imbedding for functions locally in L^p in Ω whose gradient belongs to $L^p(\Omega)$: we say that $L^{1,p}(\Omega)$ imbeds into $\mathrm{Lip}_{\alpha}(\Omega)$ if the latter supremum is bounded by $||\nabla u||_{L^p(\Omega)}$ for each such function u. For bounded domains these two imbeddings are equivalent by the results in [KR] whereas the former imbedding is weaker than the latter one for unbounded domains. For geometric criteria for these imbeddings see [KR].

Corollary 4.3. *Let $\Omega \subset \mathbb{R}^n$ be quasiconformally equivalent to a uniform domain. Then Ω is uniform if and only if the following two conditions hold:*

(1) Ω supports a Sobolev-Poincaré inequality for some (all) $1 \leq p < n$.

(2) $W^{1,p}(\Omega)$ imbeds into $C^{0,1-n/p}(\overline{\Omega})$ ($L^{1,p}(\Omega)$ into $Lip_{1-n/p}(\Omega)$ if Ω is unbounded) for some (all) $p > n$.

Proof. If Ω is uniform and bounded, then clearly Ω is a John domain. If Ω is unbounded it similarly follows that Ω is an unbounded John domain. Thus (1) holds. Moreover, (2) essentially follows from Jones' extension theorem [J] (or for unbounded domains from the results of [HeK]); see [KR] for a direct proof.

Conversely, using the results from Section 3 and Theorem 4.2, we conclude that (1) implies that Ω is a John domain (bounded or unbounded). Moreover, it is known, see e.g. [KR], that the imbedding in (2) guarantees that any pair x, y of points in Ω can be joined by a curve in Ω whose length does not exceed $C|x-y|$; that is Ω is quasiconvex. Each quasiconvex John domain (bounded or unbounded) that is quasiconformally equivalent to a uniform domain is in fact itself uniform; see e.g. [V2]. The claim follows. □

By the Riemann Mapping Theorem each simply connected plane domain is quasiconformally equivalent to a uniform domain. Moreover, a simply connected proper subdomain of the plane is uniform if and only if it is a quasidisk (the image of a disk under a quasiconformal mapping of the entire plane). Hence Corollary 4.3 gives the following new characterisation for quasidisks; see [Ge] for other characterisations.

Corollary 4.4. *Let Ω be a simply connected plane domain. Then Ω is a quasidisk if and only if the following two conditions hold:*

(1) Ω supports a Sobolev-Poincaré inequality for some (all) $1 \le p < 2$.

(2) $W^{1,p}(\Omega)$ imbeds into $C^{0,1-2/p}(\overline{\Omega})$ ($L^{1,p}(\Omega)$ into $Lip_{1-2/p}(\Omega)$ if Ω unbounded) for some (all) $p > 2$.

§5. Sharpness of Corollary 4.1

We now present an example to show that the index $(n-p)(q-p)/p^2$ in Corollary 4.1 is best possible when $p \ge 1$. Since our analysis here is similar to that employed in [BK] and elsewhere, we shall be a little sketchy at times for the sake of brevity. The interested reader should not have difficulty filling in the gaps, especially after looking at the proof of Theorem 1.5 in [BK].

We assume throughout that $1 \le p < n$, $p < q < p^* \equiv np/(n-p)$. We write a point in $\mathbb{R}^n$ in the form $x = (x_1, x')$, where $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{n-1}$. Our example consists of a sequence of cylinders attached via narrow necks to a central ball in $\mathbb{R}^n$, where each cylinder is congruent to

$$C(r;\alpha,\beta) \equiv C(r) = \{(x_1, x') \mid 0 < x_1 < r^\alpha, |x'| < r^\beta\}$$

with $0 < r < 1$, $\alpha = (n-p)(q-p)/p^2$, and

$$\beta = \frac{(n-p)q - \alpha p}{p(n-1)} = \frac{(n-p)(qp-q+p)}{p^2(n-1)}.$$

Here, r will be the radius of the neck attaching the cylinder to the central ball. Since α, β are linear in q, a little calculation shows that we always have $0 < \alpha < \beta \le 1$ for all q in the allowed range (and $\beta = 1$ precisely when $p = 1$).

Before we properly define our full domain Ω, let us first show that $C(r)$ satisfies a Poincaré inequality uniformly for all $0 < r < 1$. Here and later the exponents of this inequality are implicitly assumed to be p, q, unless otherwise stated. Also, constants below are only significant if they depend on r, or the chosen functions or subdomains (cubes, etc.), so we use words such as "comparable" and "approximately" with the implicit meaning "up to a non-significant constant." We use C to denote any non-significant constant.

$C(r)$ is a John domain but, as r tends to zero, $C(r)$ becomes very elongated and its John constant tends to zero. Therefore we first slice $C(r)$ into N cylindrical segments A_i (ordered in the natural way) whose height and radius are comparable; since the radius is r^β, N is approximately $r^{\alpha-\beta}$. Let Q_i be a "central" Whitney cube in each segment, i.e. Q_i is a cube in A_i centred on its cylindrical axis, whose sidelength and distance from ∂A_i are both comparable to r^β. We wish to show that

$$\int_{C(r)} |u - u_{Q_1}|^q \le C \left(\int_{C(r)} |\nabla u|^p \right)^{q/p}, \tag{5.1}$$

where as usual $u_{Q_i} = \fint_{Q_i} u$. To estimate the left-hand side of (5.1), we need to estimate and sum over i two types of terms: $\int_{A_i} |u - u_{Q_i}|^q$ and $|Q_i||u_{Q_i} - u_{Q_1}|^q$. After using a Poincaré inequality (uniform in i, r) on each segment, the first terms are easy to handle, so we investigate only the second sum of terms.

The intersection of the cubes Q_i and the axis of the cylinder is a union of line segments (of length approximately r^β). We now add more cubes of the same approximate size centred on this axis so as to fill in the gaps. Doing this in such a way that we minimise the number of added cubes, we have M cubes where M is approximately $r^{\alpha-\beta}$. Let us call the new sequence of cubes Q'_i, $1 \le i \le M$, (ordered in their natural way from $Q'_1 = Q_1$ to $Q'_M = Q_N$). Letting $v = |\nabla u|$, we make a crude estimate and then apply a Poincaré inequality on each cube as in [BK] to get

$$\begin{aligned} \sum_{i=1}^N |Q_i||u_{Q_i} - u_{Q_1}|^q &\le r^{\beta n} N \left(\sum_{j=2}^M |u_{Q'_j} - u_{Q'_{j-1}}| \right)^q \\ &\le C r^{\beta n(1-q/p^*)} N \left(\sum_{j=1}^M \left(\int_{Q_j} v^p \right)^{1/p} \right)^q \\ &\le C r^{\beta n(1-q/p^*)} N M^{(p-1)q/p} \left(\sum_{j=1}^M \int_{Q_j} v^p \right)^{q/p} \\ &\le C r^{\beta n(1-q/p^*)} N M^{(p-1)q/p} \left(\int_{C(r)} v^p \right)^{q/p}. \end{aligned}$$

Since N and M are both approximately $r^{\alpha-\beta}$, we see after some calculations that

$$r^{\beta n(1-q/p^*)} N M^{1+(p-1)q/p}$$

is approximately 1, and so we are done.

We now define the full domain Ω to consist of a central ball of radius 1 with a sequence of elongated cylinders Ω_k attached via narrow necks (i.e. a "ball with clown-balloons attached"). More precisely, Ω_k is congruent to $C(r_k)$, where for now r_k is any number smaller than A^{-k}, for some fixed $A > 2^{1/\beta}$; Ω_k has main axis normal to the ball and is attached to the ball via a smaller cylinder of radius and height r_k. We also assume that the cylinders do not intersect each other — the upper bound on r_k ensures there is enough room on the surface of the ball to attach all of the cylinders (even along one circular arc).

We now show that Ω satisfies a Poincaré inequality with exponents p, r for any $p < r < q$. This will justify our claim that the index $\alpha = (n-p)(q-p)/p^2$ in Corollary 4.1 is best possible since Ω is clearly not an s-John domain for any $s > \alpha$. Also note that it suffices to prove such a Poincaré inequality with u_Ω replaced by u_{Q_Ω}, where Q_Ω is a "central" Whitney cube of the ball.

Thinking of Ω as the union of a bumpy ball (the unit ball together with the necks of the balloons) and a sequence of cylinders, we see that the bumpy ball is a John domain and so satisfies a Sobolev-Poincaré inequality (with u_{Q_Ω} replacing u_Ω), while we have already seen that the cylinders satisfy a uniform Poincaré inequality. We are left only with estimating the terms $|\Omega_k||u_{Q^k} - u_{Q_\Omega}|^r$ which arise from correcting the constant subtracted from u in the cylinder Poincaré inequalities (here Q^k, which is near the neck of the balloon, refers to the cube in Ω_k which was called Q_1 when we examined a single cylinder).

Let us denote the concentric dilate of a cube Q by a factor t as tQ. Letting P^k be a Whitney cube in the neck of Ω_k, we can connect both Q_Ω and Q^k to P^k by a *Boman chain*, i.e. a chain of cubes $\{Q_k\}_{j=0}^t$ for which $Q_0 = Q_\Omega$ (or Q^k), $Q_t = P^k$, each Q_k is contained in the nine-fold concentric dilates of its immediate neighbours, and there exists some $R > 0$ such that $P^k \subset RQ_j$ for each $0 \le j \le s$. This is possible precisely because both the bumpy ball and the first segment of the balloon's main cylinder (with the neck included) are John domains. One also sees that $t = t_k$ is at most $C \log_2(1/r_k)$.

Let us consider the chain of cubes from Q_Ω to P^k (the chain inside the balloon is handled similarly). Letting $v = |\nabla u|$, it is not hard to show (following the proof of Theorem 1.5 in [BK], for example) that

$$S_k \equiv |\Omega_k||u_{P^k} - u_{Q_\Omega}|^r \le |\Omega_k| \left(\sum_{i=1}^t |u_{Q_i} - u_{Q_{i-1}}| \right)^r$$
$$\le C|\Omega_k| \left(\sum_{i=1}^t |Q_i|^{-1/p^*} \left(\int_{9Q_i} v^p \right)^{1/p} \right)^r$$

and hence

$$S_k \le C|\Omega_k| t^{r-1} \sum_{i=1}^t |Q_i|^{-r/p^*} \left(\int_{9Q_i} v^p \right)^{r/p} .$$

Since $|Q_i| \ge C r_k^n$ for all i, $|\Omega_k| = C r_k^{(n-p)q/p}$, and the power of t is essentially a power of $\log 1/r_k$, we see that for any $\epsilon < (n-p)(q-r)/p$, there exists a constant C such that for all k, $S_k \le C r_k^\epsilon \left(\int_\Omega v^p \right)^{r/p}$, and so we are done.

By adjusting the above example a little, we can in fact produce an example of a domain which is not s-John for any $s > \alpha$ but supports a Poincaré inequality with exponents p, r, where $r = q$. First note that we used the fact that $r < q$ in only two places above: to dominate the logarithmic factor t_k^{r-1} and to give a convergent series in k when we add together the bounds for all of the cylinders. To overcome these obstacles, let us further restrict the choice of $\{r_k\}$

by forcing them to decrease much faster ($r_k < 2^{-2^k}$, say). Now let the radii of the neck and main cylinder of the balloon remain the same, but shorten the length of the balloon by a factor $\log(1/r_k)^A$. Then Ω is still not s-John for any $s > \alpha$. Furthermore if A is chosen large enough that $\log(1/r_k)^A > 2^k t_k^{r-1}$, the resulting decrease in the volume of the balloon is sufficient to allow one to take care of the two problem steps in the argument.

Finally, we leave as an open question the sharpness for $p < 1$ of the index under discussion here. In this case, the above examples break down since $\beta > 1$ for all $q < p^*$; consequently, the "neck" of the balloon is now wider than its "main cylinder" and serves no useful purpose.

ADDED. We have very recently discovered characterisations for the remaining cases ($p \geq n$) of the Sobolev-Poincaré Imbedding Theorems.

References

[A] R. Andersson, "Unbounded Soboleff regions," Math. Scand. **13** (1963), 75–89.

[B] B. Bojarski, "Remarks on Sobolev imbedding inequalities," in Lecture Notes in Math. 1351, 52–68, Springer-Verlag, Berlin (1989).

[BK] S. Buckley and P. Koskela, "Sobolev-Poincaré inequalities for $p < 1$," Indiana Univ. Math. J. **43** (1994), 221–240.

[CWZ] Z.Q. Chen, R.J. Williams and Z. Zhao, "A Sobolev inequality and Neumann heat kernel estimate for unbounded domains," Mathematical Research Letters **1** (1994), 177–184.

[C] S.-K. Chua, "Weighted Sobolev's inequality on domains satisfying the chain condition," Proc. Amer. Math. Soc. **117** (1993), 449–457.

[FF] W.H. Federer, W. Fleming, "Normal and integral currents," Ann. Math. **72** (1960), 458–520.

[G] E. Gagliardo, "Properieta di alcune classi di funzioni in piu variabli," Rireche di Mat. Napli **7** (1958), 102–137.

[Ge] F.W. Gehring, "Characteristic properties of quasidisks," Presses de l'Université de Montréal, Quebec (1982).

[GM] F.W. Gehring and O. Martio, "Quasiextremal distance domains and extension of quasiconformal mappings," J. Analyse Math. **45** (1985), 181–206.

[GO] F.W. Gehring and B.G. Osgood, "Uniform domains and the quasihyperbolic metric," J. Analyse Math. **36** (1979), 50–74.

[HaK] P. Hajłasz and P. Koskela, "Isoperimetric inequalities and imbedding theorems in irregular domains," Preprint.

[HeK] D. Herron and P. Koskela, "Uniform, Sobolev extension and quasiconformal circle domains," J. Analyse Math. **57** (1991), 172–202.

[H] R. Hurri-Syrjänen, "Unbounded Poincaré domains," Ann. Acad. Sci. Fenn. Ser. A I. Math. **17** (1992), 409–423.

[J] F. John, "Rotation and strain," Comm. Pure Appl. Math. **14** (1961), 391–413.

[Jn] P.W. Jones, "Quasiconformal mappings and extendability of functions in Sobolev spaces," Acta Math. **147** (1981), 71–88.

[KR] P. Koskela and F. Reitich, "Hölder continuity of Sobolev functions and quasiconformal mappings," Math. Z. **213** (1993), 457–472.

[MS] O. Martio and J. Sarvas, "Injectivity theorems in plane and space," Ann. Acad. Sci. Fenn. Ser. A I. Math. **4** (1978-79), 383–401.

[M1] V.G. Maz'ya, "Classes of domains and imbedding theorems for function spaces," Dokl. Akad. Nauk SSSR **133** (3) (1960), 527–530 (Russian).

[M2] V.G. Maz'ya, Sobolev spaces, Springer-Verlag, Berlin (1985).

[NV] R. Näkki and J. Väisälä, "John disks," Exposition. Math. **9** (1991), 3–43.

[N] L. Nirenberg, "On elliptic differential equations," Ann. Scuola Norm. Pisa (III) **13** (1959), 1–48.

[S1] S.L. Sobolev, "On the estimates relating to families of functions having derivatives that are square integrable," Dokl. Akad. Nauk SSSR **1** (1936), 267-270 (Russian).

[S2] S.L. Sobolev, "On a theorem of functional analysis," Mat. Sbornik **46** (1938), 471–497 (Russian).

[V1] J. Väisälä, Lectures on n-dimensional quasiconformal mappings, Lecture Notes in Mathematics 229, Springer-Verlag, Berlin (1970).

[V2] J. Väisälä, "Quasimöbius maps," J. Analyse Math. **44** (1984–85), 218–234.

[V3] J. Väisälä, "Exhaustion of John domains," Ann. Acad. Sci. Fenn. Ser. A I. Math. **19** (1994), 47–54.

Stephen M. Buckley: Department of Mathematics, St. Patrick's College, Maynooth, Co. Kildare, Ireland, e-mail: sbuckley@maths.may.ie

Pekka Koskela: Department of Mathematics, University of Jyväskylä, P.O.Box 35, Fin-40351 Jyväskylä, Finland, and Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, e-mail: pkoskela@math.jyu.fi