Convergence of recursive functions on computers

Erivelton Geraldo Nepomuceno[1,2]

[1]*Intelligent Systems and Networks Group, Department of Electrical and Electronic Engineering, Imperial College London, Federal University of São João del-Rei, São João del-Rei, Brazil*
[2]*Model and Control Group, Department of Electrical Engineering, Federal University of São João del-Rei, São João del-Rei, Brazil*
E-mail: nepomuceno@ufsj.edu.br

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Abstract: A theorem is presented which has applications in the numerical computation of fixed points of recursive functions. If a sequence of functions $\{f_n\}$ is convergent on a metric space $I \subseteq \mathbb{R}$, then it is possible to observe this behaviour on the set $\mathbb{D} \subset \mathbb{Q}$ of all numbers represented in a computer. However, as $\mathbb{D}$ is not complete, the representation of f_n on $\mathbb{D}$ is subject to an error. Then f_n and f_m are considered equal when its differences computed on $\mathbb{D}$ are equal or lower than the sum of error of each f_n and f_m. An example is given to illustrate the use of the theorem.

1 Introduction

Recursive functions (RF) provides a description for a variety of problems [1]. For instance, there has been significant interest in finding electronic circuits that represent the behaviour of RF with potential applications in random number generation, frequency-hopping, ranging and spread-spectrum communications [2, 3]. These circuits are usually carefully designed to approximate one of the well-known chaos maps, for example, a tent map [4] or a logistic map [5], in order to obtain required statistical or frequency properties of the generated signal [6].

Let I be a metric space such that $I \subseteq \mathbb{R}$, with a distance between any two points x and y of I given by a real number $d(x, y) = |x - y|$. Let also $f: I \to \mathbb{R}$. An RF can be defined as

$$x_n = f(x_{n-1}) \tag{1}$$

which may be written as a result of composite functions such that

$$x_n = f_1(x_{n-1}) = f_2(x_{n-2}) = \cdots = f_n(x_0) \tag{2}$$

A discrete-time series can be generated by a simple iterative procedure of (1). It can be chaotic if f is suitably chosen [6] or it can present other behaviour such as fixed point of period 1 or period >1.

If $f(x^*) = x^*$, then x^* is a fixed point of (1). The contractive mapping principle gives a simple constructive means of finding the fixed point by starting with an arbitrary element x_0 and define a sequence $\{x_n\}$ by $x_n = f(x_{n-1})$ [7]. If this sequence is convergent, then $x_n \to x^*$ as $n \to \infty$. In many cases, this calculation is performed by digital computers. However, let $\mathbb{D} \subset \mathbb{Q}$ be the set of all numbers represented in a computer. $\mathbb{D}$ is not a complete metric space, which implies that it is not possible to have sufficient conditions to verify if a sequence is convergent. Nevertheless, if a sequence is convergent then it can be observed on $\mathbb{D}$.

Our main problem can be established as in the Definition 1.

Definition 1: Suppose $\{f_n\}$, $n = 1, 2, 3, \ldots$, is a sequence of functions defined on a set $E \subseteq \mathbb{R}$, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. Function f is defined by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x \in E) \tag{3}$$

The sequence $\{f_n\}$ converges on E and that f is the 'limit', or the 'limit function', of $\{f_n\}$. It is also possible to say that $\{f_n\}$ converges to f pointwise on E if (3) holds [7, p. 143].

For example, consider the discrete logistic model [5] given by

$$x_{n+1} = rx_n(1 - x_n) \tag{4}$$

Let us simulate (4) with $r = 327/100$ and initial condition $x_0 = 100/327$ as shown in Algorithm 1 (see Fig. 1).

The result is presented in Fig. 2. It is clear that simulation converges to fixed points of period 2 if we take into account only the numerical results offered by Algorithm 1. Although, the correct result is not that, and indeed, it must converge to only one fixed point as easily shown

$$x_2 = \frac{\cancel{327}}{\cancel{100}}\,\frac{\cancel{100}}{\cancel{327}}\left(1 - \frac{100}{327}\right) = \frac{327 - 100}{327} = \frac{227}{327} \tag{5}$$

and

$$x_3 = \frac{\cancel{327}}{100}\,\frac{227}{\cancel{327}}\left(1 - \frac{227}{327}\right) = \frac{227}{\cancel{100}}\,\frac{\cancel{100}}{327} = \frac{227}{327} \tag{6}$$

and then $x_2 = x_3, \ldots, x_n = 227/327$, which means that (4) converges to only one fixed point. This illustrates a situation in which the computation of a fixed point is wrong. This Letter presents a theorem that aims at avoiding this problem.

Algorithm 1

```
r ← 327/100;
x_0 ← 100/327;
N ← 200;
for n ← 0 to N do
  | x_{n+1} ← r x_n (1 − x_n);
end
```

Fig. 1 *Simulation of (4)*

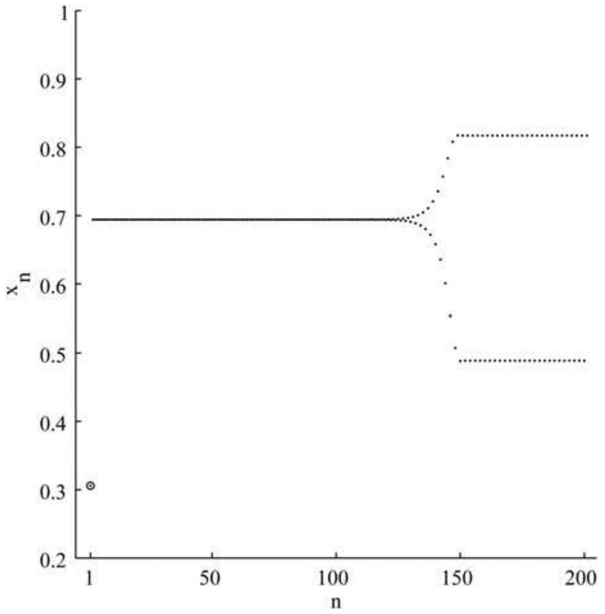

Fig. 2 *Simulation of x_n in (4) with $r = 327/100$ and $x_0 = 100/327$ (represented by ○ around the dot)*

2 Convergence on $\mathbb{D}$

Consider some basic definitions and theorems as follows [7, 8].

Definition 2: The extended real number system consists of all the real field $\mathbb{R}$ and two symbols $+\infty$ and $-\infty$. We preserve the original order in $\mathbb{R}$, and define $-\infty < x < +\infty$ for every $x \in \mathbb{R}$ [7, p. 11].

Definition 3: A sequence $\{p_n\}$ in a metric space X is said to be a Cauchy sequence if for every $\varepsilon > 0$ there is an integer such that $d(p_n, p_m) < \varepsilon$ if $n \geq N$ and $m \geq N$.

Theorem 1: In any metric space X, every convergent sequence is a Cauchy sequence.
The Cauchy criterion for uniform convergence is defined as Theorem 2.

Theorem 2: The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$ and $x \in E$ imply

$$d(f_n(x), f_m(x)) \leq \varepsilon \tag{7}$$

Demonstrations of Theorems 1 and 2 are found in [7].

Let $I \subseteq \mathbb{R}$ be an interval, let $x \in I$ and let $f: I \to \mathbb{R}$ be a function. Let $J \subseteq \mathbb{D}$ be an interval, $\hat{x} \in J$ be a representation of x and $\hat{f}$ be an approximation of f. Let $\delta \geq 0$, we have

$$d(f(x), \hat{f}(\hat{x})) \leq \delta \tag{8}$$

For iterated functions, we use the notation f_n. In case of iterated functions we can also write

$$d(f_n(x), \hat{f}_n(\hat{x})) \leq \delta_n \tag{9}$$

From (9), we can state $f_m(x) \neq f_n(x)$ only if

$$d(\hat{f}_m(\hat{x}), \hat{f}_n(\hat{x})) > \delta_n + \delta_m \tag{10}$$

This leads us to the following definition of fixed point.

Definition 4: If $d(\hat{f}_n(\hat{x}^*), \hat{f}_{n-1}(\hat{x}^*)) \leq \delta_n + \delta_{n-1}$, then $\hat{x}^*$ is a fixed point.
For any f_n there is an error δ_n associated. Let $\hat{\varepsilon}$ such that

$$\hat{\varepsilon} > \kappa = \sup(\delta_n) + \sup(\delta_m) \tag{11}$$

The main contribution of this Letter is expressed like Theorem 3.

Theorem 3: If the sequence of functions $\{f_n\}$ defined on E converges uniformly on E, then for every $\hat{\varepsilon} > \kappa$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in I$ and $\hat{x} \in J$ imply

$$\kappa < d(\hat{f}_n(\hat{x}), \hat{f}_m(\hat{x}) \leq \hat{\varepsilon} \tag{12}$$

Proof: First let us consider the trivial condition when $\kappa = 0$ for all n, $m \geq N$. In this case, we have $\hat{\varepsilon} > 0$ and $f_n(x) = \hat{f}_n(\hat{x})$ and $f_m(x) = \hat{f}_m(\hat{x})$ and Theorem 3 is equivalent to Theorem 2 for the direct way.
Second, let us consider the condition when $\kappa > 0$ for some n, $m \geq N$. Using (7) and by triangle inequality we have

$$\begin{aligned}
&|f_n(x) - f_m(x)| + |\hat{f}_n(\hat{x}) - f_n(x)| \\
&\quad + |f_m(x) - \hat{f}_m(\hat{x})| + \leq \varepsilon + \delta_n + \delta_m \\
&|f_n(x) - f_m(x) + \hat{f}_n(\hat{x}) - f_n(x) \\
&\quad + f_m(x) - \hat{f}_m(\hat{x})| \leq \varepsilon + \delta_n + \delta_m \\
&|\hat{f}_n(\hat{x}) - \hat{f}_m(\hat{x})| \leq \varepsilon + \delta_n + \delta_m
\end{aligned} \tag{13}$$

If we choose $\varepsilon = \hat{\varepsilon} - \delta_n - \delta_m$ then (13) implies (12). If $\hat{\varepsilon} > \kappa$ then $\varepsilon > 0$ and that completes the proof. □

From (12), we see that for all m and n we must have $d(\hat{f}_n(\hat{x}), \hat{f}_m(\hat{x})) > \kappa$, which has a practical result in limiting the number of iterations. This leads us to Corollary 1.

Corollary 1: The maximum number of iterations $k = \max(m, n)$ is subject to $d(\hat{f}_n(\hat{x}), \hat{f}_m(\hat{x})) \leq (\delta_n + \delta_m)$ for all n and m.

Example: Consider $r = 327/100$ and initial condition $x_0 = 100/327$ in (4). The value of δ_0 is due representation of $x_0 = 100/327$ as $\hat{x}_0$ and it is given by

$$\delta_0 = \frac{1}{2}\mathrm{ulp}(\hat{x}_0) \tag{14}$$

where ulp is the unit in the last place [9]. As $d(\hat{x}_0, x_0)) \leq \delta_0$, we also have $d(f(\hat{x}_0), f(x_0)) \leq \beta_0$. For β_0 we have

$$\begin{aligned}
\beta_0 &= \left|\frac{\mathrm{d}f(\hat{x}_0)}{\mathrm{d}\hat{x}_0}\delta_0\right| + \left|\frac{1}{2}\frac{\mathrm{d}^2 f(\hat{x}_0)}{\mathrm{d}^2\hat{x}_0}\delta_0^2\right| \\
&= |(r - 2r\hat{x}_0)\delta_0| + |(1/2)(-2r)(\delta_0)^2|
\end{aligned} \tag{15}$$

However, we also have an approximation of the f as $\hat{f}$. We split f in three arithmetic operations $a_0 = (1 - x_0)$, $b_0 = rx_0$ and $c_0 = a_0 b_0$. Thus, we have the following error

$$\gamma_0 = \frac{1}{2}\mathrm{ulp}(a_0) + \frac{1}{2}\mathrm{ulp}(b_0) + \frac{1}{2}\mathrm{ulp}(c_0) \tag{16}$$

and thus, the next value of $\delta_1 = \delta_0 + \beta_0 + \gamma_0$. By induction, the nth term of error is estimated by

$$\delta_n = \delta_{n-1} + \beta_{n-1} + \gamma_{n-1} \tag{17}$$

Table 1 Simulation of (4) for the first three iterates

n	$\hat{x}_n$	$d(\hat{x}_n, \hat{x}_{n-1})$	δ_n
0	0.305810397553517	0	$2.77555756156289 \times 10^{-17}$
1	0.694189602446483	0.388379204892966	$3.25197734863617 \times 10^{-16}$
2	0.694189602446483	$2.22044604925031 \times 10^{-16}$	$1.52284898079424 \times 10^{-15}$
3	0.694189602446483	$3.33066907387547 \times 10^{-16}$	$5.43916855498739 \times 10^{-15}$

Table 1 summarises these results. When $n = 1$, $d(\hat{x}_1, \hat{x}_0) > \delta_1 + \delta_0$ which satisfies Theorem 3. When $n = 2$, $d(\hat{x}_2, \hat{x}_1) < \delta_2 + \delta_1$ and it does not satisfy Theorem 3 and by Corollary 1 the iteration should be finished. $\{x_n\}$ is convergent to $0.694189602446483 \simeq 227/327$ as shown in (6). This conclusion is also coherent with Definition 4 of fixed point.

3 Conclusion

This Letter presents a theorem that makes it possible to observe convergence of $\{f_n\}$ on $\mathbb{D}$ by limiting the absolute difference of any two values of this sequence to the sum of each error associated to its computation. By Corollary 1, we also limit the number of iterations in order to avoid pseudo results of convergence. This result might be extended to fixed points of higher periods and other invariant sets.

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