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Application of Nonclassical Models of Shell Theory to Study Mechanical Parameters of Multilayer Nanotubes

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Abstract—Stress-strain state of multilayer anisotropic cylindrical shells under a local pressure is studied. Such a problem may model the bending of an asbestos nanotube under the action of a research probe. In earlier works, these authors showed that the application of classical shell theories yields results far from experimental data. More accurate results are obtained by taking into account additional factors, such as the change of the transverse displacement magnitude (according to the Timoshenko—Reissner theory) or the layered structure of asbestos and cylindrical anisotropy (according to the Rodinova—Titaev—Chernykh theory). In the present paper, yet another shell theory, the Palii—Spiro theory, is applied to solve the problem; this theory was developed for shall of average thickness and is based on the following assumptions: (a) the rectilinear fibers of the shell perpendicular to its middle surface before deformation remain rectilinear after deformation; (b) the cosine of the angle between the shell of such fibers and the middle surface of the deformed shell equals the averaged angle of the transverse displacement.

Deformation field are studied with the use of nonclassical (the Rodinova—Titaev—Chernykh and Palii—Spiro) shell theories; a comparison with results obtained for three-dimensional models with the use of the Ansys 11 package is performed.

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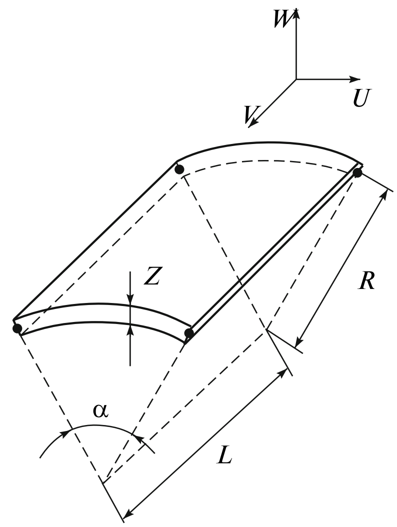
1. INTRODUCTION

In recent years, the possibility of applying methods of classical mechanics to nanoobjects have been extensively discussed. In [1, 2], it was mentioned that the mechanical characteristics corresponding to nanodimensional structural elements, such as a bar and a plate, may differ from those of the corresponding structures made of the same material but having "ordinary" geometric size. In addition to dimensional effects, the anisotropy of nanoobjects may manifest itself. In [3, 4], results of an experimental study of mechanical properties of nanotubes made of natural chrysolite asbestos with outer diameter of about 30 nm and inner diameter of about 5 nm were discussed; the inner cavity of the tube was filled by water, mercury, or tellurium under pressure. By means of scanning probe microscopy, the stiffness of the nanotube was measured. By stiffness the ratio of the applied force to the deflection of the bridge formed by the nanotube and spanning a hole in a porous substrate was understood. (The details of the experiment can be found in [3].) The experiment has shown that a tube filled by water is substantially softer than a dry tube (without any filling). Tubes filled by tellurium or mercury are somewhat stiffer than dry tubes. In [3], the experimental data were compared with results of modeling in the framework of continuum elasticity theory. The simplest classical models of an isotropic bar and nonclassical transversally isotropic models were considered. To analytically estimate the deflection of a nanotube as a bar, the Timoshenko—Reissner theory was applied, because the layered structure of asbestos nanotubes can be considered transversally isotropic. Each layer may retain its structure, but the displacement magnitude in a cross-section G' may significantly vary, depending on the filling. The softness of a tube filled by water can be explained by that the displacement magnitude in a cross-section is smaller than for a dry tube. In the isotropic case, the Timoshenko—Reissner theory, which takes into account displacements, only slightly refines classical theory, but for bodies made of a transversally isotropic material with moderately small transverse stiffness with respect to displacements, the Timoshenko—Reissner theory strongly refines the Bernoulli—Kirchhoff—Love theory and gives the following asymptotic approximation to three-dimensional theory [5]. A body with moderately small transverse stiffness with respect to displacements is a thin body for which the small parameter $g = G'/E$ (where E is the Young modulus in the tangential direction and G' is the shear modulus for any plane normal to the surface of isotropy) satisfies the relation $\mu^2 \ll g \ll 1$, where μ is the parameter of a thin-



An element of a circular cylindrical shell.

wall construction; for a cylindrical shell, $\mu \sim h/R$ (h is the thickness and R is the radius of the shell). In [4], the problem on nanotube deformation was solved by using the Rodionova–Titaev–Chernykh theory of anisotropic shells [6], which takes into account not only transverse shifts but also the layered structure of asbestos and cylindrical anisotropy. In this paper, the problem of the deformation of a layered tube under of a locally applied load is solved by using the theory of anisotropic shells of average thickness developed by Palii and Spiro in [7]. The results obtained by means of the Rodionova–Titaev–Chernykh and the Palii–Spiro theory are compared with those obtained for the same parameters by the finite-element method with the use of the ANSYS package.

2. STATEMENT OF THE PROBLEM

Suppose that α and β are cylindrical coordinates on the surface of the shell, α is the polar angle, β is the coordinate along the generatrix of the tube, $h^{(i)}$ are the thicknesses of the middle surfaces of the shell layers, $R^{(i)}$ are their radii, and L is the length of the tube. To determine coefficients, we use the notation $A_j^{(i)}$. The subscript j indicates which curvilinear coordinate corresponds to the quantity A under consideration, and the superscript i indicates to which shell it refers; thus, for $i = 1$, this quantity refers to the first inner shell, and for $i = N$, it refers to the last outer shell. By $E_1^{(i)}$, $E_2^{(i)}$, and $E_3^{(i)}$ we denote the elasticity moduli in the tangential and normal coordinate directions; the Poisson coefficients are denoted by $\nu_{jk}^{(i)}$.

Let us determine the stress-strain state of a layered tube under a local load by using the Rodionova–Titaev–Chernykh refined iteration theory of anisotropic shells [6] and the Palii–Spiro theory developed in [7].

The Rodionova–Titaev–Chernykh refined iteration theory is based on the following assumptions:

(1) the distributions of the transverse tangential and normal stresses are, respectively, quadratic and cubic with respect to thickness;

(2) the distributions of the tangential and normal components of the displacement vector are, respectively, quadratic and cubic with respect to thickness.

This theory makes it possible to take into account the rotation of fibers, their distortion, and the change of the length.

We suggest to seek the functions $u_1(\alpha, \beta, z)$, $u_2(\alpha, \beta, z)$, and $u_3(\alpha, \beta, z)$, which describe the displacements of shell layers in the Rodionova–Chernykh theory, in the form series in the Legendre polynomials P_0, P_1, P_2, and P_3 in the normal coordinate $z \in [-h/2, h/2]$:

$$u_1(\alpha, \beta, z) = u(\alpha, \beta)*P_0(z) + \gamma_1(\alpha, \beta)*P_1(z) + \theta_1(\alpha, \beta)*P_2(z) + \varphi_1(\alpha, \beta)*P_3(z),$$
$$u_2(\alpha, \beta, z) = v(\alpha, \beta)*P_0(z) + \gamma_2(\alpha, \beta)*P_1(z) + \theta_2(\alpha, \beta)*P_2(z) + \varphi_2(\alpha, \beta)*P_3(z), \quad (1)$$
$$u_3(\alpha, \beta, z) = w(\alpha, \beta)*P_0(z) + \gamma_3(\alpha, \beta)*P_1(z) + \theta_3(\alpha, \beta)*P_2(z),$$

$$P_0(z) = 1, \quad P_1(z) = \frac{2z}{h}, \quad P_2(z) = \frac{6z^2}{h^2} - \frac{1}{2}, \quad P_3(z) = \frac{20z^3}{h^3} - \frac{3z}{h}, \quad (2)$$

where u, v, and w are the components of the displacement vector of a point on the middle surface of the shell, γ_3 and θ_3 characterize the change of the length of the normal to this surface, and γ_1 and γ_2 are angles of rotation of the normal in the planes (α, z) and (β, z), respectively. The quantities θ_1, φ_1 and θ_2, φ_2 describe the normal curvature in the planes (α, z) and (β, z) of fibers perpendicular to the middle surface of the shell before deformation.

The Palii–Spiro theory of shells [7] this a theory of shells of average thickness, in which the following assumptions are made:

(1) rectilinear fibers of the shell perpendicular to its middle surface before deformation remain rectilinear after deformation;

(2) the cosine of the angle between the shell of such fibers and the middle surface of the deformed shell is equal to the averaged angle of the transverse displacement.

Mathematically, these assumptions reduce to the relations

$$u_1(\alpha, \beta, z) = u(\alpha, \beta) + \phi(\alpha, \beta)*z, \quad u_2(\alpha, \beta, z) = v(\alpha, \beta) + \psi(\alpha, \beta)*z,$$
$$u_3(\alpha, \beta, z) = w(\alpha, \beta) + F(\alpha, \beta, z),$$
$$\phi(\alpha, \beta) = \gamma_1(\alpha, \beta) + \phi_0(\alpha, \beta), \quad \psi(\alpha, \beta) = \gamma_2(\alpha, \beta) + \psi_0(\alpha, \beta), \quad (3)$$
$$\phi_0(\alpha, \beta) = -\frac{1}{A_1}\frac{\partial w(\alpha, \beta)}{\partial \alpha} + k_1 u(\alpha, \beta), \quad \psi_0(\alpha, \beta) = -\frac{1}{A_2}\frac{\partial w(\alpha, \beta)}{\partial \alpha} + k_2 v(\alpha, \beta),$$

where ϕ and ψ are the rotation angles of the normal in the planes (α, z) and (β, z); ϕ_0, ψ_0, γ_1, and γ_2 are the rotation angles of the normal to the middle surface and the displacement angles in the same planes. The function $F(\alpha, \beta, z)$ characterizes the change of the length of the normal to the middle surface.

The Lamé coefficients and the curvatures determining the geometry of the cylindrical shell have the form

$$A_1^{(i)} = R^{(i)}, \quad A_2^{(i)} = 1, \quad k_1^{(i)} = \frac{1}{R^{(i)}}, \quad k_2^{(i)} = 0. \quad (4)$$

We reduce the main quantities to dimensionless form by the formulas

$$\tilde{R}^{(i)} = \frac{A_1^{(i)}}{A_2^{(i)}}, \quad \tilde{h}^{(i)} = \frac{h}{R^{(i)}},$$
$$\{\tilde{u}^{(i)}, \tilde{v}^{(i)}, \tilde{w}^{(i)}, \tilde{\gamma}^{(i)}_{1,2,3}, \tilde{\theta}^{(i)}_{1,2,3}, \tilde{\varphi}^{(i)}_{1,2,3}, \tilde{\phi}^{(i)}_{,0}, \tilde{\psi}^{(i)}_{,0}\} = \frac{1}{h}\{u^{(i)}, v^{(i)}, w^{(i)}, \gamma^{(i)}_{1,2,3}, \theta^{(i)}_{1,2,3}, \varphi^{(i)}_{1,2}, \phi^{(i)}_{,0}, \psi^{(i)}_{,0}\},$$
$$\tilde{E}_{2,3} = \frac{E_{2,3}}{E_1}, \quad \tilde{G}_{13,12,23} = \frac{G_{13,12,23}}{E_1}, \quad \tilde{Pin}^{(i)}_{1,2,3} = \frac{Pin^{(i)}_{1,2,3}}{E_1}, \quad \tilde{Pout}^{(i)}_{1,2,3} = \frac{Pout^{(i)}_{1,2,3}}{E_1}, \quad (5)$$
$$\{\tilde{T}^{(i)}_{0,1,2}, \tilde{Q}^{(i)}_{1,2}, \tilde{m}^{(i)}_{1,2,3}\} = \frac{\{T^{(i)}_{0,1,2}, Q^{(i)}_{1,2}, m^{(i)}_{1,2,3}\}}{R^{(i)}E_1}, \quad \{\tilde{M}^{(i)}_{0,1,2}\} = \frac{\{M^{(i)}_{0,1,2}\}}{R^{(i)}E_1 h}, \quad \{\tilde{q}^{(i)}_{1,2,3}\} = \frac{\{q^{(i)}_{1,2,3}\}}{E_1},$$

where $Pout_x$ and Pin_x are the pressures on the inner and outer surfaces of the shell.

For convenience, we introduce the parameters

$$E_{11} = \frac{1}{1 - \nu_{12}\nu_{21}}, \quad E_{12} = \frac{\tilde{E}_2}{1 - \nu_{12}\nu_{21}}, \quad E_{22} = \frac{\nu_{12}}{1 - \nu_{12}\nu_{21}}, \quad E_z = \frac{\tilde{E}_3}{1 - \nu_{13}\mu_1 - \nu_{23}\mu_2},$$
$$\mu_1 = \frac{\nu_{31} + \nu_{21}\nu_{32}}{1 - \nu_{12}\nu_{21}}, \quad \mu_2 = \frac{\nu_{32} + \nu_{21}\nu_{31}}{1 - \nu_{12}\nu_{21}},$$

$$K_{11} = -E_{11}\tilde{h}^{(i)}, \quad K_{12} = E_{22}\tilde{h}^{(i)}, \quad K_{21} = \frac{3}{2}E_{11}\tilde{h}^{(i)}\mu_1, \quad K_{22} = \frac{3}{2}E_{22}\tilde{h}^{(i)}\mu_2,$$

$$K_{13} = E_{11}\frac{\tilde{h}^{(i)}}{2}(\mu_2 + 2\nu_{12}\mu_1), \quad K_{23} = E_{11}\frac{\tilde{h}^{(i)}}{2}(\nu_{12}\mu_1 + 2\mu_2), \tag{6}$$

$$\tilde{m}_x^{(i)} = \frac{\tilde{h}^{(i)}}{2}\tilde{P}out_x^{(i)}\left(1 + \frac{\tilde{h}^{(i)}}{2}\right) + \frac{\tilde{h}^{(i)}}{2}\tilde{P}in_x^{(i)}\left(1 - \frac{\tilde{h}^{(i)}}{2}\right),$$

$$\tilde{q}_x^{(i)} = \tilde{P}out_x^{(i)}\left(1 + \frac{\tilde{h}^{(i)}}{2}\right) - \tilde{P}in_x^{(i)}\left(1 - \frac{\tilde{h}^{(i)}}{2}\right) \quad (x = 1, 2, 3).$$

3. THE RELATIONSHIP BETWEEN THE RODIONOVA–TITAEV–CHERNYKH AND PALII–SPIRO THEORIES OF SHELLS

The deformation of a shell in the theories under consideration is expressed in terms of the displacement components by the following formulas:

The Rodionova–Titaev–Chernykh theory

$$\tilde{\varepsilon}_1^{(i)} = \tilde{h}^{(i)}\left(\frac{\partial \tilde{u}^{(i)}}{\partial \alpha^{(i)}} + \tilde{w}^{(i)}\right), \quad \tilde{\varepsilon}_2^{(i)} = \frac{\partial \tilde{v}^{(i)}}{\partial \beta^{(i)}},$$

$$\tilde{\eta}_1^{(i)} = \tilde{h}^{(i)}\left(\frac{\partial \tilde{\gamma}_1^{(i)}}{\partial \alpha^{(i)}} + \tilde{\gamma}_3^{(i)}\right), \quad \tilde{\eta}_2^{(i)} = \frac{\partial \tilde{\gamma}_2^{(i)}}{\partial \beta^{(i)}},$$

$$\tilde{\varepsilon}_{13}^{(i)} = \tilde{h}^{(i)}\frac{\partial \tilde{w}^{(i)}}{\partial \alpha^{(i)}} - \tilde{h}^{(i)}\tilde{u}^{(i)} + 2\tilde{\gamma}_1^{(i)},$$

$$\tilde{\varepsilon}_{23}^{(i)} = \frac{\partial \tilde{w}^{(i)}}{\partial \beta^{(i)}} + 2\tilde{\gamma}_2^{(i)},$$

$$\tilde{\omega}_1^{(i)} = \tilde{h}^{(i)}\frac{\partial \tilde{v}^{(i)}}{\partial \alpha^{(i)}}, \quad \tilde{\omega}_2^{(i)} = \frac{\partial \tilde{u}^{(i)}}{\partial \beta^{(i)}},$$

$$\tilde{\tau}_1^{(i)} = \tilde{h}^{(i)}\frac{\partial \tilde{\gamma}_2^{(i)}}{\partial \alpha^{(i)}}, \quad \tilde{\tau}_2^{(i)} = \frac{\partial \tilde{\gamma}_1^{(i)}}{\partial \beta^{(i)}},$$

The Palii–Spiro theory

$$\tilde{\varepsilon}_1^{(i)} = \tilde{h}^{(i)}\left(\frac{\partial \tilde{u}^{(i)}}{\partial \alpha^{(i)}} + \tilde{w}^{(i)}\right), \quad \tilde{\varepsilon}_2^{(i)} = \frac{\partial \tilde{v}^{(i)}}{\partial \beta^{(i)}},$$

$$\underline{\tilde{\eta}_1^{(i)} = \tilde{h}^{(i)}\left(\frac{\partial \tilde{\phi}^{(i)}}{\partial \alpha^{(i)}}\right)}, \quad \underline{\tilde{\eta}_2^{(i)} = \frac{\partial \tilde{\psi}^{(i)}}{\partial \beta^{(i)}}},$$

$$\underline{\tilde{\varepsilon}_{13}^{(i)} = 0},$$

$$\underline{\tilde{\varepsilon}_{23}^{(i)} = 0}, \tag{7}$$

$$\tilde{\omega}_1^{(i)} = \tilde{h}^{(i)}\frac{\partial \tilde{v}^{(i)}}{\partial \alpha^{(i)}}, \quad \tilde{\omega}_2^{(i)} = \frac{\partial \tilde{u}^{(i)}}{\partial \beta^{(i)}},$$

$$\underline{\tilde{\tau}_1^{(i)} = \tilde{h}^{(i)}\frac{\partial \tilde{\psi}^{(i)}}{\partial \alpha^{(i)}}}, \quad \underline{\tilde{\tau}_2^{(i)} = \frac{\partial \tilde{\phi}^{(i)}}{\partial \beta^{(i)}}},$$

$$\tilde{\tau}^{(i)} = \tilde{\tau}_1^{(i)} + \tilde{\tau}_2^{(i)}, \quad \tilde{\omega}^{(i)} = \tilde{\omega}_1^{(i)} + \tilde{\omega}_2^{(i)}.$$

Underlined are the deformation components which differ in these theories.

The dependences of momenta and forces on the deformation components which the Rodionova–Titaev–Chernykh theory gives in the case of a cylindrical shell are as follows. Substituting dependence (7) into relation (8), we obtain the following equations for the required relations:

$$\tilde{T}_1^{(i)} = E_{11}\tilde{h}^{(i)}\varepsilon_1^{(i)} + E_{12}\tilde{h}^{(i)}\varepsilon_2^{(i)} + \mu_1^{(i)}\tilde{T}_0^{(i)}, \quad \tilde{T}_2^{(i)} = E_{12}\tilde{h}^{(i)}\varepsilon_1^{(i)} + E_{22}\tilde{h}^{(i)}\varepsilon_2^{(i)} + \mu_2^{(i)}\tilde{T}_0^{(i)},$$

$$\tilde{M}_1^{(i)} = \frac{\tilde{h}^{(i)}}{6}(E_{11}\eta_1^{(i)} + E_{12}\eta_2^{(i)}) + \mu_1^{(i)}\tilde{M}_0^{(i)}, \quad \tilde{M}_2^{(i)} = \frac{\tilde{h}^{(i)}}{6}(E_{12}\eta_1^{(i)} + E_{22}\eta_2^{(i)}) + \mu_2^{(i)}\tilde{M}_0^{(i)},$$

$$\tilde{T}_{12}^{(i)} = \tilde{T}_{21}^{(i)} = \tilde{G}_{12}^{(i)}\tilde{h}^{(i)}\tilde{\tau}^{(i)}, \quad \tilde{M}_{12}^{(i)} = \tilde{M}_{21}^{(i)} = \frac{1}{6}\tilde{G}_{12}^{(i)}\tilde{h}^{(i)}\tilde{\omega}^{(i)},$$

$$\tilde{Q}_1^{(i)} = \frac{5\tilde{h}^{(i)}\tilde{G}_{13}^{(i)}}{6}\varepsilon_{13}^{(i)} + \frac{\tilde{m}_1^{(i)}}{6} - (\tilde{h}^{(i)})^2\frac{\tilde{G}_{13}^{(i)}}{6}\frac{\partial \theta_3^{(i)}}{\partial \alpha^{(i)}}, \tag{8}$$

$$\tilde{Q}_2^{(i)} = \frac{5\tilde{h}^{(i)}\tilde{G}_{23}^{(i)}}{6}\varepsilon_{23}^{(i)} + \frac{\tilde{m}_2^{(i)}}{6} - \tilde{h}^{(i)}\frac{\tilde{G}_{23}^{(i)}}{6}\frac{\partial\theta_3^{(i)}}{\partial\beta^{(i)}},$$

$$\tilde{T}_0^{(i)} = \tilde{m}_3^{(i)} + \frac{(\tilde{h}^{(i)})^2}{12}\left(\frac{\partial\tilde{q}_1^{(i)}}{\partial\alpha^{(i)}} + \tilde{R}^{(i)}\frac{\partial\tilde{q}_2^{(i)}}{\partial\beta^{(i)}}\right) - \tilde{h}^{(i)}\tilde{M}_1^{(i)},$$

$$M_0^{(i)} = \frac{(\tilde{h}^{(i)})^2}{10}\tilde{q}_3^{(i)} + \frac{\tilde{h}^{(i)}}{60}\left(\frac{\partial\tilde{m}_1^{(i)}}{\partial\alpha^{(i)}} + \tilde{R}^{(i)}\frac{\partial\tilde{m}_2^{(i)}}{\partial\beta^{(i)}}\right) - \frac{\tilde{h}^{(i)}}{60}\tilde{T}_1^{(i)}.$$

Substituting the following relations (9) for the six displacement components into (8), we obtain dependences on the five main displacement components u, v, w, γ_1, and γ_2:

$$\tilde{\theta}_1^{(i)} = \frac{\tilde{q}_1^{(i)}}{12G_{13}} - \frac{\tilde{h}^{(i)}}{6}\frac{\partial\tilde{\gamma}_3^{(i)}}{\partial\alpha^{(i)}}, \quad \tilde{\varphi}_1^{(i)} = \frac{m_1^{(i)} - \tilde{Q}_1^{(i)}}{10\tilde{h}^{(i)}\tilde{G}_{13}^{(i)}} - \frac{\tilde{h}^{(i)}}{10}\frac{\partial\tilde{\theta}_3^{(i)}}{\partial\alpha^{(i)}},$$

$$\tilde{\theta}_2^{(i)} = \frac{\tilde{q}_2^{(i)}}{12G_{23}} - \frac{\tilde{h}^{(i)}}{6\tilde{R}^{(i)}}\frac{\partial\tilde{\gamma}_3^{(i)}}{\partial\beta^{(i)}}, \quad \tilde{\varphi}_2^{(i)} = \frac{m_2^{(i)} - \tilde{Q}_2^{(i)}}{10\tilde{h}^{(i)}\tilde{G}_{23}^{(i)}} - \frac{\tilde{h}^{(i)}}{\tilde{R}^{(i)}}\frac{\partial\tilde{\theta}_3^{(i)}}{\partial\beta^{(i)}}, \tag{9}$$

$$\tilde{\gamma}_3^{(i)} = \frac{1}{2\tilde{h}^{(i)}}\frac{\tilde{T}_0^{(i)}}{\tilde{E}_z^{(i)}} - \frac{1}{2}(\mu_1\varepsilon_1^{(i)} + \mu_2^{(i)}\varepsilon_2^{(i)}), \quad \tilde{\theta}_3^{(i)} = \frac{1}{\tilde{h}^{(i)}}\frac{\tilde{M}_0^{(i)}}{\tilde{E}_z^{(i)}} - \frac{1}{6}(\mu_1\eta_1^{(i)} + \mu_2\eta_2^{(i)}).$$

Let us perform a similar transformation for the Palii–Spiro theory. Equations for relations between deformations and forces and momenta in this theory have the form

$$\tilde{T}_1^{(i)} = E_{11}\tilde{h}^{(i)}\varepsilon_1^{(i)} + E_{12}\tilde{h}^{(i)}\varepsilon_2^{(i)} + \underline{\frac{\tilde{h}^{(i)}}{12}((K_{11} - K_{12})\tilde{\eta}_1^{(i)} - K_{13}\tilde{\eta}_2^{(i)}) + \mu_1^{(i)}\frac{q_3^i}{2}\tilde{h}^{(i)}},$$

$$\tilde{T}_2^{(i)} = E_{12}\tilde{h}^{(i)}\varepsilon_1^{(i)} + E_{22}\tilde{h}^{(i)}\varepsilon_2^{(i)} + \underline{\frac{\tilde{h}^{(i)}}{12}((K_{21} - K_{22})\tilde{\eta}_2^{(i)} - K_{23}\tilde{\eta}_1^{(i)}) + \mu_2^{(i)}\frac{q_3^i}{2}\tilde{h}^{(i)}},$$

$$\tilde{T}_{12}^{(i)} = \tilde{G}_{12}^{(i)}\tilde{h}^{(i)}\left(\tilde{\omega}_1^{(i)} + \tilde{\omega}_2^{(i)} - \frac{(\tilde{h}^{(i)})^2}{\underline{12}}\tilde{\tau}_1^{(i)}\right), \quad \tilde{T}_{21}^{(i)} = \tilde{G}_{12}^{(i)}\tilde{h}^{(i)}\left(\tilde{\omega}_1^{(i)} + \tilde{\omega}_2^{(i)} + \frac{(\tilde{h}^{(i)})^2}{\underline{12}}\tilde{\tau}_2^{(i)}\right),$$

$$\tilde{M}_1^{(i)} = \frac{\tilde{h}^{(i)}}{6}(E_{11}\eta_1^{(i)} + E_{12}\eta_2^{(i)} + \underline{(K_{11} - K_{12})\tilde{\varepsilon}_1^{(i)} - K_{23}\tilde{\varepsilon}_2^{(i)}) + \mu_1^{(i)}\frac{q_3^{(i)}}{8}\tilde{h}^{(i)}},$$

$$\tilde{M}_2^{(i)} = \frac{\tilde{h}^{(i)}}{6}(E_{12}\eta_1^{(i)} + E_{22}\eta_2^{(i)} + \underline{(K_{21} - K_{22})\tilde{\varepsilon}_2^{(i)} - K_{13}\tilde{\varepsilon}_1^{(i)}) + \mu_2^{(i)}\frac{q_3^{(i)}}{8}\tilde{h}^{(i)}}, \tag{10}$$

$$\underline{\tilde{M}_{12}^{(i)} = \tilde{G}_{12}^{(i)}\frac{\tilde{h}^{(i)}}{12}(\tilde{\tau}_1^{(i)} + \tilde{\tau}_2^{(i)} - (\tilde{h}^{(i)})^2\tilde{\omega}_1^{(i)})}, \quad \underline{\tilde{M}_{21}^{(i)} = \tilde{G}_{12}^{(i)}\frac{\tilde{h}^{(i)}}{12}(\tilde{\tau}_1^{(i)} + \tilde{\tau}_2^{(i)} + (\tilde{h}^{(i)})^2\tilde{\omega}_2^{(i)})},$$

$$\underline{\tilde{Q}_1^{(i)} = \tilde{G}_{13}^{(i)}\tilde{h}^{(i)}\gamma_1^{(i)}}, \quad \underline{\tilde{Q}_2^{(i)} = \tilde{G}_{23}^{(i)}\tilde{h}^{(i)}\gamma_2^{(i)}},$$

$$\sigma_{33} = \frac{\tilde{P}out_3^{(i)}\left(1 + \frac{\tilde{h}^{(i)}}{2}\right)\left(0.5 + \frac{z}{h^{(i)}}\right) - \tilde{P}\mathrm{in}_3^{(i)}\left(1 - \frac{\tilde{h}^{(i)}}{2}\right)\left(0.5 - \frac{z}{h^{(i)}}\right)}{1 + \frac{z}{R^{(i)}}},$$

$$\tilde{F}(\alpha, \beta, z)^{(i)} = \int_0^z \frac{\sigma_{33}}{E_z}dz - (\mu_1\varepsilon_1 + \mu_2\varepsilon_2)z - \left(\mu_1\left(\eta_1 - \frac{\varepsilon_1}{R^{(i)}}\right) + \mu_2\eta_2\right)\frac{z^2}{2} + \left(\frac{\mu_1\eta_1}{R^{(i)}}\right)\frac{z^3}{3}.$$

Substituting them into the equation (7) for deformations, we obtain dependences of momenta and forces on the displacement components u, v, w, γ_1, and γ_2.

Thus, for both theories, we have obtained formulas relating forces and momenta to the displacement components. Substituting them into the equilibrium equations of a cylindrical shell, which are

$$\frac{\partial\tilde{T}_1^{(i)}}{\partial\alpha^{(i)}} + \tilde{R}^{(i)}\frac{\partial\tilde{T}_{21}^{(i)}}{\partial\beta^{(i)}} + \tilde{Q}_1^{(i)} + \tilde{q}_1^{(i)} = 0, \quad \frac{\partial\tilde{T}_{12}^{(i)}}{\partial\alpha^{(i)}} + \tilde{R}^{(i)}\frac{\partial\tilde{T}_2^{(i)}}{\partial\beta^{(i)}} + \tilde{q}_2^{(i)} = 0,$$

$$\frac{\partial\tilde{Q}_1^{(i)}}{\partial\alpha^{(i)}} + \tilde{R}^{(i)}\frac{\partial\tilde{Q}_2^{(i)}}{\partial\beta^{(i)}} - \tilde{T}_1^{(i)} + \tilde{q}_3^{(i)} = 0,$$

$$\frac{1}{\tilde{h}^{(i)}}\left(\frac{\partial\tilde{M}_1^{(i)}}{\partial\alpha^{(i)}} + \tilde{R}^{(i)}\frac{\partial\tilde{M}_{21}^{(i)}}{\partial\beta^{(i)}}\right) - \tilde{Q}_1^{(i)} + \tilde{m}_1^{(i)} = 0, \tag{11}$$

$$\frac{1}{\tilde{h}^{(i)}}\left(\frac{\partial\tilde{M}_{12}^{(i)}}{\partial\alpha^{(i)}} + \tilde{R}^{(i)}\frac{\partial\tilde{M}_2^{(i)}}{\partial\beta^{(i)}}\right) - \tilde{Q}_2^{(i)} + \tilde{m}_2^{(i)} = 0,$$

we obtain a system of five partial differential equations with five unknown functions. In the Rodionova–Titaev–Chernykh theory, this system is of order 14, while in the Palii–Spiro theory, it is of order 12.

Substituting the corresponding deformation components into the computational formulas (1)–(3), we obtain all components of the stress-strain state of the shell under consideration.

4. A NUMERICAL METHOD

We solve the problem by using the system of equations for a shell in displacements (11). The displacements of the middle surface of the shell are specified in the form

$$u^{(i)}(\alpha, \beta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} u_{nm}^{(i)} \sin[n\alpha]\sin[\bar{m}\beta],$$

$$v^{(i)}(\alpha, \beta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} v_{nm}^{(i)} \cos[n\alpha]\cos[\bar{m}\beta],$$

$$w^{(i)}(\alpha, \beta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} w_{nm}^{(i)} \cos[n\alpha]\sin[\bar{m}\beta], \quad \bar{m} = \frac{\pi m}{L}. \tag{12}$$

$$\gamma_1^{(i)}(\alpha, \beta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \gamma 1_{nm}^{(i)} \sin[n\alpha]\sin[\bar{m}\beta],$$

$$\gamma_2^{(i)}(\alpha, \beta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \gamma 2_{nm}^{(i)} \cos[n\alpha]\cos[\bar{m}\beta],$$

These formulas take into account the symmetry of the shell deformation with respect to the plane $\alpha = 0$ and ensure that the displacement components u, γ_1, and w vanish at $\beta = 0, L$. The expressions for v and γ_2 do not satisfy the zero boundary conditions, but if deformation does not reach the boundary, then these displacements are small. We represent the external and internal forces acting on the surface of the shell as a product of expansions of forces acting on sections. Let $X_1^{(i+1)}$, $X_2^{(i+1)}$, and $X_3^{(i+1)}$ be the components of the pressure on the outer surface of the ith shell, and let $X_1^{(i)}$, $X_2^{(i)}$, and $X_3^{(i)}$ be the components of the pressure on its inner surface. Then the expressions for the load and the momenta take the form

$$\tilde{m}_1^{(i)}(\alpha, \beta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\tilde{h}^{(i)}}{2}\left(X1_{nm}^{(i+1)}\left(1 + \frac{\tilde{h}^{(i)}}{2}\right) + X1_{nm}^{(i)}\left(1 - \frac{\tilde{h}^{(i)}}{2}\right)\right)\sin[n\alpha]\sin[\bar{m}\beta],$$

$$\tilde{q}_1^{(i)}(\alpha, \beta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left(X1_{nm}^{(i+1)}\left(1 + \frac{\tilde{h}^{(i)}}{2}\right) - X1_{nm}^{(i)}\left(1 - \frac{\tilde{h}^{(i)}}{2}\right)\right)\sin[n\alpha]\sin[\bar{m}\beta],$$

$$\tilde{m}_2^{(i)}(\alpha, \beta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\tilde{h}^{(i)}}{2}\left(X1_{nm}^{(i+1)}\left(1+\frac{\tilde{h}^{(i)}}{2}\right) + X1_{nm}^{(i)}\left(1-\frac{\tilde{h}^{(i)}}{2}\right)\right)\cos[n\alpha]\cos[\overline{m}\beta],$$

$$\tilde{q}_2^{(i)}(\alpha, \beta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left(X1_{nm}^{(i+1)}\left(1+\frac{\tilde{h}^{(i)}}{2}\right) - X1_{nm}^{(i)}\left(1-\frac{\tilde{h}^{(i)}}{2}\right)\right)\cos[n\alpha]\cos[\overline{m}\beta],$$

$$\tilde{m}_3^{(i)}(\alpha, \beta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\tilde{h}^{(i)}}{2}\left(X1_{nm}^{(i+1)}\left(1+\frac{\tilde{h}^{(i)}}{2}\right) + X1_{nm}^{(i)}\left(1-\frac{\tilde{h}^{(i)}}{2}\right)\right)\cos[n\alpha]\sin[\overline{m}\beta],$$

$$\tilde{q}_3^{(i)}(\alpha, \beta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left(X1_{nm}^{(i+1)}\left(1+\frac{\tilde{h}^{(i)}}{2}\right) - X1_{nm}^{(i)}\left(1-\frac{\tilde{h}^{(i)}}{2}\right)\right)\cos[n\alpha]\sin[\overline{m}\beta]; \tag{13}$$

$i = 1$ corresponds to the inner surface of the tube and $i = N + 1$ corresponds to its outer surface; the tube consists of N layers. Following [6], we impose the following rigid fixation condition between layers:

$$\tilde{u}_1^{(i)}\left(\alpha, \beta, \frac{h}{2}\right) = \tilde{u}_1^{(i+1)}\left(\alpha, \beta, \frac{-h}{2}\right),$$

$$\tilde{u}_2^{(i)}\left(\alpha, \beta, \frac{h}{2}\right) = \tilde{u}_2^{(i+1)}\left(\alpha, \beta, \frac{-h}{2}\right), \tag{14}$$

$$\tilde{u}_3^{(i)}\left(\alpha, \beta, \frac{h}{2}\right) = \tilde{u}_3^{(i+1)}\left(\alpha, \beta, \frac{-h}{2}\right).$$

We represent a load localized on a small rectangular domain as the product of the two Fourier transforms of the load functions in a longitudinal and a transverse section:

$$Pa[\alpha] = P\left(\frac{C}{L} + \frac{2}{L}\sum_{n=0}^{\infty}\frac{L}{n\pi}\sin\left(\frac{n\pi}{L}C\right)\cos\left(\frac{n\pi}{L}\alpha\right)\right). \tag{15}$$

The pressure in the longitudinal section of the tube is described as

$$Pb[\beta] = P\left(\frac{4}{L}\sum_{m=0}^{\infty}\frac{L}{m\pi}\sin\left(\frac{m\pi}{L}C\right)\sin\left(\frac{m\pi}{L}L_v\right)\sin\left(\frac{m\pi}{L}\beta\right)\right), \tag{16}$$

where L_v is the center of the loaded domain, $2C$ is the size of this domain, and P is the pressure on this domain.

The loaded domain is described by a product of series as

$$Pd[\alpha, \beta] = Pa[\alpha] * Pb[\beta]. \tag{17}$$

The load is applied to the outer surface of the tube:

$$X1_{nm}^{(N+1)} = 0, \quad X2_{nm}^{(N+1)} = 0, \quad X3_{nm}^{(N+1)} = Pd[\alpha, \beta]. \tag{18}$$

The pressure on the inner surface vanishes:

$$X1_{nm}^{(1)} = 0, \quad X2_{nm}^{(1)} = 0, \quad X3_{nm}^{(1)} = 0. \tag{19}$$

Substituting the dependences (12), (13), and (17) into the shell equilibrium equations (11) and into the rigid fixation relations (14), we obtain a system of $8N - 3$ linear algebraic equations with respect to the $5N$ deformation components and the $3N - 3$ interaction forces between the shell layers. Each of the obtained coefficients $u_{nm}^{(i)}$, $v_{nm}^{(i)}$, $w_{nm}^{(i)}$, $\gamma 1_{nm}^{(i)}$, $\gamma 2_{nm}^{(i)}$, $X_1^{(i)}$, $X_2^{(i)}$, and $X_3^{(i)}$ is a term in the Fourier expansion for the deformation and load functions.

To implement this numerical method, we developed a program based on the Mathematica 7.0 package.

5. COMPUTATIONAL RESULTS

In [4], the deformation of a nanotube with the following parameters was considered: the thickness of each of the 100 layers was $h = 0.135$ nm, the inner radius of the tube was $R = 2.5$ nm, the outer radius was $R = 16$ nm, and the length of the tube was $L = 500$ nm. For the shell elasticity moduli $E_{1,2,3} = 1.75 \times$

Table 1

L_v	250	204.5	181.8	159	136.3	90.9	68.18	45.45	22.72
Palii–Spiro theory	48.14	46.35	44.13	41.03	37.09	26.76	20.47	13.49	5.93
Rodionova–Titaev–Chernykh theory	48.35	46.55	44.32	41.21	37.24	26.87	20.55	13.55	5.96
Ansys	46.13	44.48	42.39	39.5	35.80	26.23	21.11	14.77	7.73

Table 2

h/R	1/15	1/12	1/9	1/6	1/5	1/4	1/3
Palii–Spiro theory	628.76	501.25	373.58	245.49	203.61	161.12	120.17
Rodionova–Titaev–Chernykh theory	630.3	503.3	375.8	248.22	206.58	164.4	123.81
Ansys	616	485.3	360.4	232.2	191.6	150.6	95.9

10^{11} Pa and comparatively small displacement magnitude $G_{13} = G_{12} = G_{23} = 2.3 \times 10^7$ Pa, the deflections of the tube under a local load determined by means of the Rodionova–Titaev–Chernykh theory were close to experimental data; the Poisson coefficients were $\nu_{12} = \nu_{21} = \nu_{32} = \nu_{31} = \nu_{23} = \nu_{13} = 0.3$.

Table 1 contains the deflections of the tube described above obtained by means of the Rodionova–Titaev–Chernykh and the Palii–Spiro theory. In this table, L_v denotes the coordinate of the point on the outer surface of the shell to which the force is applied. The displacement functions were calculated for the external force $F_v = 10$ nN. For comparison, deflection values for a transversally isotropic tube calculated by using the Ansys 11 package are given. A three-dimensional 20-nodal Solid 186 element was used.

Let us compare the results obtained by means of three-dimensional theory, which is used in the Ansys 11 package, with those obtained by means of the nonclassical shell theories described above. We consider a one-layer cylindrical shell with constant outer radius. Gradually increasing the thickness of the shell (and, as a consequence, decreasing the radius of the middle surface of the shell), we obtain the values of deflection at the center of the shells under consideration given in the table.

The results presented in Table 2 show that both theories give close deflections of shells. Moreover, other quantities characterizing stress-strain state of shells are close as well. As the relative thickness of the shell increases, the deflection values obtained by means of the Palii–Spiro theory approach those obtained by the finite element method.

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