Smooth Structures on Spheres

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Contents

Abstract

We provide an account of Milnor's construction of an exotic 7-sphere and the subsequent rapid development of differential topology used to produce and classify exotic manifolds. We begin by giving some necessary background, assuming only previous knowledge of real analysis and linear algebra. Smooth manifolds, vector bundles, and fibre bundles are introduced, along with some operations on manifolds not usually seen in a first course, before giving a review of all necessary algebraic topology. We pay particular attention to the homology of smooth manifolds, as this will form the basis for the constructions of the following section. We then introduce characteristic classes, which are one of the main ingredients in constructing smooth manifold invariants. Using this setup, we develop Milnor's original smooth invariant and a generalisation of it to a wider class of manifolds. We give a brief introduction to Morse theory, which we use to characterise topological spheres. Having set up the necessary background, we construct a number of examples of exotic spheres. We first present Milnor's original example, and then develop a more general tool, plumbing disk bundles, to give a much larger class of examples. Finally, we turn to the classification of smooth structures on spheres of dimension greater than four, developing the necessary background to state Milnor and Kervaire's classification results on homotopy groups of spheres, before indicating a number of future directions of study to the reader, as this thesis is ultimately intended to be an introduction to a vast field.

Introduction

A naïve goal of manifold topology would be to classify all manifolds up to homeomorphism, though this is far too optimistic. A more realistic goal would be to classify all manifolds of a certain homotopy or homology type up to homeomorphism. This idea led to one of the most fruitful problems in topology - is it possible to classify all compact manifolds with the homology of a sphere? In [Poi00], Poincaré claimed[1] that every compact n-dimensional manifold with the homology of the n-sphere is homeomorphic to S^n. However, Poincaré himself provided a counterexample in [Poi04]. He constructed the *Poincaré homology sphere.* This is a compact 3-dimensional manifold with the homology of S^3 as required. However, its fundamental group is the binary icosahedral group. Poincaré's sphere then cannot by homeomorphic to S^3, as the 3-sphere has trivial fundamental group. In the same paper Poincaré stated the *Poincaré conjecture.*

Poincaré conjecture. *Every compact 3-dimensional manifold with trivial fundamental group is homeomorphic to S^3.*

This conjecture would remain open for 98 years. The question has a natural generalisation. Define a *homotopy sphere* to be a compact manifold which is homotopy equivalent to a sphere.

Generalised Poincaré conjecture. *Is every n-dimensional homotopy sphere homeomorphic to the n-sphere?*

We can replace homeomorphism with either diffeomorphism or piecewise linear (PL) homeomorphism to obtain the *smooth Poincaré conjecture* and the *PL Poincaré conjecture* respectively.

For $n < 4$, the topological, smooth, and PL conjectures are equivalent. For $n = 0$ the conjecture is trivially true. The $n = 1$ and $n = 2$ conjectures are true by the classification of closed 1-manifolds and surfaces respectively [Mun00].

The classical Poincaré conjecture remained open until 2002. In 1983 Thurston [Thu82] stated his geometrisation conjecture. This stated that every compact oriented 3-manifold can be cut along spheres and tori in a unique way to decompose the manifold into pieces with one of eight geometric structures. This opened the door to attacking the Poincaré conjecture using methods from differential geometry and differential equations. Hamilton [Ham82] set out a program in 1982 to solve Thurston's geometrisation conjecture and the Poincaré conjecture simultaneously. This program was eventually completed in a pair of papers of Perelman [Per02], [Per03].

The four dimensional conjecture wasn't resolved until the 1980's. Freedman [Fre82] proved two closed simply connected 4-manifolds are homeomorphic if and only if they have the same intersection form and Kirby-Siebenmann invariant. In particular, homotopy 4-spheres have trivial intersection forms and Kirby-Siebenmann invariant zero. Hence, the Poincaré

[1] This is a restatement of the original problem in modern terminology.

conjecture is true in dimension 4. However, the 4-dimensional conjecture for PL and smooth manifolds has proven much more complicated than all others. The smooth and PL conjecture are equivalent in dimension 4. Most of the results come from the study of smooth manifolds in physics. Donaldson [Don83] developed a connection between gauge theory and smooth 4-manifolds. This would grow into its own field, Donaldson theory. Taubes used Donaldson theory to show that there are uncountably many distinct smooth structures on $\mathbb{R}^4$ in [Tau87]. There is very little known about the Poincaré conjecture itself in dimension 4. In fact, it is still unclear whether or not the conjecture is true in this dimension.

Surprisingly, all dimensions $n \geq 5$ can be handled simultaneously in the topological setting. The problem is actually simpler in high dimensions due to increased freedom to manipulate low dimensional submanifolds. Smale proved that if M^n is a smooth homotopy sphere of dimension $n \geq 5$, then M is homeomorphic to S^n in [Sma61]. The key idea is to build up a manifold from an n-disk, D^n, using handles. A λ-*handle* is a copy of D^n, given as a product, $H^\lambda \cong D^\lambda \times D^{n-\lambda}$. A *handle presentation* of M is then a sequence of attachments of handles to D^n to arrive at a manifold homeomorphic to M. Two manifolds with identical handle presentations will then be homeomorphic. A handle presentation can be simplified by "cancelling" handles, leading to a *minimal presentation* of M. Smale showed that all smooth homotopy n-spheres had the same minimal presentation as S^n. This resolved the generalised topological Poincaré conjecture for smooth manifolds, though the smooth requirement was eventually removed.

It was initially assumed that the smooth version of the conjecture for high dimensions would follow quickly from the topological version as homeomorphisms can be approximated with arbitrarily close diffeomorphisms. However, there is no way to ensure the two approximated diffeomorphisms are inverses of one another. Still, it was expected that these problems would be easily resolved in time. As such, when Milnor first discovered a homotopy 7-sphere which wasn't diffeomorphic to S^7, he assumed he had found a counterexample to the topological Poincaré conjecture [Mil00]. However, the Reeb sphere theorem confirmed that Milnor's manifold was homeomorphic to S^7. He was left to conclude in [Mil56] that he had discovered an *exotic sphere*, a sphere homeomorphic to S^n, but not diffeomorphic to it.

This and the work in the years following, led to the birth of differential topology as its own field and earned Milnor the Fields Medal in 1962. Many questions posed themselves immediately- how many distinct smooth structures can a topological manifold possess? Can a manifold possess no smooth structure? In the seven years following Milnor's initial discovery, an understanding of smooth structures on manifolds as an additional and interesting structure emerged. In 1957, Shimada [Shi57] extended Milnor's work to construct a number of exotic 15-spheres. Milnor went on to discover exotic spheres in all dimensions of the form $4n-1$ in [Mil59a]. Perhaps most strangely of all, Kervaire constructed a 10-manifold which does not admit any smooth structure in [Ker60]. The smooth Poincaré conjecture was finally resolved for $n \geq 5$ in 1963 with the joint work of Milnor and Kervaire in [KM63]. They made use of Smale's h-cobordism theorem to convert the problem of counting smooth structures on homotopy n-spheres to the computation of stable homotopy groups.

The goal of this paper is to give an account of Milnor's discovery and the results which followed in the years after. In particular, we will construct Milnor's exotic 7-sphere and give a generalisation to the invariant he used to other dimensions. We then define plumbing of manifolds to construct a large number of exotic spheres in various dimensions. Finally, we explore the structure of homotopy n-spheres and related structures, and outline the classification of homotopy spheres carried out by Milnor and Kervaire.

1 Smooth Manifolds

A first course on manifolds will often focus on either topological or smooth manifolds. There usually isn't much emphasis placed on the extra structure smoothness enforces. However many natural questions arise from the definition of a smooth structure, as outlined in the introduction. To explore these questions, we first recall the definition of a topological manifold, then introduce smooth structures and smooth manifolds. In section 1.3 we introduce an important structure associated to a smooth manifold, the tangent bundle. This allows us to define the notion of a smooth submanifold and define a smooth version of general position.

1.1 Topological Manifolds

A topological space, M, is an *n-dimensional topological manifold* if it is:

1. **Locally Euclidean.** Each point x of M has a neighbourhood U that is homeomorphic to an open subset of $\mathbb{R}^n$.

2. **Hausdorff.** For any pair of distinct points x, y of M there exist disjoint open subsets of M which contain x and y respectively. i.e. any two points can be separated.

3. **Second-countable.** The topology of M has a countable basis.

The second and third conditions exclude pathological examples.

Some extra terminology is useful when working with manifolds. A *chart* for M is a pair (U, φ) where U is an open subset of M and φ is a homeomorphism from U to an open subset $\varphi(U) \subset \mathbb{R}^n$. We call U a *coordinate neighbourhood* and φ a *coordinate map.* The component functions $(x^1, \ldots, x^n)$ given by

$$(x^1(p), \ldots, x^n(p)) = \varphi(p)$$

are *local coordinates* on U. An *atlas* for a manifold M is a collection of charts which cover M. We will sometimes use the notation M^n to indicate that M is an n-dimensional manifold. We will use the convention that a *function* refers to a continuous mapping into the real numbers and a *map* is a continuous mapping between topological spaces.

Example 1.1 The space $\mathbb{R}^n$ is a topological manifold of dimension n with one coordinate chart given by the identity map. It is Hausdorff as points of $\mathbb{R}^n$ are separable and has countable basis given by the collection of open balls with rational radii and rational centre coordinates.

Example 1.2 Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^k$ be a continuous function. The *graph* of f is the set

$$G(f) = \left\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : \ x \in U, \ y = f(x)\right\}.$$

Equip this set with the subspace topology. It is both second countable and Hausdorff as it is a subspace of $\mathbb{R}^n \times \mathbb{R}^k$. On $G(f)$, the map π_1 which projects to the first factor is continuous with continuous inverse given by $\pi_1^{-1}(x) = (x, f(x))$. Therefore π_1 is a homeomorphism onto U. Hence $G(f)$ is an n-dimensional manifold homeomorphic to U.

Example 1.3 The n-sphere

$$S^n := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1\}$$

is Hausdorff and second countable as it is a subspace of $\mathbb{R}^{n+1}$. Let U_i^+ be the subset of S^n with ith coordinate positive, and U_i^- be the subset with ith coordinate negative. Note every point of S^n is in at least one of these sets. Define $f : B^n \to \mathbb{R}$ by

$$f(v) = \sqrt{1 - |v|^2}.$$

Then each $U_i^\pm$ is the graph of the function

$$y^i = \pm f(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}),$$

where $\hat{x}_i$ means x_i is omitted. Therefore each of these sets is locally Euclidean and we can define coordinate maps by

$$\varphi_i^\pm(x_1, \ldots, x_{n+1}) = (y_1, \ldots, \hat{y}_i, \ldots, y_{n+1}).$$

1.2 Smooth Manifolds

A map $F : \mathbb{R}^n \to \mathbb{R}^k$ is *differentiable* at x if there is a linear map $DF_x : \mathbb{R}^n \to \mathbb{R}^k$ such that

$$\lim_{h \to 0} \frac{||F(x+h) - F(x) - DF_x(h)||_{\mathbb{R}^k}}{||h||_{\mathbb{R}^n}} = 0.$$

Recall if F is differentiable at x then all partial derivatives of F at x exist and DF_x is the Jacobian matrix of partial derivatives. A map $F : \mathbb{R}^n \to \mathbb{R}^k$ is *smooth* if its partial derivatives of all orders exist. A map is a *diffeomorphism* if it is smooth and has a smooth inverse. We would like to transfer as much of this machinery as possible to manifolds by using charts, however, we need a definition of smoothness which is well defined when changing between charts.

We define a smooth compatibility condition between charts as follows. Let (U, φ), (V, θ) be two charts with non-empty intersection. The map $\theta \circ \varphi^{-1} : \varphi(U \cap V) \to \theta(U \cap V)$ is called the *transition map* from φ to θ; see Figure 1. Note transition maps are just maps between open subsets of $\mathbb{R}^n$. We say two charts are *smoothly compatible* if either the transition map is a diffeomorphism or their intersection is empty. An atlas $\mathcal{A}$ for a manifold M is a *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible. A *smooth structure* $\mathcal{A}$ on M is a maximal smooth atlas – that is, $\mathcal{A}$ is not a proper subset of any larger smooth atlas. A *smooth manifold* is a pair $(M, \mathcal{A})$, though usually we write M, the smooth structure being clear from context.

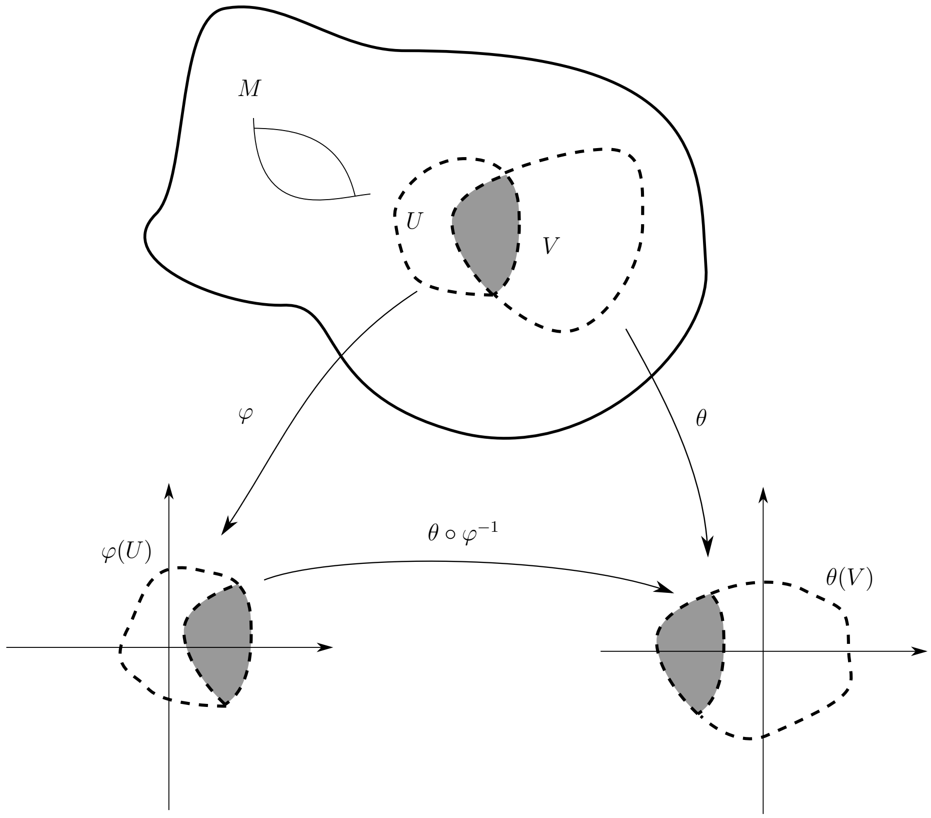


Figure 1: Transition maps between charts

Example 1.4 Consider the following 3 atlases for $\mathbb{R}$:

1. $\mathcal{A}_1 = \{(\mathbb{R}, \mathrm{Id}_{\mathbb{R}})\}$.
2. $\mathcal{A}_2 = \{(B_1(x), \mathrm{Id}_{B_1(x)}) : x \in \mathbb{R}\}$.
3. $\mathcal{A}_3 = \{(\mathbb{R}, \varphi), \varphi(x) = x^3\}$.

The first two are different atlases, but both clearly determine the same maximal atlas, therefore the same smooth structure. We call this the standard smooth structure for $\mathbb{R}$. However the smooth structure determined by $\mathcal{A}_3$ is not the standard one. We can see it is not compatible with the standard structure as the transition map

$$\mathrm{Id}_{\mathbb{R}} \circ \varphi^{-1}(x) = x^{1/3}$$

is not smooth at the origin.

In fact, if a manifold has a smooth structure, it has uncountably many distinct smooth structures. The intuitive idea is as follows. Construct a family of homeomorphisms F_s of B^n which are not diffeomorphisms. Then replace a chart (U, φ) with $(\varphi^{-1}(B^n), F_s \circ \varphi)$. The new chart will then not be compatible with the original atlas, and so this new atlas must define a new smooth structure. Some care must be taken in how we replace this chart to make the argument rigorous. This is less than ideal if we want to classify smooth manifolds. We would like an idea of equivalence between smooth manifolds which is weaker than having identical smooth structure. For this we need the concept of smooth maps.

To define a smooth map between manifolds we use the definition of smoothness in the real case.

Let M^m, N^n be smooth manifolds. A map $F : M \to N$ is *smooth* if for every point $p \in M$ there exist a pair of smooth charts (U, φ) and (V, θ) containing p and $F(p)$ respectively such that $F(U) \subset V$ and

$$\theta \circ F \circ \varphi^{-1} : \varphi(U) \to \theta(V)$$

is a smooth real function.

Note that if M, N above are subsets of $\mathbb{R}^n$ we recover the usual definition for smoothness of real functions.

Just as we define a homeomorphism to be a continuous bijection with continuous inverse, a *diffeomorphism* is a bijective smooth map with smooth inverse. We say the manifolds M, N are *diffeomorphic* to each other if there exists a diffeomorphism between them, and write $M \overset{\text{diff}}{\cong} N$. We consider two smooth manifolds to be equivalent if they are diffeomorphic.

Example 1.5 Returning to the atlases $\mathcal{A}_1$, $\mathcal{A}_3$ of example 1.1 we can see the smooth manifolds $M_1 = (\mathbb{R}, \mathcal{A}_1)$, $M_3 = (\mathbb{R}, \mathcal{A}_3)$ are diffeomorphic through the diffeomorphism $F : M_1 \to M_3$ given by

$$F(x) = x^{1/3}.$$

This is a diffeomorphism as both $(\varphi \circ F \circ \mathrm{Id}_{\mathbb{R}}^{-1})(x) = x$ and its inverse are smooth functions.

Oriented Manifolds

Recall that an orientation for a real vector space, F^n, is given by an equivalence class of bases, $\{v_1, \ldots, v_n\}$, where two bases are equivalent if any only if the change of basis between them has positive determinant. As such, every vector space has exactly two orientations. The vector space $\mathbb{R}^n$ has a canonical orientation given by $\{e_1, \ldots, e_n\}$, where e_i has a one in the ith coordinate and is zero in all other coordinates.

Let M be a smooth manifold and consider a pair of charts (U, φ), (V, θ) for M. The transition function between these charts, $\theta \circ \varphi^{-1}$, is a smooth real map, hence we may consider the determinant of the Jacobian matrix. Recall if this determinant is positive then the transition map is an orientation preserving map, and if it is negative then it is orientation reversing. We say the charts (U, φ), (V, θ) are *oriented compatibly* if the transition function is orientation preserving. An atlas of charts such that each pair of charts is oriented compatibly is called an *oriented atlas*. We say the atlas determines an *orientation* for M. If there is an oriented atlas for M we say that M is *orientable.* If M is orientable, we may compose each chart with an orientation reversing diffeomorphism of $\mathbb{R}^n$. This defines an oriented smooth manifold diffeomorphic to M but with a different orientation. We call this manifold *M with orientation reversed*, and denote it by $-M$. We will give some alternative definitions of oriented manifolds later.

Manifolds with Boundary

Our definition of manifold does not allow for any kind of boundary - every point must locally look like $\mathbb{R}^n$. There is a natural generalisation by allowing our manifolds to be locally homeomorphic to relatively open subsets the closed upper half plane,

$$\mathbb{R}^n_{\geq 0} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_n \geq 0\} .$$

We define a *manifold with boundary* to be a second countable Hausdorff space for which every point has a neighbourhood which is either homeomorphic to an open subset of $\mathbb{R}^n$ or to a relatively open subset of $\mathbb{R}^n_{\geq 0}$. Charts with image open in $\mathbb{R}^n$ are called *interior charts* and charts with image open only in $\mathbb{R}^n_{\geq 0}$ are called *boundary charts.* Similarly a point $x \in M$ is called an *interior point* of M if it is contained in some interior chart and a *boundary point* if it is in some boundary chart (U, φ) such that $\varphi(x) \in \partial \mathbb{R}^n_{\geq 0}$. The set of all boundary points of M is called the boundary of M and denoted ∂M. The set of all interior points, denoted $\mathring{M}$, is called the interior.

Lemma 1.6 *The boundary and interior of a manifold M are disjoint sets whose union is M. That is, every point of M is either an interior point or boundary point.*

1. *$\mathring{M}$ is an open subset of M and an n-manifold without boundary.*
2. *∂M is a closed subset of M and an $(n-1)$-manifold without boundary.*

Note that every manifold is a manifold with boundary. However a manifold with boundary is only a manifold if it has empty boundary. "Manifold" will always mean manifold without boundary, but if we want to emphasise that a result cannot be extended to manifolds with boundary we say a *closed manifold* is a compact manifold without boundary.

Let S be a subset of $\mathbb{R}^n$. Recall a map $f : S \to \mathbb{R}^k$ is smooth if for each point x of S there is a neighbourhood $x \in U \subset \mathbb{R}^n$ and an extension F of f such that F is smooth on U. We define a smooth structure for a manifold with boundary exactly as before - it is a maximal smooth atlas of charts whose domains cover M and whose transition maps are smooth. However now smoothness is defined in terms of open subsets of $\mathbb{R}^n$ and $\mathbb{R}^n_{\geq 0}$ as above. A manifold with boundary equipped with a smooth structure is a *smooth manifold with boundary.* Define an orientation for a manifold with boundary as an orientation of its interior. This induces an orientation on the boundary by restriction.

1.3 The Tangent Bundle

Tangent Vectors

The derivative of a map $F : \mathbb{R}^n \to \mathbb{R}^k$ at a point p is a linear approximation to F. For example, the best linear approximation of a map of one variable given by the tangent line to its graph, and a tangent plane is the best linear approximation of a map of two variables. We think of tangent vectors as existing in a copy of $\mathbb{R}^n$ attached to p. To extend this idea to manifolds we need to define a tangent space which will hold the derivatives of functions at a point.

Let $p \in \mathbb{R}^n$, to every tangent vector $v \in \mathbb{R}^n$ at p we can associate a directional derivative D_v acting on smooth real valued functions $f : \mathbb{R}^n \to \mathbb{R}$ by

$$D_v f(p) = \left.\frac{d}{dt}\right|_{t=0} f(p + tv).$$

For $p \in \mathbb{R}^n$ a *derivation* at p is a linear map $X : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ which satisfies the product rule

$$X(fg) = f(p)X(g) + X(f)g(p)$$

for all smooth functions f, g on $\mathbb{R}^n$. The directional derivate above is a derivation. Denote by $T_p\mathbb{R}^n$ the space of derivations at p.

Theorem 1.7 *The map $v_p \to D_v|_p$ is an isomorphism from the space of tangent vectors at p to $T_p\mathbb{R}^n$. We call $T_p\mathbb{R}^n$ the tangent space to $\mathbb{R}^n$ at p.*

For any $p \in \mathbb{R}^n$ the partial derivatives $\partial/\partial x^i|_p = D_{e_i}|_p$ are derivations which form a basis for $T_p\mathbb{R}^n$ at any point, hence $T_p\mathbb{R}^n$ is an n-dimensional vector space.

We now want to adapt the above to smooth manifolds. The set of smooth functions on M forms a ring under pointwise addition and multiplication. Denote this ring by $C^\infty(M)$. A *derivation* of $C^\infty(M)$ at $p \in M$ is a linear function $X : C^\infty(M) \to \mathbb{R}$ which satisfies

$$X(fg) = X(f)g(p) + f(p)X(g) \quad \forall f, g \in C^\infty(M).$$

The *tangent space* of M at p, T_pM is the set of all derivations of $C^\infty(M)$ at p. Elements of T_pM are called *tangent vectors* at p.

For M, N smooth manifolds and $F : M \to N$ a smooth map we can define a linear map $F_* : T_pM \to T_{F(p)}N$ by

$$(F_*X)(f) = X(f \circ F).$$

We call this map the *pushforward* of F at p, and sometimes use the notation dF_p. The pushforward is a linear map. It follows directly from the definition that $(F \circ G)_* = F_* \circ G_*$ and that the pushforward of the identity map is the identity on T_pM, hence if F is a diffeomorphism then F_* is an isomorphism of of tangent spaces.

Let (U, φ) be a coordinate chart centered at $p \in M^n$. The map φ induces a vector space isomorphism through the differential,

$$d\varphi : T_pM \overset{\cong}{\to} T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n.$$

Hence the tangent space at each point carries the structure of a real n-dimensional vector space; see Figure 2.

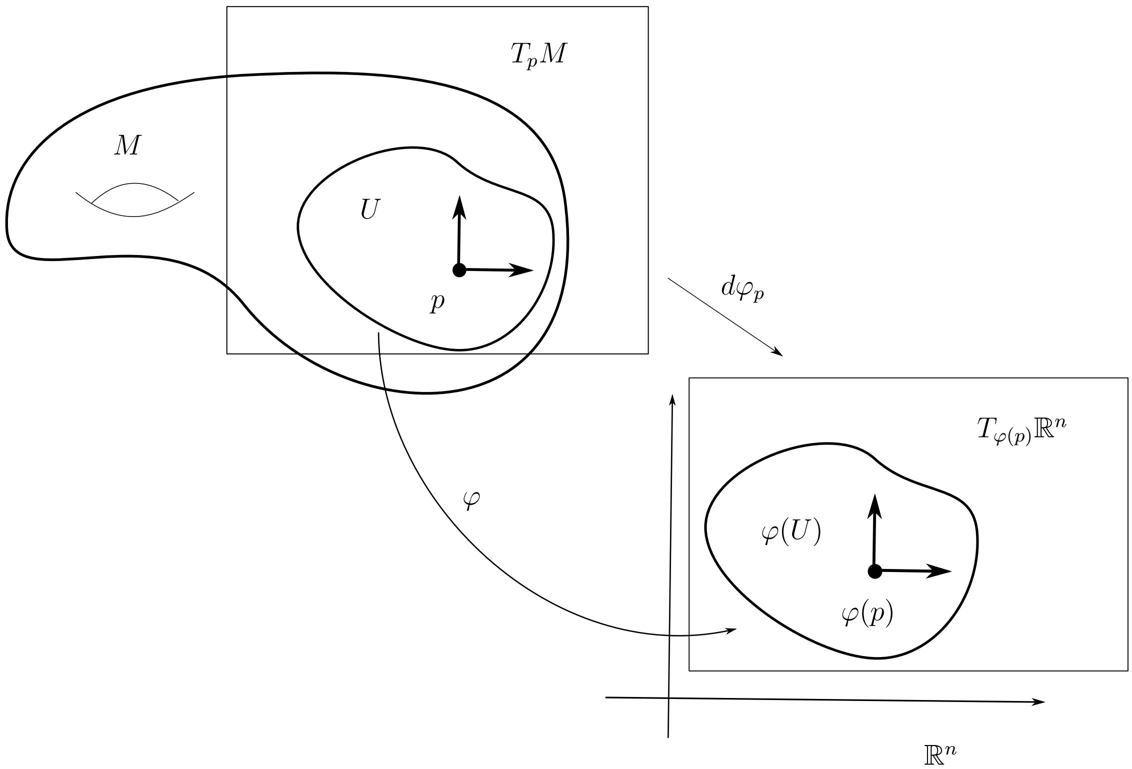


Figure 2: The tangent space to a point is a vector space.

We form a basis for T_pM as follows. Recall the partial derivatives form a basis for $T_{\varphi(p)}\mathbb{R}^n$. As $d\varphi$ is an isomorphism, the preimage of the partial derivatives under $d\varphi$ must therefore form a basis for T_pM. Define the *i-th coordinate vector at p*, $\partial/\partial x^i|p$ as

$$\left.\frac{\partial}{\partial x^i}\right|_p = (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right).$$

These vectors act on a function $f \in C^\infty(U)$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} (f \circ \varphi^{-1})$$

It follows that any tangent vector $v \in T_pM$ can be written as a linear combination

$$v = \sum_i v^i \left.\frac{\partial}{\partial x^i}\right|_p.$$

We will usually want to work with all of the tangent spaces of a manifold simultaneously. Define the *tangent bundle*, TM, as the disjoint union of the tangent spaces to every point of M,

$$TM = \bigsqcup_{p \in M} T_pM.$$

Elements of TM are denoted by pairs (p, v), $p \in M$, $v \in T_pM$. We usually do not distinguish between $(p, v) \in TM$ and $v \in T_pM$. Note that the tangent bundle has a natural projection map $\pi : TM \to M$ sending a pair (p, v) to the point p. The tangent bundle has a natural topology and smooth structure making it into a smooth manifold.

Theorem 1.8 *Let M be a smooth manifold. The tangent bundle of M, TM, has a natural topology and smooth structure such that TM is a smooth $2n$-dimensional manifold and that the projection map $\pi : TM \to M$ is smooth.*

Proof. Let (U, φ) be a chart for M and let $(x_1, \ldots, x_n)$ be the local coordinates of φ. Define a chart $(\pi^{-1}(U), \tilde{\varphi})$ by

$$\tilde{\varphi}\left(v_i \frac{\partial}{\partial x^i}\bigg|_p\right) = (x_1(p), \ldots, x_n(p), v_1, \ldots, v_n)$$

The map $\tilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n$ has smooth inverse given by

$$\tilde{\varphi}^{-1}(x_1(p), \ldots, x_n(p), v_1, \ldots, v_n) = v_i \frac{\partial}{\partial x^i}\bigg|_{\varphi^{-1}(x)}$$

It follows that TM is locally homeomorphic to $\mathbb{R}^{2n}$. Verifying that these charts define a smooth structure for TM and that TM is a manifold is largely routine. For full details see [Lee03] Lemma 4.1. $\square$

Vector Fields

A vector field X for a smooth manifold M is a choice of tangent vector $X(p) \in T_pM$ for each $p \in M$. That is, a *vector field* is a map $X : M \to TM$ such that

$$\pi \circ X = \mathrm{Id}_M.$$

A *smooth vector field* is a vector field which is a smooth map from M to the tangent bundle.

Example 1.9 Let $(U, (x_1, \ldots, x_n))$ be a smooth chart for M. Define the *ith coordinate vector field*, $\partial/\partial x_i$, as the vector field given locally by $(0, \ldots, 1, \ldots, 0)$, with a 1 in the ith position. This is a smooth vector field on U.

Integral Curves

Let I denote the unit interval $[0,1]$. On a smooth manifold M, a smooth curve $\gamma : I \to M$ defines a tangent vector $\gamma^{'}(t) \in T_{\gamma(t)}M$ at each point $\gamma(t)$ of the curve. For a general vector field, X, we can ask whether there exist a smooth curve $\gamma : I \to M$ starting at $\gamma(0) = p$ on some interval such that on I

$$\begin{aligned}\gamma^{'}(t) &= X(\gamma(t));\\ \gamma(0) &= p.\end{aligned}$$

Such a curve is called an integral curve for X.

The condition can be restated in local coordinates as the system of ordinary differential equations

$$\begin{aligned}(\gamma^i)^{'}(t) &= X^i(\gamma^1(t), \ldots, \gamma^n(t));\\ \gamma(0) &= p.\end{aligned}$$

It follows from the existence and uniqueness of solutions to ordinary differential equations there exists an integral curve γ with $\gamma(0) = p$ on some interval I for X [TP63].

An alternative perspective on integral curves is given by the notion of a flow. Suppose X is a vector field on M such that each point $p \in M$ has a unique integral curve $\theta^{(p)} : \mathbb{R} \to M$ starting at p and defined for all $t \in \mathbb{R}$. Then define a map $\theta_t : M \to M$ for each $t \in \mathbb{R}$ by

$$\theta_t : p \mapsto \theta^{(p)}(t).$$

The map moves points of the manifold along an integral curve through the point for time t. Note that $\theta_0(p) = p$. As the integral curves at a point are unique we have $\left(t \mapsto \theta^{(p)}(t+s)\right)$ is an integral curve of X starting at $\theta^{(p)}(s)$, therefore $\theta_t \circ \theta_s(p) = \theta_{t+s}(p)$.

All together we have an additive group action of $\mathbb{R}$ on M given by the map $\theta : \mathbb{R} \times M \to M$, $(t,p) \mapsto \theta_t(p)$. Define a *one-parameter group of diffeomorphisms*, or *global flow* of M to be a smooth left-$\mathbb{R}$ action on M. That is, a global flow is a smooth map $\theta : \mathbb{R} \times M \to M$ satisfying

1. $\theta(0,p) = p$.
2. $\theta(t, \theta(s,p)) = \theta(t+s, p)$.

1.4 Submanifolds

Embeddings and Isotopies

We define the smooth version of a topological embedding. A smooth map $f : M \to N$ is a *smooth embedding* if f is a topological embedding and df is injective at each point of M. Unless specified otherwise, embedding will always mean smooth embedding. We say M is a *submanifold* of N if the inclusion map is an embedding.

Note every manifold we have considered so far has been embedded in a Euclidean space. The following theorem of Whitney says that we can consider any compact n-manifold to be embedded in $\mathbb{R}^N$, for some N.

Theorem 1.10 (Whitney Embedding Theorem) *A smooth compact manifold can be embedded in a Euclidean space.*

This N can be much larger than the dimension of the manifold itself. There are a number of bounds on this dimension, however we will not need them; see [Kos72].

Similar to the notion of homotopic maps, we can define an equivalence between embeddings by isotopies. Let M, N be smooth manifolds and $f, g : M \to N$ embeddings. An *isotopy* between f and g is a smooth map $F : M \times \mathbb{R} \to N$ such that

1. $$\begin{cases} F(x,0) = f(x) \\ F(x,1) = g(x) \end{cases}$$

2. For fixed $t \in [0,1]$, F is an embedding. That is, $F(x,t)$ is an embedding for $0 \leq t \leq 1$.

Two embeddings of a smooth manifold M into N are *isotopic* if there is an *isotopy* between them. We consider isotopic embeddings to be equivalent.

Transversality

Let M^m, N^n be embedded submanifolds of a smooth manifold X^d. We say that M is *transversal* N if for all points $p \in M \cap N$,

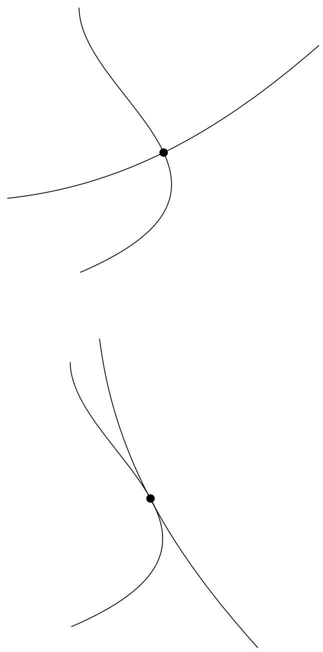
$$T_pM + T_pN = T_pX.$$

Note this relation is symmetric. This is a smooth version of the topological notion of general position. If $m + n < d$ we can move M and N through isotopies to be disjoint to each other in X, hence transversality is vacuously true. When $d = m + n$ the two manifolds must intersect in a discrete collection of points as

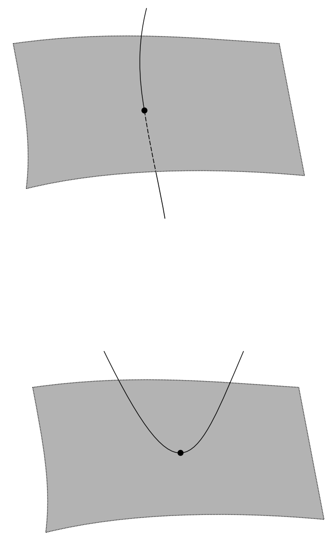
$$\dim M \cap N = \dim M + \dim N - \dim X = m + n - d = 0.$$

See Figure 3 below for some typical examples of transversality. Note that the definition relies strongly on the surrounding space. The top left example is a transversal intersection in $\mathbb{R}^2$, however it would not be transversal in $\mathbb{R}^3$.

We will need two important facts we need about transversality. First, transversal intersections are stable under small perturbations of the embeddings of M and N in X. This can be seen in Figure 3 above- perturbing either of the non-transversal intersections gives a transversal intersection. The second fact we need is that transversal intersections are *generic*. That is, given any embedding of M into X there is an arbitrarily close embedding which has M intersect N transversally. This follows from the transversality theorem. Further details and all proofs can be found in [GP74].



(a) Intersections in $\mathbb{R}^2$.



(b) Intersections in $\mathbb{R}^3$.

Figure 3: Some examples of transversal and non-transversal intersections. The intersections on the top row are transversal and the intersections on the bottom are not.

1.5 Operations on Manifolds

The boundary of a smooth manifold is well behaved in the following sense. For any smooth manifold with boundary there is a smooth embedding $\alpha : [0,1) \times \partial M \to M$ which is the identity when restricted to ∂M. Such an embedding is called a *collar embedding* of M and we call the image of α a *collar neighbourhood* of ∂M. The intuitive idea is indicated in Figure 4, in a neighbourhood of the boundary M looks like the product of ∂M and a half-open interval.

Theorem 1.11 (Collar Neighbourhood Theorem) *If M is a smooth manifold with non-empty boundary then ∂M has a collar neighbourhood.*

See [Lee03] for a proof.

This theorem allows us to paste together manifolds with diffeomorphic boundaries.

Lemma 1.12 (Gluing Lemma) *Let M, N be smooth n-dimensional manifolds with boundary and $f : \partial N \to \partial M$ be a diffeomorphism. Then the space*

$$M \cup_f N := M \sqcup N \, / \, (x \sim f(x))$$

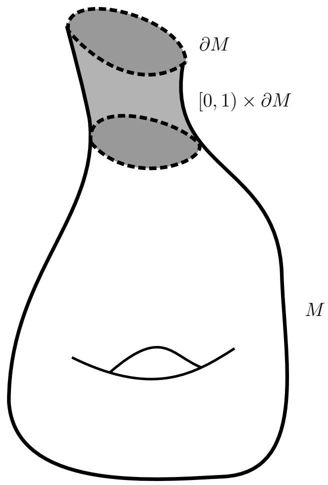


Figure 4: A collar neighbourhood of ∂M.

is a smooth manifold with submanifolds M', N' diffeomorphic to M, N such that

$$M' \sqcup N' = M \cup_f N;$$

$$M' \cap N' = \partial M' = \partial N'.$$

Furthermore, suppose M, N are oriented manifolds and f is an orientation preserving diffeomorphism. Then we define the gluing of M and N as

$$M \cup_f N := M \sqcup (-N) \,/\, (x \sim f(x)) \,.$$

The glued manifold will then be oriented such that $M^{'}$ is oriented the same way as M and $N^{'}$ is oriented the same way as $-N$.

Proof. Let $X = M \cup_f N$ and let $\pi : M \sqcup N \to X$ be the quotient map. The collar neighbourhood theorem gives a collar embedding $\alpha_M : [0,1) \times \partial M \to M$. Denote by C_M the image of this map. Similarly define α_N and C_N for N.

Define a map $F : C_M \sqcup C_N \to (-1,1) \times \partial M$ by

$$F(x) = \begin{cases} (-t, p) & x = \alpha_M(t,p) \in C_N \\ (t, f(q)) & x = \alpha_N(t,q) \in C_M. \end{cases}$$

F is an embedding on C_M and C_N with closed image by construction, hence a closed map. F is constant on fibres of π as when $t = 0$ we have $(0, f(p)) = (0, p)$. Thus, F descends to a continuous map $\hat{F} : \pi(C_M \sqcup C_N) \to (-1,1) \times \partial M$. $\hat{F}$ is bijective by construction and closed as F was closed, hence it is a homeomorphism. Therefore $\pi(C_M \sqcup C_N)$ is a topological manifold. However $\pi(\mathring{M} \sqcup \mathring{N}) \cong \mathring{M} \sqcup \mathring{N}$ is also a topological manifold. Hence X is covered by two topological manifolds and so is both locally Euclidean and second countable. Any two fibres of π can be separated by saturated open sets, so X is also Hausdorff, hence a topological manifold.

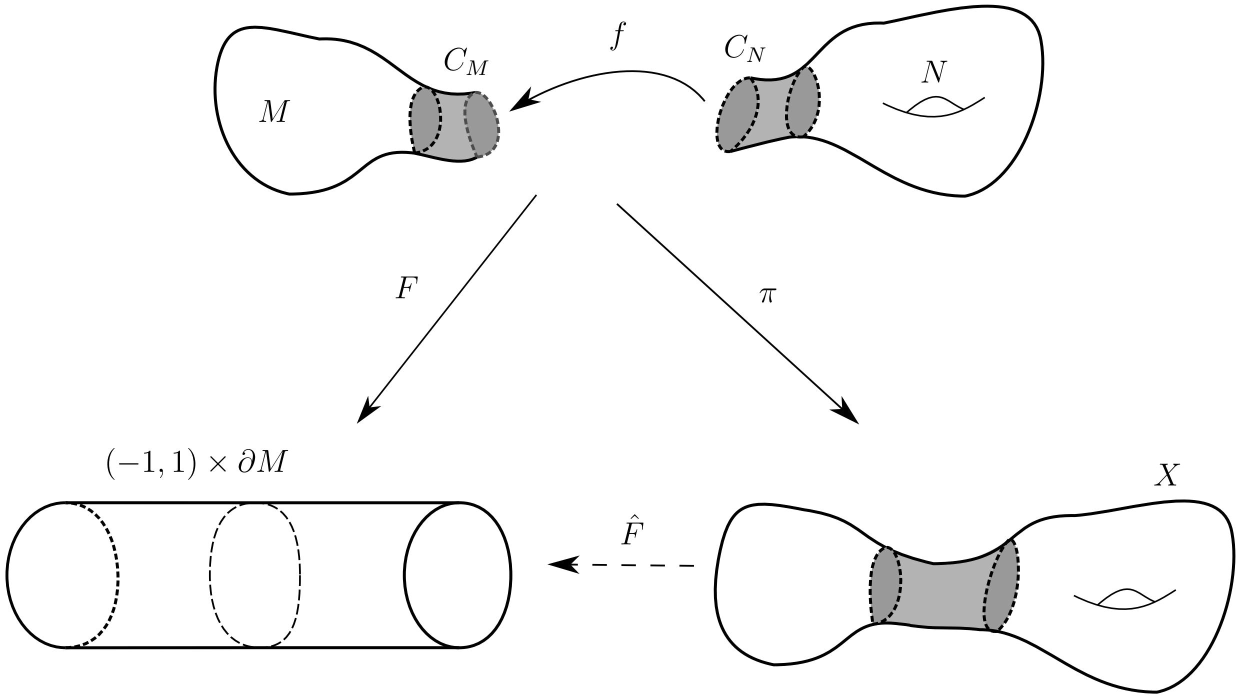


Figure 5: Gluing manifolds along common boundaries.

Define an atlas of charts on X as follows. For each smooth chart (U, φ) of the interior of M or N add the chart

$$\left(\pi(U), \varphi \circ \pi^{-1}|_{\pi(U)}\right).$$

Also for each smooth chart (U, φ) of $(-1,1) \times \partial M$ add to the atlas the chart

$$\left(\hat{F}^{-1}(U), \varphi \circ \hat{F}|_{\hat{F}^{-1}(U)}\right).$$

These are compositions of homeomorphisms which cover X and are smoothly compatible. This defines a smooth structure for X. The restriction of π to either M or N is continuous, closed, and injective. We define the image of these embeddings to be M' and N' respectively. The required properties follow by construction. $\square$

This lemma allows us to define a new construction for manifolds. Let M_1, M_2 be smooth, connected, n-dimensional manifolds. Let (V_i, φ_i) be coordinate neighbourhoods at points $p_i \in M_i$. By composing diffeomorphisms we can pass to coordinate neighbourhoods U_i such that

$$\varphi(U_i) = B_1(0), \quad \varphi(\overline{U_i}) = D^n.$$

Then the sets $M_i - U_i$ are smooth manifolds with boundary diffeomorphic to S^{n-1}. Choose a diffeomorphism f of S^{n-1} isotopic to the identity and consider it as a diffeomorphism from ∂M_1 to ∂M_2.

Define the *connected sum* of M_1 and M_2, $M_1 \# M_2$, to be the set

$$M_1 \# M_2 := M_1 \sqcup M_2 \,/\, (x \sim f(x)).$$

By the gluing lemma, this is a smooth manifold. We can similarly define the *connected sum along the boundary* of two manifolds with boundary, denoted $M_1 \#_\partial M_2$. A detailed construction can be found in [Kos72].

Under the operation of connected sum, the set of connected, oriented, and closed smooth n-dimensional manifolds forms a commutative monoid with identity S^n. Commutativity and associativity are clear from the definition. Note that S^n with a disk removed is diffeomorphic to D^n. Hence, the connected sum of a manifold M with S^n amounts to removing a disk from M and then reattaching a disk, and so S^n is the identity element. We will study this monoid in chapter 7. In particular, we will show that the invertible elements, A^n, are homotopy spheres. This will then lead to a way to count the number of smooth structures on homotopy n-spheres.

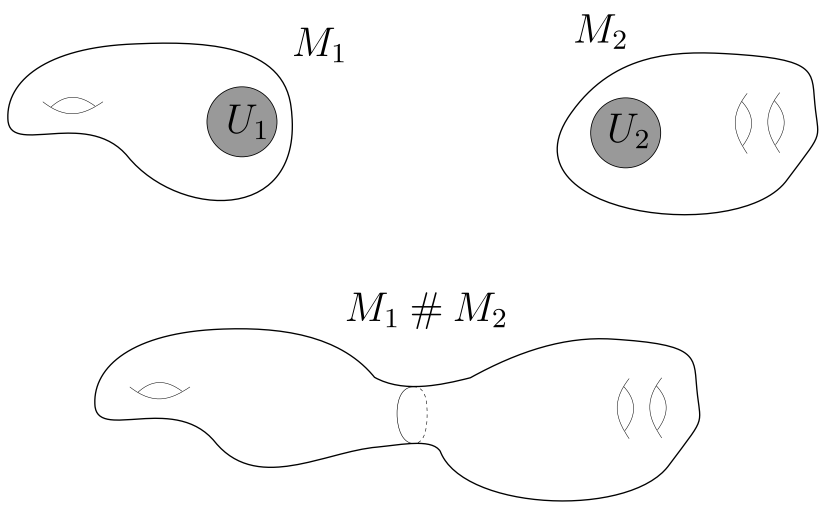


Figure 6: The connected sum of M_1 and M_2 along the sets U_i.

Cobordism

Another notion of equivalence between manifolds which will become quite useful in the following chapters is that of cobordism.

Two closed smooth n-dimensional manifolds M, N are (unoriented) *cobordant* if their disjoint union is the boundary of a compact smooth $(n+1)$-dimensional manifold W, $\partial W = M \sqcup N$. We say M and N belong to the same *cobordism class*, $[M]$. The triple $\{M, N; W\}$ is called a *cobordism* between M and N.

The set of unoriented cobordism classes of dimension n, $\mathfrak{N}_n$, forms an abelian group with addition given by disjoint union. The identity is given by the cobordism class of boundaries, we denote the identity by $[\emptyset]$. Clearly $[M]+[M] = [\emptyset]$, as $M \sqcup M = \partial(M \times [0,1])$. Additionally the topological product of manifolds descends to an associative, bilinear product on the cobordism groups

$$\mathfrak{N}_m \times \mathfrak{N}_n \to \mathfrak{N}_{m+n}.$$

Therefore the unoriented cobordism groups form a graded algebra $\mathfrak{N}^*$.

For oriented manifolds M, N we can similarly define an oriented cobordism between M and N as a cobordism between M and N such that the bounding manifold W is oriented with oriented boundary given by $M \sqcup (-N)$. Denote the set of oriented cobordism classes of dimension n as Ω_n. As in the unoriented case, Ω_n forms a group under disjoint union and products of manifolds defines a graded ring structure with identity $\{pt\} \in \Omega_0$. However, the oriented cobordism ring is graded commutative as

$$M^m \times N^n \cong (-1)^{mn} N^n \times M^m.$$

Lemma 1.13 *Let M_1, M_2 be smooth, oriented manifolds of dimension $n > 0$. The disjoint union of M_1 and M_2 is cobordant to their connected sum.*

Proof. Consider the disjoint union of the cylinders $M_1 \times I$ and $M_2 \times I$. Let W be the boundary connected sum of these cylinders along $M_i \times \{1\}$,

$$W = M_1 \times I \,\#_\partial\, M_2 \times I.$$

At the upper boundary, this boundary connected sum is just the connected sum $M_1 \# M_2$. At the lower boundary, we have $M_1 \sqcup M_2$ by construction. Hence W is the required cobordism. □

It follows that for $n > 0$ the group operation of disjoint union can be replaced with connected sum. This is occasionally a more convenient operation to work with.

Both notions of cobordism are very coarse relations. As 0-dimensional manifolds are collections of points we can pair them off as in Figure 7, hence $\mathfrak{N}_0 = \mathbb{Z}/2\mathbb{Z}$. For oriented 0-manifolds each point is assigned ± 1. We pair off points with opposite sign to ensure the orientations are compatible. It follows that $\Omega_0 = \mathbb{Z}$.

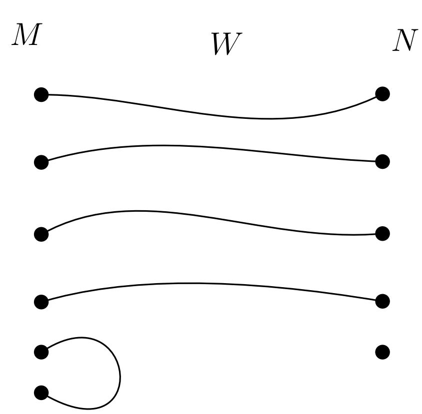


Figure 7: Unoriented collections of points can be paired off and connected by a line segment.

Any compact 1-dimensional manifold is a disjoint union of circles, which can clearly be given as the boundary of a surface; see Figure 8. Hence $\mathfrak{N}_1 = \Omega_1 = 0$.

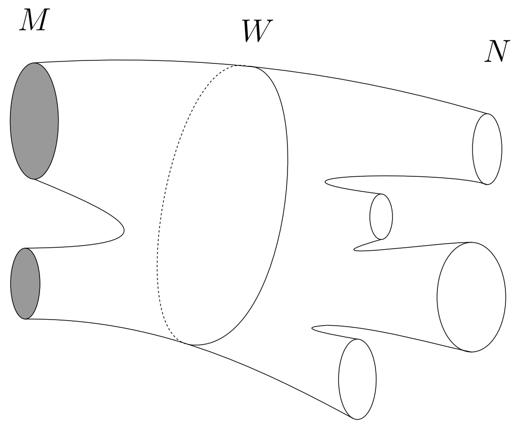


Figure 8: A collection of circles bounds a surface.

Higher cobordism groups are considerably harder to compute. The main tool to compute these groups is a connection between homotopy groups and cobordism groups, developed by Pontrjagin and Thom independently. We will describe a particular example of this connection in chapter 7. See Table 1 below for a list of the first eight groups.

n	0	1	2	3	4	5	6	7	8
$\mathfrak{N}_n$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$\mathbb{Z}/2/\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^5$
Ω_n	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	0	$\mathbb{Z} \oplus \mathbb{Z}$

Table 1: Table of cobordism groups.

2 Fibre Bundles

The concept of attaching some structure to every point of a topological space turns out to be useful in a wide variety of contexts. We first define vector bundles as the result of "attaching" a vector space to every point of a manifold, generalising tangent bundles to manifolds. We then extend this idea to fibre bundles, attaching instead arbitrary topological spaces, called the fibre, to a manifold. These spaces will then locally be products of the manifold and the fibre. We will see that they can be quite twisted and non-trivial globally.

2.1 Vector Bundles

A *real vector bundle*, ξ, over a topological space X is a triple (E, X, π), usually denoted $\pi : E \to X$ where:

1. X called the *base space.*
2. E is a topological space, called the *total space.*
3. π is a continuous surjective map, called the *projection map.*
4. For every point $x \in X$ the *fibre* $F_x = \pi^{-1}(x)$ is a finite dimensional real vector space.
5. The bundle satisfies the *local trivility condition.*

Local triviality condition. For every point $x \in X$ there exists a local coordinate system (U, φ) for ξ. That is, a pair, (U, φ), where U is an open neighbourhood of p and φ is a homeomorphism

$$\varphi : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

such that for all $p \in U$:

1. $\pi \circ \varphi(p, v) = p$ for all $v \in \mathbb{R}^n$.
2. $v \mapsto \varphi(p, v)$ is a vector space isomorphism between $\mathbb{R}^k$ and $F_p = \pi^{-1}(p)$.

We call such a coordinate system a *local trivialisation.*

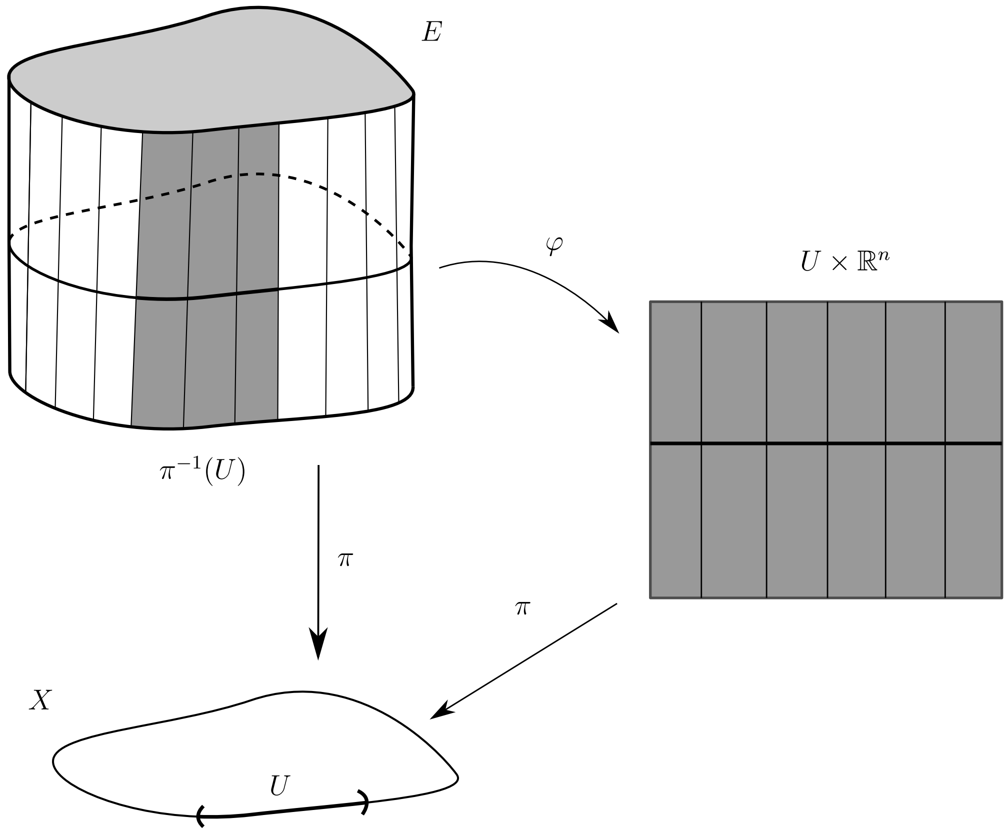


Figure 9: Vector bundles are locally a product of a manifold with $\mathbb{R}^n$.

We will sometimes use the total space to refer to the whole vector bundle when the rest of the structure is clear from context.

If it is possible to cover the entire bundle ξ with a single coordinate system (U, φ) then ξ is called a *trivial bundle.*

The dimension of the fibres is locally constant due to the triviality condition. It follows that vector bundles over connected manifolds have fibres of fixed dimension, called the *rank* of the vector bundle. We will usually be interested in spaces with constant fibre dimension.

A vector bundle is a *smooth vector bundle* if the base and total space are smooth manifolds, π is a smooth map, and the trivialisations in the definition above are diffeomorphisms.

Example 2.1 The rank k trivial bundle, ε^k, over X is the vector bundle with total space $X \times \mathbb{R}^k$. The projection map is given by projection onto the first factor and the vector space structure is inherited from $\mathbb{R}^k$.

Example 2.2 The tangent bundle, TM, of a smooth manifold M^n is a rank n smooth vector bundle by construction.

Example 2.3 The *normal bundle*, NM, of a manifold M embedded in $\mathbb{R}^N$ is defined to have total space the set of all pairs (x, v) with $x \in M$ and $v \perp T_xM$.

Example 2.4 We will construct a rank one bundle over the real projective space, $\mathbb{R}P^n$, called the *tautological line bundle.* Let $E(\gamma_n^1) \subset \mathbb{R}P^n \times \mathbb{R}^{n+1}$ be the set of all pairs

$$E(\gamma_n^1) = \{(\{\pm x\}, tx) : x \in S^n, t \in \mathbb{R}\}.$$

Define $\pi : E(\gamma_n^1) \to \mathbb{R}P^n$ as

$$\pi(\{\pm x\}, v) = \{\pm x\}.$$

We give each fibre the usual vector space structure on $\mathbb{R}$. To show the bundle is locally trivial, let $U \subset S^n$ contain no pairs of antipodal points. Denote the image of this set in $\mathbb{R}P^n$ by U'. Then the map

$$h : (\{\pm x\}, t) \mapsto h(\{\pm x\}, t) = (\{\pm x\}, tx)$$

is a homeomorphism $h : U' \times \mathbb{R} \to \pi^{-1}(U')$.

The resulting vector bundle has base $\mathbb{R}P^n$ and fibre at x the line through x and $-x$. Denote this bundle by γ_n^1. When $n = 1$ the total space of this bundle is the open Möbius band, which is non-trivial. This bundle is in fact non-trivial in each dimension.

Two vector bundles ξ_1, ξ_2 over X are *isomorphic* if there exists a homeomorphism of total spaces $f : E_1 \to E_2$ which maps each fibre $F_p(\xi_1)$ isomorphically into the corresponding fibre $F_p(\xi_2)$.

More generally, we can define morphisms between bundles. Let ξ_1, ξ_2 be vector bundles. A *bundle map* from ξ_1 to ξ_2 is a continuous function $f : E_1 \to E_2$ which carries each fibre $F_x(\xi_1)$ isomorphically onto one of the fibres $F_y(\xi_2)$.

Given a topological space Y and a continuous map $f : Y \to X$ we can pull back bundles over X to bundles over Y. If ξ is a bundle over X then the *pullback bundle*, $f^*\xi$, is defined as follows. The total space, $f^*(E)$, is given by

$$f^*(E) := \{(x, p) \in Y \times E : f(x) = \pi(p)\}.$$

The projection map π^* is defined by $\pi^*(x, v) = x$. Given local coordinates (U, h) for ξ we define local coordinates (U^*, h^*) for $f^*\xi$ by

$$U^* = f^{-1}(U);$$

$$h^*(x, v) = (x, h(f(x), v)).$$

Consider two bundles ξ_1 and ξ_2. The *product bundle* $\xi_1 \times \xi_2$ is defined as follows. The total space is the product of the total spaces of ξ_1 and ξ_2, similarly for the base. The projection is given by the product of the projection maps. Each fibre is given the structure of the product vector space $F_{p_1}(\xi_1) \times F_{p_2}(\xi_2)$. All together we have a triple $(E_1 \times E_2, B_1 \times B_2, \pi_1 \times \pi_2)$.

Consider two bundles ξ and η over the same base B. Let $d : B \to B \times B$ be the diagonal map. The bundle $d^*(\xi \times \eta) = \xi \oplus \eta$ over B is the *Whitney sum* of ξ and η. The fibres of the Whitney sum are isomorphic to the direct sum of the fibres:

$$F_x(\xi \oplus \eta) = F_x(\xi) \oplus F_x(\eta).$$

Example 2.5 Let M^n be a smooth manifold embedded in $\mathbb{R}^N$ with tangent bundle TM and normal bundle NM. The tangent bundle to $\mathbb{R}^N$ is trivial, hence

$$TM \oplus NM \cong \varepsilon^N.$$

We call NM is the *orthogonal complement* of TM.

We extend the idea of vector fields to general bundles as sections of a bundle. A *section* of a vector bundle is a continuous function $s : X \to E$ which takes each $x \in X$ into its fibre. That is,

$$\pi \circ s = \mathrm{Id}_M.$$

Example 2.6 A section of a tangent bundle is a vector field.

Example 2.7 Let $\pi : E \to X$ be a vector bundle. Define the *zero section* to be

$$s(p) = (p, 0) \in E.$$

The image of this section, $S(X)$, is homeomorphic to X. As the fibres are contractible, it follows that E deformation retracts onto X.

A collection of k sections is called a *linearly independent k-frame* if its image is linearly independent at each point $x \in X$.

Theorem 2.8 *A rank n bundle ξ is trivial if and only if ξ admits n linearly independent cross sections.*

A smooth manifold M^n is *parallelisable* if its tangent bundle is trivial. This is equivalent to admitting a collection of n linearly independent sections. A slightly weaker condition is sometimes useful. A smooth manifold M is *stably parallisable*[2] if its tangent bundle TM is *stably trivial.* That is,

$$TM \oplus \varepsilon^k \cong \varepsilon^{n+1}.$$

Example 2.9 All n-spheres are stably parallelisable. Let S^n be the n-sphere in $\mathbb{R}^{n+1}$. The normal bundle to a sphere in $\mathbb{R}^{n+1}$ is trivial. Hence, as in Example 2.5 we have

$$TS^n \oplus NS^n \cong TS^n \oplus \varepsilon^1 \cong \varepsilon^{n+1}.$$

Note that a bundle ξ is stably trivial if $\xi \oplus \varepsilon^k$ is trivial for any $k \geq 0$. Every parallelisable manifold is stably parallelisable. Stable parallelisability is generally better behaved than parallelisability. In particular, the product of two manifolds is stably parallelisable if and only if each factor is stably parallelisable. A proof of this and conditions for the two notions to coincide is in [Kos72].

[2] Sometimes also called s-parallelisable or a π-manifold.

Consider a vector bundle ξ with a continuous function $g : E \to \mathbb{R}$ for which the restriction of g to each fibre is a positive definite quadratic form. Then this function defines a continuously varying inner product on the fibres of ξ. We call ξ a *vector bundle with inner product*, or *Euclidean vector bundle,* and g is called the *metric* on ξ. When ξ is the tangent bundle to a smooth manifold M and g is smoothly varying then g is called a *Riemannian metric* and M is called a *Riemannian manifold.*

Given this we can define the *normal bundle* to a submanifold X of a Riemannian manifold Y, $N(X;Y)$, as the bundle over Y with total space

$$N(X;Y) = \{(x,v) \in T_xY : \langle v,w\rangle = 0, \forall w \in T_pX\}.$$

Smoothly embedded submanifolds lay within a smooth manifold in a particularly nice way, described by the following theorem.

Theorem 2.10 (Tubular Neighbourhood Theorem) *Let X be a submanifold of Y. There exists a neighbourhood U of X in Y diffeomorphic to a neighbourhood of X in $N(X,Y)$.*

See [Lee03] Theorem 10.19 for a proof.

An *orientation* for a rank n real vector bundle ξ is a function which assigns an orientation to each fibre F and obeys the following local compatibility condition. For every $x_0 \in X$, there must exist local coordinates (U,h) around x_0 such that for each fibre over $x \in U$ the isomorphism $v \mapsto h(x,v)$ is orientation preserving. If there is an orientation for a bundle ξ we say ξ is *orientable.*

Let M be a smooth manifold. An orientation for M gives rise to an orientation for the tangent bundle, TM, using the natural coordinates on TM. Similarly, an orientation for TM induces an oriented atlas on M through the local trivialisations. Hence M is an oriented manifold if and only if TM is an oriented vector bundle.

2.2 Fibre Bundles

We can generalise the idea of vector bundles by replacing the fibres $\mathbb{R}^n$ with an arbitrary topological space F.

A *fibre bundle with fibre* F is a triple (E,X,π) such that $\pi : E \to X$ is a continuous surjective map and the bundle satisfies the following condition.

Local triviality condition. Every point $x \in X$ has a neighbourhood U_x and a homeomorphism

$$h_x : \pi^{-1}(U_x) \to U_x \times F,$$

taking $\pi^{-1}(b)$ to $\{b\} \times F$ for every $b \in U_x$.

As before we call X the *base space* and E the *total space.* A vector bundle is then a fibre bundle with fibre $\mathbb{R}^n$ such that each fibre has the structure of a finite dimensional vector space. We can generalise the notion of the fibres having additional structure using the following construction.

For a point $p \in X$ choose two different trivialisations (U_α, h_α), (U_β, h_β). Define the *transition map*, $g_{\alpha\beta} : U_\alpha \cap U_\beta \times F \to U_\alpha \cap U_\beta \times F$, to be the composition

$$g_{\alpha\beta} = h_\alpha \circ h_\beta^{-1}.$$

By the local triviality condition, this map takes

$$g_{\alpha\beta} : (p, f) \mapsto (p, g_{\alpha\beta}(p)f).$$

That is, $g_{\alpha\beta}$ are homeomorphisms of F. The smallest subgroup G of Homeo(F) which contains each possible $g_{\alpha\beta}$ is called the *structure group* of the fibre bundle.

We have already seen a number of examples. A rank n vector bundle has structure group $GL(n)$. Suppose that we give this vector bundle a metric. The transition maps must respect this metric, and so the structure group reduces to $O(n)$. Furthermore, if the vector bundle is oriented, the structure group will reduce to $SO(n)$. Any covering space over a connected base is a fibre bundle. Conversely, any fibre bundle with discrete fibre is a covering space.

Consider a vector bundle ξ with inner product $\langle\ ,\ \rangle$. Then the subspace $S(E)$ of vectors of length one is a fibre bundle with fibre the unit sphere.

$$S(E) := \{(x, v) \in E : \langle v, v \rangle = 1\} \subset E$$

We call this the *sphere bundle* of E. Choosing local trivialisations for E which are isometries gives local trivialisations for $S(E)$, hence the structure group for a sphere bundle reduces is $SO(n)$.

Similarly define the *disk bundle* $D(E)$ to be the bundle with fibres the unit disks in the fibres of ξ.

$$D(E) := \{(x, v) \in E : \langle v, v \rangle \leq 1\} \subset E.$$

Example 2.11 One notable example of a sphere bundle is the *Hopf fibration*

$$S^1 \hookrightarrow S^3 \xrightarrow{p} S^2.$$

Here the map p is the *Hopf map,* defined as follows. Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$. Then S^3 is the set of unit vectors in $\mathbb{C}^2$. Define p by

$$p(z_0, z_1) = (2z_0z_1^*, |z_0|^2 - |z_1|^2).$$

If $p(z_0, z_1) = p(w_0, w_1)$ then we must have $(z_0, z_1) = u(w_0, w_1)$ for u a unit complex number. The set of unit complex numbers in $\mathbb{C}$ is a circle, so for each $x \in S^2$ we have

$$p^{-1}(x) \cong S^1.$$

It follows that this is a fibre bundle over the sphere with fibre the circle and total space S^3.

A more geometric construction of the bundle can be given by considering rotations of S^2 in $\mathbb{R}^3$. The group of rotations $SO(3)$ has a double cover, $SU(2)$, which is constructed as follows. Identify $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with the imaginary quaternion $x = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$. Recall quaternion multiplication is defined as follows. Let $u = a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$, $v = a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}$ then

$$\begin{aligned} u \cdot v = a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 \\ + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\mathbf{i} \\ + (a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2)\mathbf{j} \\ + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)\mathbf{k}. \end{aligned}$$

Let q be a unit length quaternion. Then the map

$$x \mapsto qxq^*$$

is a rotation in $\mathbb{R}^3$. $SO(3)$ can be identified with the group of these mapping modulo the identity $q = -q$. The set of quaternions which fix a given x is a subgroup homeomorphic to a circle. The Hopf bundle can then be defined as the map which sends a quaternion q to qxq^* for a fixed x.

2.3 Complex Bundles and Complex Manifolds

Complex Bundles

So far we have only considered real manifolds and vector bundles with real fibres, however much of the work so far can be carried out with complex structures. We sketch the basic constructions here.

A *complex rank n vector bundle*, ω over X is a triple, (π, E, X), where X, E are topological spaces and $\pi : E \to X$ is a projection map. Furthermore, each fibre $\pi^{-1}(x)$ has the structure of a complex vector space and we have the following local triviality condition. For every point $x \in X$ there exists a neighbourhood U of x such that $\pi^{-1}(U)$ is homeomorphic to $U \times \mathbb{C}^n$ with each fibre mapped complex linearly to $\{x\} \times \mathbb{C}^n$. That is, a complex rank n vector bundle is a fibre bundle with fibre $\mathbb{C}^n$ and structure group $GL_n(\mathbb{C})$.

We can construct a complex vector bundle from a real rank $2n$ bundle ξ by adding a complex structure to each real fibre. A *complex structure* on a bundle E is a continuous map $J : E \to E$ which maps each fibre complex linearly to itself. That is, J is $\mathbb{R}$-linear and

$$J^2(v) = -v.$$

With a complex structure we can make each fibre a complex vector space by defining

$$(x + iy)v = xv + J(yv).$$

This makes ξ a complex vector bundle.

Example 2.12 Let U be an open subset of $\mathbb{C}^n$. Then the tangent bundle $TU = U \times \mathbb{C}^n$ has a canonical complex structure

$$J_0(u, v) = (u, iv).$$

From any complex bundle ω we can form an oriented rank $2n$ real vector bundle ω_R by forgetting the complex structure. We call this the *underlying real vector bundle.*

Lemma 2.13 *The underlying real vector bundle ω_R for any complex vector bundle ω has a canonical orientation.*

Proof. Let F be any fibre of the bundle ω. Choose a basis $v_1, \ldots, v_n$ for F. Now a basis for the underlying real $2n$-dimensional fibre F_R is given by $v_1, iv_1, v_2, \ldots, v_n, iv_n$. As $GL_n(\mathbb{C})$ is connected we can pass continuously from any given basis for F to $v_1, \ldots, v_n$. Hence this uniquely determines an oriented basis for F_R. Applying this construction to each fibre we obtain an orientation for ω_R. $\square$

Lemma 2.14 *For complex bundles ω, θ there is an oriented bundle isomorphism*

$$\omega_R \oplus \theta_R \cong (\omega \oplus \theta)_R.$$

Proof. Let $v_1, \ldots, v_n$ be a basis of F determining the orientation $v_1, iv_1, v_2, \ldots, v_n, iv_n$ in ω. Similarly let $w_1, \ldots, w_m$ be a basis of F' determining the orientation $w_1, iw_1, w_2, \ldots, w_m, iw_m$ in θ. Then the preferred orientation for $F_R \oplus F'_R$ is given by

$$v_1, iv_1, \ldots, v_n, iv_n, w_1, iw_1, \ldots, w_m, iw_m,$$

which is the preferred orientation of $(F \oplus F')_R$. $\square$

Complex Manifolds

We define a complex structure on a manifold M similarly to a smooth structure, replacing diffeomorphisms with holomorphic maps.

Let $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^k$ be open sets. Recall a smooth map $F : U \to V$ is *holomorphic* if the differential,

$$dF_z : T_zU \to T_{F(z)}V,$$

is a complex linear map at all points $z \in U$. That is, dF is $\mathbb{R}$-linear and

$$dF \circ J_0 = J_0 \circ dF.$$

Define a *complex structure* on a manifold M^{2n} to be a complex structure J on the tangent bundle TM such that M is *locally holomorphic* to an open subset of $\mathbb{C}^n$. That is, every point $z \in M$ has a neighbourhood U and a holomorphic diffeomorphism $h : U \to V \subset \mathbb{C}^n$ such that

$$dh \circ J = J_0 \circ dh.$$

We say (M, J) is an *n-dimensional complex manifold*. As with smooth manifolds we usually omit reference to the complex structure.

Example 2.15 The n-dimensional complex space, $\mathbb{C}^n$, is a complex manifold with its usual complex structure.

Example 2.16 The complex projective space, $\mathbb{C}P^n$, is a complex manifold. Define charts (U_i, φ_i) by

$$U_i = \{z \in \mathbb{C}P^n : z_i \neq 0\} ;$$
$$\varphi_i([z_0, \dots, z_n]) = \left(\frac{z_0}{z_i}, \dots, \hat{z}_i, \dots, \frac{z_n}{z_i} \right).$$

The open sets U_i clearly cover $\mathbb{C}P^n$ and it is easy to check that the transition maps are holomorphic.

Example 2.17 Similarly to Example 2.4, we can define the *tautological complex line bundle* over $\mathbb{C}P^n$, $\gamma^1_{n\mathbb{C}}$. This is the bundle with base space $\mathbb{C}P^n$ and total space all pairs (ℓ, v) where ℓ is a complex line through the origin and $v \in \ell$ with obvious projection.

Corollary 2.18 *Every complex manifold is orientable.*

Proof. The tangent space TM of a complex manifold is a complex vector bundle, hence orientable. □

A smooth map F between two complex manifolds is *holomorphic* if dF is complex linear, that is

$$dF \circ J = J \circ dF.$$

3 Algebraic Topology

Topology is the study of spaces up to continuous deformation, that is, spaces up to homotopy. Let f, g be continuous maps between topological spaces X and Y. A *homotopy* between f and g is a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. We then say that f and g are homotopic. Define X and Y to be *homotopy equivalent* if there exists maps $f : X \to Y$, $g : Y \to X$ such that $f \circ g$ is homotopic to id_X and $g \circ f$ is homotopic to id_Y.

Algebraic topology aims to solve topological problems through the use of algebraic invariants associated to a space. The first such invariant usually introduced in an algebraic topology course is the fundamental group. Intuitively, the fundamental group is the set of loops at a point in a space up to continuous deformation. For a topological space X, define the *fundamental group* of X at $x_0 \in X$, $\pi_1(X, x_0)$, to be the set of based homotopy classes of maps from the circle into X,

$$\pi_1(X, x_0) := [S^1, X].$$

We will often omit the basepoint from the notation. A space with trivial fundamental group is called *simply connected*. We recall some basic facts about the fundamental group.

Theorem 3.1 *Let X, Y be path connected topological spaces.*

1. *The fundamental group of the circle is infinite cyclic, $\pi_1(S^1) \cong \mathbb{Z}$.*
2. *The fundamental group of S^n is zero for $n \geq 2$.*
3. *The fundamental group of the product of two spaces is the product of their fundamental groups, $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.*

Theorem 3.2 (Seifert-Van Kampen Theorem) *Let X be the union of two open, path connected subspaces U_1, U_2. Furthermore suppose $U_1 \cap U_2$ is path connected, simply connected, and non-empty. Then $\pi_1(X) \cong \pi_1(U_1) * \pi_1(U_2)$, where $*$ denotes the free product of groups.*

The natural generalisation to maps from S^n to X is explored in section 3.3, however, this is often difficult to work with. We introduce the much more easily computed homology theory in section 3.1, and the dual notion of cohomology theory in section 3.2. We assume some familiarity with basic algebraic topology, all necessary background can be found in [Hat00]. Finally, in section 3.4 we explore the homology and cohomology of manifolds.

3.1 Homology

Singular Homology

We would like to model a space using simple n-dimensional building blocks, similarly to a triangulation of a surface. For this we will need an n-dimensional generalisation of the triangle. Let $v_0, \dots, v_n$ be affinely independent points of $\mathbb{R}^m$. Define the *n-simplex* $[v_0, \dots, v_n]$ to be the smallest convex set in $\mathbb{R}^m$ containing $v_0, \dots, v_n$. The points $v_0, \dots, v_n$ are called the *vertices* of $[v_0, \dots, v_n]$. Note that we include the ordering of the vertices as part of the definition.

Recall the *standard n-simplex* $\Delta^n \subset \mathbb{R}^{n+1}$ is the set

$$\left\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \quad \sum t_i \leq 1\right\}.$$

This allows us to more explicitly define the n-simplex $[v_0, \dots, v_n]$ as the image of the homeomorphism $\varphi : \Delta^n \to [v_0, \dots, v_n]$

$$\varphi : (t_0, \dots, t_n) \mapsto \varphi(t_0, \dots, t_n) = \sum t_i v_i.$$

The coefficients t_i are called the *barycentric coordinates* of the point $\sum t_i v_i$. We distinguish one point, the *barycentre*, as the point of $[v_0, \dots, v_n]$ with all barycentric coordinates equal to $1/(n+1)$. Intuitively, the barycentre is the centre of gravity of the simplex.

Deleting the ith vertex of an n-simplex leaves an $(n-1)$-simplex with ordering inherited from the n-simplex. We call this simplex the *i-face* of $[v_0, \dots, v_n]$ and denote it by $[v_0, \dots, \hat{v}_i, \dots, v_n]$. The union of all the faces of a simplex is the *boundary*, denoted $\partial[v_0, \dots, v_n]$.

We use these simplices to detect features of a topological space X. Define a *singular n-simplex* of X to be a continuous map $\sigma : \Delta^n \to X$. Let $C_n(X)$ be the free abelian group generated by the set of all singular n-simplices of X. Elements of $C_n(X)$, called *n-chains*, are finite formal sums of n-simplices with coefficients in $\mathbb{Z}$. We define a boundary map from an n-simplex to an $(n-1)$-simplex by

$$\partial_n(\sigma(v_0, \dots, v_n)) = \sum_i (-1)^i \sigma(v_0, \dots, \hat{v}_i, \dots, v_n)$$

That is, we take a signed sum of the faces of σ. We can extend this map linearly to a boundary map on the chain groups

$$\partial_n : C_n(X) \to C_{n-1}(X).$$

We will sometimes write ∂ when n is clear from context.

Lemma 3.3 *The composition of the boundary map with itself is zero, that is $\partial^2 = 0$.*

Proof. Computing ∂^2 for an n-chain σ from the definition

$$\begin{aligned}\partial_{n-1}\partial_n(\sigma) &= \sum_{j<i} (-1)^i (-1)^j \sigma(v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n) \\ &+ \sum_{i<j} (-1)^i (-1)^{j-1} \sigma(v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n) \\ &= 0,\end{aligned}$$

as switching i, j in the second sum makes it the the negative of the first. □

The n-th *singular homology group* $H_n(X)$ is defined to be

$$H_n(X) = \frac{\text{Ker } \partial_n}{\text{Im } \partial_{n+1}}.$$

Intuitively these groups detect n-dimensional "holes" in a space. We consider the set of n-dimensional boundaries in a space and quotient out anything which is the boundary of an $(n+1)$-dimensional chain in the space. For example, $\text{rank}(H_0(X))$ is the number of connected components of X. Also note that when X is path connected, $H_1(X)$ is isomorphic to the abelianisation of the fundamental group, $\pi_1(X)$, see [Hat00] Theorem 2A.1 for a proof. The following example illustrates the intuitive idea of homology.

Example 3.4 Consider the chains on T^2 indicated in Figure 10 below.

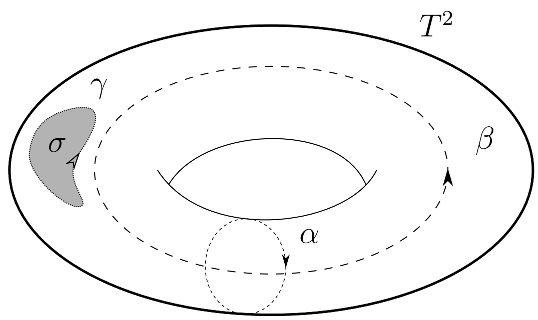


Figure 10: Examples of chains on the torus.

The 1-chain γ is the boundary of the 2-chain σ. Hence this is represented by the zero homology class. Similarly any closed curve which does not enclose a latitudinal or longitudinal circle will bound a 2-chain. The 1-chains α, β clearly do not bound a 2-chain in T^2. Any other 1-chain can be represented as a sum of these two curves. Also note that α and β do not satisfy any algebraic relations. Hence

$$H^1(T^2) = \mathbb{Z} \times \mathbb{Z}.$$

Any pair of points is the boundary of a path between them. Then, as T^2 is path connected,

$$H_0(T^2) \cong \mathbb{Z}.$$

Clearly $\ker \partial_2$ is generated by a 2-chain with image T^2. Note also that $\partial_3 = 0$ as $\dim T^2 = 2$. Hence

$$H_2(T^2) \cong \mathbb{Z}.$$

In summary,

$$H^k(T) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z} \times \mathbb{Z} & k = 1 \\ 0 & \text{else.} \end{cases}$$

Example 3.5 For a point x there is a unique singular n-simplex σ_n for each n and $\partial\sigma_n = \sum_{i=0}^{n}(-1)^i\sigma_{n-1}$ which is 0 for n odd and σ_{n-1} for n even. Therefore the chain complex of chain groups for x is

$$\ldots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} 0$$

Hence $H_0(x) = \mathbb{Z}$ and all other homology groups are 0.

Example 3.6 Both $\mathbb{R}^n$ and the n-dimensional ball have the homology of a point as they deformation retract to a point.

Example 3.7 For the n-sphere we have

$$H^k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \text{else.} \end{cases}$$

This requires some work, though we point out that this can be proven directly from the homology axioms below.

For non-empty spaces X it is sometimes useful to work with the *reduced homology groups*, $\widetilde{H}_n(X)$ defined as follows. We introduce the homomorphism $\varepsilon : C_0(X) \to \mathbb{Z}$ which acts on chains by

$$\varepsilon\left(\sum_i n_i\sigma_i\right) = \sum_i n_i.$$

The reduced singular homology of X is then given by the homology of the complex

$$\ldots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

This gives an identical collection of groups to ordinary homology, except in dimension zero, where we have

$$H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}.$$

Reduced homology has the benefit that a point has trivial homology in all dimensions.

We can use the homology groups of a space to define a useful invariant. The *Euler characteristic* of X, $\chi(X)$, is defined as the alternating sum

$$\chi(M) = \sum_{i=0}(-1)^i \operatorname{rank} H_i(X).$$

It is easy to check that for polyhedra this is equivalent to the usual definition of the Euler characteristic as the alternating sum of vertices, edges, and faces.

It is often useful to ignore all chains contained in some subspace. Let X be a topological space and $A \subset X$. Define the *relative n-chain group*, $C_n(X, A)$, as the quotient

$$C_n(X, A) = C_n(X)/C_n(A).$$

Chains contained in A are then zero in $C_n(X, A)$. The boundary map descends to a boundary map on the relative chain groups. Hence we may define the *relative homology groups* of X with respect to A as the homology groups of the chain complex of $C_n(X, A)$. Note that $H_n(X, \emptyset) = H_n(X)$.

In the definition of $C_n(X)$ we could replace the integer coefficients with any abelian group. For G an abelian group, define $C_n(X; G)$ to be the set of finite formal sums of n-chains with coefficients in G. We then define *singular homology with G-coefficients* to be the homology of the chain complex for $C_n(X; G)$ with ∂ as before. Different coefficient groups have various advantages. For example, we will often work with rational coefficients in later chapters to avoid torsion elements.

Homology Axioms

There are a wide variety of tools available to compute singular homology groups, though constructing them usually takes a lot of work. We will instead introduce axioms for a general homology theory which singular homology satisfies. Singular homology is just one example of a more general idea of homology theory. Instead of restricting ourselves to just singular homology we introduce a homology theory to be an assignment from topological spaces to abelian groups as follows.

A *homology theory* is a map $(X, A) \mapsto H_*(X, A)$ which assigns to each pair of topological spaces, (X, A), a sequence of abelian groups, $H_n(X, A)$, and to each map $f : (X, A) \to (Y, B)$ between pairs, a sequence of homomorphisms $f_* : H_n(X, A) \to H_n(Y, B)$. Furthermore there is a homomorphism, $\partial : H_k(X, A) \to H_{k-1}(A)$, called the boundary map, which commutes with maps between pairs and induced maps of homology groups. This can be restated in categorical language as a functor $H_*(\cdot)$ from topological pairs to abelian groups along with a natural transformation ∂. We require the following axioms to hold.

1. **Homotopy:** Homotopic maps induce equal maps in homology. That is, if $f \sim g$ then $f_* = g_*$.
2. **Excision:** For a pair (X, A), if $U \subset A$ is such that the closure of U is in the interior of A then $i : (X \backslash U, A \backslash U) \to (X, A)$ induces a homology isomorphism.
3. **Dimension:** For a point x, $H_k(x) = 0$ for all $k \neq 0$.
4. **Additivity:** If (X, A) is the disjoint union of pairs $\sqcup_\alpha (X_\alpha, A_\alpha)$ then
$$H_n(X, A) \cong \bigoplus_\alpha H_n(X_\alpha, A_\alpha).$$
5. **Exactness:** Each pair (X, A) with inclusions $i : A \hookrightarrow X$, $j : X \to (X, A)$ induces a long exact sequence
$$\ldots \longrightarrow H_k(A) \xrightarrow{i^*} H_k(X) \xrightarrow{j^*} H_k(X, A) \xrightarrow{\partial} H_{k-1}(A) \longrightarrow \ldots$$

Our definition of relative singular homology satisfies these axioms. There are many properties to check, with some steps being very lengthy. Full detail can be found in [Hat00].

3.2 Cohomology

Singular Cohomology

For a given ring R we define the n-th *singular cochain group*, $C^n(X;R)$, to be the dual of the n-th singular chain group:

$$C^n(X;R) := \mathrm{Hom}_R(C_n(X), R).$$

That is, $C^n(X;R)$ is the group of R-linear maps from $C_n(X)$ to R, we call these maps *cochains.* Note we can pair any chain α with a cochain f by applying f to α to obtain an element of $\mathbb{R}$. Denote this pairing by $\langle f, \alpha \rangle = f(\alpha)$.

With this we can define the n-th *coboundry map* δ_n by the relation

$$< \delta_n f, \alpha >= (-1)^{n+1} < f, \partial_n \alpha > .$$

That is, $\delta_n f = f\partial_{n-1}$. As for the boundary map we will sometimes write δ when n is clear from context. It follows from the definition that $\delta^2 = 0$.

The n-th *singular cohomology group* $H_n(X)$ is defined to be

$$H^n(X;R) = \frac{\text{Ker } \delta_n}{\text{Im } \delta_{n-1}}.$$

There is a close connection between cohomology groups and homology, though the situation isn't usually as simple the the cohomology groups being the dual of the homology groups. However, in the following situations cohomology is the dual of homology.

Theorem 3.8 *Let R be a principal ideal domain. If $H_{n-1}(X)$ is zero or a free R-module then*

$$H^n(X;R) \cong \mathrm{Hom}_R(H_n(X), R).$$

Theorem 3.9 *Let F be a field. Then*

$$H^n(X;F) \cong \mathrm{Hom}_F(H_n(X), F).$$

Both of these are consequences of the *Universal Coefficient Theorem*, see [Hat00] Chapter 3.1 for a proof.

Example 3.10 The cohomology of a point x is given by $H^0(x) = \mathbb{Z}$ and 0 in other dimensions.

Given two cochains $f \in C^m(X;R)$, $g \in C^n(X;R)$ define the *cup product* $f \smile g \in C^{m+n}(X;R)$ to be the cochain whose value on a singular simplex σ is

$$(f \smile g)(\sigma) = f(\sigma|_{[v_0,\dots,v_m]}) \cdot g(\sigma|_{[v_m,\dots,v_{m+n}]}) \in R.$$

Lemma 3.11 *For f, g as above we have*

$$\delta(f \smile g) = \delta f \smile g + (-1)^m f \smile \delta g.$$

Proof. Compute the action of each term on a $(m+n+1)$-chain σ

$$(\delta f \smile g)(\sigma) = \sum_{i=0}^{m+1} (-1)^i f(\sigma|_{[v_0,\dots,\hat{v_i},\dots,v_{m+1}]}) g(\sigma|_{[v_{m+1},\dots,v_{m+n+1}]});$$

$$(-1)^m (f \smile \delta g)(\sigma) = \sum_{i=m}^{m+n+1} (-1)^i f(\sigma|_{[v_0,\dots,v_m]}) g(\sigma|_{[v_m,\dots,\hat{v_i},\dots,v_{m+n+1}]}).$$

Adding these two expressions, the last term of $(\delta f \smile g)(\sigma)$ cancels the first term of $(-1)^m(f \smile \delta g)(\sigma)$. What remains is $(f \smile g)(\partial\sigma) = \delta(f \smile g)(\sigma)$, by definition. $\square$

This implies that the cup product descends to a well defined multiplication on relative cohomology

$$H^m(X,A;R) \times H^n(X,B;R) \xrightarrow{\smile} H^{m+n}(X, A \cup B; R).$$

Cohomology groups of a space X together with the cup product forms a graded ring, $H^*(X;R)$, called the *cohomology ring* of X. The multiplication of $H^*(X;R)$ is *graded commutative* when the ring R is commutative. That is, for $f \in H^m(X;R)$, $g \in H^n(X;R)$

$$f \smile g = (-1)^{mn} g \smile f.$$

See [Hat00] for a proof. The main idea is to note that the two cup products only differ by a permutation of the vertices of a simplex, then confirming that permuting the vertices produces the required change of sign.

The cup product allows us to define an external product, called the cross product, which relates the cohomology of a product to the cohomology of the factors. Let $f \in H^k(X;R)$, $g \in H^l(Y;R)$. Denote by p_X the inclusion of X into $X \times Y$, similarly for p_Y. We then define the *cross product* of f and g,

$$\times : H^k(X;R) \times H^l(Y;R) \to H^{k+l}(X \times Y;R)$$

as $f \times g = p_X^*(f) \smile p_Y^*(g)$. This is a bilinear map which descends to a ring homomorphism

$$\times : H^k(X;R) \otimes_R H^l(Y;R) \to H^{k+l}(X \times Y;R)$$

This map is actually an isomorphism is many cases.

Theorem 3.12 (Künneth Isomorphism Theorem) *Let X, Y be CW complexes and suppose $H^i(Y;R)$ is a finitely generated free R-module for all i. Then the cross product $H^k(X;R) \otimes_R H^l(Y;R) \to H^{k+l}(X \times Y;R)$ is a ring isomorphism.*

For a proof see [Hat00] Theorem 3.15.

Given an n-chain $\alpha \in C_n(X)$ and a k-cochain $f \in C^k(X)$ define the *cap product*,

$$\frown : C^k(X) \otimes C_n(X) \to C_{n-k}(X),$$

to be the unique element of $C_{n-k}(X)$ such that for all cochains $g \in C^{n-k}(X)$

$$\langle g, f \frown \alpha \rangle = \langle f \smile g, \alpha \rangle.$$

It follows from the definition and the corresponding property for cup products that

$$\partial(f \frown \alpha) = (\delta f) \frown \alpha + (-1)^k f \frown (\partial \alpha).$$

The cap product then descends to a bilinear pairing on homology and cohomology

$$\frown : H^k(X) \otimes H_n(X) \to H_{n-k}(X).$$

Cohomology Axioms

Just as for homology, singular cohomology is just one type of more general cohomology theory.

A *cohomology theory* is map $(X, A) \mapsto H^*(X, A; R)$ which assigns to each pair of topological spaces (X, A) a sequence of abelian groups $H^n(X, A; R)$ and to each map $f : (X, A) \to (Y, B)$ between pairs a sequence of homomorphisms $f^* : H^n(X, A; R) \leftarrow H^n(Y, B; R)$. Furthermore, there is a homomorphism, $\delta H^k(X, A; R) \to H^{k+1}(X, A; R)$, called the boundary map, which commutes with maps between pairs and induced maps of groups. In categorical language, we have a contravariant functor, $H^*(\cdot)$, and a natural transformation, δ. We require the following axioms to hold.

1. **Homotopy:** Homotopic maps induce equal maps in cohomology. That is, $f \sim g$ then $f^* = g^*$.
2. **Excision:** For a pair (X, A), if $U \subset A$ is such that the closure of U is in the interior of A then $i : (X \backslash U, A \backslash U) \to (X, A)$ induces an isomorphism on cohomology groups.
3. **Dimension:** For a point x, $H^k(x) = 0$ for all $k \neq 0$.
4. **Additivity:** If (X, A) is the disjoint union of pairs $\sqcup_\alpha (X_\alpha, A_\alpha)$ then

$$H^n(X, A) \cong \bigoplus_\alpha H^n(X_\alpha, A_\alpha).$$

5. **Exactness:** Each pair (X, A) with inclusions $i : A \hookrightarrow X$, $j : X \to (X, A)$ induces a long exact sequence

$$\ldots \longrightarrow H^k(X, A) \xrightarrow{i^*} H^k(X) \xrightarrow{j^*} H^k(A) \xrightarrow{\delta} H^{k+1}(X, A) \longrightarrow \ldots$$

3.3 Homotopy

The definition of the fundamental group leads to a natural extension. For $n \geq 0$ define $\pi_n(X)$ to be the set of based homotopy classes of maps from S^n to X. That is,

$$\pi_n(X) = [S^n, X].$$

The set $\pi_0(X)$ is given by based maps from $\{-1, 1\}$ to X. These maps correspond to a choice of one point in X. It is clear that any two maps to points in the same path component of X will be homotopic through a path between the points, so elements of $\pi_0(X)$ correspond to path components of X.

For $n \geq 2$, $\pi_n(X)$ is an abelian group called the *n-th homotopy group of X.* The group action is concatenation, defined similarly to the fundamental group. Details and proof can be found in [Hat00]. A space with $\pi_i(X) = 0$ for $i \leq n$ is called *n-connected.*

All proofs of the results in the remainder of the section may be found in [Hat00].

The following theorem gives the main connected between homotopy and homology.

Theorem 3.13 (Hurewicz) *If a topological space X is $(n-1)$-connected for $n > 1$, then $\widetilde{H}_k(X) = 0$ for $k < n$ and $\pi_n(X) \cong H_n(X)$.*

This can be restated as saying the first nonzero homotopy and first nonzero homology groups of a simply connected space occur in the same dimension and are isomorphic.

This has a very important consequence for what follows. Recall a homotopy n-sphere is a closed n-manifold of the same homotopy type as S^n. Define a *homology n-sphere* to be a closed n-manifold with the same homology as S^n. By the Hurewicz theorem, every simply connected homology sphere is a homotopy sphere. It is generally far easier to compute a manifold's homology groups and fundamental group than it is to find homotopy type directly.

Theorem 3.14 *For $i, n \geq 1$*

1. *The sphere S^n is $(n-1)$-connected. That is, $\pi_i(S^n) = 0$, $i < n$.*
2. *The nth homology group of the n-sphere, S^n, is isomorphic to the integers, $\pi_n(S^n) \cong \mathbb{Z}$.*

For $i > n$, far less is known and the situation is in general quite complicated. However, the groups $\pi_{n+k}(S^k)$ have an important property. Considering a table of homotopy groups of spheres it appears that these groups eventually become independent of n. This is a consequence of the Freudenthal suspension theorem, though we will not go into the details here. We define the *nth stable homotopy group*, Π_n, as

$$\Pi_n = \pi_{n+k}(S^k), \quad k > n+1.$$

Serre proved that for $n > 0$ these groups are finite in [Ser53]. These groups are one of the most important objects in algebraic topology, though we will not need them until chapter 7.

Homotopy of Fibre Bundles

Each fibre bundle has an associate long exact sequence in homotopy. See [Hat00] Theorem 4.41 for a proof.

Theorem 3.15 *Let X be a path connected space and $\pi : E \to X$ a fibre bundle over X. Choose basepoints $x \in X$, $v \in F = \pi^{-1}(x)$. Then there is a long exact sequence*

$$\cdots \longrightarrow \pi_n(F,v) \longrightarrow \pi_n(E,v) \xrightarrow{p_*} \pi_n(X,x) \longrightarrow \pi_{n-1}(F,v) \longrightarrow \ldots$$

Example 3.16 Consider the covering space $\mathbb{Z}/2\mathbb{Z} \to S^n \to \mathbb{R}P^n$. The long exact sequence in homotopy for $k \geq 2$ gives

$$\cdots \longrightarrow 0 \longrightarrow \pi_k(S^n, s) \xrightarrow{p_*} \pi_k(\mathbb{R}P^n, x) \longrightarrow 0 \longrightarrow \ldots$$

Hence $\pi_k(\mathbb{R}P^n) \cong \pi_k(S^n)$, $k \geq 2$. We obtain a similar result for any covering space with discrete fibre.

Example 3.17 Consider the Hopf fibration, $S^1 \to S^3 \xrightarrow{p} S^2$, of example 2.11. Consider the long exact sequence in homotopy for $k \geq 3$.

$$\cdots \longrightarrow 0 \longrightarrow \pi_k(S^3) \xrightarrow{p_*} \pi_k(S^2) \longrightarrow 0 \longrightarrow \ldots$$

This gives an isomorphism

$$\pi_k(S^3) \cong \pi_k(S^2), \quad k \geq 3.$$

For $k = 3$, this gives $\pi_3(S^2) \cong \mathbb{Z}$. The Hopf fibration is a generator of $\pi_3(S^2)$.

Example 3.18 Consider the Lie group of orthogonal transformations of $\mathbb{R}^n$, $O(n)$. This is the set of symmetries of S^{n-1}. By restricting to elements which fix the first basis vector, we obtain an inclusion $O(n-1) \subset O(n)$. Denote by p the evaluation of an element of $O(n)$ on S^{n-1}. There is a fibre bundle

$$O(n-1) \longrightarrow O(n) \xrightarrow{p} S^{n-1}$$

The homotopy long exact sequence of this bundle links the homotopy groups of spheres and homotopy groups of $O(n)$. From this, we obtain an isomorphism $\pi_k(O(n-1)) \cong \pi_k(O(n))$ for $k < n-2$. The groups $\pi_k(O(n))$ must then be independent of n for large values of n. Surprisingly, these groups are periodic of period 8.

Theorem 3.19 (Bott Periodicity Theorem) *Let $k < n-2$. Then the following holds.*

$k \mod 8$	0	1	2	3	4	5	6	7
$\pi_k(O(n))$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Define the *stable orthogonal group*, O, as the limit of the inclusions

$$O(0) \subset O(1) \subset \ldots \subset \bigcup_{i=0}^{\infty} O(i) = O.$$

This is a topological space using the direct limit topology [Hat00]. The above theorem then gives the homotopy groups of the stable orthogonal group, $\pi_k(O)$, by replacing $O(n)$ with O. A version of Bott periodicity holds for both $U(n)$ and $Sp(n)$, though we will not need them here.

Recall that $O(n)$ consists of two homeomorphic connected components. The special orthogonal group, $SO(n)$, is the connected component of $O(n)$ consisting of elements whose matrices have positive determinant. The homotopy groups $\pi_k(SO(n))$ are then isomorphic to those of $O(n)$ for $k > 0$ and $\pi_0(SO(n)) = 0$.

We have seen that the homotopy groups of $O(n)$ are closely linked with homotopy groups of spheres. We would like to use the Bott Periodicity Theorem above to learn more about $\pi_k(S^n)$. We will construct a homomorphism from $\pi_n(SO(k))$ to the stable homotopy group Π_n. The image of this map is a very important subgroup of Π_n.

Elements of the special orthogonal group $SO(k)$ may be considered as symmetries $S^{k-1} \to S^{k-1}$. Furthermore, the group $\pi_n(SO(k))$ consists of homotopy classes of maps from S^n to $SO(k)$. Hence, an element of $\pi_n(SO(k))$ may be represented by a map $S^n \times S^{k-1} \to S^{k-1}$. From such a map, we may obtain a map $S^{n+k} \to S^k$ using a construction of Hopf. This construction defines a homomorphism,

$$J_n : \pi_n(SO(k)) \to \pi_{n+k}(S^k),$$

called the *Hopf-Whitehead J-homomorphism.* Taking the limit as k goes to infinity, we have a homomorphism $J_n : \pi_n(SO) \to \Pi_n$, where SO is the stable special orthogonal group. Adams gave a complete description of the image of J_n in [Ada63] and following papers. We will construct a map between the set of homotopy n-spheres and a quotient of Π_n, isomorphic to the cokernel of J_n, in chapter 7.

3.4 Homology of Manifolds and Orientation

Lemma 3.20 *Let x be a point of an n-dimensional manifold M. Then the groups $H_i(M, M-x;\mathbb{Z})$ are given by*

$$H_i(M, M-x;\mathbb{Z}) \cong \begin{cases} 0 & i \neq n \\ \mathbb{Z} & i = n. \end{cases}$$

Proof. By excision $H_i(M, M-x) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - x)$. Then applying the long exact sequence for the pair $(\mathbb{R}^n, \mathbb{R}^n - x)$ we have

$$\begin{aligned} H_i(M, M-x) &\cong H_i(\mathbb{R}^n, \mathbb{R}^n - x) \\ &\cong H_{i-1}(\mathbb{R}^n - x) \\ &\cong H_i(S^{n-1}). \end{aligned}$$

□

We can then choose a preferred generator μ_x of $H_n(M, M-x)$. We call this preferred generator a *local orientation* for M at x. A local orientation at x determines a local orientation for all points y in a neighbourhood V of x through the isomorphisms

$$H_k(M, M-x) \xleftarrow{r_x} H_k(M, M-V) \xrightarrow{r_y} H_k(M, M-y)$$

An *orientation* for M is a function which assigns a local orientation μ_x to each point of M which varies continuously with x. That is, for each x there exists a compact neighbourhood U of x and a class $\mu_U \in H_n(M, M-U)$ such that $r_y(\mu_U) = \mu_y$. A manifold which admits an orientation is said to be *orientable* and a manifold equipped with an orientation is called an *oriented manifold.*

Theorem 3.21 *For a compact oriented manifold M there is exactly one class $\mu = \mu_M \in H_n(M)$ such that $r_x(\mu) = \mu_x$ for all $x \in M$. This is called the fundamental homology class of M.*

The proof is quite long and technical, see appendix A of [MS16] for details.

We have now given two different definitions of an orientation of a manifold. Let us confirm they are equivalent.

Theorem 3.22 *An orientation for a manifold M induces an orientation on the tangent bundle TM. Similarly any orientation for the tangent bundle TM gives rise to an orientation for the manifold M.*

Proof. An orientation for TM gives rise to a generator μ_p of $H_n(T_pM, T_pM - 0)$ which varies continuously in p as follows. Let Δ^n be the standard n-simplex with standard ordering of its vertices. Choose an orientation preserving linear embedding $\sigma : \Delta^n \to T_pM$ mapping the barycentre of Δ^n to zero. Then the homology class of σ is a generator μ_p of $H_n(T_pM, T_pM-0)$.

Now note that there is a pair of canonical isomorphisms

$$H_n(T_pM, T_pM - 0) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \cong H_n(M, M-p),$$

hence a generator of $H_n(T_pM, T_pM - 0)$ varies continuously in p if and only if the corresponding generator of $H_n(M, M-p)$ varies continuously in p. Now note that this is exactly the homological definition of an orientation of M. That is, an orientation of M is a function which assigns a local orientation $\mu_p \in H_n(M, M-p)$ to each point p which varies continuously in p. □

Note that we can reinterpret the vector bundle definition of orientation in terms of cohomology. Let $F_0 = F - \{0\}$. An orientation of a bundle ξ corresponds to a choice of preferred generator $u_F \in H^n(F, F_0; \mathbb{Z})$ in each fibre. The local compatibility condition ensures that for each point p in the base there is a neighbourhood U of p and a class

$$u \in H^n(\pi^{-1}(U), \pi^{-1}(U) - \{0\}; \mathbb{Z})$$

such that for every $x \in U$ the restriction of u to $\pi^{-1}(x) = F$ is u_F.

The homology and cohomology groups of compact, oriented manifolds are closely related to one another.

Theorem 3.23 (Poincaré duality) *For M a compact, n-dimensional oriented manifold the correspondence $f \mapsto D(f) = f \frown \mu_M$ defines an isomorphism*

$$D : H^k(M) \xrightarrow{\cong} H_{n-k}(M).$$

The element $D(f) \in H_{n-k}(M)$ is called the *Poincaré dual* of f. Similarly, the dual of $\alpha \in H_k(M)$ is the unique cohomology class $D(\alpha)$ such that

$$D(\alpha) \frown \mu_M = \alpha.$$

There is a corresponding version of Poincaré duality for manifolds with boundary, called *Lefschetz duality,* providing isomorphisms

$$D : H^k(M, \partial M) \xrightarrow{\cong} H_{n-k}(M);$$

$$D : H_k(M, \partial M) \xrightarrow{\cong} H^{n-k}(M).$$

Proofs and details can be found in both [Hat00] and appendix A of [MS16].

Intersection numbers and the Intersection Form

Let M^m, N^n be closed, oriented submanifolds of the oriented manifold X^{m+n} intersecting transversally at points $\{p_1, \ldots, p_k\}$. By transversality, there is a tubular neighbourhood of M such that the fibres F_{p_i} are open neighbourhoods of p_i in N [Kos72] IV 1.7. Orient the tubular neighbourhood of M such that the orientation of M followed by the orientation of its fibre agrees with the orientation of X.

We can compare the local orientation of F_{p_i} and the local orientation of N at p_i. Set $\operatorname{sgn}(p_i) = 1$ if the orientations agree and -1 otherwise. Define the *intersection number* of M and N to be

$$[N : M] = \sum_{p_i \in M \cap N} \operatorname{sgn}(p_i).$$

It is not immediately clear that this is invariant in any way. However, consider the following construction.

The orientation of M gives rise to a generator $\mu_M \in H_n(X, X-M)$ and similarly for N. Let $\mu_i \in H_n(N, N-\{p_i\})$ be the local orientation of N at p_i. The inclusion of $(N, N-\{p_i\})$ into $(X, X-M)$ takes μ_i to $\text{sgn}(p_i)\mu_M$.

Let j_* be the composition

$$j_* : H_n(N) \to H_n(N, N - \cup_i p_i) \to H_n(X, X-M).$$

Then by definition we must have

$$j_*\mu_N = [N : M]\mu_M.$$

Note that we can also express j_* as the composition

$$j_* : H_n(N) \to H_n(X) \to H_n(X, X-M),$$

and this map does not depend on the embedding of M and N in X. It follows that the intersection number is an isotopy invariant. We may then consider intersection numbers of manifolds which do not intersect transversally by isotoping them to be transverse.

Note we could have worked with a tubular neighbourhood of N instead of M throughout to obtain $[M : N]$. The following lemma follows directly from the definitions.

Lemma 3.24 *Let M^m, N^n be closed, oriented submanifolds of X^{m+n}. Then*

$$[M : N] = (-1)^{nm}[N : M].$$

There is an equivalent formulation of the intersection number in terms of homology and cohomology which will be useful later.

We first define the *intersection product* of homology classes. Keeping the notation of above let $\alpha \in H_m(X)$, $\beta \in H_n(X)$. Denote by $D(\alpha) \in H^n(X)$ the Poincaré dual of α and similarly for β. Then the *intersection product* of α and β is

$$\alpha \cdot \beta := \langle D(\alpha) \smile D(\beta), \mu_X \rangle.$$

Note we can also consider the intersection product to be a product on cohomology classes. We will freely switch between these perspectives.

This product is easily computed in terms of intersection numbers if we can represent the homology classes by submanifolds.

Theorem 3.25 *Let M^m, N^n be closed, oriented submanifolds of X^{m+n}, and let i_M, i_N denote the inclusions of M and N into X. Then*

$$(i_{M*}(\mu_M)) \cdot (i_{N*}(\mu_N)) = [N : M].$$

We will mostly be interested in the case $n = m$ and homology with rational coefficients. In this case, the dimension of X is $2n$ and $H^n(X;\mathbb{Q})$ is a vector space. Furthermore, under taking Poincaré duals, the intersection product pairs with itself. Hence the intersection product (when interpreted as acting on either homology or cohomology classes) is a non-degenerate rational valued bilinear form, called the *intersection form* of X. By Lemma 3.24, the intersection form is symmetric when n is even and skew-symmetric when n is odd.

Example 3.26 Consider the torus T^2. Submanifolds α, β of T^2 which represent a basis for $H^1(T^2;\mathbb{Q})$ are indicated in Figure 11 below.

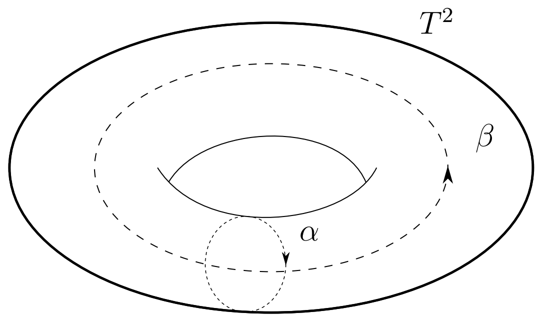


Figure 11: The longitudinal and latitudinal curves α and β generate a basis of the first homology group of T^2.

The submanifolds α and β intersect at a single point. At this point we have

$$[\alpha : \beta] = \pm 1,$$

depending on the orientation chosen for T^2. Choose the orientation such that $[\alpha : \beta] = 1$. Then the intersection number of β with α will be -1. Hence the intersection form of T^2 has matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which is a skew-symmetric matrix, as expected. Note that perturbing either of the curves to create additional intersections will result in cancelling pairs of intersections being formed, and so the intersection number will be fixed.

The Euler characteristic of a manifold can be given in terms of a certain intersection number. Let M be a compact, oriented manifold. Define the *self intersection number* of M to be the intersection number of the diagonal of $M \times M$ with itself in $M \times M$.

Lemma 3.27 *The Euler characteristic of a compact oriented M is the self intersection number of M.*

Proofs of this and alternate perspectives on intersection numbers can be found in [Bro63] and [Kos72].

4 Characteristic Classes

Let $\pi : E \to X$ be a vector bundle. The zero section of this bundle is a deformation retract of X in E. Hence,

$$\pi_*(E) \cong \pi_*(X);$$
$$H_*(E) \cong H_*(X);$$
$$H^*(E) \cong H^*(X).$$

We then cannot use the homotopy or homology of the total space to distinguish vector bundles over X from one another. We will need other tools to study vector bundles. For brevity, by bundle we will mean a vector bundle unless specified otherwise.

A characteristic class of a bundle, $\pi : E \to X$, is a distinguished cohomology element of $H^*(X)$ which depends on properties of the bundle.

We sketch here a construction of a particularly simple characteristic class which detects orientability of a bundle. Let M^n be a path connected Riemannian manifold and $p \in M$. Consider an oriented basis $\{v_1, \ldots, v_n\}$ of T_pM and a smooth loop γ at p. Slide the vectors of the basis along the loop γ such that the vectors are parallel to themselves using parallel transport.[3] Moving the frame around the entire loop, we then obtains a new basis at p, $\{v_1', \ldots, v_n'\}$. The intuitive idea is indicated in Figure 12.

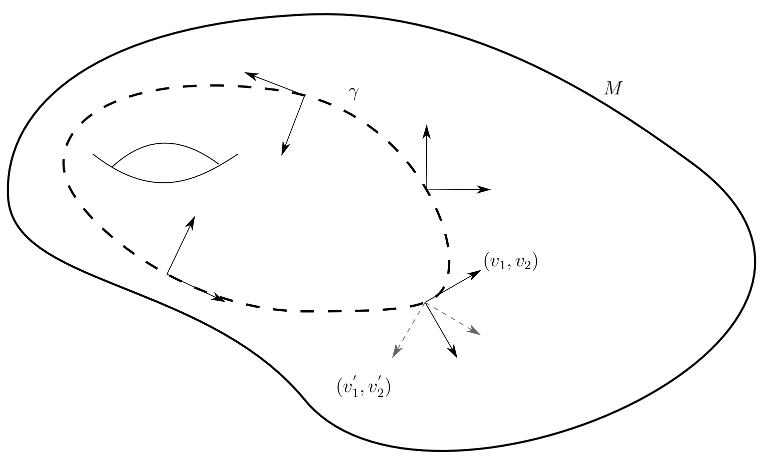


Figure 12: The frame (v_1, v_2) is transported around the loop γ, giving a new frame (v_1', v_2') denoted with dashed grey arrows. Some intermediate steps are drawn around the curve.

Define a map w as follows. If the orientation determined by the two bases match, let $w(\gamma) = 0$, otherwise, let $w(\gamma) = 1$. This definition actually depends only on the homotopy class of the loop γ, hence we obtain a homomorphism $w : \pi_1(M) \to \mathbb{Z}/2\mathbb{Z}$.

[3]This requires a choice of connection, though the construction is independent of the choice of connection. For a formal definition see [Lee19].

Note M is orientable if and only if there is no loop γ which reverses orientation. Hence we have an element,

$$w \in \operatorname{Hom}(\pi_1(M), \mathbb{Z}/2\mathbb{Z}),$$

which is zero if and only if M is orientable. As $\mathbb{Z}/2\mathbb{Z}$ is abelian, we can pass to the abelianisation of the fundamental group to obtain

$$\operatorname{Hom}(\pi_1(M), \mathbb{Z}/2\mathbb{Z}) \cong \operatorname{Hom}(H_1(M), \mathbb{Z}/2\mathbb{Z}).$$

However, as $\mathbb{Z}/2\mathbb{Z}$ is a field, $\operatorname{Hom}(H_1(M), \mathbb{Z}/2\mathbb{Z}) \cong H^1(M; \mathbb{Z}/2\mathbb{Z})$ by theorem 3.9. The element w then descends to a well defined cohomology class

$$w_1(TM) \in H^1(M; \mathbb{Z}/2\mathbb{Z}).$$

Define the *first Steifel-Whitney class* of the tangent bundle to M to be $w_1(TM)$. This class is zero if and only if the manifold is orientable. As an immediate application, this class can distinguish between the tangent bundle to $\mathbb{R}P^1$ and the tautological line bundle γ_1^1 as follows. Recall that $\mathbb{R}P^1$ is a circle, hence $T\mathbb{R}P^1 = TS^1$ is orientable, and so $w_1(\mathbb{R}P^1) = 0$. However, note that the total space of γ_1^1 is the open Mobius band, which is non-orientable, and so $w_1(\gamma_1^1) = 1$.

There are further Steifel-Whitney classes $w_k(TM) \in H^k(M; \mathbb{Z}/2\mathbb{Z})$ which detect further features of a manifold, though we will not need them.

We first introduce the Euler class for oriented vector bundles. This class is closely related to the Euler characteristic of a manifold. The Euler class can also be viewed as an obstruction to a bundle possessing an everywhere non-zero section. In fact, characteristic classes in general can be defined as obstructions to the existence of certain linearly independent frames, however we will not use this perspective.

4.1 The Euler Class

The following theorem on the cohomology groups of oriented bundles will be key to most of our results on characteristic classes. Let $\pi : E \to X$ be an oriented n-plane bundle. Define the *deleted space,* E_0, to be the set of non-zero vectors in E. That is, E_0 is the total space less the 0-section of the bundle,

$$E_0 = \{(x, v) \in E : v \neq 0\}.$$

Denote the restriction of the projection to the deleted space by $\pi_0 = \pi|_{E_0}$.

Theorem 4.1 (Thom Isomorphism Theorem) *Let $\pi : E \to X$ be a rank n oriented bundle.*

1. *The relative cohomology groups $H^k(E, E_0; \mathbb{Z})$ are zero for $k < n$.*

2. *The top cohomology group, $H^n(E, E_0; \mathbb{Z})$, contains exactly one element u, the Thom class, such that the restriction of u to any fibre F is equal to the preferred generator of the fibre given by the orientation,*

$$u|_{(F,F_0)} = u_F \in H^n(F, F_0; \mathbb{Z}).$$

3. *The map $x \mapsto x \smile u$ defines an isomorphism for all k,*

$$H^k(E; \mathbb{Z}) \xrightarrow{\cong} H^{k+n}(E, E_0; \mathbb{Z}).$$

The proof of the Thom isomorphism theorem is both long and not useful for what follows, a proof may be found in Chapter 10 of [MS16].

As the base space of a vector bundle is a deformation retract of the total space we have an isomorphism

$$\pi^* : H^k(X) \xrightarrow{\cong} H^k(E).$$

Combining this with the Thom isomorphism theorem we can define the *Thom isomorphism*, $\phi : H^k(X; \mathbb{Z}) \to H^{k+n}(E, E_0; \mathbb{Z})$, by

$$\phi : x \mapsto \phi(x) = \pi^*(x) \smile u.$$

We will use the Thom class to define a characteristic class in the top cohomology group of the base space. This class will be used to construct each other characteristic class in this paper. The inclusion

$$(E, \emptyset) \xhookrightarrow{i} (E, E_0)$$

induces a restriction homomorphism to the zero section in cohomology. For an oriented vector bundle ξ we can map the Thom class into $H^n(E)$ using this restriction. Define the *Euler class* to be the inverse image of this element under π^*. That is, $e(\xi) \in H^n(X; \mathbb{Z})$ is given by

$$e(\xi) = (\pi^*)^{-1} \circ\ i^*(u).$$

Consider the following diagram.

$$\begin{array}{ccccc} H^n(E, E_0) & \xrightarrow{i^*} & H^n(E) & \xrightarrow{(\pi^*)^{-1}} & H^n(X) \\ \\ u & \longmapsto & i^*(u) & \longrightarrow & e(\xi). \end{array}$$

From this we see that the Euler class is the restriction of the Thom class to the zero section. The Euler class has many useful properties.

Lemma 4.2 *Let ξ be an oriented rank n bundles $\pi : E \to X$. Then*

1. ***Naturality.*** *If $f : Y \to X$ is covered by an orientation preserving bundle map then $e(f^*\xi) = f^*e(\xi)$.*

2. ***Orientation.*** *If the orientation of ξ is reversed then $e(\xi)$ changes sign. That is, $e(-\xi) = -e(\xi)$.*

3. ***Triviality.*** *If ξ is trivial then $e(\xi) = 0$.*

Proof. The first two follow directly from the definition of the Euler class and the properties of the Thom class. Triviality follows from naturality by pulling back to a bundle over a point. □

Corollary 4.3 *If the rank of the bundle is odd then $2e(\xi) = 0$.*

Proof. Every odd dimensional vector bundle has an orientation reversing automorphism f given by $(x, v) \mapsto (x, -v)$. This map covers the identity map on the base. Then, by naturality and part 2 of Lemma 4.2,

$$e(\xi) = e(-\xi) = -e(\xi).$$

□

Lemma 4.4 *If an oriented vector bundle ξ possesses a non-vanishing section then the Euler class of ξ is zero.*

Proof. Consider a section $s : X \to E$ and suppose it is non-vanishing. Then we can consider it to be a map $s : X \to E_0$. Let $j : E_0 \to E$ be the inclusion. Note the composition $\pi \circ j \circ s$ is the identity as s is non-vanishing. We then have the following commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "s"'] & & \\ E_0 \arrow[rr, hook, "j"] & & E \arrow[ul, "\pi"'] \arrow[r, "i"] & (E, E_0) \end{tikzcd}$$

Pass to cohomology and note that the bottom row is the segment of the cohomology long exact sequence of (E, E_0). We obtain

$$\begin{tikzcd} & & & H^n(X) \arrow[dl, "\pi^*"'] & & \\ \dots \arrow[r] & H^n(E, E_0) \arrow[r, "i^*"] & H^n(E) \arrow[rr, "j^*"] & & H^n(E_0) \arrow[ul, "s^*"'] \arrow[r] & \dots \end{tikzcd}$$

Then, by exactness and commutativity, we have

$$e(\xi) = (\pi^*)^{-1} i^*(u) = s^* j^* i^*(u) = 0.$$

□

Lemma 4.5 (Whitney Product Formula) *The Euler class of a Whitney sum of two oriented bundles, ξ, η, is given by the cup product of the Euler classes of the bundles. That is,*

$$e(\xi \oplus \eta) = e(\xi) \smile e(\eta).$$

Proof. The Thom class of the sum $\xi \oplus \eta$ is given by

$$u(\xi \oplus \eta) = u(\xi) \times u(\eta).$$

Applying the restriction to the zero section to both sides we get

$$e(\xi \oplus \eta) = e(\xi) \times e(\eta).$$

Now pulling back along the diagonal mapping gives the required result. □

We can compare Euler classes of oriented manifolds using the fundamental class. Let M be a smooth oriented manifold with fundamental class $\mu \in H_n(M)$. Define the *Euler number* of M, $e[M]$, to be the integer

$$e[M] = \langle e(TM), \mu \rangle.$$

The following theorem explains the connection between the Euler characteristic and Euler class promised in the introduction to this chapter.

Theorem 4.6 *Let M be a smooth oriented manifold. The Euler number of M is equal to the Euler characteristic of M,*

$$e[M] = \chi(M).$$

A rather technical proof is given in [MS16] as Corollary 11.12.

4.2 Gysin Sequence

We introduce an exact sequence for oriented vector bundles which proves to be a powerful computational device. In particular, this sequence can be used to compute cohomology rings of $\mathbb{R}P^n$, $\mathbb{C}P^n$, and $\mathbb{H}P^n$. We will use it in the following section to construct a family of characteristic classes.

Let $\pi : E \to X$ be an oriented rank n bundle. As before, denote the restriction of the projection to the deleted space by $\pi_0 = \pi|_{E_0}$.

Theorem 4.7 *For every oriented rank n bundle there is an exact sequence*

$$\ldots \longrightarrow H^i(X) \xrightarrow{\smile e} H^{i+n}(X) \xrightarrow{\pi_0^*} H^{i+n}(E_0) \longrightarrow H^{i+1}(X) \xrightarrow{\smile e} \ldots$$

Proof. Recall the cohomology exact sequence of the pair (E, E_0)

$$\dots \longrightarrow H^i(E, E_0) \longrightarrow H^i(E) \longrightarrow H^i(E_0) \xrightarrow{\delta} H^{i+1}(E, E_0) \longrightarrow \dots$$

Now we use the isomorphism $\smile u : H^{i-n}(E) \to H^i(E, E_0)$ obtained from multiplication by the Thom class u to replace $H^i(E, E_0)$ with $H^{i-n}(E)$.

$$\dots \longrightarrow H^{i-n}(E) \longrightarrow H^i(E) \longrightarrow H^i(E_0) \longrightarrow H^{i-n+1}(E) \longrightarrow \dots$$

Finally we can replace $H^i(E)$ with $H^i(X)$ as E is a deformation retract of X. Note that the map on the left above is given by $x \mapsto x \smile u|_E$, so when we map $\pi^* : H^i(X) \mapsto H^i(E)$ this becomes $x \mapsto x \smile e$, by the definition of the Euler class. We then have

$$\dots \longrightarrow H^{i-n}(X) \xrightarrow{\smile e} H^i(X) \longrightarrow H^i(E_0) \longrightarrow H^{i-n+1}(X) \longrightarrow \dots$$

Note that this is the required sequence, shifted down n steps. □

4.3 Chern Classes

Recall every complex bundle is oriented, and so the Euler class $e(\omega_R) \in H^{2n}(X; \mathbb{Z})$ is defined. We will use the Euler class to inductively define a collection of characteristic classes for complex bundles,

$$c_k(\omega) \in H^{2k}(X; \mathbb{Z}).$$

To do this we will need to associate a rank $(n-1)$ complex bundle, ω_0, to a rank n complex bundle, ω.

We construct ω_0 as a bundle with base the deleted space E_0. Define the fibre of ω_0 over v to be

$$F_0 = F/\{\lambda v;\ \lambda \in \mathbb{C},\ v \in F - \{0\}\}.$$

That is, the quotient of F by all non-zero multiples of v. This is the orthogonal complement to the line spanned by v, an alternate construction which makes this clear can be given by introducing a Hermitian metric on ω.

Consider the Gysin sequence for ω

$$\dots \longrightarrow H^{i-2n}(X) \xrightarrow{\smile e} H^i(X) \xrightarrow{\pi_0^*} H^i(E_0) \longrightarrow H^{i-2n+1}(X) \longrightarrow \dots$$

For $i < 2n-1$, $H^{i-2n}(B)$ and $H^{i-2n+1}(B)$ are zero. Hence this reduces to a collection of isomorphisms

$$0 \longrightarrow H^i(X) \xrightarrow{\cong} H^i(E_0) \longrightarrow 0.$$

For a rank n complex vector bundle ω, define the *Chern classes* $c_k(\omega) \in H^{2k}(X;\mathbb{Z})$ to be zero for $k > n$ and inductively as follows for $k \leq n$.

$$c_n(\omega) = e(\omega_R)$$
$$c_k(\omega) = \pi_0^{*-1} c_k(\omega_0).$$

Define the *total Chern class*, $c(\omega)$, to be the sum

$$c(\omega) = 1 + c_1(\omega) + \ldots + c_n(\omega).$$

The Chern classes satisfy similar properties to the Euler class.

Lemma 4.8 (Naturality) *If $f : Y \to X$ is covered by a bundle map then $c(f^*\omega) = f^*c(\omega)$.*

It follows that the Chern classes of a trivial bundle are zero.

Lemma 4.9 (Stability) *Taking a Whitney sum with a trivial bundle preserves the total Chern class. That is,*

$$c(\omega \oplus \varepsilon) = c(\omega).$$

Proof. The bundle $\omega \oplus \varepsilon$ has a non-vanishing section, denote it by s. Hence, by Lemma 4.4,

$$c_{n+1}(\omega \oplus \varepsilon) = e((\omega \oplus \varepsilon)_R) = 0.$$

The class $c_{n+1}(\omega)$ is similarly zero as ω is a rank n bundle.

The section s is covered by a bundle map $\omega \to (\omega \oplus \varepsilon)_0$. By naturality we have

$$s^* c_k((\omega \oplus \varepsilon)_0) = c_k(\omega).$$

Finally, using that $s \circ \pi_0 = \mathrm{id}$, we have

$$c_k(\omega) = s^* c_k((\omega \oplus \varepsilon)_0) = s^* \circ \pi_0^* \left(c_k(\omega \oplus \varepsilon)\right) = c_k(\omega \oplus \varepsilon).$$

□

It follows that the total Chern class of a stably trivial bundle is 1.

Theorem 4.10 (Whitney Product Formula) *Let ω, θ be two complex vector bundles over the same base. Then the total Chern class of their Whitney sum is given by*

$$c(\omega \oplus \theta) = c(\omega)c(\theta).$$

The proof follows from the definition and the corresponding property of the Euler class using an induction argument.

Example 4.11 Consider the canonical line bundle $\gamma^1_{n\mathbb{C}}$ over $\mathbb{C}P^n$. Denote by E_0 the deleted space of $\gamma^1_{n\mathbb{C}}$. The Gysin sequence for this bundle is

$$\cdots \longrightarrow H^{k+1}(E_0) \longrightarrow H^k(\mathbb{C}P^n) \xrightarrow{\smile e} H^{k+2}(\mathbb{C}P^n) \xrightarrow{\pi_0^*} H^{k+2}(E_0) \longrightarrow \cdots$$

The deleted space is the set of pairs of lines through the origin in $\mathbb{C}^{n+1}$ and non-zero vectors in those lines. We may identify this space with S^{2n+1}. Note also that the Euler class is given by $e(\gamma^1_{n\mathbb{C}}) = c_1(\gamma^1_{n\mathbb{C}})$. For $0 \leq k \leq 2n-2$, the sequence reduces to a collection of short exact sequences

$$0 \longrightarrow H^k(\mathbb{C}P^n) \xrightarrow{\smile c_1} H^{k+2}(\mathbb{C}P^n) \longrightarrow 0$$

It follows that $H^{2k}(\mathbb{C}P^n)$ is an infinite cyclic group generated by $c_1(\gamma^1_{\mathbb{C}n})^k$. Similarly, the odd dimensional cohomology groups are all zero. Hence

$$H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[a]/(a^{n+1}),$$

where $a = c_1(\gamma^1_{\mathbb{C}n})$.

A similar computation gives

$$H^*(\mathbb{H}P^n;\mathbb{Z}) = \mathbb{Z}[b]/(b^{n+1}),$$

where $b = c_2(\gamma^1_{n\mathbb{H}}) \in H^4(\mathbb{H}P^n)$ is the second Chern class of the tautological line bundle over $\mathbb{H}P^n$.

Example 4.12 The total Chern class of the tangent space to complex projective space, $T\mathbb{C}P^n$, is given by

$$c(T\mathbb{C}P^n) = (1+a)^{n+1},$$

where $a = c_1(\gamma^1_{n\mathbb{C}}) \in H^2(\mathbb{C}P^n)$. Hence

$$c_k(T\mathbb{C}P^n) = \binom{n+1}{k} a^k.$$

Similarly, for the quaternion projective space, we have

$$c(T\mathbb{H}P^n) = (1+b)^{n+1},$$

where $b = c_2(\gamma^1_{n\mathbb{H}}) \in H^4(\mathbb{H}P^n)$. See [MS16] for a proof.

For any complex bundle, ω, we can define the *conjugate bundle*, $\bar{\omega}$, to be the complex vector bundle with the same underlying real bundle but with opposite complex structure. In each fibre we map $v + iw \mapsto v - iw$. The Chern classes of a bundle are closely linked to the Chern classes of its conjugate.

Lemma 4.13 *Let ω be a complex vector bundle. The kth Chern class of the conjugate bundle, $c_k(\bar{\omega})$, is given by $(-1)^k c_k(\omega)$.*

Proof. Choose a basis $v_1, \dots, v_n$ for any fibre F of ω. This basis gives rise to an oriented basis for the underlying fibre F_R,

$$v_1, iv_1, \dots, v_n, iv_n.$$

Similarly, we obtain an oriented basis for the fibre $\bar{F}_R$ of $\bar{\omega}_R$ given by

$$v_1, -iv_1, \dots, v_n, -iv_n.$$

These bases will have the same orientation if n is even, hence

$$c_n(\omega) = e(\omega_R) = (-1)^n c_n(\bar{\omega}).$$

Now by induction we have

$$c_k(\omega) = \pi_0^{*-1} c_k(\omega_0) = (-1)^k \pi_0^{*-1} c_k(\bar{\omega_0}) = (-1)^k c_k(\bar{\omega}).$$

In the second equality we used that $\overline{\omega_0} \cong (\bar{\omega})_0$. □

4.4 Pontrjagin Classes

Complexification

Given a real n-dimensional vector space, F, we can form an n-dimensional complex vector space by taking a tensor product with $\mathbb{C}$. Define the *complexification* of F to be

$$F \otimes \mathbb{C} := F \otimes_{\mathbb{R}} \mathbb{C}.$$

This vector space is isomorphic to the direct sum $F \oplus iF$.

This construction allows us to associate to any real vector bundle a complex vector bundle with a natural complex structure. Let ξ be a real rank n vector bundle. The *complexification of the bundle*, $\xi \otimes \mathbb{C}$, is the complex bundle obtained by replacing each fibre F of ξ with the complexification, $F \otimes \mathbb{C}$. This is a rank n complex vector bundle with underlying real bundle

$$(\xi \otimes \mathbb{C})_R \cong \xi \oplus \xi,$$

and complex structure $J(v, w) = (-w, v)$.

Lemma 4.14 *The complexification of a vector bundle is isomorphic to its conjugate.*

Proof. Consider the map $f : E(\xi \otimes \mathbb{C}) \to E(\overline{\xi \otimes \mathbb{C}})$ given by

$$f(v + iw) = v - iw.$$

This is a conjugate linear homeomorphism, hence a bundle isomorphism. □

Pontrjagin Classes

As the complexification of a vector bundle is isomorphic to its conjugate bundle, by the naturality of Chern classes we have

$$c(\xi \otimes \mathbb{C}) = c(\overline{\xi \otimes \mathbb{C}}).$$

Using Lemma 4.13, we find

$$1 + c_1(\xi \otimes \mathbb{C}) + \ldots + c_n(\xi \otimes \mathbb{C}) = 1 - c_1(\overline{\xi \otimes \mathbb{C}}) + \ldots \pm c_n(\overline{\xi \otimes \mathbb{C}}).$$

This can be true only if

$$2c_{2k+1}(\xi \otimes \mathbb{C}) = 0.$$

For a rank n real vector bundle, ξ, define the kth *Pontrjagin class* $p_k(\xi) \in H^{4k}(X;\mathbb{Z})$ as

$$p_k(\xi) = (-1)^k c_{2k}(\xi \otimes \mathbb{C}).$$

The *total Pontrjagin class* is defined to be

$$p(\xi) = 1 + p_1(\xi) + \ldots + p_{[n/2]}(\xi),$$

where $[n/2]$ is the largest integer less than or equal to $n/2$.

Lemma 4.15 (Properties of Pontrjagin Classes) *Let ξ be a rank n real vector bundle, and θ a rank m real vector bundle*

1. ***Dimension.*** $p_k(\xi) = 0$ *for* $k > \frac{n}{2}$
2. ***Naturality.*** *If $f : Y \to X$ is covered by a bundle map then $p_k(f^*\xi) = f^*p_k(\xi)$.*
3. ***Triviality.*** *The Pontrjagin classes of a trivial bundle are zero.*
4. ***Stability.*** *If ε^j is a trivial bundle then $p(\xi \oplus \varepsilon^j) = p(\xi)$.*
5. ***Whitney product formula.*** *$p(\xi \oplus \theta)$ is equal to $p(\xi) \smile p(\theta)$ modulo elements of order 2.*

Proof. All but the last follow directly from the properties of Chern classes. For the product formula expand the definitions to obtain

$$\begin{aligned}
p_k(\xi \oplus \theta) &= (-1)^k c_{2k}((\xi \oplus \theta) \otimes \mathbb{C}) \\
&= (-1)^k \sum_{i+j=2k} c_i(\xi \otimes \mathbb{C}) c_j(\theta \otimes \mathbb{C}) \\
&= \sum_{i+j=k} (-1)^i c_{2i}(\xi \otimes \mathbb{C}) \cdot (-1)^j c_{2j}(\theta \otimes \mathbb{C}) + \{\text{products of odd Chern classes}\} \\
&= \sum_{i+j=k} p_i(\xi) p_j(\theta) + \{\text{2-torsion elements}\}
\end{aligned}$$

□

Example 4.16 Consider the sphere S^n. The sphere is stably parallelisable, as proven in Example 2.9, and so, by part 4 of Lemma 4.15 above,

$$p(TS^n) = p(TS^n \oplus \varepsilon^1) = p(\varepsilon^{n+1}) = 1.$$

Hence the Pontrjagin classes of TS^n are trivial.

Similarly, it follows that the total Pontrjagin class of a stably trivial bundle is 1.

Given a real, oriented, or complex vector bundle we can take the complexification of the underlying real bundle to obtain a complex bundle of twice the (real) dimension. The Pontrjagin classes of complex or oriented bundles have nice properties due to the added structure on the fibres before complexifying.

Lemma 4.17 *Let ω be a complex vector bundle. The complexification of the underlying bundle, ω_R, is isomorphic to the Whitney sum of ω and its conjugate.*

$$\omega_R \otimes \mathbb{C} \cong \omega \oplus \bar{\omega}.$$

Corollary 4.18 *For a complex bundle, ω, the Chern and Pontrjagin classes satisfy the relation*

$$\sum_{i=0}^{n} (-1)^i p_i(\omega) = c(\bar{\omega})c(\omega)$$

The Pontrjagin classes are then given by

$$p_k(\omega) = c_k(\omega)^2 - 2c_{k-1}(\omega)c_{k+1}(\omega) + \ldots \pm 2c_1(\omega)c_{2k-1}(\omega) \mp 2c_{2k}(\omega).$$

See [MS16] for a proof of both.

Example 4.19 For the tangent bundle to complex projective space, $T\mathbb{C}P^n$, by Example 4.12 we have

$$c(T\mathbb{C}P^n) = (1+a)^{n+1},$$

where $a = c_1(\gamma_n^1) \in H^2(\mathbb{C}P^n)$. Therefore

$$\sum_{i=0}^{n} (-1)^i p_i = (1-a)^{n+1}(1+a)^{n+1} = (1-a^2)^{n+1}.$$

Hence

$$p(T\mathbb{C}P^n) = (1+a^2)^{n+1}.$$

Lemma 4.20 *For an oriented real n-plane bundle, ξ, the real rank $2n$ bundle $(\xi \otimes \mathbb{C})_R$ is isomorphic to $\xi \oplus \xi$ through an isomorphism which either preserves or reverses the orientation of ξ depending on whether the sign of $n(n-1)/2$ is even or odd.*

Proof. Let $v_1, \ldots, v_n$ be a basis determining the orientation of a fibre F of ξ. Then the vectors

$$v_1, iv_1, \ldots, v_n, iv_n$$

determine the canonical orientation for $(F \otimes \mathbb{C})_R \cong F \oplus iF$. Identify $F \oplus iF$ with $F \oplus F$. Now a basis determining the orientation for the fibre $(F \oplus F)$ of $\xi \oplus \xi$ is given by

$$v_1, \ldots, v_n, iv_1, \ldots, iv_n.$$

These bases can be changed into each other through a sequence of $n(n-1)/2$ transpositions. Therefore the orientations will agree if $n(n-1)/2$ is even and will disagree otherwise. □

Corollary 4.21 *For an oriented rank $2n$ real vector bundle, ξ, the top Pontrjagin class is the square of the Euler class. That is,*

$$p_n(\xi) = e(\xi) \smile e(\xi).$$

Proof. Expanding the definition of $p_n(\xi)$

$$\begin{aligned} p_n(\xi) &= (-1)^n c_{2n}(\xi \otimes \mathbb{C}) \\ &= (-1)^n e((\xi \otimes \mathbb{C})_R) \\ &= (-1)^n (-1)^n e(\xi \oplus \xi) \\ &= e(\xi)^2. \end{aligned}$$

In the second last equality we used Lemma 4.20 and in the final equality we used Lemma 4.5. □

It follows that Pontrjagin classes are orientation invariant. This property will be very important later in our construction of a smooth invariant for homotopy spheres.

4.5 Chern and Pontrjagin Numbers

Making use of the characteristic classes of tangent bundles of compact oriented manifolds we can define a collection of smooth manifold invariants. We saw a first example of this in the Euler number, $e[M]$. We now define a similar construction for Chern and Pontrjagin classes.

Recall a compact oriented manifold M^n possesses a unique fundamental class $\mu \in H_n(M)$. Hence for any cohomology class $a \in H^n(M)$ the pairing $a[M] = \langle a, \mu \rangle \in \mathbb{Z}$ is a well-defined homotopy invariant.

Define a *partition*, I, of the natural number n to be a tuple of non-negative integers numbers,

$$I = (i_1, \ldots, i_k),$$

such that $i_1 + \cdots + i_k = n$. Denote the number of partitions of n by $p(n)$.

Let M^n be a compact complex n-manifold. Recall that $c_i(M) \in H^{2i}(M)$. Hence for a partition I of n the product

$$c_I(TM) = c_{i_1}(TM) \cdots c_{i_k}(TM)$$

is in the top cohomology group of M, $H^{2n}(M)$. Define the *Chern number* for the partition I, $c_I[M]$, as

$$c_I[M] = \langle c_{i_1}(TM) \cdots c_{i_k}(TM), \mu_{2n} \rangle \in \mathbb{Z}.$$

These form a collection of integers, $c_I[M]$, associated to a manifold M as I ranges over all partitions of n. We are usually concerned with every possible Chern number for M. We say M and N have the same Chern numbers if $c_I[M] = c_I[N]$ for every partition I of n.

Example 4.22 Consider $\mathbb{C}P^n$. By Example 4.12 we have

$$c_i(T\mathbb{C}P^n) = \binom{n+1}{i} a^i.$$

Hence

$$\begin{aligned} c_I[\mathbb{C}P^n] &= \binom{n+1}{i_1} \cdots \binom{n+1}{i_k} \langle a^n, \mu \rangle \\ &= \binom{n+1}{i_1} \cdots \binom{n+1}{i_k}. \end{aligned}$$

A one dimensional complex manifold has one Chern number- the Euler characteristic $c_1[M] = \chi(M)$. In general, an n-dimensional complex manifold will have $p(n)$ distinct Chern numbers.

Similarly, for M^n a compact oriented manifold and I a partition of n, we define the *Pontrjagin number* $p_I[M]$ to be

$$p_I[M] = \langle p_{i_1}(TM) \cdots p_{i_k}(TM), \mu_{4n} \rangle \in \mathbb{Z}.$$

Example 4.23 The Pontrjagin classes of the tangent bundle to $\mathbb{C}P^{2n}$ are given by

$$p_k(\mathbb{C}P^{2n}) = \binom{2n+1}{k} a^{4k},$$

by Example 4.19. Hence the Pontrjagin numbers are given by

$$p_I[\mathbb{C}P^{2n}] = \binom{2n+1}{i_1} \cdots \binom{2n+1}{i_k}.$$

Lemma 4.24 *Let M be a compact oriented manifold with Pontrjagin numbers $p_I(M)$. Then M with orientation reversed, $-M$, has Pontrjagin numbers*

$$p_I[-M] = -p_I[M].$$

Proof. The map $M \mapsto -M$ fixes the Pontrjagin classes $p_k(M)$ as Pontrjagin classes are orientation invariant. However, reversing orientation changes the sign of the fundamental class. Therefore

$$\begin{aligned} p_I[-M] &= \langle p_{i_1}(-TM)\cdots p_{i_k}(-TM), -\mu\rangle \\ &= -\langle p_{i_1}(TM)\cdots p_{i_k}(TM), \mu\rangle \\ &= -p_I[M]. \end{aligned}$$

□

Theorem 4.25 *If a smooth, compact, orientable manifold M^{4n} is the boundary of a smooth, compact, orientable manifold B^{4n+1} then the Pontrjagin numbers of M are all zero.*

Proof. Let B be a smooth, compact, orientable manifold with $\partial B = M$ and let μ_B, μ_M denote the fundamental classes of B and M respectively, the orientation on B chosen such that the boundary homomorphism $\partial H_{4n+1}(B, M) \to H_{4n}(M)$ maps μ_B to μ_M.

Then for any Pontrjagin number $p_I \in H^{4n}(M)$ we have

$$p_I = \langle p_I, \mu_M\rangle = \langle p_I, \partial\mu_B\rangle = (-1)^{4n+2}\langle \delta p_I, \mu_B\rangle.$$

We need only show $\langle \delta p_I, \mu_B\rangle = 0$. Choose a Euclidean metric on B. Then the outward facing normal vector field along M decomposes the tangent bundle to B restricted to M as $\tau_B|_M = \tau_M \oplus \varepsilon^1$. Therefore $p_j(\tau_B)|_M = p_j(\tau_M)$. The short exact sequence of the pair (B, M) then gives

$$\delta(p_j(\tau_M)) = \delta(p_j(\tau_B)|_M) = 0.$$

Hence $p_I = \langle p_I, \mu_M\rangle = \langle \delta p_I, \mu_B\rangle = 0$ as required. □

This implies that Pontrjagin numbers are a cobordism invariant. Recall the set of oriented cobordism classes of n-manifolds, Ω_n, forms a group under disjoint union. Then, as a corollary of the above, we get the following.

Corollary 4.26 *For M^{4n} a smooth, compact, orientable manifold and I a partition of n the correspondence*

$$M \mapsto p_I[M]$$

defines a homomorphism between the oriented cobordism group Ω_{4n} and $\mathbb{Z}$.

4.6 Multiplicative Sequences

Multiplicative sequences of polynomials provide a number of useful relations between characteristic numbers. The main result of this section is the *Hirzebruch signature theorem* which will be the key tool used to construct a smooth invariant for spheres. However we must present a number of algebraic results to build up to this. Omitted proofs and details can be found in [Hir66] and [MS16].

Let R be a commutative ring with unit and $A^* = (A^0, A^1, \ldots)$ be a graded R-algebra which is commutative (*not* graded commutative). Consider the commutative ring A^Π of formal sums

$$A^\Pi = \left\{ \sum_{i=0}^{\infty} a_i : \; a_i \in A^i \right\}.$$

We will mostly be interested in the group of elements with leading term $a_0 = 1$, denoted A_1^Π. That is, an element $a \in A_1^\Pi$ is of the form

$$a = 1 + a_1 + a_2 + \ldots$$

Let x_i have degree i and $\{K_1(x_1), K_2(x_1, x_2), \ldots\}$ be a sequence of polynomials in R such that each K_i is homogeneous of degree i. Then for any $a \in A_1^\Pi$ define $K(a) \in A_1^\Pi$ by

$$K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + \cdots.$$

A sequence, $\{K_n\}$, of polynomials as above forms a *multiplicative sequence* if for all commutative graded R-algebras A^* we have

$$K(ab) = K(a)K(b) \quad \forall a, b \in A_1^\Pi.$$

Example 4.27 Given a constant $r \in R$, the collection K_i defined by

$$K_i(x_1, \ldots, x_i) = r^i x_i$$

form a multiplicative sequence with

$$K\left(\sum a_i\right) = \sum r^i x_i.$$

It is possible to classify all multiplicative sequences using power series. Let A^* be the graded polynomial ring $R[t]$ with t of degree 1. Then an element of A^Π is a formal power series

$$f(t) = \sum r_i t^i.$$

Lemma 4.28 *To each formal power series with coefficients in R, $f(t)$, there is a unique multiplicative sequence with coefficients in R, $\{K_i\}$, such that*

$$K(1+t) = f(t).$$

We say that $\{K_i\}$ belongs to $f(t)$.

See [Hir66] for a proof.

Example 4.29 The multiplicative sequence K_i of Example 4.27 belongs to $f(t) = 1 + rt$.

Let M be a smooth, oriented, compact manifold and consider a multiplicative sequence $\{K_i\}$ with coefficients in $\mathbb{Q}$. We define a characteristic number for the sequence in terms of Pontrjagin classes. Suppose the dimension of M is divisible by 4. Define the K-*genus* of M, $K[M]$, to be

$$K_n[M^{4n}] = \langle K_n(p_1, \dots, p_n), \mu\rangle \in \mathbb{Q}.$$

If the dimension of M is not divisible by 4 we adopt the convention that $K[M] = 0$.

The notion of K-genus allows us to define a number of new smooth invariants for manifolds. First we prove a stronger version of Theorem 4.25, constructing a ring homomorphism from the oriented cobordism ring, rather than a group homomorphism.

Lemma 4.30 *For every multiplicative sequence $\{K_i\}$ with coefficients in $\mathbb{Q}$ the map $M \mapsto K[M]$ is a ring homomorphism from the oriented cobordism ring, Ω_*, to $\mathbb{Q}$.*

Proof. Theorem 4.25 states that the K-genus of a boundary is zero, hence the map is well defined. The map is clearly additive. We need only prove that the map is a homomorphism with respect to multiplication.

A product manifold $M \times N$ has total Pontrjagin class $p(M) \times p(N)$ modulo 2 torsion. However there are no torsion elements as we are working with rational coefficients, hence

$$p(M \times N) = p(M) \times p(N).$$

As $\{K_i\}$ is multiplicative it follows that

$$K(p(M) \times p(N)) = K(p(M)) \times K(p(N)).$$

Therefore

$$\begin{aligned} \langle K(p(M) \times p(N)), \mu_M \times \mu_N\rangle &= (-1)^{16mn} \langle K(p(M))\mu_M\rangle \langle K(p(N))\mu_N\rangle \\ &= K[M]K[N], \end{aligned}$$

as required. □

We will now introduce a homotopy invariant which will be related to a K-genus of a certain multiplicative sequence.

Let M^{4n} be a smooth, compact, oriented $4n$-dimensional manifold and consider cohomology with rational coefficients. Recall in this case the intersection form of M is a non-degenerate symmetric bilinear form

$$\langle \cdot, \cdot\rangle : H^{2n}(M; \mathbb{Q}) \otimes H^{2n}(M; \mathbb{Q}) \to \mathbb{Q}.$$

Define the *signature* of M, $\sigma(M)$, to be the number of negative eigenvalues minus the number of positive eigenvalues of the matrix of this form.

We can compute this directly as follows. As the intersection form is a symmetric bilinear form in dimension $4n$, the matrix of the intersection form is diagonalisable. Let $b_1, \dots, b_k$

be a basis of $H^{2n}(M;\mathbb{Q})$ which diagonalises the matrix. The signature is then the number of positive entries minus the number of negative entries on the diagonal of

$$[\langle b_i \smile b_j, \mu\rangle]\,.$$

We adopt the convention that the signature is zero when the dimension of M is not divisible by 4.

The signature has a number of convenient properties.

Lemma 4.31 *Let M, N be smooth, compact, oriented manifolds.*

1. $\sigma(M \cup N) = \sigma(M) + \sigma(N)$.
2. $\sigma(-M) = -\sigma(M)$.
3. $\sigma(M \times N) = \sigma(M)\sigma(N)$.
4. *If $M = \partial W^{4n+1}$ is an oriented boundary then the signature of M is* 0.

Proof. The first two follow directly from the definition. The third is proven using the Künneth isomorphism theorem. The final property follows from linking the cohomology and homology long exact sequences of the pair (M, N) using Poincaré duality of manifolds with boundary. Details can be found in [Hir66]. □

The following theorem of Hirzebruch links together each of the concepts in this section.

Theorem 4.32 (Hirzebruch Signature Theorem) *Let $\{L_i\}$ be the multiplicative sequence belonging to the series*

$$\frac{\sqrt{t}}{\tanh(\sqrt{t})} = \sum_{i=0} \frac{(-1)^{i-1}2^{2i}B_i}{2i!}\, t^i,$$

where B_i is the i-th Bernoulli number. The signature $\sigma(M)$ of any smooth, compact, oriented manifold M^{4n} is equal to the L-genus of M.

By Lemma 4.30 and Lemma 4.31 the correspondences $M \mapsto L[M]$ and $M \mapsto \sigma(M)$ both define algebra homomorphisms from $\Omega_* \otimes \mathbb{Q}$ to $\mathbb{Q}$. Therefore we need only verify the theorem on a set of generators of Ω_*. The proof uses the complex projective spaces $\mathbb{C}P^{2n}$ as basis. However, proving this is a basis and computing the L-genus of these manifolds requires complex integration and a number of other tools not relevant to the rest of the paper; see [MS16] Theorem 19.4 for a complete proof.

The first two L_i are given by

$$L_1(p_1) = \frac{1}{3}p_1;$$
$$L_2(p_1, p_2) = \frac{1}{45}(7p_2 - p_1^2).$$

We will use the signature and L-genus to construct smooth invariants of homotopy spheres in the following section. The following two corollaries are the first steps to this construction.

Corollary 4.33 *The L-genus of any manifold is an integer.*

Corollary 4.34 *The L-genus of a manifold depends only on the oriented homotopy type of M.*

These results are somewhat surprising. There is no reason to expect the L-genus would be an integer from the definition itself. Furthermore, the signature theorem gives an oriented homotopy invariant in terms of a rational linear combination of numbers built from smooth structures on a manifold. This is a significant link between possible smooth structures on a manifold and its homotopy type.

4.7 A Smooth Invariant for Spheres

We will construct the smooth invariant for homotopy 7-spheres, used by Milnor in [Mil56], to prove his 7-sphere was exotic. We follow Milnor's construction closely before generalising the invariant to a class of $(4n-1)$-dimensional manifolds. We first sketch the motivation behind the invariant.

Consider a $4n$-dimensional manifold B with $\partial B = M^{4n-1}$. Were B a closed manifold we could use the signature theorem to show $\sigma(B) = L_n[B]$. Note, as each term of L_n has degree n, the top Pontrjagin class p_n appears linearly in L_n. Let s_n be the coefficient of p_n in L_n. Hence we can express $L_n(p_1, \ldots, p_n)$ as

$$L_n(p_1, \ldots, p_n) = L_n(p_1, \ldots, p_{n-1}, 0) + s_n p_n,$$

Using this form, the Hirzebruch signature theorem may be expressed as

$$\frac{1}{s_n}\left(\sigma(B) - L_n(p_1, \ldots, p_{n-1}, 0)[B]\right) = p_n(B) \in \mathbb{Z}.$$

However, B is not closed, so the signature theorem is not necessarily true. We will show that although the above relation does not hold, it does define a residue class in $\mathbb{Q}/\mathbb{Z}$ independent of the choice of B, i.e. is a smooth invariant of W^{4n-1}.

Milnor's Invariant

Let M be a closed, smooth, oriented 7-manifold satisfying the condition

$$H^3(M) = H^4(M) = 0. \tag{$*$}$$

Recall the 7-th oriented cobordism group is zero, and so every closed 7-manifold, M^7, is the boundary of some smooth 8-manifold, B^8. The invariant, $\lambda_7(M^7)$, will be defined as a mod 7 residue class of a function of the signature and the first Pontrjagin class of B.

Lemma 4.35 *The inclusion $i : H^4(B, M) \to H^4(B)$ is an isomorphism.*

Proof. Take the long exact sequence of the pair (B, M) and note the outer terms are 0 by $(*)$.

$$\begin{array}{ccccccccc} \dots \longrightarrow & H^3(M) & \xrightarrow{\partial} & H^4(B,M) & \xrightarrow{i} & H^4(B) & \longrightarrow & H^4(M) & \longrightarrow \dots \\ & \| & & & & & & \| & \\ & 0 & & & & & & 0 & \end{array}$$

□

Let $p_1 = p_1(TB) \in H^4(B)$ be the first Pontrjagin class of B. Using the lemma we can define a relative Pontrjagin number $q(B)$ by

$$q(B) = \langle i^{-1}(p_1)^2, \mu_B \rangle.$$

Define the invariant $\lambda_7(M) = 2q(B) - \sigma(B) \mod 7$.

Theorem 4.36 *$\lambda_7(M)$ is a diffeomorphism invariant of M. Furthermore, $\lambda_7(M)$ does not depend on the choice of manifold B.*

Proof. Let M_1, M_2 be closed, smooth, oriented manifolds diffeomorphic to M, and let $f : M_1 \to M_2$ be a diffeomorphism. Furthermore, suppose B_1 is a smooth manifold with boundary M_1, similarly for B_2. Then, by the gluing lemma, $C = B_1 \cup_f B_2$ is a closed, smooth, oriented 8-manifold.

Let $q(C) = \langle p_1^2(C), \mu_C \rangle$. The signature theorem applied to C gives

$$\sigma(C) = \langle \frac{1}{45}(7p_2(C) - p_1^2(C)), \mu_C \rangle.$$

Hence

$$45\sigma(C) + q(C) = 7\langle p_2(C), \mu_C \rangle \equiv 0 \mod 7.$$

This implies $3\sigma(C) + q(C) \equiv 45\sigma(C) + q(C) \equiv 0 \mod 7$. Therefore we have

$$\begin{aligned} 2q(C) - \sigma(C) &\equiv 2q(C) - \sigma(C) - 3(3\sigma(C) + q(C)) \mod 7 \\ &\equiv -q(C) - 10\sigma(C) \mod 7 \\ &\equiv 3\sigma(C) + q(C) \mod 7 \\ &\equiv 0 \mod 7. \end{aligned} \tag{1}$$

We want to show that $2q(B_1) - \sigma(B_1) \equiv 2q(B_2) - \sigma(B_2) \mod 7$. To do this we will relate the signature of C to that of B_1 and B_2 and $q(C)$ to $q(B_1)$ and $q(B_2)$.

Let $i_m : H^4(B_m, M_m) \to H^4(B_m)$ and $j : H^4(C, M) \to H^4(C)$ be the homomorphisms induced by the inclusions. Consider the diagram

$$\begin{tikzcd} H^4(B_1, M_1) \oplus H^4(B_2, M_2) \arrow[d, "i_1 \oplus i_2"] & H^4(C, M) \arrow[l, "h"'] \arrow[d, "j"] \\ H^4(B_1) \oplus H^4(B_2) & H^4(C) \arrow[l, "k"'] \end{tikzcd}$$

Here k is the map from the Mayer-Vietoris sequence for (C, B_1, B_2) and h is the map from the relative Mayer-Vietoris sequence for (C, B_1, B_2) relative to $M \cong M_1 \cong M_2$.

Both h and k are isomorphisms by the Mayer-Vietoris sequence and condition $(*)$. Also, $i_1 \oplus i_2$ is an isomorphism by Lemma 4.35. This diagram is commutative by the naturality of Mayer-Vietoris sequences; hence j is an isomorphism.

Take $\alpha_1 \in H^4(B_1, M_1)$, $\alpha_2 \in H^4(B_2, M_2)$ and consider

$$\alpha = jh^{-1}(\alpha_1 \oplus \alpha_2) \in H^4(C).$$

Then applying commutativity and duality

$$\begin{aligned}\langle \alpha^2, \mu_C \rangle &= \langle jh^{-1}(\alpha_1^2 \oplus \alpha_2^2), \mu_C \rangle \\ &= \langle k^{-1}(i_1 \oplus i_2)(\alpha_1^2 \oplus \alpha_2^2), \mu_C \rangle \\ &= \langle (i_1 \oplus i_2)(\alpha_1^2 \oplus \alpha_2^2), k(\mu_C) \rangle \\ &= \langle \alpha_1^2 \oplus \alpha_2^2, \mu_{B_1} \oplus -\mu_{B_2} \rangle \\ &= \langle \alpha_1^2, \mu_{B_1} \rangle - \langle \alpha_2^2, \mu_{B_2} \rangle.\end{aligned} \tag{2}$$

Hence the intersection form of C is the direct sum of the intersection forms of B_1 and $-B_2$. It follows from the definition of the signature that

$$\sigma(C) = \sigma(B_1) - \sigma(B_2). \tag{3}$$

Note that $k(p_1(C)) = p_1(B_1) \oplus p_1(B_2)$, since k is the restriction of $p_1(C)$ to B_i. Therefore using that the above diagram commutes

$$\begin{aligned}jh^{-1}(i_1^{-1}p_1(B_1) \oplus i_2^{-1}p_1(B_2)) &= k^{-1}(i_1 \oplus i_2(i_1^{-1}p_1(B_1) \oplus i_2^{-1}p_1(B_2))) \\ &= k^{-1}(p_1(B_1) \oplus p_1(B_2)) \\ &= p_1(C).\end{aligned}$$

Choosing $p_1(C)$ for α in (2) we find

$$\begin{aligned}q(C) &= \langle p_1(C)^2, \mu_C \rangle \\ &= \langle i_1^{-1}p_1(B_1)^2, \mu_{B_1} \rangle - \langle i_2^{-1}p_1(B_2)^2, \mu_{B_2} \rangle \\ &= q(B_1) - q(B_2).\end{aligned} \tag{4}$$

Now substitute (3) and (4) into (1) to obtain

$$\begin{aligned}0 &\equiv 2q(C) - \sigma(C) \mod 7 \\ &\equiv 2q(B_1) - 2q(B_2) - \sigma(B_1) + \sigma(B_2) \mod 7\end{aligned}$$

$$\implies 2q(B_1) - \sigma(B_1) \equiv 2q(B_2) - \sigma(B_2) \mod 7.$$

Hence λ_7 is well defined smooth invariant of M, independent of the choice of B. $\square$

Corollary 4.37 *If $\lambda_7(M^7) \neq 0$ then M is not the boundary of any 8-manifold B with $H^4(B) = 0$.*

Proof. As $H^4(B) = 0$, $\sigma(B)$ and $q(B)$ must both be zero by definition. □

Lemma 4.38 *If the orientation of M^7 is reversed then $\lambda_7(M)$ is multiplied by -1.*

Proof. Reversing the orientation of M reverses the induced orientation on B, hence μ_B is mapped to $-\mu_B$. However Pontrjagin classes and cup products are unchanged under change of orientation. Hence the sign of $q(B)$ and $\sigma(B)$ are reversed by definition. □

Corollary 4.39 *If $\lambda_7(M^7) \neq 0$ then M^7 possesses no orientation reversing diffeomorphism.*

Proof. Suppose $f : M \to M$ is an orientation reversing diffeomorphism. As λ_7 is a diffeomorphism invariant

$$\lambda_7(M) = \lambda_7(f(M)) = \lambda_7(-M).$$

However, by Lemma 4.38

$$\lambda_7(M) = \lambda_7(-M) = -\lambda_7(M).$$

Hence $\lambda_7(M) = 0$, a contradiction. □

Now the 7-sphere does possess an orientation reversing diffeomorphism, as do all spheres, and so if $\lambda_7(M^7) \neq 0$ then M^7 is not diffeomorphic to the 7-sphere.

The generalised λ-invariant

Milnor noted in his original paper that his invariant could be generalised to higher dimensions. Shimada, [Shi57], defined a 15-dimensional version of the invariant to prove a collection of manifolds were exotic 15-spheres using similar methods to Milnor. Milnor himself defined the general invariant in [Mil59b].

We need slightly different conditions to define the general invariant. Let M be a closed, oriented $(4n-1)$-dimensional smooth manifold satisfying the following conditions.

1. M has the same rational homology as S^{2n-1}.
2. M is the boundary of a smooth $4n$-manifold B.

Note the second condition is not trivial, as generally Ω_{4n-1} is non-zero. However both conditions hold for homotopy spheres.

Then, as above, $i : H^j(B, M; \mathbb{Q}) \to H^j(B; \mathbb{Q})$ is an isomorphism for $0 < j < 4n-1$. Hence the Pontrjagin classes $p_1(B), \ldots, p_{n-1}(B)$ can be lifted to the relative classes $i^{-1}(p_k(B)) \in H^{4k}(B, M; \mathbb{Q})$.

Define the invariant $\lambda(M^{4n-1})$ as the rational number

$$\lambda(M^{4n-1}) := \frac{1}{s_n}\left(\sigma(B) - L_n(i^{-1}p_1(B), \ldots, i^{-1}p_{n-1}(B), 0)\right) \mod 1.$$

Note in the $n = 2$ case that the L-polynomial is given by

$$L_2(p_1, p_2) = \frac{1}{45}(7p_2 - p_1^2).$$

Hence,

$$\lambda(M^7) = \frac{45}{7}\left(\sigma(B) + \frac{1}{45}i^{-1}p_1^2[B]\right) \mod 1.$$

Multiplying by 14 we obtain the mod 7 invariant of Theorem 4.36.

Theorem 4.40 *$\lambda(M) \in \mathbb{Q}/\mathbb{Z}$ is a smooth invariant of M.*

Proof. The proof is very similar to the seven dimensional case. Let B_1, B_2 be two smooth manifolds such that $\partial B_1 = \partial B_2 = M$ and define $C := B_1 \cup_M B_2$. As before this is a smooth, compact, oriented manifold with $\sigma(C) = \sigma(B_1) - \sigma(B_2)$ and each of the Pontrjagin numbers are equal to the difference of the Pontrjagin numbers of B_1 and B_2, other than $\langle p_n(C), \mu_C\rangle$.

Therefore

$$\begin{aligned}
&\frac{1}{s_n}\left(\sigma(B_1) - L_n(i^{-1}p_1(B_1), \ldots, i^{-1}p_{n-1}(B_1), 0)\right) - \\
&\left(\frac{1}{s_n}\left(\sigma(B_2) - L_n(i^{-1}p_1(B_2), \ldots, i^{-1}p_{n-1}(B_2), 0)\right)\right) \\
&= \langle p_n(C), \mu_C\rangle \\
&= 0 \mod 1.
\end{aligned}$$

In the final line we used that $\langle p_n(C), \mu_C\rangle$ is an integer. □

The following corollaries are proven identically to the corresponding claims in dimension 7.

Corollary 4.41 *For M a closed, oriented $(4n-1)$-dimensional smooth manifold which is a homology sphere and the boundary of a smooth $4n$-dimensional manifold:*

1. *If $\lambda(M^{4n-1}) \neq 0$ then M is not the boundary of any $4n$-manifold B with $H^{2n}(B) = 0$.*
2. *If the orientation of M is reversed then $\lambda(M)$ is multiplied by -1.*
3. *If $\lambda(M) \neq 0$ then M possesses no orientation reversing diffeomorphism.*

We will use this invariant and the third part of Corollary 4.41 in chapter 6 to prove certain smooth homotopy spheres are not diffeomorphic to S^n.

5 Morse Theory

Morse theory involves the study of the relationship between smooth real valued functions on a space and the topology of that space. A simple example of this idea is the extreme value theorem- a continuous function from a compact space to $\mathbb{R}$ attains a maximum and a minimum, whereas there are functions on $\mathbb{R}$ which obtain values of arbitrary size, such as $f(x) = x^3$. Morse theory decomposes a manifold M in terms a the critical points of a smooth function, $f : M \to \mathbb{R}$, which behave like a height function for M. We first give a rough sketch of the main ideas.

Define $M^c = f^{-1}((-\infty, c])$ as the set of points $p \in M$ with $f(x) \leq c$. If $df_c \neq 0$ then, by the implicit function theorem, M^c is a smooth manifold with boundary given by $f^{-1}(c)$.

Let $a, b \in \mathbb{R}$ be the maximum and minimum values of f respectively. Then

$$M^c = \begin{cases} \emptyset & c < a; \\ \{pt\} & c = a; \\ M & c \geq b. \end{cases}$$

Between a and b the homotopy type of M^c changes from a point to that of M. We claim that the homotopy type of M^c changes exactly at the *critical points* of f, that is, points c where $df_c = 0$. Furthermore, we claim that if there are k "decreasing directions" at a critical point x, then $M^{x+\varepsilon}$ has the homotopy type of $M^{x-\varepsilon}$ with a k-cell attached.

For example, embed the torus T^2 upright in $\mathbb{R}^3$ with its base on the xy-plane as in Figure 13. Let $f : T^2 \to \mathbb{R}$ be given by the z-value of T^2. Then the critical values of T^2 are at the base and top of the torus, and at the opening, and closing of the hole. Denote these points by p, q, r, s as in Figure 13.

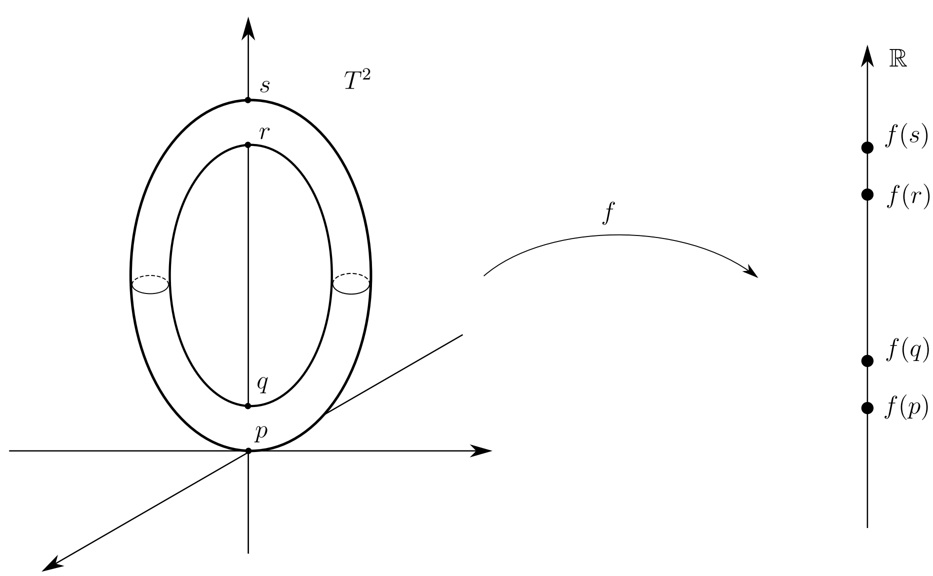


Figure 13: An embedding of the torus in $\mathbb{R}^3$ and its height function.

Note near the point p that T^2 is bowl shaped, and so there are no decreasing directions of f at p. Hence, we begin with a 0-cell. Around the point q the torus is saddle shaped, hence there is one decreasing direction. We then attach a 1-cell to obtain a manifold homotopic to a cylinder. Similarly, at r there is one decreasing direction, so we attach a 1-cell to give a torus with a disk removed. Finally, at s there are two decreasing directions, and so we attach a 2-cell along the boundary. This has the effect of capping off the boundary to form T^2. Figure 14 depicts this process.

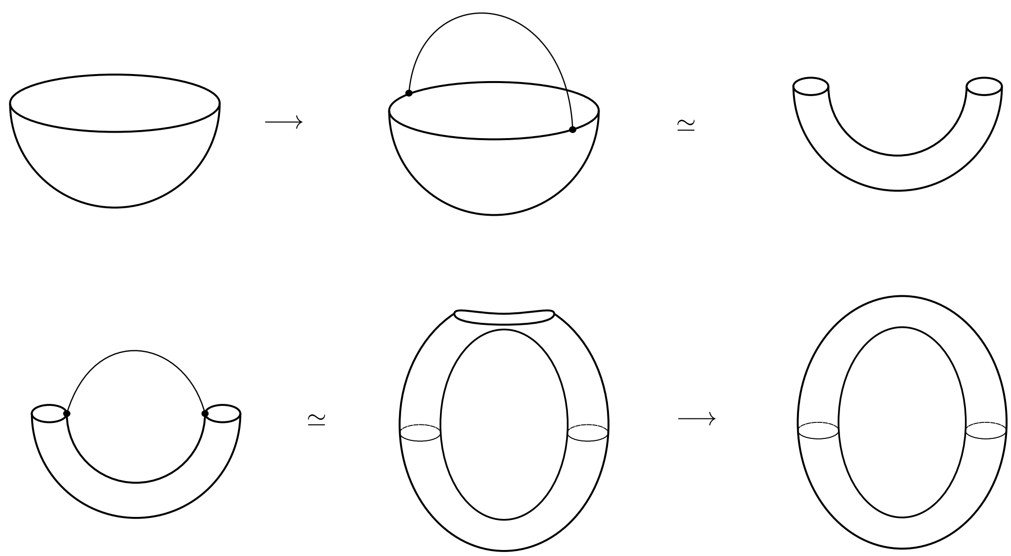


Figure 14: A Morse decomposition of the torus.

5.1 Morse Functions

Let $f : M \to \mathbb{R}$ be a smooth real valued function on a manifold M. A point $p \in M$ is called a *critical point* and $f(p)$ a *critical value* if the differential $df_p : T_pM \to T_{f(p)}\mathbb{R}$ is zero. Equivalently, in local coordinates $(x^1, \dots, x^n)$ around p we have

$$\left.\frac{\partial f}{\partial x^1}\right|_p = \cdots = \left.\frac{\partial f}{\partial x^n}\right|_p = 0.$$

The Hessian matrix at p,

$$(H_{ij})(p) = \left.\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)\right|_p,$$

determines the behaviour of f at a critical point p. It is a symmetric bilinear form on T_pM [Mil63]. A critical point p is *non-degenerate* if the Hessian is non-degenerate, that is, $\det H(p) \neq 0$.

The *index* of f at p is defined as the dimension of the maximal subspace of T_pM on which $H(p)$ is negative definite. The Morse lemma states that around a non-degenerate critical point, a function is determined by its index.

Lemma 5.1 (Morse lemma) *Let f be a smooth real valued function and p be a non-degenerate critical point of index λ. Then there is a local coordinate system $(U, (y^1, \ldots, y^n))$ in a neighbourhood of p with $y^i(p) = 0$ and*

$$f(y_1, \ldots, y_n) = f(p) - \sum_{i=1}^{\lambda} (y_i)^2 + \sum_{i=\lambda+1}^{n} (y_i)^2.$$

Proof. Suppose there is a local coordinate system $(y^1, \ldots, y^n)$ around p such that

$$f(y_1, \ldots, y_n) = f(p) - \sum_{i=1}^{\lambda} (y_i)^2 + \sum_{i=\lambda+1}^{n} (y_i)^2.$$

In these coordinates, the entries of the Hessian matrix of f at p in these coordinates are given by

$$\frac{\partial^2 f}{\partial y^i \partial y^j} = \begin{cases} -2 & i = j \leq \lambda, \\ 2 & i = j > \lambda, \\ 0 & \text{else.} \end{cases}$$

Therefore there is a λ-dimensional space on which $H(p)$ is negative definite and an $(n - \lambda)$-dimensional space where $H(p)$ is positive definite. Any negative definite subspace of dimension greater than λ would intersect this positive definite subspace, a contradiction, so the index of f at p must be λ.

Given arbitrary local coordinates $(x^1, \ldots, x^n)$ of f around p, translate them such that p and $f(p)$ are both zero in these coordinates. As $f(0) = 0$, we can use the n-dimensional fundamental theorem of calculus to write f as

$$f(x^1, \ldots, x^n) = \sum_{i=1}^{n} x^i g_i(x^1, \ldots, x^n),$$

where g_i are smooth functions defined on a neighbourhood of p with $g_i(0) = \frac{\partial f}{\partial x^i}(0) = 0$. We can then apply the fundamental theorem of calculus to the g_i to obtain smooth functions h_{ij} such that

$$f(x^1, \ldots, x^n) = \sum_{i=1}^{n} x^i x^j h_{ij}(x^1, \ldots, x^n).$$

Note that if we replace (h_{ij}) with the symmetric matrix $(h_{ij})^{\text{sym}} = \frac{1}{2}((h_{ij})+(h_{ji}))$ we still have $f(x) = \sum x^i x^j h_{ij}^{\text{sym}}(x)$, as the above expression is symmetric in x, also $(h_{ij}(0))^{\text{sym}} = (h_{ij}(0))$. Hence we may assume h_{ij} is symmetric. Now note the matrix $(h_{ij}(0))$ at zero is non-degenerate as

$$(h_{ij}(0)) = \left(\frac{1}{2} \frac{\partial^2 f}{\partial x^i \partial x^j}\bigg|_0 \right).$$

Then $h_{ij}(0)$ is a symmetric, real valued, non-degenerate matrix, and so we can find a coordinate system in which it is diagonal. Note that in this coordinate system f takes the required form. That the standard diagonalisation process defines a smooth transformation of the coordinates $(x^1, \ldots, x^n)$ can be seen by explicitly writing out the coordinate transformations. Full details can be found in [Mil63]. $\square$

Corollary 5.2 *Non-degenerate critical points are isolated.*

A *Morse function* $f \in C^\infty(M)$ is a smooth real valued function on a manifold M with only non-degenerate critical points. For the remainder of the chapter we will restrict our attention to Morse functions. This is not a very strong restriction, due to the following corollary of Sard's theorem. Recall that $C^\infty(M)$ may be given the structure of a topological space, see [Mat02] for a construction of this space.

Theorem 5.3 *Let M be a smooth closed manifold. The set of Morse functions of M is dense in $C^\infty(M)$.*

See [Mat02] for a proof of both Sard's theorem and this result.

5.2 Morse Functions and Handle Attachments

We will now describe how the homotopy type of M^c changes with c and prove the claims made in the opening of this chapter.

Theorem 5.4 *Let $f : M \to \mathbb{R}$ be a smooth function and a, $b \in M$ be two points such that $a < b$ and $f^{-1}([a,b])$ is compact and contains no critical points of f. Then M^a is diffeomorphic to M^b, and M^a is a deformation retract of M^b.*

Proof. The intuitive idea of the proof is indicated in Figure 15. We push M^b along the trajectories of f. Choose a Riemannian metric on M and denote the inner product by $\langle \cdot, \cdot \rangle$. Define the gradient of f as the vector field ∇f given by

$$\langle X, \nabla f \rangle = X(f).$$

Note that ∇f vanishes at the critical points of f.

Let $r : f^{-1}([a,b]) \to \mathbb{R}$ be the smooth function

$$r(x) = \frac{1}{\langle \nabla f, \nabla f \rangle}.$$

Extend this to a smooth function on M which vanishes outside a neighbourhood of $f^{-1}([a,b])$. Then the smooth vector field X given by

$$X(p) = r(p)\nabla f(p)$$

generates a 1-parameter group of diffeomorphisms $\varphi_t : M \to M$ given by the integral curves of X.

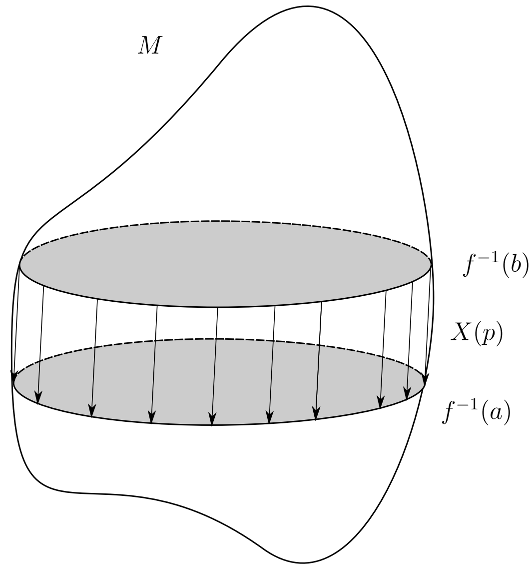


Figure 15: The flow lines generated by the gradient vector field form a retraction.

If $\varphi_t(p) \in f^{-1}([a,b])$ then

$$\frac{d}{dt} f(\varphi_t(p)) = \langle \frac{d\varphi_t(p)}{dt}, \nabla f \rangle = \langle X, \nabla f \rangle = 1.$$

The integral curve then climbs at constant speed from the level set $f^{-1}(a)$ at $t = 0$ to $f^{-1}(b)$ at $t = b - a$. Therefore the map $\varphi(p,t)$ is a diffeomorphism of $f^{-1}(a) \times [0, b-a]$ onto $f^{-1}([a,b])$. It follows that the diffeomorphism φ_{b-a} maps M^a diffeomorphically onto M^b.

Define a deformation retract of M^b to M^a by

$$r_t(p) = \begin{cases} p & f(p) \leq a \\ \varphi_{t(a-f(p))}(p) & a \leq f(p) \leq b. \end{cases}$$

□

Theorem 5.5 *Suppose $p \in M$ is a non-degenerate critical point of f with index λ and set $f(p) = c$. Let $\varepsilon > 0$ be such that $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact and contains no other critical points of f. Then $M^{c+\varepsilon}$ is homotopic to $M^{c-\varepsilon}$ with a λ-handle attached.*

Proof. We give the main idea of the proof; for details refer to [Mil63]. By the Morse lemma, there is a coordinate neighbourhood U around p such that f is of the form

$$f(x_1, \cdots, x_n) = -\sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2.$$

In these coordinates, the level sets of f are as depicted in Figure 16.

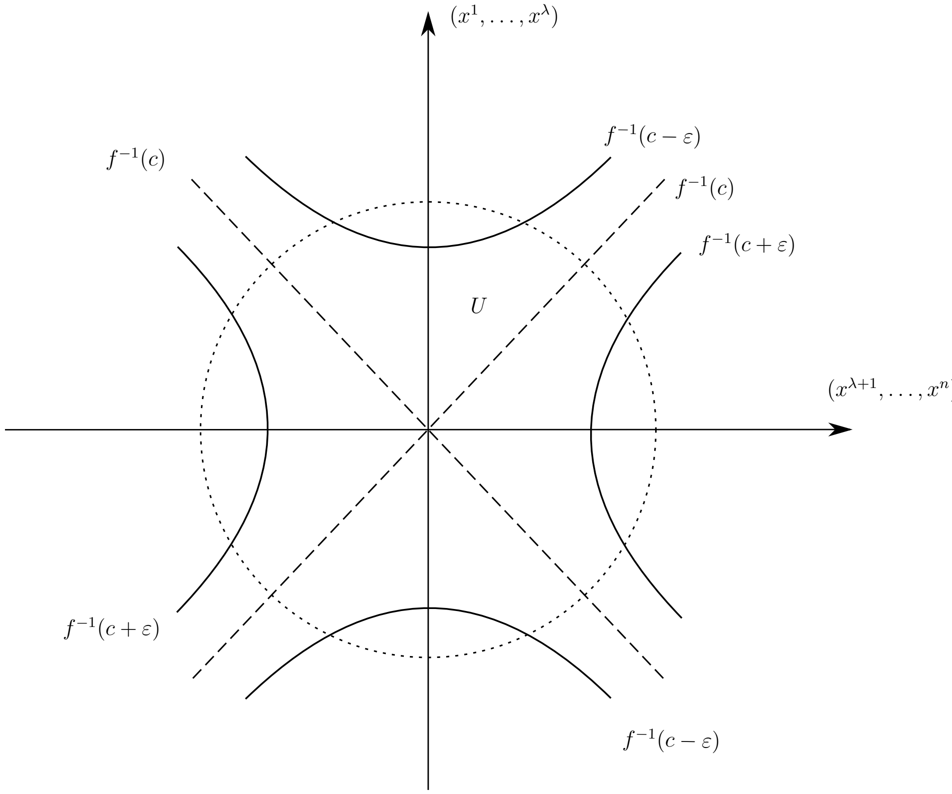


Figure 16: A schematic representation of M in Morse coordinates.

As f is of the above form around p, the tangent space of M at p, T_pM, is split into a λ-dimensional decreasing subspace and an $(n-\lambda)$-dimensional increasing subspace.

Consider the sets $M^{c+\varepsilon} \cap U$ and $H = (M^{c+\varepsilon} \cap U) - M^{c-\varepsilon}$. Observe that H is homotopy equivalent to $D^\lambda \times D^{n-\lambda}$, a λ-handle. Figure 17 below gives a schematic representation.

We can construct a deformation retraction between $M^{c+\varepsilon}$ and $M^{c-\varepsilon} \cup H$ by introducing a new function $F : M \to \mathbb{R}$ which agrees with f everywhere but in the neighbourhood U where $F < f$. We then show that F contains no critical points in U and use Theorem 5.4 to finish. $\square$

With this we have a direct correspondence between the critical points of a Morse function on M and a handlebody presentation of M. A full account of this connection is given in [Mat02].

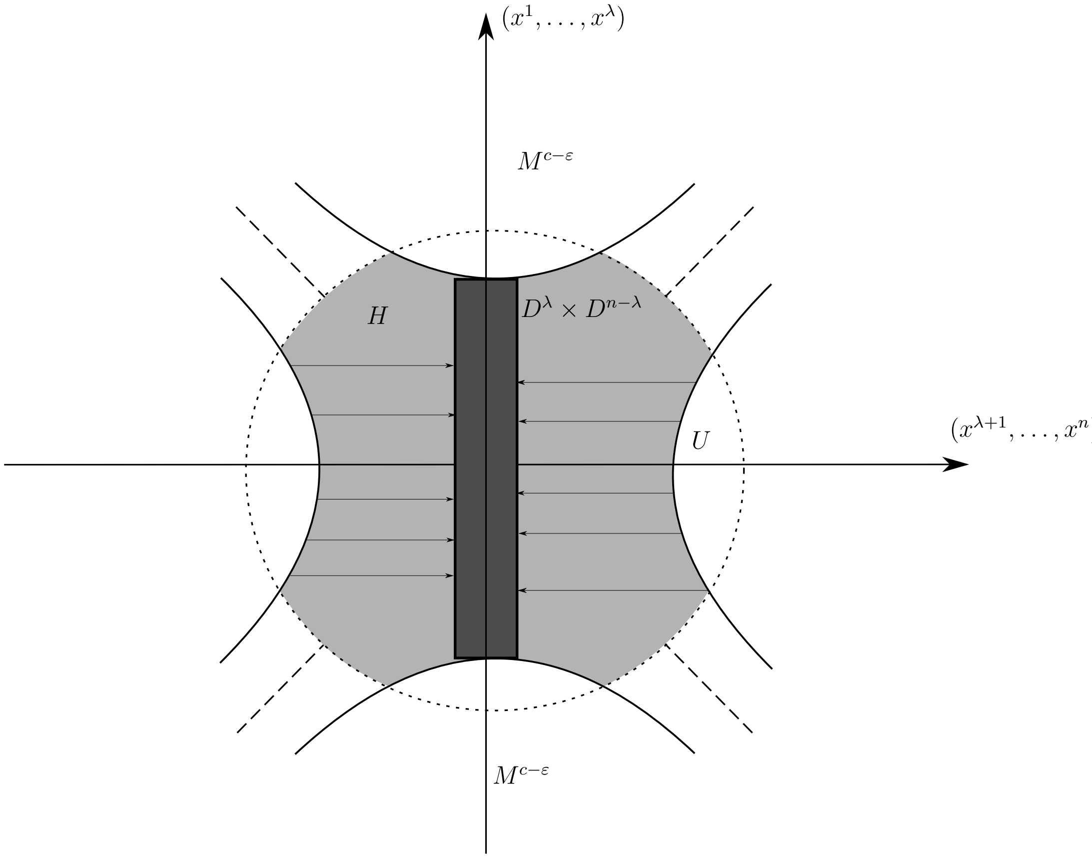


Figure 17: The Handle H is shaded lightly. The deformation retract of H onto $D^\lambda \times D^{n-\lambda}$ is indicated with arrows.

5.3 A characterisation of topological spheres

We can now give a criterion to determine if a given manifold is homeomorphic to a sphere.

Theorem 5.6 (Reeb Sphere Theorem) *If M^n is a compact manifold which admits a Morse function f with exactly two critical points then M is homeomorphic to the n-sphere S^n.*

Proof. As M is compact, the two critical points a, b must be the maximum and minimum of f. Scale f such that $f(a) = 0$ and $f(b) = 1$. Both $f^{-1}(0) = a$ and $f^{-1}(1) = b$ are homotopy equivalent to D^n. By the Morse lemma, $f^{-1}([0, \varepsilon])$ and $f^{-1}([1-\varepsilon, 1])$ are homeomorphic to D^n for some $\varepsilon > 0$.

Remove disks D_a, D_b around a and b. Then f restricted to $M^{'} = M - (D_a \cup D_b)$ has no critical points. Hence, by Theorem 5.4 above, $M^{'}$ is homeomorphic to $S^{n-1} \times [\varepsilon, 1-\varepsilon]$. This provides an explicit homeomorphism between the boundary of the top and bottom disks of M, therefore M is homeomorphic to two copies of D^n matched along their boundary, a sphere. $\square$

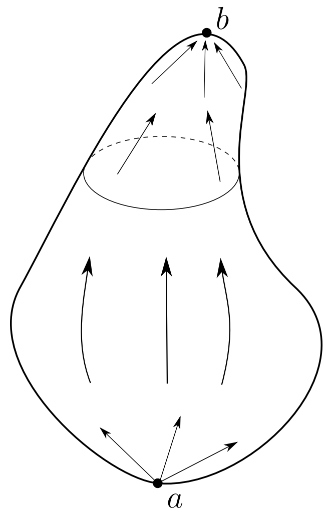


Figure 18: The critical points a and b are marked. The arrows denote the retraction between the level sets of M.

Note that this lemma only tells us that M is homeomorphic to S^n, not diffeomorphic to it. However we do know that M is the union of two n-disks matched along their boundary. Hence M less a point is diffeomorphic to S^n less a point.

This theorem does not imply M is diffeomorphic to S^n. However, M is diffeomorphic to a manifold obtained by gluing two n-disks along S^{n-1}. If the gluing is a diffeomorphism isotopic to the identity then we can conclude that $M = S^n$. This raises the question, can there be diffeomorphisms

$$g : S^{n-1} \to S^{n-1}$$

such that g is not isotopic to the identity? If so, we could produce a manifold homeomorphic to S^n but not diffeomorphic to it by attaching two copies of D^n along the boundary using g. In chapter 7 we will study $\mathrm{Diff}(S^{n-1})$, the set of diffeomorphisms of S^{n-1} up to isotopy, and show that it can contain non-trivial elements. This is very different to the topological setting, where all homeomorphisms of S^n can be extended to D^{n+1}, and hence are isotopic to the identity map [Mat02].

6 Some Constructions of Exotic Spheres

In this chapter we will construct a number of examples of exotic spheres. We first describe a method for easily generating fibre bundles over spheres. We then follow Milnor's original construction as in [Mil56]. Finally, we introduce one of the two main methods of explicitly constructing exotic spheres.

6.1 Clutching Functions

We introduce a simple way to generate fibre bundles over an n-sphere using *clutching functions.* Write S^n as the union of the northern and southern hemispheres D^n_N, D^n_S. Note the intersection is the equator, $D^n_N \cap D^n_S = S^{n-1}$. Let $D^n_N \times \mathbb{R}^k$ be a trivial $\mathbb{R}^k$ bundle with structure group G over D_N. Similarly let $D^n_S \times \mathbb{R}^k$ be a trivial bundle over D_S. Furthermore, let $f : S^{n-1} \to G$ be a map from the equator into G. Define a new bundle over the whole sphere as follows. Let

$$E_f = \left(D^n_N \times \mathbb{R}^k\right) \sqcup \left(D^n_S \times \mathbb{R}^k\right) / \sim,$$

where $(x, v) \in \partial D^n_S \times \mathbb{R}^k$ is identified with $(x, f(x)v) \in \partial D^n_N \times \mathbb{R}^k$. This forms an $\mathbb{R}^k$-bundle over S^n, $\pi : E_f \to S^n$, with structure group G. The map f is called a *clutching function* for E_f.

Lemma 6.1 *If two functions f and g are homotopic then $E_f \cong E_g$.*

Proof. Let $F : S^{n-1} \times [0, 1] \to G$ be a homotopy from f to g. By a similar construction as above we may define a fibre bundle $E_F \to S^n \times [0, 1]$ that restricts to E_f on $S^n \times \{0\}$ and to E_g on $S^n \times \{1\}$. It follows that E_f is isomorphic to E_g, see [Hat] Proposition 1.7 for a proof of this fact. □

The resulting bundle will then only depend on the homotopy class of f. As such, we have a correspondence between fibre bundles with structure group G and homotopy classes of maps $S^{n-1} \to G$, that is, elements of $\pi_{n-1}(G)$. Oriented vector bundles are particularly well behaved. Recall that the structure group of an oriented rank k vector bundle is $SO(k)$, hence a clutching function will define a homotopy class of maps in $\pi_{n-1}(SO(k))$. Denote by $\mathrm{Vect}^k_+(X)$ the set of isomorphism classes of oriented rank k vector bundles over X.

Theorem 6.2 *The map $f \mapsto E_f$ defines a one-to-one correspondence*

$$\mathrm{Vect}^k_+(S^n) \xleftrightarrow{1-1} [S^{n-1}, SO(k)] = \pi_{n-1}(SO(k)).$$

Sketch proof. Define an inverse map as follows. Let ξ be an oriented rank k bundle, $\pi : E \to S^n$. Denote by E_N, E_S the restrictions of the bundle over D_N and D_S respectively. As D^n is contractible, these restrictions are trivial bundles. Let $h_N : E_N \to D^n_N \times \mathbb{R}^k$ be a trivialisation, similarly for h_S. Then $h_N \circ h_S^{-1}$ will be a map $S^{n-1} \to SO(k)$ by construction. Define

$$\varphi(E) = h_N \circ h_S^{-1} \in \pi_{n-1}(SO(k)).$$

We claim this is the required inverse. Clearly $\varphi(E)$ is the clutching function for E, we need only check the map is well defined. Note, as D^n is contractible, any choice of maps h_N, h_S differ by a map homotopic to a constant. To finish, use that $SO(k)$ is path connected to conclude that $h_N \circ h_S^{-1}$ is unique up to homotopy. For details see [Hat]. $\square$

Example 6.3 Consider S^1 as a subset of $\mathbb{C}$ and let $f : S^1 \to SO(2)$ be given by

$$f(u)v = u \cdot v.$$

The resulting bundle from this clutching function, E_f, is the tautological line bundle over $\mathbb{C}P^1 \cong S^2$. That is, $E_f \cong \gamma^1_{1\mathbb{C}}$. This bundle generates $\mathrm{Vect}^2_+(S^2)$ as

$$\mathrm{Vect}^2_+(S^2) \cong \pi_1(SO(2)) = \pi_1(S^1) \cong \mathbb{Z}.$$

Note that the sphere bundle of $\gamma^1_{1\mathbb{C}}$ is the Hopf fibration of Example 2.11.

Similarly, the tautological line bundle over $\mathbb{R}P^1 \cong S^1$, γ^1_1, may be realised as the bundle resulting from the clutching function $f : S^0 \to O(1)$ given by $f(u)v = uv$.

Finally, consider S^3 as a subset of the quaternions, $\mathbb{H}$. Then define the clutching function $f : S^3 \to SO(4)$ as $f(u)v = u \cdot v$. The bundle E_f is then the tautological line bundle over the quaternion projective plane, $\mathbb{H}P^1$.

6.2 Milnor's Construction

We will construct a family of homotopy 7-spheres out of 3-sphere bundles over S^4, and then use Milnor's invariant to prove that some of these spheres are exotic. We construct these bundles as follows. Recall we defined sphere bundles in terms of oriented vector bundles with inner product. By Theorem 6.2, oriented $\mathbb{R}^3$-bundles with inner product over S^4 are in bijection with elements of $\pi_3(SO(4))$. The space $SO(4)$ is the Lie group of rotations in $\mathbb{R}^4$. We may decompose this space as

$$SO(4) \cong SO(3) \times SU(2) \cong \mathbb{R}P^3 \times S^3.$$

Hence

$$\pi_3(SO(4)) \cong \pi_3(\mathbb{R}P^3) \times \pi_3(S^3) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

That is, orientable 3-sphere bundles over S^4 are in bijection with pairs $(h, j) \in \mathbb{Z} \oplus \mathbb{Z}$. To explicitly represent these bundles, recall that, using quaternion multiplication, the map $f(u)v = u \cdot v$ generates the tautological line bundle over $\mathbb{H}P^1 \cong S^4$. Similarly, define the map $g : S^3 \to SO(4)$ by $g(u)v = vu$. The maps f and g define a basis of $\pi_3(SO(4))$. We may then represent any such vector bundle by a clutching function

$$f_{h,j}(u)v = u^h v u^j.$$

Let $\xi_{h,j}$ be the vector bundle with clutching function $f_{h,j}$.

We now want to compute Milnor's invariant for the total space of these bundles. Let α denote a generator of $H^4(S^4)$.

Lemma 6.4 *The first Pontrjagin class of $\xi_{h,j}$, $p_1(\xi_{h,j})$, is given by*

$$p_1(\xi_{h,j}) = \pm 2(h-j)\alpha.$$

Proof. The first Pontrjagin class is a linear function of h and j taking values in $H^4(S^4)$. That is,

$$p_1(\xi_{ah,bj}) = (ah+bj)\alpha.$$

Pontrjagin classes are independent of the orientation of the fibre. However, $\xi_{h,j}$ with reversed orientation is the bundle $\xi_{-j,-h}$, and so

$$p_1(\xi_{-j,-h}) = (-aj-bh)\alpha = p_1(\xi_{h,j}) = (ah+bj)\alpha.$$

Hence, $p_1(\xi_{h,j}) = \ell(h-j)\alpha$, for some constant ℓ.

Now to compute ℓ, consider the case $h=1, j=0$. Then

$$f_{1,0}(u)v = u \cdot v,$$

and so $\xi_{1,0}$ is the tautological line bundle over $\mathbb{H}P^1$. This is a complex vector bundle, hence we may apply Corollary 4.18 to compute

$$\ell\alpha = p_1(\xi_{0,1}) = c_1(\xi_{0,1})^2 - 2c_2(\xi_{0,1}).$$

By Example 4.12, $c_1(\xi_{0,1}) = 0$ and $c_2(\xi_{0,1}) = \pm\alpha$. Hence $\ell = \pm 2$. $\square$

Let k be an odd integer and suppose $h+j=1$, $h-j=k$. We define the 7-manifold M_k to be the total space of the sphere bundle for this $\xi_{h,j}$. We may define Pontrjagin classes for a sphere bundle as the corresponding Pontrjagin class for the underlying vector bundle. Then, by the above lemma,

$$p_1(M_k) = p_1(\xi_{h,j}) = \pm 2k\alpha.$$

Lemma 6.5 *For M_k as above, Milnor's λ invariant is given by*

$$\lambda(M_k^7) = \frac{3}{7} + \frac{4}{7}k^2 \mod 1.$$

Proof. Recall the definition of the invariant in dimension 7. Let B_k be a smooth compact manifold such that $\partial B_k = M_k$. Then

$$\lambda(M_k^7) = \frac{45}{7}\left(\sigma(B_k^8) + \frac{1}{45}p_1^2[B_k^8]\right) \mod 1.$$

We then must construct such a manifold B_k and compute its signature and first Pontrjagin class. Construct B_k as follows.

Replace each fibre of M_k with a 4-disk to obtain a 4-disk bundle

$$\rho_k : B_k^8 \to S^4.$$

The total space, B_k, is a smooth manifold with boundary M_k. Furthermore, $H^4(B_k)$ is generated by $\beta = \rho_k^*(\alpha)$. Choose orientations for M_k, B_k such that

$$\sigma(B_k) = \langle (i^{-1}\beta)^2, \mu_{B_k} \rangle = 1.$$

Embed the manifold B_k in a Euclidean space $\mathbb{R}^N$. The tangent bundle of B_k, TB_k, is then the Whitney sum of the bundle of vectors tangent to the fibre induced from TM_k and the bundle of vectors normal to the fibre. However, the normal bundle is trivial as it is induced from TS^4. By the Whitney product theorem,

$$p_1(B_k) = p_1(B_K|_{M_k} \oplus S^4) = \rho_k^*(p_1(M_k)) = \pm 2k\beta.$$

Using this, we compute

$$p_1^2[B] = \langle (i^{-1}(\pm 2k\beta))^2, \mu_{B_k} \rangle = 4k^2 \langle i^{-1}(\beta)^2, \mu_{B_k} \rangle = 4k^2.$$

Combining our results,

$$\begin{aligned} \lambda(M_k^7) &= \frac{45}{7}\left(1 + \frac{4}{45}k^2\right) \mod 1 \\ &= \frac{3}{7} + \frac{4}{7}k^2 \mod 1, \end{aligned}$$

as required. □

Multiplying by 14 we obtain that the invariant of Theorem 4.36 is given by

$$\lambda_7(M_k) = 2q - \sigma = k^2 - 1 \equiv k^2 - 1 \mod 7,$$

as in [Mil56].

In either case, we have that the invariant is non-zero for $k^2 \neq 1 \mod 7$. Then M_k^7 possesses no orientation reversing diffeomorphism, and so it is not diffeomorphic to S^7 for $k^2 \neq 1 \mod 7$.

Corollary 6.6 *The manifold M_k^7 is not diffeomorphic to S^7 when $k^2 \neq 1 \mod 7$.*

We will now prove that M_k^7 is homeomorphic to S^7 using the Reeb sphere theorem. Recall the theorem required a Morse function on M_k with exactly two critical points.

Lemma 6.7 *There is a smooth function, $h : M_k \to \mathbb{R}$, possessing exactly two critical points, both non-degenerate.*

Proof. The main idea is as follows. We use the effect of the clutching construction on the stereographic coordinates of M_k to construct explicit coordinates for M_k. We then define a Morse function in these coordinates.

Denote by (U_N, x_N) the chart given by stereographic projection from the north pole of S^4. Similarly denote by (U_S, x_S) the chart given by stereographic projection from the south pole. Recall the transition map between these two charts is given by

$$x_S = \frac{x_N}{|x_N|^2}.$$

Consider the vector bundle associated to M_k. Let $V_N \cong \mathbb{R}^4 \times \mathbb{R}^4$ be the chart corresponding to the trivialisation of the southern hemisphere of S^4 used in the clutching construction with stereographic coordinates, similarly for V_S. As the sphere with a point removed is contractible, we may extend these charts to S^4 minus the north and south pole respectively. The transition map is then

$$(x_S, v_S) = \left(\frac{x_N}{|x_N|^2}, \frac{x_N^i v_N x_N^j}{|x_N|} \right).$$

This extends the clutching function to $V_N \cap V_S$. Restrict these charts and maps to the total space of M_k.

Define a function, $h : M^k \to \mathbb{R}$, in these coordinates by

$$h_N(x_N, v_N) = \frac{\mathrm{Re}(v_N)}{\sqrt{1+|x_N|^2}};$$
$$h_S(x_S, v_S) = \frac{\mathrm{Re}(x_S v_S^{-1})}{\sqrt{1+|x_S|^2}}.$$

The real part and absolute value of a quaternion is invariant under conjugation, hence, on the intersection of the charts we have $h_N(x_N, v_N) = h_S(x_S, v_S)$. That is, h is well defined on M_k. Note h_S is an increasing function, hence has no critical points. It follows that all critical points of h must be of the form $(0, v_N)$. However, $h_N(0, v_N)$ is just the height function of S^3, which has exactly two critical points at $(0, -1)$ and $(0, 1)$, both non-degenerate. □

By the Reeb sphere theorem we find.

Corollary 6.8 *The manifold M_k^7 is homeomorphic to S^7.*

Combining Corollary 6.6 and Corollary 6.8 we obtain our main result.

Theorem 6.9 *For $k^2 \neq 1 \mod 7$ the manifold M_k^7 is homeomorphic to S^7 but not diffeomorphic to it.*

We have finally completed our first goal of constructing an exotic sphere. However, this leads to many questions. Are there infinitely many exotic 7-spheres? Are these the only possible exotic 7-spheres? Computing $\lambda(M_k)$ for $k = 0, 2, 3$ we get $3/7, 5/7, 4/7$, and so we have at least three exotic 7-spheres. We will construct an infinite collection of exotic spheres in the following section, and then explore how many possible exotic spheres there are in each dimension in chapter 7.

6.3 Plumbing of Disk Bundles

Here we describe a process of obtaining a large number of exotic spheres in various dimensions.

Let $\pi_i : E_i \to M_i^n$ be two n-disk bundles over n-dimensional, smooth, oriented manifolds M_i. Choose a point $x_i \in M_i$ in each manifold. There exist trivialisations of E_i in a neighbourhood U_i of each point. By restricting to a small disk and composing with diffeomorphisms we obtain trivialisations over unit n-disks, D_i. The trivialisations of E_i then provide diffeomorphisms

$$\pi_i^{-1}(D_i) \cong D_i \times D_i^{'}.$$

Now we glue E_1 and E_2 along these neighbourhoods by attaching the fibre of E_1 to the base of E_2 and vice versa. In detail, let $h_\pm : D_1 \to D_2^{'}$, $k_\pm : D_1^{'} \to D_2$ be diffeomorphisms which either preserve or reverse orientation according to sign. We then define the *plumbing* of E_1 and E_2 with sign ± 1 to be

$$E_1 \,\square\, E_2 \ := E_1 \sqcup E_2 / \sim,$$

where $\sim$ is the relation $D_1 \times D_1^{'} \ni (x, y) \sim (k_\pm(y), h_\pm(x)) \in D_2 \times D_2^{'}$. A schematic idea of this construction is given in Figures 19 and 20 below.

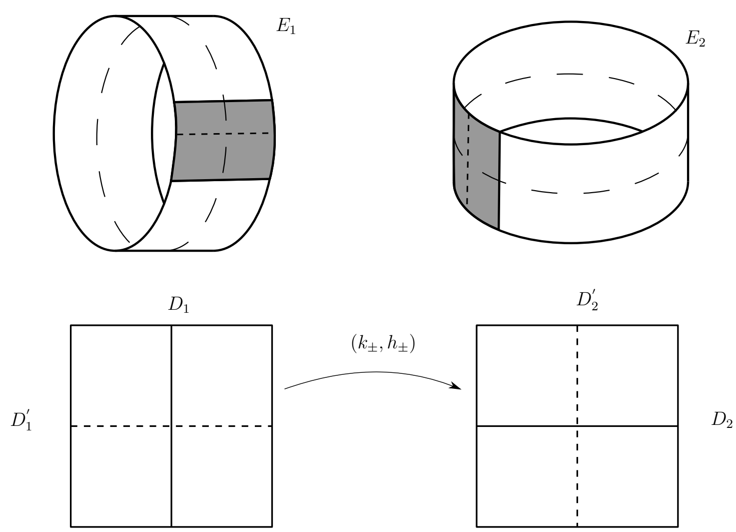


Figure 19: Plumbing of two trivial disk bundles over a circle. The zero sections are given by the long dashed lines and a fibre is shown in each with a dense dashed line. The plumbed neighbourhoods are shaded.

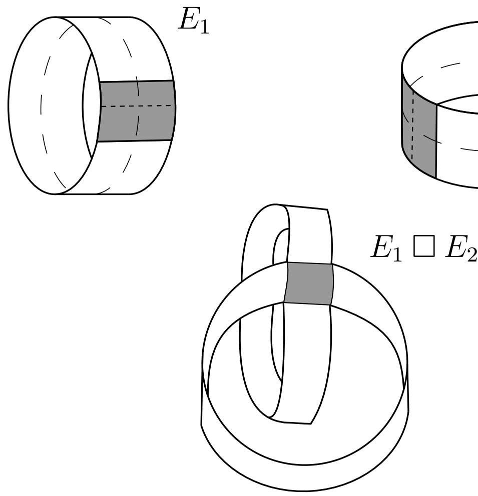
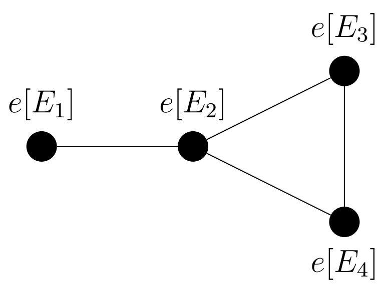


Figure 20: The result of plumbing of two trivial disk bundles over a circle.

The space $E_1 \square E_2$ will not be a smooth manifold by construction, however we can straighten the corners to give $E_1 \square E_2$ the structure of a smooth disk bundle in a unique way, up to diffeomorphism. This straightening is described explicitly in [Kos72].

Note that we may choose several points on M_1, M_2 and plumb on each of these points with sign ± 1. Away from the plumbed points $E_1 \square E_2$ is still locally a product of either M_1 or M_2 with D^n. Hence we can continue to plumb more bundles to $E_1 \square E_2$, choosing new basepoints in the base spaces each time. It is convenient to represent the results of plumbing a number of bundles together in a weighted multigraph. We add a vertex for each bundle E_i and label it with the Euler number of E_i. We then attach an edge between vertex v_i and v_j for each plumbing between the bundles E_i and E_j.

From any graph define the *adjacency matrix* as follows. Let the off-diagonal entries, m_{ij}, be the number of (signed) edges between vertex v_i and vertex v_j, and let the diagonal entries be zero. It will be convenient for our purposes to define a slightly different matrix. Let m_{ij} be the number of edges between v_i and v_j, as before. On the diagonal, let $m_{ii} = e[E_i]$, the Euler number of the bundle E_i.



$$\rightsquigarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Figure 21: An example of a graph arising from plumbing and its associated matrix.

Theorem 6.10 *Let M be a symmetric, integer valued $k \times k$ matrix with even entries on the diagonal. Then, for any $n > 1$, there is a $4n$-dimensional smooth manifold W such that*

1. *W is $(2n-1)$-connected and ∂W is $(2n-2)$-connected.*

2. *$H_{2n}(W)$ is free abelian.*

3. *M is the matrix of the intersection form $H_{2n}(W) \otimes H_{2n}(W) \to \mathbb{Z}$.*

Proof. We begin by constructing a manifold V according to the plumbing prescribed by the matrix M and showing that it has some of the above properties. We then perform surgery on each connected component of V to kill off its fundamental group. Finally, the required manifold W will be obtained by taking the boundary connected sum of these altered connected components.

Let S_i be a $2n$-sphere. The sphere S_i is orientable, and so by Theorem 6.2 we may associate to TS^{2n} an element $\tau \in \pi_{2n-1}(SO(2n))$. Let E_i be the sphere bundle over S^{2n} associated to

$$\frac{1}{2} m_{ii} \tau \in \pi_{2n-1}(SO(2n)).$$

Recall the Euler characteristic of S^{2n} is 2. It follows that the self intersection number of S^{2n} in E_i will be $2m_{ii}/2 = m_{ii}$. That is, the Euler number of E_i is m_{ii}.

Plumb these bundles together by plumbing E_i to E_j in $|m_{ij}|$ points with sign $\operatorname{sgn}(m_{ij})$. The resulting manifold, V, contains as deformation retract the union of the S_i's joined together in $|m_{ij}|$ points.

The union of two $2n$-spheres intersecting in l points is homotopic to

$$S^{2n} \vee S^{2n} \vee_{l-1} S^1,$$

where $\vee_{l-1}$ denotes $l-1$ wedges of circles. As the total space contains the base as deformation retract, it follows that each component of V is homotopic to a wedge of $2n$-spheres and circles.

We will prove that the boundary resulting from plumbing k disk bundles over $2n$-spheres is a union of $(2n-2)$-connected components ∂E_i with $(2n-2)$-connected intersections, hence is $(2n-2)$-connected itself. For a plumbing at one point the boundary is given by

$$\partial(E_i \,\square\, E_j) = \left(\partial E_i - (D_i^{2n} \times S^{2n-1})\right) \cup \left(\partial E_j - (D_j^{2n} \times S^{2n-1})\right).$$

Note $\partial E_i - (D_i^{2n} \times S^{2n-1})$ is homotopic to $\partial E_i - S^{2n-1}$. As S^{2n-1} has codimension $2n$, the inclusion

$$\pi_k(\partial E_i - (D_i^{2n} \times S^{2n-1})) \to \pi_k(\partial E_i)$$

is an isomorphism for $k \leq 2n-2$. It then follows that ∂E_i is $(2n-2)$-connected. The intersection of E_i and E_j is given by

$$\left(\partial E_i - (D_i^{2n} \times S^{2n-1})\right) \cap \left(\partial E_j - (D_j^{2n} \times S^{2n-1})\right) = S^{2n-1} \times S^{2n-1},$$

which is $(2n-2)$-connected. Hence $\partial(E_1 \,\square \ldots \square\, E_k)$ is the union of $(2n-2)$-connected components along $(2n-2)$-connected intersections, as required.

Let $n > 1$ and consider $E_1 \square \ldots \square E_k$. Each component C of $E_1 \square \ldots \square E_k$ is the union of simply connected spaces along simply connected intersections, denote these intersections by A_j. Similarly for ∂C, denote the corresponding intersections by B_j. Therefore $\pi_1(C)$ and $\pi_1(\partial C)$ are free by the Seifert-Van Kampen theorem. Furthermore, each $B_j \in \partial C$ is a copy of $S^{2n-1} \times S^{2n-1}$. This space is the boundary of a copy of $D^{2n} \times D^{2n}$ which corresponds to an $A_k \subset C$. Similarly, each A_j has boundary corresponding to a B_l. That is, the components of the intersections for the boundary are in one to one correspondence with components of the intersections for C. Hence,

$$\pi_1(C) \cong \pi_1(\partial C).$$

Similarly, as C is the union of $(2n-2)$-connected components along $(2n-2)$-connected intersections, applying the Mayer-Vietoris sequence for each component we obtain

$$H_i(\partial C) \cong H_i(C) = 0, \quad 1 < i < 2n-1.$$

Let $S_1 \subset \partial C$ represent a free generator of $\pi_1(\partial C)$. We can kill off[4] this element by attaching a copy of D^2 along S_1^1. Note the resulting space, C_1, is homotopic to $C \cup_{S_1} D^2$. Furthermore, as $\pi_1(C)$ is free,

$$\begin{aligned} \pi_1(C_1) &\cong \pi_1(C)/[S_1]; \\ H_i(C_1) &\cong H_i(C), \quad i \neq 1; \\ H_i(\partial C_1) &\cong H_i(\partial C), \quad 1 < i < 4n-2. \end{aligned}$$

As S_1 is a free generator of the free group $\pi_1(\partial C) \cong \pi_1(C)$, it follows that $\pi_1(\partial C_1) \cong \pi_1(C_1)$ are both free groups on one less generator. We may continue this process for each generator $[S_i]$ of $\pi_1(C)$. Denote the resulting space by C_l. Then

$$\begin{aligned} \pi_1(C_l) &\cong \pi_1(\partial C_l) = 0; \\ H_i(C_l) &\cong H_i(C), \quad i \neq 1; \\ H_i(\partial C_l) &\cong H_i(\partial C), \quad 1 < i < 4n-2. \end{aligned}$$

Define W to be the boundary connected sum of the C_l's. Then $V \subset W$, W is connected, simply connected, and $H_i(W) \cong H_i(V)$, $i \neq 1$. We proved $H_i(V) = 0$ for $1 < i < 2n-1$. Hence, W is $(2n-1)$-connected. Similarly ∂W is $(2n-2)$-connected.

Finally, as $V \subset W$, we have a set of embedded spheres $S_i^{2n} \in W$ which generate a basis for $H_{2n}(V) \cong H_{2n}(W)$. Hence, by construction, the intersection matrix of W is M. Further details can be found in [Bro63].

$\square$

Theorem 6.11 *Let the manifold W^{4n} come from the plumbing prescribed by a symmetric, integer valued matrix M with even entries on the diagonal. Then ∂W is a homotopy sphere if and only if* $\det M = \pm 1$.

[4] This does not work in general, there are certain obstructions to performing surgery to kill of homotopy groups, see [Bro63] for exact conditions.

Proof. Consider the following part of the long exact sequence of the pair $(W, \partial W)$.

$$\cdots \longrightarrow H_{2n}(\partial W) \xrightarrow{i_*} H_{2n}(W) \xrightarrow{j_*} H_{2n}(W, \partial W) \xrightarrow{\partial} H_{2n-1}(\partial W) \longrightarrow \cdots$$

The manifold W is $(2n-1)$-connected, hence by the Hurewicz theorem we have

$$H_i(W) = 0, \quad i \leq 2n-1.$$

This gives a zero on the right. By Poincaré duality with boundary,

$$H_{2n+1}(W, \partial W) \cong H^{2n-1}(W) = 0,$$

which gives a zero on the left. The sequence then reduces to

$$0 \longrightarrow H_{2n}(\partial W) \xrightarrow{i_*} H_{2n}(W) \xrightarrow{j_*} H_{2n}(W, \partial W) \xrightarrow{\partial} H_{2n-1}(\partial W) \longrightarrow 0$$

$H_{2n}(W)$ and $H_{2n}(W, \partial W)$ are free, hence the intersection form $H_{2n}(W) \otimes H_{2n}(W, \partial W) \to \mathbb{Z}$ is non-singular. Therefore the map

$$\alpha : H_{2n}(W) \otimes H_{2n}(W) \to \mathbb{Z}$$

is injective if and only if $\ker j_* = 0$ and is surjective if and only if $\partial = 0$. By exactness, $\ker j_* = \operatorname{Im} i_*$. However, i_* is injective. It follows that α is injective if and only if $H_{2n}(\partial W) = 0$. Similarly, ∂ is surjective, and so α is surjective if and only if $H_{2n-1}(\partial W) = 0$.

All together we have α is an isomorphism if and only if

$$H_{2n}(\partial W) = H_{2n-1}(\partial W) = 0. \tag{*}$$

However, α is an isomorphism exactly when $\det M = \pm 1$.

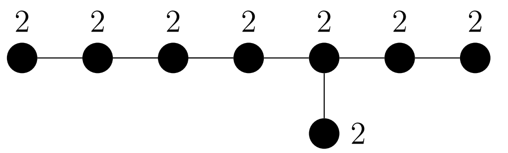
Recall ∂W is a closed, $(2n-2)$-connected manifold, hence simply connected. Now if $(*)$ holds then we have

$$H_k(\partial W) = \begin{cases} \mathbb{Z} & k = 0, 4n-1; \\ 0 & \text{else.} \end{cases}$$

That is, ∂W has the homology of a sphere. Then ∂W is a simply connected homology sphere, hence a homotopy sphere by the Hurewicz theorem. □

We then have ∂W is a homotopy $(4n-1)$-sphere when M has determinant ± 1. Then by the Generalised Poincaré Conjecture, Theorem 7.9 of the following chapter, ∂W is homeomorphic to S^{4n-1} if and only if M has determinant ± 1.

Example 6.12 Consider the manifold W obtained from the graph of the Dynkin diagram of E_8.



$$\rightsquigarrow \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

We diagonalise the matrix associated to W to obtain the matrix

$$\mathrm{diag}\left(2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{7}{10}, \frac{4}{7}, \frac{1}{4}, 2\right).$$

This matrix has determinant 1 and, as all entries are positive, its signature is 8. Therefore W has signature 8 and ∂W is a homotopy sphere.

Now we have a homotopy sphere, $\Sigma^{4n-1} = \partial W$, and a $(2n-1)$-connected manifold, W, which bounds Σ and has signature 8. We want to compute

$$\lambda(\Sigma) = \frac{1}{s_n}\left(\sigma(W) - L_n(i^{-1}p_1(W), \ldots, i^{-1}p_{n-1}(W))\right) \mod 1$$

Note that as W is $(2n-1)$-connected all induced Pontrjagin classes in this expression other than $i^{-1}p_{n/2}$ are zero by the Hurewicz theorem and Poincaré duality.

We claim W is stably parallelisable. We proceed by proving the deformation retract V of W from Theorem 6.10 is stably parallelisable, and hence W is. Note each sphere bundles in the construction of W is equivalent. As such, consider the associated vector bundle to E_1. As the Euler number of this bundle is 2, we may take TS^{2n} as representative of this bundle. Recall, TS^{2n} was shown to be stably trivial in Example 2.9. Then V is the union of stably parallelisable manifolds, and so is stably parallelisable itself. Hence W is also stably parallelisable. Recall that Pontrjagin classes of a stably parallelisable manifold are zero, and so $i^{-1}p_{n/2} = 0$.

All together we have

$$\lambda(\Sigma) = 8/s_n \mod 1.$$

The coefficient s_n is given by

$$s_n = \frac{2^{2n}(2^{n-1}-1)B_n}{(2n)!}$$

Then for $n = 2, 3, 4$ we have $\lambda(\Sigma^{4n-1}) = \frac{3}{7}, \frac{8}{31}, \frac{8}{127}$. It follows that Σ is exotic in these dimensions. We have confirmed numerically that s_n does not divide 8 for $n < 12000$, however, we have not been able to directly prove that this holds for all n.[5] Later results in surgery

[5] Milnor himself was not sure whether this was true in [Mil59b]

theory guarantee that Σ^{4n-1} is exotic for all $n > 2$. This implies that s_n never divides 8, though we are not sure if this identity has any application outside of this. These spheres are known as the *Milnor spheres.* Milnor constructed them and proved some were exotic spheres through similar methods in [Mil59a].

Example 6.13 Let k be odd and consider the $2k$-dimensional manifold, $K(2k)$, obtained from plumbing k-spheres according to the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The boundary $\partial K(2k)$ is known as the *Kervaire sphere.* For $k = 1, 3, 7$ it is diffeomorphic to S^{2k-1}. In [Ker60], Kervaire proved that $\partial K(10)$ is not diffeomorphic to S^9, and used this to show $K(10)$ does not admit any smooth structure. Determining for which k the homotopy sphere $\partial K(2k)$ is diffeomorphic to S^{2k-1} turns out to be very important in counting exotic spheres, we will discuss this in the following chapter. The strongest result for these manifolds was proven by Browder in [Bro69], where he proved $\partial K(2k)$ is not diffeomorphic to S^{2k-1} unless $k = 2^j - 1$.

We have now established exotic spheres exist in many dimensions and constructed a number of examples. We now move on to the question of exactly how many smooth structures a topological sphere can have.

7 Groups of Homotopy Spheres

Having established that exotic spheres exist and constructed a number of examples of them, the next natural question is to ask how many distinct smooth structures on spheres there can be in each dimension, and whether they have any additional structure. Milnor and Kervaire resolved the question for spheres of almost all dimensions in [KM63]. They proved the surprising result that there are a finite number of distinct smooth structures on spheres in all dimensions other than three and four[6]. We introduce a number of structures which are closely related to the set of smooth structures of S^n. We also define the notion of h-cobordism and and state a number of powerful theorems. Finally, we sketch the main ideas of Milnor and Kervaire's paper and state the main results on the classification of homotopy spheres.

7.1 Homotopy Spheres and Connected Sum

Consider the monoid of closed, connected, oriented n-manifolds under connected sum. Denote by A^n the group of invertible elements of this monoid. We claim all elements of this group are n-dimensional homotopy spheres.

Lemma 7.1 *Let M, N be closed, connected, oriented n-manifolds. Then*

1. *For $0 < k < n$, $H_k(M \# N) \cong H_k(M) \oplus H_k(N)$.*
2. *For $n \geq 3$, $\pi_1(M \# N) \cong \pi_1(M) * \pi_1(N)$, where $*$ is the free product.*

Proof. Let $M^{'}$ be the image of $M - \{pt.\}$ in $M \# N$, similarly for $N^{'}$. Note $M^{'} \cap N^{'} \cong S^{n-1}$. Hence, the Mayer-Vietoris sequence of the pair $(M^{'}, N^{'})$ gives the first claim. Similarly, the Seifert-Van Kampen theorem applied to $(M^{'}, N^{'})$ gives the second claim. $\square$

Corollary 7.2 *Let $n \geq 3$. The connected sum of two n-manifolds is a homotopy sphere if and only if they are both homotopy spheres.*

As S^n is the identity of the monoid, every invertible element must be a homotopy sphere. In dimensions $n \geq 5$, we have that every invertible element of A^n must be a topological sphere by the generalised Poincaré conjecture. Hence each element of this group must carry a distinct smooth structure on the sphere. The invertible elements are actually topological spheres in all dimensions, though proving this requires more machinery.

Lemma 7.3 *The connected sum of two n-manifolds is homeomorphic to a sphere if and only if they are both homeomorphic to a sphere.*

We prove the following lemma for use later.

[6]The dimension three case has since been resolved with the proof of the Poincaré conjecture. As the topological and smooth case coincide in dimension three, there is a unique smooth structure on the 3-sphere.

Lemma 7.4 *Let Σ be a homotopy n-sphere. The connected sum $\Sigma \# (-\Sigma)$ bounds a contractible manifold.*

Proof. Denote by Σ' the manifold Σ with the interior of a disk removed. Then $\Sigma \# (-\Sigma)$ can be considered as the quotient space $\Sigma' \times \{0,1\}/\sim$, where $(x,0) \sim (x,1)$ for all $x \in \partial\Sigma'$. This is the boundary of $\Sigma' \times [0,1]$, which contains Σ' as deformation retract. However, Σ' is a homotopy sphere with the interior of a disk removed, and so homeomorphic to the n-disk, which is contractible. Hence, $\Sigma \# (-\Sigma)$ bounds a contractible manifold. □

We will show later that for $n \geq 5$ all topological spheres are invertible under connected sum. However, this will require more advanced machinery than just connected sum.

7.2 The h-Cobordism Theorem

We define a *h-cobordism* to be a cobordism $\{M, N; W\}$ such that the inclusions $M \hookrightarrow W$, $N \hookrightarrow W$ are homotopy equivalences. This is equivalent to M and N being deformation retracts of W. This is considerably stronger than standard cobordism. Directly checking that a given cobordism is a h-cobordism is difficult. We instead use the following lemma.

Lemma 7.5 *Let $\{M, N; W\}$ be an oriented cobordism such that M, N, W are connected and simply connected. Then $\{M, N; W\}$ is a h-cobordism is and only if $H_*(W, M) = 0$.*

Proof. Suppose $H_*(W, M) = 0$. Then, by the Hurewicz theorem, the inclusion $M \hookrightarrow W$ is a homotopy equivalence. Conversely, suppose the inclusions $M \hookrightarrow W$, $N \hookrightarrow W$ are homotopy equivalences. A version of Poincaré duality for cobordism states that $H_i(W, M) \cong H^{m-i}(W, N)$, where m is the dimension of W [Kos72] VII 5.1. The claim then follows by the Hurewicz theorem. □

From this it follows that h-cobordism is an equivalence relation. The main use of the notion of h-cobordism is the following result of Smale, which he proved in [Sma62], following his work on the Poincaré conjecture.

Theorem 7.6 (h-Cobordism Theorem) *Let $n \geq 5$, and M^n, N^n, W^{n+1} be connected and simply connected manifolds such that $\{M, N; W\}$ a h-cobordism between M and N. Then W is diffeomorphic to $M \times [0,1]$.*

This theorem has a number of important corollaries.

Corollary 7.7 *Let W^n be a contractible, simply connected manifold with simply connected boundary, $n \geq 6$. The W is diffeomorphic to D^n.*

It follows that there is a unique smooth structure on D^n for $n \geq 6$. A weaker theorem holds for $n = 5$.

Corollary 7.8 *Let W^5 be a contractible manifold bounded by S^4. Then W is diffeomorphic to D^5.*

For proofs of both results see [Kos72]. The h-cobordism theorem implies the generalised Poincaré conjecture in dimensions $n \geq 5$.

Corollary 7.9 (Generalised Poincaré Conjecture) *Let Σ^n be a homotopy sphere, $n \geq 5$. Then Σ is homeomorphic to S^n.*

Proof. By Lemma 7.4, $\Sigma \# (-\Sigma)$ bounds a contractible manifold of dimension $n \geq 6$. Hence, by Corollary 7.7, $\Sigma \# (-\Sigma)$ bounds D^{n+1}, and so $\Sigma \# (-\Sigma)$ is homeomorphic to S^n. Finally, by Lemma 7.3, Σ is homeomorphic to S^n. $\square$

7.3 Diffeomorphisms of Spheres

We take a small detour to describe a group closely connected to the set of homotopy n-spheres. Let M be a smooth oriented manifold. Denote the set of orientation preserving diffeomorphisms of M by $\mathrm{Diff}(M)$. This is both a topological space and a group under composition of diffeomorphisms. It is usually a very large and non-abelian group. However, a certain quotient of $\mathrm{Diff}(S^{n-1})$ is very well behaved.

We will need the following result of Palais, [Pal60].

Theorem 7.10 (Disk Theorem) *Let M^n be a connected manifold and $f, g : D^k \to M$ be two embeddings of the disk into the interior M. If $n = k$ then assume f and g are oriented the same way. Then f is isotopic to g. Furthermore, if $f = g$ on a disk $D^l \subset D^k$ then the isotopy can be made stationary on this disk.*

Denote by $\mathrm{Diff}_0(S^{n-1})$ the subgroup of diffeomorphisms isotopic to the identity.

Lemma 7.11 *The subgroup $\mathrm{Diff}_0(S^{n-1})$ contains the commutator subgroup of $\mathrm{Diff}(S^{n-1})$.*

Proof. Let $f, g \in \mathrm{Diff}(S^{n-1})$. Let D_N, D_S be the northern and southern hemispheres of S^{n-1} respectively. We will construct diffeomorphisms f_N, g_S isotopic to f and g respectively such that f_N is the identity on D_N and g_S is the identity on D_S.

By the disk theorem, $f|_{D_N}$ is isotopic to the inclusion of D_N into S^{n-1}. Extend this isotopy over S^{n-1} to obtain an isotopy between f and a diffeomorphism f_N. By construction, f_N is the identity on D_N. We similarly construct a map g_S which is the identity on D_S.

Now f_N and g_S clearly commute, hence their commutator is the identity map. However, as f and g are respectively isotopic to f_N and g_S, it follows that the commutator of f and g is isotopic to the commutator of f_N and g_S, and hence to the identity. $\square$

As $\text{Diff}_0(S^{n-1})$ contains the commutator subgroup, the quotient $\text{Diff}(S^{n-1})/\text{Diff}_0(S^{n-1})$ is abelian.

Consider the diffeomorphisms of S^{n-1} which can be extended over D^n. That is, the image

$$\partial : \text{Diff}(D^n) \to \text{Diff}(S^{n-1}).$$

By the disk theorem, $\text{Diff}_0(S^{n-1}) \subset \partial\text{Diff}(D^n)$. Hence the quotient

$$\Gamma^n = \text{Diff}(S^{n-1})/\partial\text{Diff}(D^n)$$

is an abelian group.

Let $f \in \text{Diff}(S^{n-1})$. Define $\Sigma(f)$ to be

$$\Sigma(f) = D^n \cup_f (-D^n).$$

This is a smooth manifold by the gluing lemma. Furthermore, this manifold is obtained from gluing together two n-disks by a diffeomorphism of their common boundary. As any homeomorphism of S^{n-1} is isotopic to the identity, this manifold is homeomorphic to an n-sphere. However $\Sigma(f)$ will be diffeomorphic to S^n if and only if f can be extended over D^n, that is, $f \equiv 0 \in \Gamma^n$. Note that any manifold obtainable from this construction can be given an atlas with exactly two charts.

These groups are closely connected with A^n, the group of invertible homotopy spheres under connected sum.

Lemma 7.12 *Define a map $G : \Gamma^n \to A^n$ by*

$$G : f \to \Sigma(f).$$

Then, for $n \geq 6$.

1. *Let $f, g \in \Gamma^n$, then $\Sigma(fg) \overset{\text{diff}}{\cong} \Sigma(f) \# \Sigma(g)$.*
2. *The map G is a well defined homomorphism into A^n.*
3. *The map G is injective.*

Proof. Removing the interior of a disk from $\Sigma(f)$ leaves an n-disk, D_f, similarly for $\Sigma(g)$. Hence, we may view the connected sum $\Sigma(f) \# \Sigma(g)$ as

$$\Sigma(f) \# \Sigma(g) \overset{\text{diff}}{\cong} D_f \sqcup_f \left(S^{n-1} \times I\right) \sqcup_g D_g,$$

where we identify $\partial D^n \times \{0\}$ with ∂D_f via f and similarly $\partial D^n \times \{1\}$ with ∂D_g via g. Note that $\Sigma(fg)$ is diffeomorphic to $\Sigma(fg) \# \Sigma(Id)$, so we can similarly write

$$\Sigma(fg) \overset{\text{diff}}{\cong} D_{fg} \sqcup_{fg} \left(S^{n-1} \times I\right) \sqcup_{Id} D_{Id}.$$

We will construct a diffeomorphism between these two manifolds. By Corollary 7.7, D_f is diffeomorphic to D_{fg} and D_g is diffeomorphic to D_{Id}. The two copies of the cylinder are of course diffeomorphic. By the disk theorem, we may construct an isotopy between these diffeomorphisms to ones which match on the common boundaries. Hence the compositions of these maps form a diffeomorphism between the two manifolds, as required.

We show that G actually maps into A^n as follows. Let $f \in \Gamma^n$. Then the map f^{-1} provides an inverse to $\Sigma(f)$ as

$$\Sigma(f) \# \Sigma(f^{-1}) \overset{\text{diff}}{\cong} \Sigma(ff^{-1}) \overset{\text{diff}}{\cong} S^n.$$

Hence, $\Sigma(f)$ is an invertible homotopy sphere, that is, $\Sigma(f) \in A^n$.

Now we show G is well defined. Let $f : S^{n-1} \to S^{n-1}$ be a representative of the identity element of Γ^n. This map can then be extended to a diffeomorphism $F : D^n \to D^n$. We claim $\Sigma(f) = S^n$. Denote the two disks in the construction of $\Sigma(f)$ by D_1 and D_2. The restriction $F|_{D_1}$ is an embedding of the disk in S^n. Hence, by the disk theorem, it is isotopic to a map sending D_1 to the southern hemisphere of S^n. As such, assume F sends D_1 to the southern hemisphere of S^n. Then F sends the common boundary of D_1 and D_2 to the equator of S^n by f. Extend f over the disk D_2 to a diffeomorphism from D_2 to the northern hemisphere of S^n. These maps match on the boundary by construction, hence $\Sigma(f) = S^n$. Combining this and the first part of the lemma gives that G is a well defined homomorphism.

Finally, we show G is injective. Let $f \in \ker G$, that is, $\Sigma(f) \overset{\text{diff}}{\cong} S^n$. We show that f may be extended to a diffeomorphism of D^n, and hence $f \sim Id \in \Gamma^n$. Let F be a diffeomorphism between $\Sigma(f)$ and S^n. As above, let F map D_1 to the southern hemisphere of S^n. Then F must map the common boundary of D_1 and D_2 to the equator of S^n. It follows that $F|_{D_2}$ is a diffeomorphism onto the northern hemisphere of S^n extending f, as required. $\square$

Note this theorem actually holds for all n, however the proof of the first claim must be altered.

7.4 The h-Cobordism Group

As with ordinary cobordism, equivalence classes of manifolds up to h-cobordism form a group. These groups will connect both A^n, Γ^n, and the set of homotopy n-spheres.

Theorem 7.13 *Let $n \geq 3$. Consider the set of h-cobordism equivalence classes of simply connected, closed, oriented n-manifolds. This set forms a commutative monoid under connected sum. The identity element is given by the class of manifolds which bound a contractible manifold. Furthermore, the group of invertible elements, denoted Θ_n, consists of homotopy spheres.*

We will prove this in a series of lemmas. First we prove that connected sum is well defined on h-cobordism classes.

Lemma 7.14 *Suppose M_1 is h-cobordant to M_2. Then $M_1 \# N$ is h-cobordant to $M_2 \# N$.*

Sketch proof. Let $\{M_1, M_2; W\}$ be a h-cobordism between M_1 and M_2. Consider the h-cobordism between N and itself, $\{N, -N; N \times I\}$. Choose paths L_1, L_2 in W and $N \times I$ respectively such that they have one endpoint in each boundary component. The manifolds W and $N \times I$ can be "pasted" along these paths to form a simply connected manifold W' by a construction similar to a connected sum; see [Kos72] VI.4. This pasted manifold W' will have boundary $(M_1 \# N) \sqcup (-M_2 \# N)$, and so will be a cobordism between $M_1 \# N$ and $M_2 \# N$. To show it is a h-cobordism, we must only check $H_*(W', M_1 \# N) = 0$. However, this follows from Lemma 7.5 applied to W and $N \times I$. □

The following lemma states that S^n represents the class of simply connected manifolds which bound a contractible manifold. Note the simple connectivity assumption is essential here.

Lemma 7.15 *A manifold M^n is h-cobordant to S^n if and only if it bounds a contractible manifold.*

Proof. Suppose $M = \partial W^{n+1}$. Remove the interior of an $(n+1)$-disk from W to obtain the manifold

$$W' = W - \mathrm{int}(D^{n+1}).$$

The boundary of W' is then $M \sqcup S^n$. Hence $\{M, S^n; W'\}$ is a h-cobordism between M and S^n.

Conversely, let $\{M, S^n; W\}$ be a h-cobordism between M and S^n. The n-sphere is h-cobordant to the empty set through $\{S^n, \emptyset; D^{n+1}\}$. Hence,

$$\{M, \emptyset; W \cup_{id_{S^n}} D^{n+1}\}$$

is a h-cobordism between M and $\emptyset$. Then $(W \cup_{id_{S^n}} D^{n+1})$ is a contractible manifold with M as boundary, as required. □

Finally, we show that M is invertible if and only if M is a homotopy sphere.

Lemma 7.16 *There exists a manifold N such that $M \# N$ bounds a contractible manifold if and only if M is a homotopy sphere.*

Proof. Assume there exists a manifold N such that $M \# N$ bounds a contractible manifold. By Lemma 7.15, $M \# N$ is h-cobordant to S^n. Then, by Corollary 7.2, both M and N are h-cobordant to homotopy spheres.

Conversely, let M be a homotopy sphere. By Lemma 7.4, $M \# (-M)$ bounds a contractible manifold. □

With this, the proof of Theorem 7.13 is complete. In summary, for $n \geq 3$ we have a group, Θ_n, consisting of h-cobordism classes of homotopy spheres. The identity element is represented by the standard sphere S^n and the group operation is connected sum. By the generalised Poincaré conjecture, this group can be identified with the group of smooth structures on the topological n-sphere for $n \geq 5$.

Recall the group A^n was defined to be the group of invertible smooth structures on the topological n-sphere under connected sum. The map taking each element of A^n to its equivalence class in Θ_n is a homomorphism. Recall also we had an injective map $G : \Gamma^n \to A^n$. The following theorem links each of these groups.

Theorem 7.17 *Let $n \geq 5$. The homomorphisms $G : \Gamma^n \to A^n$ and $i : A^n \to \Theta_n$ are isomorphisms.*

Proof. Consider $\ker i$. This is the set of homotopy spheres which bound contractible manifolds. However, by Corollary 7.7, the only such homotopy sphere is S^n, hence i is injective for $n \geq 5$. We proved G is injective in Lemma 7.12.

We then need only show the composition $\Gamma^n \to A^n \to \Theta_n$ is surjective. Let $\Sigma \in \Theta_n$ represent a homotopy sphere and consider an n-disk D_1 embedded in Σ. Denote the complement of D_1 by D_2. That is,

$$D_2 = \Sigma - D_1.$$

Note D_2 is a contractible manifold with boundary diffeomorphic to S^{n-1}. Hence, if $n \geq 6$, we can use Corollary 7.7 to conclude D_2 is diffeomorphic to D^n. For $n = 5$ we apply Corollary 7.8. Denote by h the identification of the boundary of D_1 with D_2. We then have

$$\Sigma = D_1 \cup_h D_2 \overset{\text{diff}}{\cong} D^n \cup_h D^n = \Sigma(h).$$

Hence $\Sigma \overset{\text{diff}}{\cong} \Sigma(h)$ and the map is surjective, as required. $\square$

For $n \geq 5$, it follows that each of these groups can be interpreted as the group of smooth structures on the topological n-sphere, that each element is invertible under connected sum, and each element can be given an atlas with exactly two charts.

We have now given three different ways to interpret the set of smooth structures on the topological n-sphere for $n \geq 5$. Each of these groups are useful in their own right, and each can be more convenient to work with depending on the context. The objective of the next section is to prove that Θ_n is finite for all $n \neq 3$, and hence that there are a finite number of smooth structures on the n-sphere for all $n \geq 5$.

7.5 Classification of Homotopy Spheres

In [KM63], Milnor and Kervaire proved that Θ_n is finite in all dimensions but three. The key step was to use the Pontrjagin-Thom construction to turn problem of computation of cobordism groups into a homotopy problem and to use the recently developed surgery theory to kill homotopy groups.

The main result of [KM63] is the following.

Theorem 7.18 *For $n \neq 3$ the group Θ_n is finite.*

The proof proceeds by defining a subgroup, $bP_{n+1} \subset \Theta_n$, and proving that Θ_n/bP_{n+1} and bP_{n+1} are finite. We define the subgroup bP_{n+1} as those homotopy n-spheres Σ which bound parallelisable $(n+1)$-manifolds. It is not immediately clear that this is a subgroup. We sketch the main ideas of the construction, leaving all proofs to [KM63].

We will show bP_{n+1} is the kernel of a homomorphism from Θ_n to a subgroup of Π_n, and so is a normal subgroup of Θ_n. First we need the following.

Theorem 7.19 *Homotopy spheres are stably parallelisable.*

This appears as theorem 3.1 in [KM63]. The proof uses Adams' work on the stable J-homomorphism and the Bott Periodicity Theorem to prove there is no obstruction to the triviality of $T\Sigma \oplus \varepsilon$.

The following construction, known as the Pontrjagin-Thom construction, associates to a stably parallelisable manifold, M, a set of elements, $p(M) \subset \Pi_n$, of the n-th stable homotopy group of spheres. Embed the manifold M into $\mathbb{R}^{n+k}$. For $k \gg n$, there is a unique such embedding [Lee03]. As the tangent bundle to $\mathbb{R}^{n+k}$ is trivial, a stably trivial n-frame of M gives rise to a k-frame of the normal bundle to M in $\mathbb{R}^{n+k}$ which gives a trivialisation of NM, denote this k-frame by φ. Consider a tubular neighbourhood U of M. Recall we may consider S^n as the one point compactification of $\mathbb{R}^n$, $S^n = \mathbb{R}^n \cup \{\infty\}$. We define a map $\mathbb{R}^{n+k} \to S^k = \mathbb{R}^k \cup \{\infty\}$ as follows. Map the complement of U to ∞. Recall that, as U is a tubular neighbourhood of M, we may identify it with a neighbourhood of M in the normal bundle NM. We then map points of U to points of $\mathbb{R}^k$ using the k-frame φ. Extend this map to $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\}$ by sending ∞ to ∞. We then have a map

$$p(M, \varphi) : S^{n+k} \to S^k.$$

Intuitively, this map measures asymptotic normal distance from points of $\mathbb{R}^{n+k}$ to points of M^n. That is, points outside the tubular neighbourhood are at infinite distance from M and points inside the tubular neighbourhood are at distance given by the perpendicular direction to M, using the frame φ. The homotopy class of this map is a well defined element of the stable homotopy group Π_n depending on both M and φ. Further details and a proof this map is well defined may be found in [Bre93]; see Figure 22 below for a schematic picture of the construction. Varying φ over all possible k-frames, we obtain a set of elements, $p(M) \in \Pi_n$.

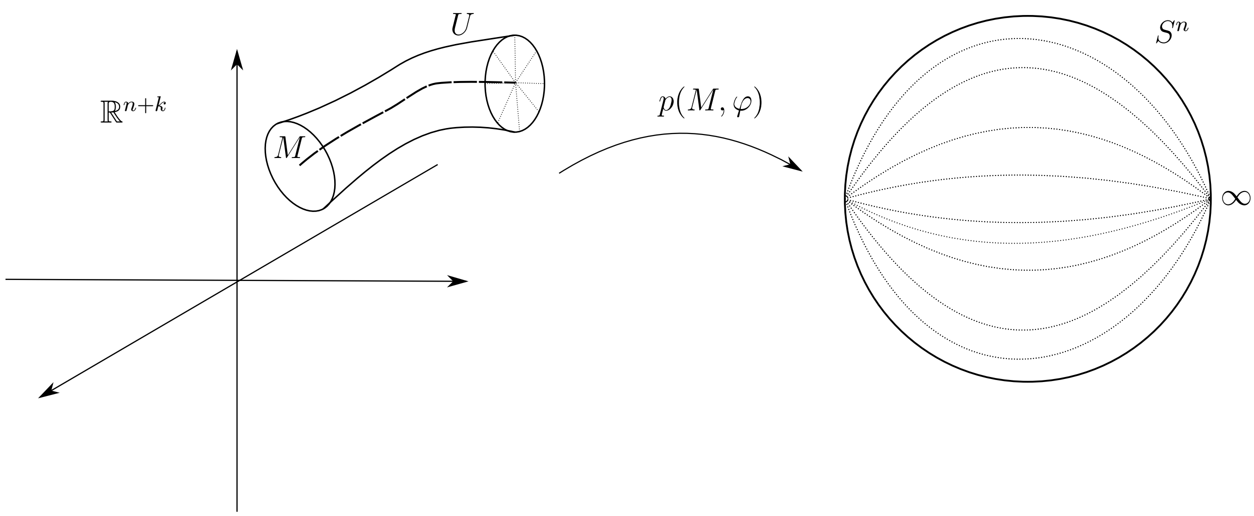


Figure 22: The Pontrjagin-Thom construction. The fibres of NM are indicated on the left and right by dotted lines and M is indicated by a heavily dashed line.

Lemma 7.20 *Let M^n, N^n be stably parallelisable manifolds.*

1. *The set $p(M)$ contains the zero element if and only if M bounds a parallelisable manifold.*
2. *If M is h-cobordant to N, then $p(M) = p(N)$.*
3. *$p(M) + p(N) \subset p(M \# N)$.*

Combining this with our results on connected sums of homotopy spheres of the previous section we get.

Theorem 7.21 *The set $p(S^n)$ is a subgroup of Π_n. For Σ a homotopy sphere, the set $p(\Sigma)$ is a coset of this subgroup. It follows that the map $\tilde{p} : \Theta_n \to \Pi_n / p(S^n)$ given by*

$$\tilde{p} : \Sigma \mapsto \tilde{p}(\Sigma) = p(\Sigma)$$

is a homomorphism.

By the first part of Lemma 7.20, the kernel of this map is the set of homotopy n-spheres which bound parallelisable $(n+1)$-manifolds, that is, bP_{n+1}. Then bP_{n+1} is a normal subgroup of Θ_n and Θ_n / bP_{n+1} is isomorphic to a subgroup of Π_n. As Π_n is finite, this implies the following.

Theorem 7.22 *The group Θ_n / bP_{n+1} is finite.*

We then have the following exact sequence.

$$0 \longrightarrow bP_{n+1} \longrightarrow \Theta_n \longrightarrow \Theta_n / bP_{n+1}$$

Milnor and Kervaire give an alternate description of the subgroup $p(S^n)$ as the image of the J-homomorphism. Hence, $\Pi_n/p(S^n)$ is the cokernel of J_n, giving the exact sequence

$$0 \longrightarrow bP_{n+1} \longrightarrow \Theta_n \longrightarrow \operatorname{coker}(J_n)$$

As Θ_n/bP_{n+1} is finite, Θ_n will be finite whenever bP_{n+1} is finite. To prove bP_{n+1} is finite, let $\Sigma \in bP_{n+1}$ and suppose W is an $(n+1)$-dimensional parallelisable manifold with boundary Σ. Our goal is then to construct a simpler manifold, W', such that $\partial W'$ is in the same h-cobordism class as Σ. We obtain this simplification by a series of surgeries on W which kill the homotopy groups of W. Ideally, we would end up with a manifold W' homeomorphic to D^{n+1}. We could then apply Corollary 7.7 and Lemma 7.15 to conclude Σ is diffeomorphic to S^n. However, there are certain obstructions to performing this simplification depending on n. We summarise the main results.

Theorem 7.23 *The group bP_{n+1} is a finite cyclic group. Let a_m be* 1 *if m is even and* 2 *if m is odd and denote by* $\operatorname{num}(x)$ *the numerator of x in reduced form. Then*

$$|bP_{n+1}| = \begin{cases} 1 & n = 2m \geq 4; \\ 1 \text{ or } 2 & n = 4m+1,\ m \geq 1; \\ 2^{2m-2}(2^{2m-1}-1)a_m \operatorname{num}(4B_m/m) & n = 4m-1. \end{cases}$$

For the groups bP_{4n+2}, the order is 2 exactly when the boundary of the manifold $K(4n+2)$ of Example 6.13 is diffeomorphic to S^{4n+1}. This is known as the *Kervaire invariant problem.* The bulk of Milnor and Kervaire's paper is dedicated to this case.

The problem of counting the number of smooth structures on homotopy spheres of a given dimension is then converted to a computation of the order of stable homotopy groups. See Table 2 below for a summary of results in the first eight dimensions.

n	1	2	3	4	5	6	7	8
$\lvert\Theta_n\rvert$	1	1	1	1	1	1	28	2
$\lvert bP_{n+1}\rvert$	1	1	1	1	1	1	28	1
$\lvert\Theta_n/bP_{n+1}\rvert$	1	1	1	1	1	1	1	2

Table 2: Order of the first eight Θ_n, bP_{n+1}, and the quotient Θ_n/bP_{n+1}.

8 Conclusion

To proceed any further into Milnor and Kervaire's paper would require developing a large amount of surgery theory and homotopy theory. We instead conclude with some comments. Although we have effectively solved the problem of counting the number of exotic spheres in each dimension, the method was not constructive. A natural problem is then to find explicit constructions of all homotopy spheres in a given dimension. The Milnor spheres Σ^{4n-1} of Example 6.12 are generators of bP_{4n}, for a proof see [Kos72]. Similarly, the Kervaire spheres $\partial K(2n)$ of Example 6.13 generate bP_{4n+2}. An alternative construction of homotopy spheres was given by Brieskorn in [Bri66]. He considered the boundary of neighbourhoods of isolated singularities of complex varieties. Using this construction, all elements of bP_{2n} can be constructed. As $\Theta_7/bP_8 = 0$, every exotic sphere can be obtained from Brieskorn's construction. These are currently the only known methods of explicitly constructing exotic spheres.

We also note that although we have converted the problem of counting homotopy spheres to the problem of computing stable homotopy groups, this problem is far from solved. Much active research in algebraic topology today is dedicated to exploring the structure of stable homotopy groups. Furthermore, almost all of the methods of the previous chapter fail completely in dimension four. The smooth Poincaré conjecture is entirely open, it is still debated whether the conjecture should be true in this case. Many results on smooth 4-manifolds have been obtained using Donaldson theory and Seiberg-Witten theory, both of which arise from theoretical physics. There are also a large number of candidates for possible exotic 4-spheres, such as those produced by Gluck twists. However, it seems we are still quite far from any resolution of the conjecture.

Although we have produced a number of exotic spheres and counted them, we have not explored their properties. It is natural to ask whether exotic spheres may possess interesting geometry. There are a number of results on curvature of S^n which do not apply to exotic spheres, though we do not have the space to explore them here.

Finally, we note that we may also consider the classification of other manifolds, such as homotopy tori. Questions of this type require a generalisation of the h-cobordism Theorem to non-simply connected manifolds, knows as the s-cobordism theorem. This states that a h-cobordism $\{M, N; W\}$ is trivial if and only if a certain well defined invariant, $\tau(W, M)$, known as the *Whitehead torsion*, is zero.

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