NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

A consistent Mathematical Framework for linear feedback systems using distributions

A dissertation
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by

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A mio padre, perchè prosegua.

A Eliza, bo Dudu jestes.

To Bill, mathematical samurai.

As I understand it, there is no reality more independent of our perception and more true to itself than mathematical reality.

Ratner Star, Don DeLillo

Contents

Glossary

$\mathbb{Z}_+$	positive integers, including 0
$C^\infty(\mathbb{R})$	smooth functions on the real line
$L^p[0,\infty)$	space of the p integrable functions on $[0,\infty)$
$L^p(\mathbb{R})$	space of the p integrable functions on $\mathbb{R}$
$l_p(\mathbb{Z}_+)$	space of the p summable sequences on $\mathbb{Z}_+$
$l_p(\mathbb{Z})$	space of the p summable sequences on $\mathbb{Z}$
$\mathcal{H}^2(\mathbb{C}_+)$	Hardy space of the right half plane
UBV	functions of uniform bounded variation
L	subclass of the locally integrable functions (see Definition 53)
L_S	subclass of L (see Definition 64)
L^T	subclass of the double-sided sequences (see Definition 19)
L_S^T	subclass of L^T (see Definition 30)
$\mathcal{T}$	linear space of linear functional with domain containing a shift-invariant subspace of $C^\infty(\mathbb{R})$ (see Section 4.4)
$\mathcal{Q}_T$	class of operators on $\mathcal{T}_\Delta$ (see 4.5)
$\bar{\mathcal{Q}}_T$	class of maximal extensions of the operators in $\mathcal{Q}_T$ (see Definition 36)
$\mathcal{Q}$	class of operators on $\mathcal{T}_\Delta$ (see 5.4)
$\bar{\mathcal{Q}}$	class of maximal extensions of the operators in $\mathcal{Q}$ (see Definition 70)
R	locally integrable functions
S	good functions (see Definition 12)
D	good functions with bounded support
$\mathcal{D}$	space of distributions (see Definition 13)
$\mathcal{D}_S$	space of tempered distributions (see Definition 13)
$\mathcal{R}$	regular functionals, functionals in $\mathcal{D}_S$ defined by members of R

$\mathcal{M}$	multipliers in the distributions (see Section 3.5)
$\mathcal{M}_S$	multipliers in the tempered distributions (see Section 3.5)
$\mathcal{M}^{\mathcal{T}}$	periodic multipliers in the distributions (see Section 3.5)
$\mathcal{F}\{f(t)\}$	Fourier transform of $f(t)$
$\mathcal{L}\{f(t)\}$	Laplace transform of $f(t)$
$\delta(x)$	delta functional
$(A/D)_T$	analogue digital converter with sampling period T
$(D/A)_T$	zero order hold
$[P, C]$	feedback system with plant P and controller C
causality	the condition that the output of a system depends only on the current and previous inputs (see ([13])). More technical Definitions of causality are dependent on the Formalism chosen, and can be found in the corresponding Sections.
stability	a performance measure of a system (see [1]). More technical Definitions of stability are dependent on the Formalism chosen, and can be found in the corresponding Sections.
T	it can denote both an operator and a sampling period. The difference can be easily inferred from the context.

$\mathcal{T}_D$, $\mathcal{T}_S$ and $\mathcal{T}_\Delta$ are subspaces of $\mathcal{T}$ (see Section 4.4).

$\mathcal{D}_E$, $\mathcal{D}_{EN}$, $\mathcal{D}_B$, $\mathcal{D}_{BN}$, $\mathcal{D}_V$, $\mathcal{D}_{VN}$, $\mathcal{D}^T$, $\mathcal{D}^T_E$, $\mathcal{D}^T_{EN}$, $\mathcal{D}^T_B$, $\mathcal{D}^T_{BN}$ are subclasses of $\mathcal{D}$ (see Section 3.5).

$\mathcal{U}_S$, $\mathcal{U}_E$, $\mathcal{U}_{EN}$, $\mathcal{U}_B$, $\mathcal{U}_{BN}$, $\mathcal{U}_V$, $\mathcal{U}_{VN}$, $\mathcal{U}^T$, $\mathcal{U}^T_E$, $\mathcal{U}^T_{EN}$, $\mathcal{U}^T_B$, $\mathcal{U}^T_{BN}$ are the Fourier transforms of the corresponding subclasses of $\mathcal{D}$ (see Section 3.5).

Chapter 1

Introduction

A Mathematical Formalism is a rigorous mathematical method and Systems Theory is a set of mathematical statements and principles devised to explain a system. The elements for a Mathematical Formalism in Systems Theory are the class of signals and the class of systems. In this work, the term Mathematical Framework is used when the elements of a Mathematical Formalism satisfy certain requirements.

Systems theory is applied to many branches of engineering. Recently, the boundaries between the traditional disciplines have become blurred with, for example, the application of control ideas to communication systems such as the internet and the use of feedback in signal processing. Consequently, the engineering systems, to which system theory is applied, have become more varied and complex. The extension of the classes of signals and systems to cater for this trend requires a careful choice of Mathematical Framework. When inadequate, inconsistencies can arise. One such inconsistency, the Georgiou Smith Paradox, that has recently been discussed widely, occurs when double sided signals are considered.

In this thesis, when the signals are either discrete time or continuous time signals, three different Mathematical Formalisms for feedback systems are investigated. The first one, the Standard Formalism, uses mathematical elements that are adopted from the conventional analysis for feedback systems (conventional analysis as in [1]). It is shown how consistency can be regained, but with the effect of severely restricting the class of signals. Moreover, it is shown under which conditions the Standard Formalism becomes a consistent Framework. The second one, the Generalised Formalism, extends the class of systems but lacks of a transform domain analysis. The third one, a Framework using Distributions, is shown to be consistent. Moreover, the class of signals does not need any restriction and a transform domain analysis can be performed. The class of signals is the space of distributions, or generalised functions. Being those an extension of the concept of ”classical” function, the traditional class of signals is largely increased. The class of systems on the distributions are, in time domain the convolutes on the distributions, and in transform domain the multipliers on the Fourier transforms of distributions. Convolutes and multipliers are a broader class of systems than the traditional class of convolutions and algebraic functions, in time and transform domain, respectively. Since this is a consistent Framework, paradoxes and inconsistencies, such as the Georgiou Smith paradox, do not occur. Hence, it is proved that the Framework using Distributions is suitable for the analysis and design of feedback systems.

The same conclusion, consistency and suitability for analysis and design, is reached when the feedback system is hybrid single rate (a feedbacks system that mixes continuous time and discrete time components). That is done showing the well posedness of sampling formulas in a distributions context.

The outline of the thesis is the following:

- in Chapter 2 a review of the literature is given;

- in Chapter 3 the requirements for a Mathematical Formalism to be a consistent Mathematical Framework are discussed. Furthermore, the historical background and the idea of Distributions are introduced, together with their mathematical notation;

- in Chapter 4 the three Mathematical Formalisms for discrete time feedback systems are investigated;

- in Chapter 5 the three Mathematical Formalisms for continuous time feedback systems are investigated;

- in Chapter 6 the analysis is shifted on hybrid single rate feedback systems.

A technical appendix proving a sampling theorem for distributions concludes the thesis.

Chapter 2

Literature Review

2.1 Introduction

While the consistency of Mathematical Frameworks for discrete LTI feedback systems, when the signals are single-sided, has never been questioned, some discrepancies have been found in the last few years, when there have been attempts to expand the same Framework to double-sided signals. Despite the attempts to solve the inconsistencies of the double-sided Framework, those have been unresolved.

Regaining consistency becomes even more necessary when dealing with hybrid feedback systems, in which continuous time and discrete time components are related together. Their mathematical relation is established using sampling formulas.

The notation used in this chapter is adopted from the corresponding papers.

2.2 Graph Theory and $L^2[0, \infty)$

A Mathematical Framework for feedback systems, when the signals are continuous, and the systems are linear operator on $L^2[0, \infty)$, possibly unbounded, is developed in [7]. This detailed analysis is explicit and is developed within an input-output approach.

A dynamical system is considered to be the linear operator

$$P : D_P \subseteq L_2[0, +\infty) \rightarrow R_P \subseteq L_2[0, +\infty)$$

where D_P is the domain of P and R_P is its range. Its graph, $\mathcal{G}_P$ is defined by

$$\mathcal{G}_P = \begin{pmatrix} I \\ P \end{pmatrix} D_P \subset L_2[0,\infty) \times L_2[0,\infty) \tag{2.1}$$

The properties of the dynamical system P, from a control perspective, are investigated in relation to the mathematical properties of its graph $\mathcal{G}_P$. In this thesis the term Graph Theory is used when referring to this kind of approach and it is not related with graph theory in mathematics and computer science (the same terminology approach that is adopted in the literature). Consider the feedback configuration of Figure 2.1 and denote it with $[P, C]$. A system P is said to be stable

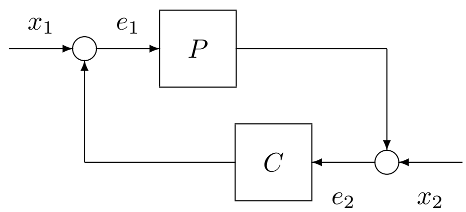

Figure 2.1: $[P, C]$

if $D_P = L^2[0,\infty)$ and

$$\sup_{u \neq 0} \|Pu\| \,/\, \|u\| = \|P\| < \infty$$

The feedback system $[P, C]$ is stable if the operators $x_i \to e_j$, $i, j = 1, 2$, are well defined and bounded. The system P is stabilisable if there exists a C such that $[P, C]$ is stable (this property is denoted as stabilisability). In other words, there exists a C such that the operator

$$F = \begin{pmatrix} I & C \\ P & I \end{pmatrix} : D_P \times D_P \to L_2[0,\infty) \times L_2[0,\infty) : \begin{pmatrix} e_1 \\ -e_2 \end{pmatrix} \to \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$$

has a bounded inverse. If $\mathcal{G}_C^I$ is the graph, defined by

$$\mathcal{G}_P = \begin{pmatrix} C \\ I \end{pmatrix} D_C$$

then necessary and sufficient conditions for the stability of $[P, C]$ in terms of the mathematical properties of the graph of P are introduced in the following Theorem.

Theorem 1. *Let P and C be linear operators defined as above. The $[P, C]$ is stable if and only if $\mathcal{G}_P$, $\mathcal{G}_C$ are closed subspaces,*

$$\mathcal{G}_P \cap \mathcal{G}_C = \{0\},$$

and

$$\mathcal{G}_P + \mathcal{G}_C^I = L^2[0,\infty) \times L^2[0,\infty).$$

A discrete time version of these results, when the signal space is $l_2(\mathbb{Z}^+)$ is given in [13]. With minimal adjustments the two versions are completely equivalent.
When the class of signals is $L^2[0,\infty)$, as in [7], the analysis of feedback systems in transform domain is done using the Fourier-Laplace transform of signals and systems. When $x \in L^2[0,\infty)$, $\hat{x} \in \mathcal{H}^2(\mathbb{C}_+)$ is the Fourier transform of x and $\mathcal{H}^2(\mathbb{C}_+)$ is the Hardy space of the right half plane. Since the Fourier transform so defined is a Hilbert space isomorphism, then the time domain analysis and the transform domain analysis are completely equivalent.
Similarly, in [13], the transform domain analysis is performed using the z transform of signals and systems. Since that is an isometric isomorphism from $l_2(\mathbb{Z}_+)$ to the Hardy space $\mathcal{H}_2(D)$, where D is the interior of the unit disc, the two analyses are completely equivalent. Moreover, when P is a causal LTI system on $l_2(\mathbb{Z}_+)$, or, more generally, on any space of sequences $\{x_k\}$ with $x[k] = 0$ for $k < 0$, it is shown in [12] that there exists a sequence $\{g_k\}$ such that

$$(Px)[k] = \sum_{h=-\infty}^{\infty} g[k-h]u[h]$$

The result shows that any causal LTI system on the class of single sided sequences has a convolution representation. However, the same does not apply when the class of sequences is double sided or the system is not causal. Two Examples, one for a noncausal system on the class of single-sided sequences, one for a causal systems on the class of double-sided sequences, are shown in [12].

2.3 Georgiou Smith paradox

In [8], Graph Theory is applied to signals that have support on the doubly-infinite time axis. There the continuous-time system

$$P_1 : D_{P_1} \subset L^2(-\infty,\infty) \to L^2(-\infty,\infty) \tag{2.2}$$

is discussed within the context of graph theory. P_1 is defined by the convolution

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau \tag{2.3}$$

with $u \in L_2(-\infty, \infty)$ and

$$h(t) = \begin{cases} e^t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

It is proved that the graph of P_1 is not closed and therefore the condition for stabilisability is not satisfied.

On the restricted domain, $D_{P_1} \cap L_2[0, \infty)$, P_1 coincides with the system

$$P_2 : D_{P_2} \subset L_2(-\infty, \infty) \to L_2(-\infty, \infty) \tag{2.4}$$

defined by 2.3 but with

$$h(t) = \begin{cases} 0 & t > 0 \\ -e^t & t \leq 0 \end{cases}$$

The graph for P_2 in the Fourier domain is the closure of $\bigcup_{T \leq 0} e^{sT} G \mathcal{H}_2(\mathbb{C}_+)$, where

$$G = \begin{pmatrix} \frac{s-1}{s+1} \\ \frac{1}{s+1} \end{pmatrix}$$

and $\mathcal{H}_2(\mathbb{C}_+)$ is the Hardy space on the right half plane. It is also proved that the graph of P_1 contains $\bigcup_{T \leq 0} e^{sT} G \mathcal{H}_2(\mathbb{C}_+)$. Hence, if the graph of P_1 is closed it must contain the graph of P_2. That is a contradiction, since the non-causal input output pair

$$u(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases} \quad y(t) = -e^{|t|}/2$$

is contained in the graph of P_2 but not P_1. As a consequence, the graph of P_1 cannot be closed. Moreover, the closure of the graph for P_1 must contain the graph for P_2 and be noncausal, [8]. This phenomena is known as Georgiou Smith paradox. A corresponding system in discrete-time domain is shown in [10]:

$$P : D_P \subset l_2(\mathbb{Z}) \to l_2(\mathbb{Z})$$

$$Pu[i] = \sum_{n=-\infty}^{i} 2^{i-n} u[n], \ u \in D_P \tag{2.5}$$

The fact that the closure of the graph for a a system may not be causal raises the need to find necessary and sufficient conditions for the graph of a system to be causally closable.

2.4 Convoluted Double Troubled

In [19] the analysis of feedback systems with double-sided signals is not performed using Graph Theory. However, the author shows a contradiction between time domain and transform domain analysis with an open loop unstable system.
The discrete-time first order convolution system

$$\begin{cases} y[i] = b\sum_{n\geq 0} a^n(u[i-n-1]+v[i-n+1])+d[i] \\ u[i] = -ky[i] \end{cases} \tag{2.6}$$

is investigated, where u is the input, y is the output, C a causal controller and v and d are external signals. Moreover, all the terms in the equations are possibly double sided. The following Theorem is proved by a time domain argument.

Theorem 2 (Mäkilä, [19]). *Consider the feedback configuration 2.6. Let* $(a-kb)=0$. *There exists a (unique) solution* $y[i]$ *for any* i *iff*

$$\lim_{i\to-\infty} a^{-i}(-kd[i]+v[i]) = 0$$

The feedback system 2.6 is further discussed in transform domain. Let $b \neq 0$ and v and d be double-sided square summable real sequences. 2.6 becomes

$$\begin{cases} Y(z) = G(z)[U(z)+V(z)]+D(z) \\ U[z] = -kY(z) \end{cases} \tag{2.7}$$

where $Y(z)$, $U(z)$, $V(z)$ and $D(z)$ are the bilateral z-transform of y, u, v and d.

$$G(z) = bz^{-1}/(1-az^{-1})$$

is the usual transfer function for the open loop system. Eliminating $U(z)$

$$[1+K(z)][Y(z)-D(z)] = G(z)W(z) \tag{2.8}$$

Hence,

$$Y(z) = [1+K(z)]^{-1}G(z)W(z)+D(z) \tag{2.9}$$

where $W(z) = -kD(z)+V(z)$. When $|a|>1$ the closed loop system is stable provided $|kb-a|<1$, including $|kb-a|=0$, generating a contradiction with the time domain analysis.

2.5 Early attempts of resolutions

Three possible remedies are already discussed in [8]. However, the authors conclude that all of them are unsatisfactory.

The first remedy is to consider system 2.2 to be non stabilisable on $L^2(-\infty,\infty)$. Consequently, the analysis should be restricted to systems that are open loop stable. This would imply that signals with doubly infinite support are meaningless for control purposes, since open loop unstable systems would have to be excluded.
The second remedy is to give a different definition of closed loop stability for signals in $L^2(-\infty,\infty)$, such that it would not disagree with the common practise of stabilizing P_1 with a proportional negative feedback of gain greater than one. It would be possible to restrict the class of signals to $L^2[T,\infty)$, for some arbitrary finite T. With this restriction the graph of P_1 would be

$$\bigcup_{T\geq 0} G^{sT} G\mathcal{H}_2$$

and the class of signals for the feedback system would become $\bigcup_{T\geq 0} e^{sT}\mathcal{H}_2$, which is not a closed subspace of $L^2(-j\infty,j\infty)\times L^2(-j\infty,j\infty)$. However, this is not really an improvement, since an analysis of feedback systems with signals with support $[T,\infty)$ is more or less the same as the one with signals with support the half line.
Considering the $L^2(-\infty,\infty)$ graph of P_1 would be more natural. As in the development of the Georgiou Smith paradox, the graph of P_1 is the same as the graph of P_2, but with the restriction that the inputs satisfy

$$\int_{-\infty}^{\infty} e^{-\tau}u(\tau)d\tau = 0$$

Consequently, the graph of P_1 is not a closed subspace of $L^2(-\infty,\infty)\times L^2(-\infty,\infty)$. Therefore, the problem is to find a suitable subspace for the signals of the feedback system. The choice of the subspace of signals in $L^2(-\infty,\infty)$ that decay sufficiently fast towards minus infinity does not seem to be satisfactory.

The third remedy is to identify P_1 in 2.2 with P_2 in 2.4. The authors judge this option not as outrageous as it would appear. In fact, in the system represented by the differential equation

$$\dot{y} = y + u$$

the trajectories of P_1 are represented by solving it forwards in time, the trajectories of P_2 by solving it backwards in time. However, this imply that the notion of causality is abandoned, and that is not possible when considering questions of control.

2.6 Replacement of the system

A more general version of the Georgiou Smith paradox is considered in [20]. Consider the feedback system

$$\begin{cases} y = G(u+v) + d \\ u = C(r-y) \end{cases} \tag{2.10}$$

where y is the output, u is the input, G a causal system, v, d and r are external signals. The class of signals is $l_p(\mathbb{Z})$, $1 \leq p < \infty$. If v is such that $(v, Gv) \in \mathcal{G}(G)$, take $r = 0$ and $Gv = -d$, then the feedback system has the solution $y = 0$, $u = 0$ and $G(u+v) = Gv = -d$. Consider any sequence $\{v_i\}$ such that $(v_i, Gv_i) \in \mathcal{G}(G)$ and $v_i \to v \in l_p$ and $G(v_i) \to H \in l_p$. As above, consider $Gv_i = -d_i$. A system is l_p gain stable if it maps l_p signals into l_p signals. If the feedback system is l_p gain stable, then $Gv = H$, implying that G is a closed operator. However, when G is a causal but unstable convolution operator, it is not a closed operator. Therefore, l_p gain stabilisability cannot be achieved when the plant G enclosed in a feedback system is an unstable convolution system.

A possible remedy that is discussed in [20] is whether or not the system G could be replaced in a meaningful sense by the operator $\bar{G}$, the l_p closure of G. In order to do so the closability of G must be investigated first. In Theorem 3.3 it is established that the causal but unstable convolution operator

$$Gv = \Phi * v \tag{2.11}$$

where $\Phi[k] = ba^{k-1}$, $k \geq 1$, has an l_p closure, proving its closability. (The notion of closability was investigated also in [21]. There the convolution systems considered are unstable and infinite dimensional. It is shown that many infinite dimensional linear systems are non stabilisable in an l_p sense also on the singly infinite time axis).

However, is $\bar{G}$ a meaningful replacement? Unfortunately the answer is no. In fact a similar conclusion to the one of the Georgiou Smith paradox is reached, $\bar{G}$ is l_p

gain stabilisable, but it is a noncausal operator.

The attempt is to find a refined argument for closability. Since

$$l_p(\mathbb{Z}) = l_p(\mathbb{Z}_-) \oplus l_p(\mathbb{N})$$

it is proven that a necessary condition for l_p gain stabilisability is the closability of G on $l_p(\mathbb{Z}_-)$. Not only is the system 2.11 not closable on $l(\mathbb{Z}_-)$, but any possible extension G_E fails to be an operator. It is established that, using the refined argument, in order to perform a meaningful approach to the stabilisability of unstable systems on the doubly infinite time axis, the class of signals must be restricted to signals that decay to zero at least exponentially, when the time index reaches $-\infty$.

In [22] the attempt is to treat the case without the restriction on signals in $l_2(\mathbb{Z})$. Instead of 2.10, with G and V possible unstable convolution operators, the following feedback system is considered

$$\begin{cases} Dy = N(u+v) + Dd \\ Yu = X(r-y) \end{cases} \tag{2.12}$$

where N, D, X and Y are causal, discrete time, shift invariant convolution operators in l_1 on $l_\infty(\mathbb{Z})$. Furthermore, it is required that $\hat{N}(0) = 0$, $\hat{D}(0) \neq 0$ and $\hat{Y}(0) \neq 0$, where $\hat{N}$ is the power series $\sum_{k\geq 0} n_k z^k$, similarly for $\hat{D}$ and $\hat{Y}$. The feedback system is replaced by the compact notation

$$\begin{pmatrix} D & -N \\ X & Y \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} Nv + Dd \\ Xr \end{pmatrix}$$

If $U = NX + DY$, it is also required that its inverse U_W^{-1} is a causal, shift invariant convolution operator in l_1. If

$$\Gamma = \begin{bmatrix} D & -N \\ X & Y \end{bmatrix}$$

it is easy to verify that it has a bounded inverse

$$\Gamma_W^{-1} = U_W^{-1} \begin{bmatrix} Y & N \\ -X & D \end{bmatrix}$$

and that the solution to 2.12 is given by

$$\begin{pmatrix} y_s \\ u_s \end{pmatrix} = \Gamma_W^{-1} \begin{pmatrix} Nv + Dd \\ Xr \end{pmatrix}$$

The problem is that on the space of all double sided sequences, the null space of Γ is not empty, therefore making the solutions non-unique. Consequently, when the system inputs are zero, the systems output may not be zero, making the system nonlinear. To avoid it, 2.12 is considered as a particular element of the set F, the set of all models of the form

$$\begin{cases} DHy = NH(u+v) + DHd \\ YVu = XV(r-y) \end{cases}$$

where H and V are causal, linear and shift invariant convolution operators in l_1. Moreover, (y_s, u_s) is a solution of the feedback system F if it solves the feedback equations for all N and V. With this modifications the uniqueness of the solution is regained. However, if modelling a feedback system as a set of models resolves the Georgiou Smith paradox, it remains unclear how to directly apply the usual transform methods.

2.7 Restriction of the class of signals

In the previous section the methodology used in order to resolve the Georgiou Smith paradox is the modification of the mathematical operators describing the feedback system. In [12] the attention is focused on the class of signals.

The investigation is first centred on the requirements of causality. In fact, if the class of signals is the set of single sided sequences then an LTI system is causal if and only if it is the convolution operator

$$(Pu)(t) = \sum_{j=0}^{t} g(t-j)u(j)$$

or if and only if $\hat{P}z^{n_0}/z^{n_0}$ is a power series in z.

If P is an LTI closed operator on $l_2(\mathbb{N}_0)$ and the graph of $\hat{P}$, the z transform of P, is $\begin{pmatrix} m \\ n \end{pmatrix}$, then P is causal if and only if

$$gcd\left(\frac{m}{gcd(m,n)}, z\right) = 1$$

The extension to $l_2(\mathbb{Z})$ is done introducing the Definition of Smirnoff class.

Definition 3 (Jacob, Partington [12]). The Smirnoff class $\mathcal{N}_+$ consist of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$, for which there exist functions f_1 and $f_2 \in H_1(\mathbb{D})$, such that f_2 is an outer function and $f = f_1/f_2$.

The Definition of outer function implies that outer functions never have zeroes in the unit disc (for Definition and properties of outer function see [13]). The conclusion is that a closed LTI system on $l_2(\mathbb{Z})$ is causal if and only if n/m is a member of the Smirnoff Class $\mathcal{N}_+$.

The investigation is then shifted to the idea of closability. It is proven that any causal LTI system on the space of single-sided sequences is closable. Clearly, it becomes more complicated when the class of signals becomes $l_2(\mathbb{Z})$. In fact, an LTI system P on $l_2(\mathbb{Z})$ given by the convolution sum

$$(Pu)(t) = \sum_{n=-\infty}^{\infty} g(t-n)u(n)$$

is closable if $\bar{D}_P = l_2(\mathbb{Z})$ or, equivalently, $D_P \cap l_2(\mathbb{N}_0) \neq \{0\}$ or $D_P \cap l_2(-\mathbb{N}_0) \neq \{0\}$. An LTI system that is not affected by the Georgiou Smith paradox must have a causal closure. Therefore, condition for the causal closability of the system must be found. An LTI system P on $l_2(\mathbb{Z})$, given by

$$(Pu)(t) = \sum_{n=-\infty}^{t} g(t-n)u(n) \tag{2.13}$$

is causally closable if $D_P \cap l_2(\mathbb{N}_0)$ is dense in $l_2(\mathbb{N}_0)$. Sufficient and necessary condition are given in the following Theorem

Theorem 4 (Jacob, Partington [12]). *Let P be an LTI system on $l_2(\mathbb{Z})$, given by 2.13. Further, let $D_P \cap l_2(\mathbb{Z}) \neq \varnothing$, then the following statements are equivalent*
i) P causally closable;
ii) $\tilde{G}(q) = \sum_{i\in\mathbb{Z}_+} g[i]q^i$ belongs to the Smirnoff class.

However, an holomorphic function belonging to the Smirnoff class is given by f_1/f_2, with f_1 and $f_2 \in H_1(\mathbb{D})$ and f_2 an outer function. Since an outer function can have no zeroes strictly inside the unit disc, a function belonging to the Smirnoff class cannot have any poles in the unit disc. Given that $G(z) = \tilde{G}(q)_{q^{-1}=z}$ is the usual transfer function the graph of an exponentially unstable system cannot be causally closable.

2.8 Replacement of the system, further development

The idea of modifying the formulation of the operators describing the system, as in [22], in order to solve the Georgiou Smith paradox, is further developed in [23], in a MIMO context.
A discrete time linear system is the quadruple (A, B, Y^p, X^m), where

$$A : D(A; Y^p) \to Y^p$$

and

$$B : D(B; X^m) \to Y^p$$

are linear operators. The quadruple consists of the set of trajectories

$$T(A, B, Y^p, X^m) = \{(u, v) \in D(B; X^m) \times D(A; Y^p) : \; Ay = Bu\}$$

where Y^p and X^m are linear spaces of sequences on $\mathbb{Z}$. Since the quadruple is a linear system it follows that T is also linear. Moreover, it is proved that if the operator A and B are closed then the system is closed too.
Consider the feedback system

$$\begin{cases} Hu = Fy + \Lambda_1 w \\ Ay = Bu + \Lambda_2 w \end{cases} \tag{2.14}$$

enclosing the plant (A, B, Y^p, X^m) with the controller (H, F, X^m, Y^p). 2.14 can be written as

$$\Gamma x = \Lambda w$$

where

$$\Gamma = \begin{pmatrix} H & -F \\ -B & A \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix}$$

Obviously $(\Gamma, \Lambda, X^m \times Y^p, V^q)$ is a linear system and it is called the feedback system associated to the plant and the controllers. It is proved that if $(\Gamma, \Lambda, X^m \times Y^p, V^q)$ is gain stable then it must be a closed system. Moreover, since the Georgiou Smith paradox implies that it is impossible to gain stabilize the plant if either A or B is strictly unstable, it is required to restrict to bounded A and B. Hence, Γ must be bounded. The condition is mandatory for such analysis. In fact, similarly to [22], it is proved that when A or B is an unbounded causal convolution operator, then the set of trajectories $T(A, B, X_E^p, X_E^m)$, X_E^p and X_E^m linear spaces equipped with a seminorm, cannot be defined in a meaningful sense.

Necessary and sufficient conditions for the gain stabilisability of $(\Lambda, I, X^m \times Y^p)$ are stated in the following Theorem.

Theorem 5 (Mäkilä, Partington [23]). *There is a linear controller* (H, F, X^m, Y^p) *such that* H*,* F *and* H^{-1} *are bounded operators, gain stabilizing the feedback system* $(\Lambda, I, X^m \times Y^p, X^m \times Y^p)$*, if and only if there exist bounded operators* $C : Y^p \to Y^p$ *and* $D : Y^p \to X^m$ *such that* $C^{-1} : Y^p \to Y^p$ *is bounded and*

$$(AC + BD)x = x \;\; x \in Y^p$$

The standard case in the input output approach, generating the Georgiou Smith paradox, corresponds to

$$\Gamma = \begin{pmatrix} I & -F \\ -B & I \end{pmatrix}$$

Obviously, if Λ is bounded, then its closure $\bar{\Lambda}$, when it exists, is also bounded. The system is affected by the Georgiou Smith paradox when $\bar{\Gamma}$ is not bounded. As in the previous literature, that is the case of an unstable convolution plant.

Example 1 *(Makila, Partington [23]).* Consider the feedback system

$$\begin{cases} y(t) = \sum_{k\geq 0} u(t-k-1) + w_2(t) \\ \sum_{k=-q}^{r} h_k(t+k) = \sum_{k=-q}^{r} f_k y(t+k) + w_1(t) \end{cases} \tag{2.15}$$

where q and r are nonnegative integers and not all h_k and f_k are zero. If the class of signals is $l_\infty(\mathbb{Z})$ the feedback system is affected by the Georgiou Smith paradox. However, if 2.15 is replaced by

$$\begin{cases} y(t) - y(t-1) = u(t-1) + w_2(t) \\ \sum_{k=-q}^{r} h_k(t+k) = \sum_{k=-q}^{r} f_k y(t+k) + w_1(t) \end{cases} \tag{2.16}$$

since there are no unbounded convolution operators the feedback system is gain stabilized by the controller $u(t) = -y(t)$.

With this approach, as in [22], it remains unclear how to perform transform domain analysis.

2.9 $l_2(\mathbb{N}_0)$ or $l_2(\mathbb{Z})$?

The author's goal in [11] is to solve the problem concerning the stabilisability for systems over the signal space $l_2(\mathbb{Z})$. Since the Georgiou Smith paradox does not occur for systems over the signal space $l_2(\mathbb{N}_0)$, the approach is to relate the two mathematical formalism. Compared to [22] and [23], the approach used always considers the necessity of a transform domain analysis. In order to do so, the z transform of an element u,

$$\bar{u}(z) = \sum_{j \in \mathbb{Z}} u(j) z^j$$

is an isometric isomorphism from $l_2(\mathbb{N}_0)^n$ onto $H_2(\mathbb{D})^n$, and from $l_2(\mathbb{Z})^n$ onto $L^2(\mathbb{T})^n$.

The feedback system $[P, C]$ on $l_2(\mathbb{N}_0)^{m+p}$, enclosing the plant P and the controller C is stable if the operator

$$F_{[P,C]} = \begin{pmatrix} I & C \\ P & I \end{pmatrix} : D_P \times D_C \to l_2(\mathbb{N}_0)^{m+p}$$

has a bounded inverse, $H_{[P,C]}$. An LTI system P is defined maximal if, for any system $\tilde{P}$ with $\mathcal{G}(P) \subset \mathcal{G}(\tilde{P})$, $\mathcal{G}(P) = \mathcal{G}(\tilde{P})$. Since any maximal system P defines a function $\mathcal{P}$ in the transform domain, $\mathcal{P}$ has a right coprime factorization (rcf) over $H_\infty(\mathbb{D})$ if there exist $M \in H_\infty(\mathbb{D})^{m \times m}$ with $detM \neq 0$, and $N \in H_\infty(\mathbb{D})^{p \times m}$ such that $\mathcal{P} = NM^{-1}$ and $\begin{pmatrix} M \\ N \end{pmatrix}$ is left invertible over $H_\infty(\mathbb{D})$. With this Definitions it is proved that P is stabilisable if and only if P is maximal and $\mathcal{P}$ possess a rcf over $H_\infty(\mathbb{D})^{m \times m}$.

In an $l_2(\mathbb{Z})$ context the system P is defined closed if the operator P is a closed operator, that is, if its graph $\mathcal{G}(P)$ is a closed subspace of $l_2(\mathbb{Z})^{m+p}$. As usual, closable means that there exists a closed operator $T : D_T \subseteq l_2(\mathbb{Z})^m \to l_2(\mathbb{Z})^p$ such that $D_P \subset D_T$ and $Tu = Pu$ for every $u \in D_P$. The system P is stable if P is closed and $D_P = l_2(\mathbb{Z})^m$. Moreover, as a consequence of the Closed Graph Theorem, P is stable if and only if P is a linear bounded operator from $l_2(\mathbb{Z})^m$ to $l_2(\mathbb{Z})^p$. The notion of stabilisability for the feedback system $F_{[P,C]}$ is the same as the one in an $l_2(\mathbb{N}_0)^{m+p}$ context. Necessary condition for the stability of $[P, C]$ is that P and C must be closed systems. Moreover, a necessary condition for the causality of a stable feedback system $[P, C]$ is that $\mathcal{P}$ and $\mathcal{C}$ possess both a normalized rcf and

normalized lcf (left coprime factorization) over $H_\infty(\mathbb{D})$.
Since this Definition of stabilisability does not rule out the Georgiou Smith paradox, the author introduces a modified Definition in order to solve the problem.

Definition 6 (Jacob [11]). An $LTI(\mathbb{Z})$-system P is called stabilisable if P is closable and there exists an $LTI(\mathbb{Z})$-system C such that the feedback system $[\bar{P}, C]$ is stable and causal.

Adopting Definition 6 implies that a closable LTI$(\mathbb{Z})^{p\times m}$ is stabilisable if and only if the transfer function $\mathcal{P}$ of $\bar{P}$ possesses a rcf over $H_\infty(\mathbb{D})$.
Consider $P_\mathbb{N}$ the restriction to $l_2(\mathbb{N}_0)^{p\times m}$ of the closure of a closable system P on $l_2(\mathbb{Z})^{p\times m}$. With the modified Definition 6 it is established that necessary condition for the stabilisability of $P_\mathbb{N}$ is the stabilisability of P. Moreover, the two systems have the same transfer function and are stabilized by the same controllers. This Theorem follows.

Theorem 7 (Jacob [11]). *Let P be a closable LTI$(\mathbb{Z})^{p\times m}$ system. Then P is stabilisable if and only if $P_\mathbb{N}$ is stabilisable. Moreover, both systems have the same transfer function and they are stabilized by the same controllers.*

Therefore, as the author states, the Georgiou Smith paradox is solved by introducing Definition 6, a modified Definition of stabilisability.

The discussions of sections 2.8 and 2.9 are further developed in the last section of Chapter 4.

2.10 Sampling Theorems

Consider the hybrid system of Figure 2.2, where $x(t)$ and $y(t)$ are input and output,

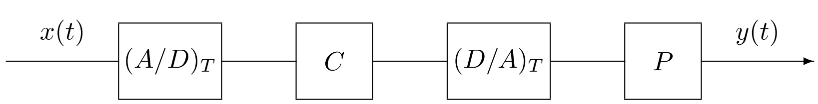

Figure 2.2: Hybrid System

$(A/D)_T$ is an A/D converter with sampling period T, $(D/A)_T$ is a zero-order hold (ZOH) and P and C are the plants of a continuous time system and a discrete time system, respectively. In order to perform the transform domain analysis of the hybrid system of Figure 2.2, the transform domain response of a sampled signal must be related to the transform response of its correspondent continuous time signal. This is done by building the transform response of the sampled signal upon the superposition of infinitely many copies of its continuous time transform response, using the formula

$$G_d(e^{st}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G(s + jk\omega_s) \tag{2.17}$$

where G is the Laplace transform of a continuous time signal g, G_d is the z transform of the sequence of its samples $\{g(kT)\}_{k=0}^{\infty}$ and T and $\omega_S = 2\pi/T$ are the sampling period and the sampling frequency, respectively.

Till 1997, with the publication of [3], 2.17 was mathematical folklore. In fact, it was very often used in the digital control literature ([24], [15], [27]), [18], [29], [30] and [28]) and it appeared in many control textbooks ([16], [26], [9]), [2] and [17]), but it was not established by a rigorous proof that indicated the relevant classes of signals considered.

The first attempt to provide 2.17 with a proof is in [5]. The author bases his proof on the use of impulse trains, those defined as the function

$$\sum_{k=-\infty}^{\infty} \delta(x - nT)$$

where $\delta(x)$ is the impulse function or Dirac function or Dirac impulse such that

$$\delta(x) = \begin{cases} +\infty & x = 0 \\ 0 & otherwise \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

However, the proof lacks rigour, since the impulse function, and hence the impulse trains, cannot be defined as functions. The proofs in [16] and [26] are similar and rely on the same concept.

In [14] it is shown the similarity between 2.17 and the Poisson Summation Formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(s)e^{-2\pi iks}ds$$

Consequently 2.17 is often indicated as the Poisson Sampling Formula. In [6] a rigorous proof,that avoids the use of the impulse trains, for

$$G_d(e^{st}) = \frac{g(0^+)}{2} + \frac{1}{T} \sum_{k=-\infty}^{\infty} G(s + jk\omega_s)$$

is derived under the assumption that the series $\sum_k G(s + jk\omega_s)$ is uniformly convergent. However, since this condition is a transform domain condition, it is not obvious when a time domain function satisfies it.

In [3] it is pointed that for 2.17 to hold, it is not enough to require that the Laplace transform G of g and its sampled version, G_D, are well defined. It is shown that, for $n_p = 2^{2^{2^p}}$ and the continuous function

$$g(t) = sin((2n_p + 1)t),\ t \in [p\pi, (p+1)p],\ p \in \mathbb{N}$$

2.17 does not hold, despite the fact that $G_d(e^{st})$ and its sampled version with period $T = \pi$, are both well defined in the open right-half plane. In fact, it is proved that

$$\lim_{n=\infty} \sum_{k=-n}^{n} G(s + jk\omega_s)$$

does not converges for any $s \geq 0$. Because of the rapid oscillations of g as $t \to \infty$ the class of signals is restricted to functions with bounded and uniform bounded variation.

Definition 8 (Braslavsky et. al, [3]). A function g defined on the closed real interval $[a, b]$ is of bounded variation (BV) when the total variation of g on $[a, b]$,

$$V_g(a, b) = \sup_{a=t_0<t_1<\ldots<t_{n-1}<t_n=b} \sum_{k=1}^{n} |g(t_k) - g(t_{k-1})|$$

is finite. The supremum is taken over every $n \in \mathbb{N}$ and every partition of the interval $[a, b]$ into subintervals $[t_k, T_{k+1}]$ where $k = 0, 1, ..., n-1$ and $a = t_0 < t_1 < ... < t_{n-1} < t_n = b$.

A function g defined on the positive real axis is of uniform bounded variation (UBV) if for some $\Delta > 0$ the total variation $V_g(x, x + \Delta)$ on intervals $[x, x + \Delta]$ of length Δ is uniformly bounded, that is, if

$$\sup_{x \in \mathbb{R}_0^-} V_g(x, x + \Delta) < \infty$$

With the class of signals restricted to UBV functions, a proof for

$$G_d(e^{st}) = \frac{g(0^+)}{2} + \sum_{k=1}^{\infty} \frac{g(kT^+) - g(kT^-)}{2} e^{-skT} + \frac{1}{T} \sum_{k=-\infty}^{\infty} G(s + jl\omega_s)$$

a more general formulation of 2.17, is provided.

Note that the well posedness of 2.17 is proved for an open loop context, when the system considered is stable. Despite the fact that it is rather common to analyse a hybrid feedback system with the help of 2.17, even if the class of signals is restricted to UBV functions, there is no proof of the well posedness of the feedback when applying 2.17.

Chapter 3

General Considerations

3.1 Systems requirements

The consistency of a Mathematical Formalism is the absence of contradictions in its mathematical formulation. When a contradiction occurs, it is denoted as a paradox. Consider the feedback system in Figure 3.1 where T represents the system, x and

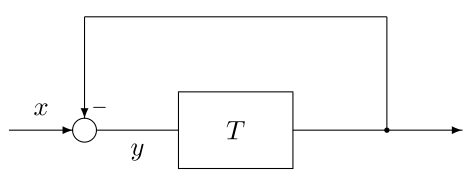

Figure 3.1: Feedback system.

y input and output, respectively. In a Mathematical Framework describing the feedback system in Figure 3.1, T is an operator such that

$$T : D_T \subseteq \mathcal{X} \to R_T \subseteq \mathcal{Y}$$

$$x \mapsto y$$

where x and y belong to $\mathcal{X}$ and $\mathcal{Y}$, respectively, the class of inputs and the class of output, and D_T and R_T are the domain and the range of the operator T. The mathematical relationship

$$[I + T]y = x, \ (x - y) \in R_T, \ y \in D_T \tag{3.1}$$

describes the feedback system of Figure 3.1.

In what follows, the fundamental requirements for the consistency of a Mathematical Framework for feedback systems are introduced.

Requirement 1. The class of inputs and the class of outputs must be the same. This class, denoted as the class of signals, must constitute a linear space.

Requirement 2. The class of systems must constitute an algebra of linear operator mapping the class of signals into itself.

Requirement 3. The inverses of the return difference operators must exist and themselves belong to the chosen class of systems.

Requirement 4. The class of signals must be a Banach space.

When one or more of the above requirements is not satisfied, the Mathematical Framework is denoted as a Mathematical Formalism.

Example 2. Consider the simple feedback system

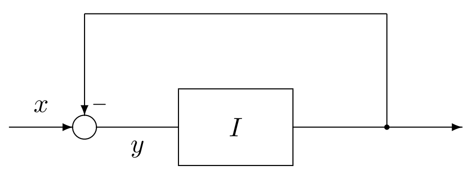

Figure 3.2: Feedback system with integers.

where I is the identity operator. Consider the class of signals to be $\mathbb{Z}$, the integers. The linear operator describing the feedback system would appear to be $\frac{1}{2}I$. Therefore the inputs are related to the outputs by the relation

$$output = \frac{input}{2}$$

It is obvious that the output might not belong to $\mathbb{Z}$. If the equation 3.1, with $T = I$ describes the feedback system of Figure 2, then that is well posed if there exists a solution, a y in the linear space, for all x belonging to the linear space. That is, the inverse operator for $(I + I)$ must exist. Clearly this does not happen in this situation. A natural solution to the problem is easily found, it consists in enlarging the linear space of inputs and outputs, from $\mathbb{Z}$ to $\mathbb{Q}$.

The analysis of a feedback system can be performed in two different domains. The time domain analysis is the analysis of the solutions for the equation 3.1.The transform domain analysis is the analysis of the solutions of the algebraic equation

$$[I + K]Y = X \tag{3.2}$$

where K, Y and X are algebraic functions, obtained when mapping T, y and x to K, Y and X, respectively. Consequently, a further requirement must be introduced.

Requirement 5. The results of the time domain analysis and the transform domain analysis must not be in contradiction.

A list of definitions for transform functions for discrete time systems and continuous time systems is provided in the next two sections.

3.2 Discrete system analysis

Definition 9. The Laurent series of a discrete time signal $x[k]$ is defined by

$$\mathcal{X}(q) = \mathcal{L}^T\{x[k]\} = \sum_{k=-\infty}^{\infty} x[k]q^k$$

with $\mathcal{X}(q)$ analytic for $R_1 < |q| < R_2$, provided the summation exists for $R_1 < |q| < R_2$.

The inverse of the Laurent series, provided that exists, is defined as

$$\{x[k]\} = \mathcal{L}^{-1}\{X(q)\} = \left\{\frac{1}{2\pi j}\oint_C \frac{X(q)}{q^{k+1}}dq\right\} \tag{3.3}$$

where C is the contour in the complex plane defined by the circle, centred on the origin, and inside the region of convergence of the corresponding Laurent transform, traversed in the anti-clockwise direction.

Changing the notation, when $R_1 < 1 < R_2$,the Fourier series is defined by

$$X(\omega) = \mathcal{P}\{x[k]\} = \mathcal{X}(q)_{q=e^{-j\omega T}} = \sum_{k=-\infty}^{\infty} x[k]e^{-jk\omega T} \tag{3.4}$$

with $X(\omega)$ a periodic function with period $2\pi/T$. Its inverse, provided the sums converges, is defined by

$$\{x[k]\} = \mathcal{P}^{-1}\{X(\omega)\} = \left\{\frac{1}{2\pi j}\oint_C \frac{\mathcal{X}(q)}{q^{k+1}}dq\right\} = \left\{\frac{T}{2\pi}\int_0^{2\pi/T} X(\omega)e^{jk\omega T}d\omega\right\} \tag{3.5}$$

where C is the contour in the complex plane defined by the circle, centred on the origin with unit radius, traversed in the anti-clockwise direction.

3.3 Continuous system analysis

Definition 10. The Laplace transform of a continuous time signal $x(t)$, when $x(t) = 0$ for $t < 0$, provided the integral exists for $s \geq \sigma$, is defined by

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)e^{-st}dt$$

$s,\ \sigma \in \mathbb{C}$.

If $X(u + j\nu)$ is the analytic continuation of $X(s)$, analytic for all $u \geq \sigma$, then, provided the integral exists for $t \geq 0$, the Bromwich integral is defined by

$$\hat{x}(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega \tag{3.6}$$

with $x(t) = \hat{x}(t)$ almost everywhere. The Bromwich integral is the inverse of the Laplace transform.

Definition 11. The Fourier transform of a continuous time signal $x(t)$ is defined by

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

when the integral exists. Its inverse is defined by

$$x(t) = \mathcal{F}^{-1}\{X(\omega\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

3.4 Distributions, historical background

The necessity to generalise the concept of a classical function started to appear in the mathematics of the nineteenth century. In some disconnected parts of mathematical analysis emerged some early ideas of generalised functions. Examples can be found in the definition of Green's function, of Laplace transform in the Riemann's theory of trigonometric series, when they are not necessarily the Fourier series of integrable functions. In engineering, the intensive use of Laplace transform led to the operational calculus, an heuristic use of symbolic methods. An example is *Electromagnetic Theory* of O.Heaviside, 1899. However, the mathematical justifications of operational calculus was based on the use of divergent series and hence it had a bad reputation in the pure mathematics community.

With introduction of Lebesgue integral appears the first pure mathematical notion of generalising a function. Since a Lebesgue integrable function is equivalent to any other which is the same almost everywhere, the value of a function at a given point of a function is not its most important feature.

During the late 1920's and 1930's the idea was developed further. The Dirac delta function was defined by Paul Dirac, in order to mathematically formalise the physical concept of density (like charge densities). Sergei Lvovich Sobolev in 1935, working in partial differential equations theory, defined the first adequate theory of generalised functions, in order to allow the definition of weak derivative and work with weak solutions of PDEs.

A systematic and rigorous description, entirely based on abstract functional analysis and on the idea of duality, that became the definitive accepted theory of generalised functions, denoted as Theory of Distributions, was due to Laurent Schwartz in the late 1940s. The first publication in which he presented the theory of distributions was *Generalisation de la notion de fonction, de derivation, de transformation de Fourier et applications mathematiques et physiques*, which appeared in 1948. Not only Schwartz's development of the Theory of Distributions put methods of this type onto a rigorous mathematical basis, but also greatly extended their range of application, providing powerful tools for applications in numerous areas. For his work in the Theory of Distributions, Laurent Schwartz was presented with a Fields Medal by Harald Bohr at the International Congress in Harvard on 30 August 1950.

3.5 Distributions

Definition 12 (Champeney [4]). Suppose a real or complex valued function $f(x)$, defined for all real x and everywhere infinitely differentiable, and suppose that each differential tends to zero as $x \to \pm\infty$ faster than any positive power of x^{-1}, or in other words, suppose that for each positive integer m and each positive integer n,

$$\lim_{x \to \pm\infty} x^m f^{(n)}(x) = 0$$

then we say that f is a good function.

Denote the set of good functions by S. An ordinary function is of bounded support if there exists a number $a > 0$ such that $f(x) = 0$ whenever $|x| \geq a$. The class of all good functions of bounded support is denoted by D. Clearly, $D \subset S$.

Definition 13 (Champeney [4]). A distribution is a functional x that assigns to each $\phi(t) \in D$, the class of good functions with bounded support, the value denoted by $x[\phi(t)]$. A tempered distribution is a functional x acting on the class of good functions, S. Denote the class of distributions by $\mathcal{D}$ and the class of tempered distributions by $\mathcal{D}_S$.

Tempered distributions can be interpreted as a sub-class of the distributions (by continuity). The members of $\mathcal{R}$, the class of regular functionals in $\mathcal{D}_S$, are defined by the members of R, the class of locally integrable ordinary functions, $f(t)$, such that $|f(t)|/(1+|t|^N)$ is integrable for some N.
The symbol for a regular functional in $\mathcal{D}$ and the ordinary function by which it is defined, x and $x(t)$, are distinguished by the explicit presence in the latter of the variable. The following subclasses of $\mathcal{D}$ are required

$$\begin{aligned}
\mathcal{D}_E &= \{x \in \mathcal{D} : x \text{ regular with } x(t)/(1+|t|)^N \text{ square integrable} \\
&\quad \text{for some } N \geq 0\} \\[2ex]
\mathcal{D}_{EN} &= \{x \in \mathcal{D} : x \text{ regular with } x(t)/(1+|t|)^N \text{ square integrable }\}; \\
&\quad N \geq 0 \\[2ex]
\mathcal{D}_B &= \{x \in \mathcal{D} : x \text{ regular with } x(t) \text{ of bounded variation on each finite} \\
&\quad \text{interval and } |x(t)| \leq c(1+|t|)^N \text{ for some } c > 0\};\ N \geq 0 \\[2ex]
\mathcal{D}_{BN} &= \{x \in \mathcal{D} : x \text{ regular with } x(t) \text{ of bounded variation on each finite} \\
&\quad \text{interval and } |x(t)| \leq c(1+|t|)^N \text{ for some } N \geq 0 \text{ and } c > 0\} \\[2ex]
\mathcal{D}_V &= \{x \in \mathcal{D} : x \text{ regular with } Var_{[a+t,b+t]}\{x(t)\} \leq c(1+|t|)^N \text{ for each} \\
&\quad \text{finite interval } [a,b] \text{ for some } N \geq 0 \text{ and } c > 0\} \\[2ex]
\mathcal{D}_{VN} &= \{x \in \mathcal{D} : x \text{ regular with } Var_{[a+t,b+t]}\{x(t)\} \leq c(1+|t|)^N \text{ for each} \\
&\quad \text{finite interval } [a,b] \text{ for some } c > 0\};\ N \geq 0 \\[2ex]
\mathcal{D}^T &= \{x \in \mathcal{D} : x = \textstyle\sum_{-\infty}^{\infty} a_k \delta_{kT}\};\ T > 0 \\[2ex]
\mathcal{D}_E^T &= \{x \in \mathcal{D} : x = \textstyle\sum_{-\infty}^{\infty} a_k \delta_{kT} \text{ with } a_k/(1+|k|)^N \text{ square summable}
\end{aligned}$$

for some $N \geq 0\}$; $T > 0$

$$\mathcal{D}_{EN}^{T} = \{x \in \mathcal{D} : x = \textstyle\sum_{-\infty}^{\infty} a_k \delta_{kT} \text{ with } a_k/(1+|k|)^N \text{ square summable } \}; \; N \geq 0,\, T > 0$$

$$\mathcal{D}_{B}^{T} = \{x \in \mathcal{D} : x = \textstyle\sum_{-\infty}^{\infty} a_k \delta_{kT} \text{ with } |a_k| \leq (1+|k|)^N \text{ for some } c > 0 \text{ and } N \geq 0\}; \; T > 0$$

$$\mathcal{D}_{BN}^{T} = \{x \in \mathcal{D} : x = \textstyle\sum_{-\infty}^{\infty} a_k \delta_{kT} \text{ with } |a_k| \leq (1+|k|)^N \text{ for some } c > 0\}; \; N \geq 0,\, T > 0$$

where $Var_{[a,b]}\{x(t)\}$ is the variation of $x(t)$ on the interval $[a, b]$ and the functional δ_τ is the delta functional in $\mathcal{D}$ defined by

$$\delta_\tau[\phi(t)] = \phi(\tau)$$

The definitions of $\mathcal{D}^T$ and its subclasses are specific to some value of the parameter, T. $\mathcal{D}_E$, $\mathcal{D}_B$, $\mathcal{D}_E^T$ and $\mathcal{D}_B^T$ are subclasses of $\mathcal{D}_S$, the class of tempered distributions.

Each functional $x \in \mathcal{D}$ is related by a linear bijections to a functional $X \in \mathcal{U}$, the class of ultradistributions (see [4]), such that

$$x[\phi(t)] = 2\pi X[\Phi(\omega]$$

for all $\phi(t) \in D$ with

$$\Phi(\omega) = \mathcal{F}[\phi(t)](\omega)$$

The functionals x and X constitutes a Fourier transform pair with

$$X = \mathcal{F}\{x\} \text{ and } x = \mathcal{F}^{-1}\{X\}$$

The subclasses $\mathcal{U}_S, \mathcal{U}_E, \mathcal{U}_{EN}, \mathcal{U}_B, \mathcal{U}_{BN}, \mathcal{U}_V, \mathcal{U}_{VN}, \mathcal{U}^T, \mathcal{U}_E^T, \mathcal{U}_{EN}^T, \mathcal{U}_B^T$ and $\mathcal{U}_{BN}^T$ are the Fourier transforms of the the corresponding subclass of $\mathcal{D}$. The members of $\mathcal{U}^T$ and its subclasses are periodic with period $2\pi/T$.

Definition 14 (Champeney [4]). A multiplier in $\mathcal{D}_S$ is an ordinary function $f(x)$ that is infinitely differentiable at all real x and such that f and each derivative is bounded by a polynomial, the polynomial being not necessarily the same for each derivative. The multipliers in $\mathcal{D}_S$ are denoted by $\mathcal{M}_S$.

Definition 15 (Champeney [4]). A multiplier in $\mathcal{D}$ is an ordinary function $f(x)$ that is infinitely differentiable at each real value of x. The multipliers in $\mathcal{D}$ are denoted by $\mathcal{M}$. The subclass $\mathcal{M}^T$ is the class of periodic multipliers with period $2\pi/T$.

Chapter 4

Discrete time feedback systems

4.1 Introduction

In this chapter, the set of LTI feedback systems with polynomially bounded inputs, together with the properties of stability and causality, is analysed. In particular, the consistency of three different Mathematical Formalisms is investigated. With the first and standard one, it is shown how consistency can be regained, but with the side effect of severely restricting the class of signals. The second Formalism is more general but it does not have a transform domain. The third is restricted to stable systems but it satisfies all the Requirements for a consistent Framework. Moreover, it does have a transform domain.

4.2 Eigenvectors and System Function

Consider the operator

$$T : D_T \subset \mathcal{X} \to R_T \subset \mathcal{Y}$$

where D_T,R_T are the domain and the range of T, $\mathcal{X}$ and $\mathcal{Y}$ are mathematical classes representing double sided discrete time signals. It is required that T satisfies

$$T(\lambda x) = \lambda T(x),\ \forall x \in D_T,\ \lambda \in \mathbb{C}$$

(hence, that T is homogeneous) and that $Tx = y$ implies $Tx_{kT} = y_{kT}$, where x_{kT} is x delayed by k times T, $k \in \mathbb{Z}$. The operator T, the classes $\mathcal{X}$ and $\mathcal{Y}$ are the elements for the Mathematical Framework for a feedback system as in Figure 4.1.

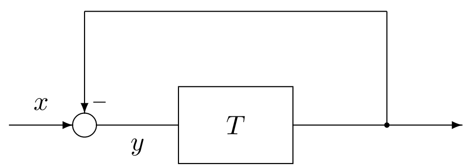

Figure 4.1: Feedback system.

Generic properties about eigenvectors and eigenvalues for the operator T can be deduced.

Lemma 16. *If $a^n \in D_T$ and corresponds to the discrete time signal a^{-k}, a^{-k+1}, ..., a^{-2}, a^{-1}, 1, a^1, a^2, ..., a^{k-1}, a^k, $k \in \mathbb{Z}_+$, then a^n is an eigenvector of T.*

Proof. Consider y such that $Ta^n = y$. Define $a^{-n}y$ the member of D_T corresponding to the discrete time signal $a^k y_{-k}$, $a^{k-1}y_{-k+1}$, ..., $a^2 y_{-2}$, $a^1 y_{-1}$, y_0, $a^{-1}y_1$, $a^{-2}y_2$, ..., $a^{-k+1}y_{k-1}$, $a^{-k}y_k$, where $y \in D_T$ corresponds to the discrete time signal y_{-k}, y_{-k+1}, ..., y_{-2}, y_{-1}, y_0, y_1, y_2, ..., y_k, y_{k+1}, $k \in \mathbb{Z}_+$. In addition, suppose $w = a^{-n}y$, with w the member of D_T corresponding to the discrete time signal w_{-k}, w_{-k+1}, ..., w_{-2}, w_{-1}, w_0, w_1, w_2, ..., w_{k-1}, w_k, $k \in \mathbb{Z}_+$. $a^n(w)$ is the member of D_T corresponding to the discrete time signal $a^{-k}w_{-k}$, $a^{-k+1}w_{-k+1}$, ..., $a^{-2}w_{-2}$, $a^{-1}w_{-1}$, w_0, $a^1 w_1$, $a^2 w_2$, ..., $a^{k-1}w_{k-1}$, $a^k w_k$, $k \in \mathbb{Z}_+$. Hence, $y = a^n(a^{-n}y)$.
Define $h \in D_T$ by

$$h = a^{-n}y$$

Hence,

$$T(a^{n-kT}) = y_{-kT} = a^{n-kT}h_{-kT}$$

From the definition of h it follows that $h_{-kT} = h$, since $a^{kT}y_{-kT} = y$ for any k. Therefore,

$$y = \lambda_a a^n$$

for some $\lambda_a \in \mathbb{C}$ dependent only on a. ■

Assume that λ_a exists for any a on some segment in the complex plane. The system function of the operator T is defined in what follows.

Definition 17. The meromorphic function $K(z)$, $z \in \mathbb{C}$, is the analytic continuation of λ_a, if

$$K(z)_{z=a} = \lambda_a$$

for any a on the segment of existence of λ_a. Furthermore, $K(z)$ is required to be bounded on any contour encircling the origin and lying between the circles of radius k and $k+1$, for any $k \geq 0$. Define $K(z)$ the system function of the operator T.

For the Definition of meromorphic function see [25]. According to that Definition, a meromorphic function can have only a countable number of poles. Moreover, the set of meromorphic functions is a field.

4.3 Standard Formalism for double-sided signals and LTI systems

4.3.1 Time domain analysis

The operator T, mapping double-sided sequences to double-sided sequences, is defined by the convolution sum

$$x = \{x_n\} \mapsto y = Tx = \{y_n\} : y_n = \sum_{m=-\infty}^{\infty} g_{n-m} x_m$$

By Lemma 16, when the convolution sum and $K(z)$, as in Definition 17, exists, then the sequence $\{a^n\}$ is an eigenvector of the operator T with eigenvalue $K(a)$.

Theorem 18. *(a) Let T_r be the mapping*

$$T_r : x = \{x_n\} \mapsto y = \{y_n\} = \Phi_r * x,\ \ \Phi_r = \{n^r g_n\},\ \ r \in \mathbb{Z}_+$$

Suppose a an interior point of the domain of $K(z)$, the system function for T_0 and $\Phi_{k,a} = \{n^k a^{-n} g_n\} \in l_1$, $k \in \mathbb{Z}_+$, then $\{a^n\}$ is an eigenvector of T_j, for $j = 0, ..., k$, with eigenvalue $K_j(a) = [(-z\frac{d}{dz})^j K(z)]_{z=a}$.

(b) Let T_S and $T_{i,r}$, $i = 1, ..., N$, respectively, be the mappings

$$T_S : x = \{x_n\} \mapsto y = \{y_n\} = \Phi * x,\ \ \Phi = \{g_n\}$$

$$T_{i,r} : x = \{x_n\} \mapsto y = \{y_n\} = \Psi_{i,r} * x,\ \ \Psi_{i,r} = \{n^r a_i^n\},\ \ |a|_i \neq 0$$

Provided $\Phi * x \in l_\infty$ *and* $x_{k_i,a_i} \in l_1$, *for* $i = 1, ..., N$, *with* $x_{j,a} = \{n^j a^{-n} x_n\}$ *then*

$$Tx = T_S x + \sum_{i=1}^{N} \sum_{j=0}^{k_i} c_{i,j} T_{i,j} x$$

where $Tx = (\Phi + \sum_{i=1}^{N} \sum_{j=0}^{k_i} c_{i,j} \Psi_{i,j}) * x$.

(c) Let $T_{S,r}$ *and* $T_{R,r}$, *respectively, be the mappings*

$$T_{S,r} : x = \{x_n\} \mapsto y = \{y_n\} = \Phi_r * x,\ \ \Phi_r = \{n^r g_n\}$$

$$T_{R,r} : x = \{x_n\} \mapsto y = \{y_n\} = \Psi_r * x,\ \ \Psi_r = \{n^r a^n\}, r \geq 0,\ \ |a| \neq 0$$

and $K_r(z)$ *the system function for* $T_{S,r}$. *Provided* $\Phi_{k,a} = \{n^k a^{-n} g_n\} \in l_1$ *and* $x_{k,a} \in l^1$, *then*

(i) $\Psi_j * (\Phi_0 * x)$ *exists for* $j = 0, ..., k$;

(ii) $(\Psi_j * \Phi_0) * x$ *exists for* $j = 0, ..., k$;

(iii) $T_{R,k}(T_{S,0}x) = \sum_{r=0}^{k} (-1)^r \begin{pmatrix} k \\ r \end{pmatrix} K_r(a) T_{R,k-r} x$.

(d) Let T_S, T_P *and* T_G, *respectively be the mappings*

$$T_S : x = \{x_n\} \mapsto y = \{y_n\} = \Phi * x,\ \ \Phi = \{g_n\}$$

$$T_P : x = \{x_n\} \mapsto y = \{y_n\} = \Omega * x$$

$$T_G x = (\Omega + \sum_{i=1}^{N} \sum_{j=0}^{k_i} c_{i,j} \Psi_{i,j}) * x$$

where $\Psi_{i,r} = \{n^r a_i^n\}$, $|a_i| \neq 0$. *Suppose* $\Omega * (\Phi * x) \in l^\infty$, $\{n^j a_i^{-n} g_n\} \in l_1$ *and* $\{n^j a_i^{-n} x_n\} \in l_1$, *for* $i = 1, ..., N$, $j = 0, ..., k_i$, *and* a *is an interior point of the domain of* $K(z)$, *the system function for* T_S, *then*

$$(T_G(T_S x) - T_P(T_S x)) = \sum_{i=1}^{N} \sum_{j=1}^{k_i} c_{i,j} \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} ((z \frac{d}{dz})^r K(z))_{z=a_i} (T_{i,j} x)$$

where $T_{i,j} x = \Psi_{i,j} * x$.

Proof. (a) For $j = 0, ..., k$, $\Phi_{j,a} \in l_1$, since $\left|n^j a^{-n} g_n\right| \leq \left|n^k a^{-n} g_n\right|$. Hence,

$$\Phi_j * v = a^n \sum_{m=-\infty}^{\infty} (n-m)^j a^{-(n-m)} g_{n-m}$$

with $v = \{a^n\}$, exists and v is an eigenvector of Φ_j with eigenvalue

$$K_j(a) = \sum_{m=-\infty}^{\infty} m^j a^{-m} g_m$$

In addition, for $j = 0, ..., k$,

$$\frac{d^j}{da^j}K(a) = \sum_{m=-\infty}^{\infty} \frac{d^j}{da^j}(a^{-m}g_m) = \sum_{m=-\infty}^{\infty} (-1)^j m(m-1)\cdots(m-j+1)a^{-(m-j)}$$

since a is an interior point of the domain of $K(a)$ and $\{\frac{d^j}{da^j}(a^{-m}g_m)\} \in l_1$. It follows immediately that $K_j(a) = (-a\frac{d}{da})^j K(a)$, $j = 0, ..., k$.

(b) For $i = 1, ..., N$, $j = 0, ..., k_i$, $x_{j,a_i} \in l_1$, since $\left|n^j a_i^{-n} x_n\right| \leq \left|n^{k_i} a_i^{-n} x_n\right|$. Hence, $\Psi_{i,j} * x$ exists, since the absolute value of its n-th element is bounded by

$$|a_i|^n \left\| \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} (-1)^{j-r} n^r x_{j-r,a_i} \right\|_1$$

and $\sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} (-1)^{j-r} n^r x_{j-r,a_i} \in l_1$. It follows immediately that

$$Tx = T_S x + \sum_{i=1}^{N}\sum_{j=0}^{k_i} c_{i,j} T_{i,j} x$$

(c)(i) For $j = 0, ..., k$, $x_{j,a} \in l_1$, since $\left|n^j a^{-n} x_n\right| \leq \left|n^k a^{-n} x_n\right|$, and $\Phi_{j,a} \in l^1$, since $\left|n^j a^{-n} g_n\right| \leq \left|n^k a^{-n} g_n\right|$. Hence, by Young's Theorem (see [4] for Young's Theorem), $\Phi_{i,a} * x_{j-r,a} \in l_1$ for $i, j = 0, ..., k$ and, since

$$\Omega_{j,a} = \{n^j v_n\} = \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} \Phi_{r,a} * x_{j-r,a}$$

where $\{v_n\} = \Phi_{0,a} * x_{0,a}$, $\Omega_{j,a} \in l_1$, for $j = 0, ..., k$. It follows that $\Phi_0 * x = \{u_n\}$ exists, since $\Omega_{0,a} \in l^1$ and $u_n = a^n v_n$ and $\Psi_j * (\Phi_0 * x)$ exists for $j = 0, ..., k$ as required, since the absolute value of its n-th element is bounded by $|a|^n \left\| \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} n^r (-1)^{j-r} \Omega_{j-r,a} \right\|_1$ and $\sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} n^r (-1)^{j-r} \Omega_{j-r,a} \in l_1$.

(ii) For $j = 0, ..., k$, $\Psi_j * \Phi_0$ exists, since the absolute value of its n-th element is bounded by

$$|a|^n \left\| \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} n^r (-1)^{j-r} \Omega_{j-r,a} \right\|_1$$

and $\sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} n^r (-1)^{j-r} \Omega_{j-r,a} \in l_1$. In addition, for $j = 0, ..., k$, $\Phi_{j,1} * \Psi_0$ exists, since the absolute value of its n-th element is bounded by $|a|^n \left\|\Phi_{j,a}\right\|_1$, and

Ψ_0 is an eigenvector of $T_{S,j}$ with eigenvalue $K_j(a)$; that is, $\Phi_{j,1} * \Psi_0 = K_j(a)\Psi_0$. Hence $\{\sum_{m=-\infty}^{\infty} a^{n-m} m^j g_m\} = \{K_j(a)a^n\}$ and

$$\begin{aligned} &\Psi_j * \Phi_0 \\ &\quad = \{\sum_{m=-\infty}^{\infty} (n-m)^j a^{n-m} g_m\} \sum_{r=0}^{i} (-1)^r \begin{pmatrix} j \\ r \end{pmatrix} \{n^{j-r} \sum_{m=-\infty}^{\infty} a^{n-m} m^r g_m\} \\ &\quad = \sum_{r=0}^{j} (-1)^r \begin{pmatrix} j \\ r \end{pmatrix} \{K_r(a) n^{j-r} a^n\} = \sum_{r=0}^{j} (-1)^r \begin{pmatrix} j \\ r \end{pmatrix} K_r(a) \Psi_{j-r} \end{aligned}$$

for $j = 0, ..., k$. Furthermore, for $j = 0, ..., k$, $\Psi_j * x$ exists, since the absolute value of its n-th element is bounded by $|a|^n \left\| \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} n^r (-1)^{j-r} x_{j-r,a} \right\|_1$ and $\sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} n^r (-1)^{j-r} x_{j-r,a} \in l_1$. It follows that $(\Psi_j * \Phi_0) * x$ exists for $j = 0, ..., k$.

(iii) By (i) and (ii),

$$\Psi_k * (\Phi_0 * x) = (\Psi_k * \Phi_0) * x = \sum_{r=0}^{k} (-1)^r \begin{pmatrix} k \\ r \end{pmatrix} K_r(a) (\Psi_{k-r} * x)$$

It follows that

$$T_{R,k}(T_{S,0}x) = \sum_{r=0}^{k} (-1)^r \begin{pmatrix} k \\ r \end{pmatrix} K_r(a) T_{R,k-r} x$$

(d) For $i = 1, ..., N$, $j = 0, ..., k_i$ $\{n^j a_i^{-n} g_n\}$, $\{n^j a_i^{-n} x_n\} \in l_1$, since

$$\left|n^j a_i^{-n} g_n\right| \leq \left|n^{k_i} a_i^{-n} g_n\right| \text{ and } \left|n^j a_i^{-n} x_n\right| \leq \left|n^{k_i} a_i^{-n} x_n\right|$$

and $\{n^j a_i^{-n} y_n\} \in l_1$, where $y = \{y_n\} = T_S x$, since

$$\{n^j a_i^{-n} y_n\} = \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} \{n^j a_i^{-n} g_n\} * \{n^{j-r} a_i^{-n} x_n\}$$

Hence, by part (b),

$$T_G(T_S x) - T_P(T_S x) = \sum_{i=1}^{N} \sum_{j=0}^{k_i} c_{i,j} \Psi_{i,j} * y$$

since $\Omega * y \in l_\infty$. The result follows immediately from part (a) and (c). ■

Definition 19. Define the class L^T of double-sided sequences by

$$L^T = \{\{a_n\} : \exists I_a \in I_0 \cup I_+ \cup I_\infty \text{ such that } \sum_{n=-\infty}^{\infty} a_n q^n \text{ converges for } |q| \in I_a,\ q \in \mathbb{C}\}$$

where

$$I_0 = \{[R_1, R_2),\ R_1 = 0,\ \text{for some } R_2 > 0\}$$

$$I_+ = \{(R_1, R_2) \text{ for some } 0 \leq R_1 < R_2 \leq \infty\}$$

$$I_\infty = \{(R_1, R_2],\ R_2 = \infty,\ \text{for some } R_1 \geq 0)\}$$

The class L^T is the class of double-sided sequences $\{a_n\}$ that have a Laurent Series, $\mathcal{X}(q) = \mathcal{L}\{a_n\} = \sum_{n=-\infty}^{\infty} a_n q^n$, analytic for $|q| \in I_a$. a_n are the Laurent coefficients, such that

$$a_n = \mathcal{L}^{-1}\{\mathcal{X}(q)\} = \frac{1}{2\pi j} \oint_C \frac{\mathcal{X}(q)}{q^{n+1}} dq$$

$\forall n$, where C is the anti-clockwise contour in the complex plane defined by the circle, centred on the origin with radius, R, $R_1 < R < R_2$.

If $\{g_n\} \in L^T$, consider the operator

$$T : D_T \subseteq L^T \to R_T \subseteq L^T$$

$$x \equiv \{x[n]\} \mapsto y \equiv \{y[n]\} :\ y[n] = \sum_{m=-\infty}^{\infty} g[m]x[n-m]$$

the domain of T is the sequences for which the summation exists.

Lemma 20. *Consider the operator T, defined by the convolution sum above. Let $G(z)$ be the system function of T, as in Definition 17, then $G(z) = \mathcal{X}(q)_{q=z^{-1}}$.*

Proof. The result follows immediately form the definitions of system function and Laurent series. ■

Consider $X(z)$ the maximal analytic extension of $\mathcal{X}_{q=z^{-1}}$. For disjoint analytic domains, the algebraic function, $X(z)$, has different Laurent series. In order to recover a one-to-one relationship, the domain $D_X \subseteq \mathbb{C}$, on which the Laurent series exists, must be specified. Therefore, the doublet notation, $\{X(z), D_X\}$, is preferred. Consequently, the doublet $\{G(z), D_G\}$ is the system function for T. The domain D_G is an open annular region, centred on the origin with inner and outer radius the modulus of singular points of $G(z)$.

Lemma 21. *Let T_1 and T_2 operators on L^T such that $T_1 x = \Phi_1 * x$ and $T_2 x = \Phi_2 * x$, where $\Phi_1 \equiv \{f_n\} \in L^T$ and $\Phi_2 \equiv \{g_n\} \in L^T$. Consider their system functions $\{G_1(z), D_{G_1}\}$ and $\{G_2(z), D_{G_2}\}$, respectively. Provided $D_{G_1} \cap D_{G_2} \neq \emptyset$, then*

(i) $\{G_1(z) + G_2(z), D_{G_1+G_2} \supset D_{G_1} \cap D_{G_2}\}$ is the system function for a system T, such that $Tx = T_1 x + T_2 x$, when $T_1 x$ and $T_2 x$ exist;

(ii) $\{G_1(z)G_2(z), D_{G_1 G_2} \supset D_{G_1} \cap D_{G_2}\}$ is the system function for a system T, such that $Tx = T_1(T_2 x)$, when $T_1 x$ and $T_1(T_2 x)$ exist.

Proof. (i) Since,

$$\forall a \in D_{G_1}, \quad \sum_{n=-\infty}^{\infty} f_n a^{-n} = G_1(a) < \infty$$

and,

$$\forall a \in D_{G_2}, \quad \sum_{n=-\infty}^{\infty} g_n a^{-n} = G_2(a) < \infty$$

then,

$$\sum_{n=-\infty}^{\infty} (f_n + g_n) a^{-n} = G_1(a) + G_2(a)$$

for all $a \in D_{G_1} \cap D_{G_2}$. Define $G_T(z) = (G_1(z) + G_2(z))$, with $D_{G_T} \supseteq D_{G_1} \cap D_{G_2}$ and Laurent coefficients $(f_n + g_n)$. It follows that G_T is the system function of the operator T on L^T, with $Tx = \Phi * x$, $\Phi = (f_n + g_n)$.

(ii) Similar to (i). ■

The domain in

$$\{G_1(z) + G_2(z), D_{G_1+G_2} \supset D_{G_1} \cap D_{G_2}\}$$

is greater than $D_{G_1} \cap D_{G_2}$ only when the removal of singular points through additive cancellations occurs. Similarly, the domain in

$$\{G_1(z)G_2(z), D_{G_1 G_2} \supseteq D_{G_1} \cap D_{G_2}\}$$

is greater than $D_{G_1} \cap D_{G_2}$ only when the removal of singular points through multiplicative cancellation with zeros occurs.

Definition 22. L^T is the class of signals and the convolution sums on L^T are the systems for a Standard Formalism in time domain analysis.
The doublet, $\{G(z), D_G\}$, is the system function for the system.

A system on L^T, defined by the sequence $\{g_n\}$ is causal if $g_n = 0$ for $n < 0$, acausal if $g_n = 0$ for $n > 0$. It is stable if there exists a $k > 0$ and a $0 < c < 1$ such that $|g_n| < kc^{|n|}$, $\forall n$. The relationship between a causal or stable system and its system function is shown in the next Theorem.

Theorem 23. *(a)(i) The system $\{G(z), D_G\}$ is causal provided $\exists R > 0$ such that*

$$D_G = \{z \in \mathbb{C} : \ |z| > R\}$$

(ii) the system $\{G(z), D_G\}$ is acausal provided $\exists R > 0$ such that

$$D_G = \{z \in \mathbb{C} : \ |z| < R\}$$

(b) The system $\{G(z), D_G\}$ is stable provided $z \in D_G$ when $|z| = 1$.

Proof. (a)(i) The sequence $\{g_n\} \in L^T$ is the inverse Laurent series for $\{G(z), D_G\}$,

$$g_n = \frac{1}{2\pi j} \oint_C \frac{G(q^{-1})}{q^{n+1}} dq = \frac{1}{2\pi j} \oint_{\hat{C}} G(z) z^{n-1} dz$$

where $\hat{C}$ is a circle in D_G centered in the origin. By Cauchy's Residue Theorem, and by the Definition of system function, $g_n = 0$, $\forall n < 0$;

(ii) similar to (i).

(b) Since D_G is an open neighborhood of the unit circle centred on the origin, it follows immediately that the system is stable. ∎

Some properties of stable systems are proved in the next two Lemmas.

Lemma 24. *Let T_1 and T_2 operators on L^T such that $T_1 x = \Phi_1 * x$ and $T_2 x = \Phi_2 * x$, where $\Phi_1 \equiv \{f_n\} \in L^T$ and $\Phi_2 \equiv \{g_n\} \in L^T$. If $x \in l_p$, $1 \leq p \leq \infty$ then*

(i) $y = T_1 x + T_2 x = Tx \in l_p$, where T is a stable system;

(ii) $y = T_1(T_2 x) = Tx \in l_p$, where T is a stable system.

Proof. (i) Since T_1 and T_2 are stable, Φ_1, $\Phi_2 \in l_1$ and so $\Psi = \Phi_1 + \Phi_2 \in l_1$. Hence, by Young's Theorem, $\forall x \in l_p$, $1 \leq p \leq \infty$, $\Phi_1 * x$, $\Phi_2 * x$, $\Psi * x \in l_p$ and it follows that $Tx = T_1 x + T_2 x$. There exist $k_1, k_2 > 0$ and $0 < c_1, c_2 < 1$ such that $|f_n| < k_1 c_1^{|n|}$ and $|g_n| < k_2 c_2^{|n|}$. Therefore, $|f_n + g_n| < (k_1 + k_2)(max\{c_1, c_2\})^{|n|}$ and Tx is stable;

(ii) similar to (ii). ∎

Lemma 25. *(i) Let $\{g_n\}$ be the sequence for a stable causal system, then $\{n^k p^{-n} g_n\} \in l_1$, for $k \geq 0$, $|p| \geq 1$.*

(ii) Let $\{g_n\}$ be the sequence for a stable acausal system, then $\{n^k p^{-n} g_n\} \in l_1$, for $k \geq 0$, $|p| \leq 1$.

Proof. (i) Since $g_n = 0$, $\forall n < 0$, and $|g_n| < kc^{|n|}$, $\forall n$, for some $k > 0$ and $c < 1$, the result follows immediately;

(ii) similar to (i). ∎

Since the analysis is centred on feedback systems the existence of the inverse of the return difference operator must be discussed. That is addressed in what follows.

Lemma 26. *Let $G(z) \neq -1$ be meromorphic on $\mathbb{C}$ such that the singular points at p_i, $i = 1, ..., N$ of order $k_1, ..., k_N$ are in the domain of the doublet $\{(1 + G(z))^{-1}, D_{(1+G)^{-1}})\}$, then $\{(1 + G(z))^{-1}, D_{(1+G)^{-1}})\}$ is the system function for a system with zero eigenvectors, $\{n^j p_i^n\}$, $i = 1, ..., N$, $j = 1, ..., k_i$.*

Proof. The proof is a consequence of the properties of the meromorphic functions. In fact, since they constitute a field, it follows immediately. ∎

Theorem 27. *Let T_P and T_S be stable systems with system functions, respectively, $\{G(z), D_G\}$ and $\{(1 + G(z))^{-1}, D_{(1+G(z))^{-1}}\}$. Then*

$$T_S x + T_P(T_S x) = (I + T_P)(T_S x) = T_S((I + T_P)x) = x, \ \forall x \in l_p, \ 1 \leq p \leq \infty$$

Proof. Since T_P and T_S are both stable, then $D_G \cap D_{(1+G)^{-1}} \neq \emptyset$. The proof is consequence of Lemma 21, Lemma 24 and Lemma 26. ∎

The system T_S, defined as above, is not necessarily the inverse of the system $T_{(1+G)}$, defined by $\{(1 + G(z)), D_{(1+G)}\}$.

The relation in time-domain between system functions is established when their domain of existence are disjoint.

Theorem 28. *Let $\{g_n\}$ and $\{f_n\} \in L^T$ be the double-sided sequences of two systems with system functions $\{G_1(z), D_1\}$ and $\{G_2(z), D_2\}$, respectively, such that $D_1 \cap D_2 = \emptyset$ and that $G_1(z) = G_2(z)$, with $G(z) = G_1(z) = G_2(z)$. The singular points* *of $G(z)$ are $z = p_1, ..., p_N$ with order $k_1, ..., k_N$, respectively. The difference between the Laurent coefficients of $\{G(z), D_1\}$ and $\{G(z), D_2\}$ is*

$$g_n - f_n = \sum_{i=1}^{N} \sum_{j=1}^{k_r} c_{i,j} n^{k_i - 1} p_i^n$$

where $c_{i,j} = 1, ..., N$, $j = 1, ...k_i$ are constants.

Proof. Define the region $D_{1\backslash 2}$ as the region between D_1 and D_2. Since $G(z)$ is meromorphic, the region $D_{1\backslash 2}$ contains only a finite number of singular points with finite order. The rest of the proof is an application of the Cauchy's Residue Theorem to the complex function $G(z)z^{n-1}$. In fact, since

$$g_n = \{\frac{1}{2\pi j} \oint_{C_1} G(z) z^{n-1} dz\}$$

and

$$f_n = \{\frac{1}{2\pi j} \oint_{C_2} G(z) z^{n-1} dz\}$$

where C_1 is an anticlockwise contour in D_1 and C_2 is an anticlockwise contour in D_2, with the radius of C_1 greater than the radius of C_2, then

$$\{\frac{1}{2\pi j} \oint_{C_1} G(z) z^{n-1} dz\} - \{\frac{1}{2\pi j} \oint_{C_2} G(z) z^{n-1} dz\} = h_n$$

with h_n the residues of the singular points in $D_{1\backslash 2}$. The rest follows immediately. ■

Consider the systems T_G, T_P and T_S with system functions $\{G(z), D_G\}$, $\{G(z), G_P\}$ and $\{(1 + G(z))^{-1}, D_{(1+G)^{-1}}\}$ respectively. Suppose that T_P is stable but not necessarily causal , that T_G is causal but not necessarily stable and that T_S is stable and causal. Let $T_G x = \Psi * x$, $T_P x = \Omega * x$ and $T_S x = \Phi * x$, with $\Psi = \{g_n\}$, $\Omega = \{f_n\}$ and $\Phi = \{h_n\}$.
The singular points of $G(z)$ are $p_1, ..., p_N$ with order $k_1, ..., k_N$, respectively. Consider the feedback systems of T_G and T_P,

$$\begin{cases} y = T_G u \\ u = r - y \end{cases} \tag{4.1}$$

$$\begin{cases} y = T_P u \\ u = r - y \end{cases} \tag{4.2}$$

when $r, u, y \in l_p$, $1 \leq p \leq \infty$. Define $\bar{p}$, a singular point of $G(z)$, such that

$$\bar{p} = max\{|p_i|\,,\ i = 1, ..., N\}$$

and

$$\bar{k} = max\{k_i,\ i \in \{j : |p_j| = |\bar{p}|\}\}$$

and define

$$\bar{p}l_p = \{x \in l^p :\ \{n^{\bar{k}}\bar{p}^{-n}x_n\} \in l^p\},\ 1 \leq p \leq \infty$$

Then the response of 4.1 is related to the response of 4.2 by the following Theorem.

Theorem 29. *Consider the systems T_G, T_P and T_S defined as above and the two feedback systems 4.1 and 4.2. Then*

(i) The feedback system 4.2 has, $\forall r \in D_{T_S}$, the solution

$$\begin{cases} y = T_P(T_S r) \\ u = T_S r \end{cases}$$

(ii) The feedback system 4.1 has, $\forall r \in \bar{p}l_p$, the solution

$$\begin{cases} y = T_P(T_S r) \\ u = T_S r \end{cases}$$

Proof. (i) By Theorem 27 $T_P(T_S x) = x - T_S x$, $\forall x \in D_{T_S}$. Therefore, $\forall r \in D_{T_S}$

$$\begin{cases} y = T_P u = T_P(T_S r)\ \in l_p \\ u = r - y = r - T_P(T_S r) = r - (r - T_S r) = T_S r\ \in l_p \end{cases}$$

as required.

(ii) By Theorem 28

$$T_G x = (\Omega + \sum_{i=1}^{N}\sum_{j=0}^{k_i} c_{i,j}\Psi_{i,j}) * x$$

with $\Psi_{i,j} = \{n^j p_i^n\}$ where p_i is a singular point of $G(z)$ with order k_i.
By Theorem 23 (a)(i) and (b), since T_P is stable and T_G is causal, $c_{i,j} = 0$, when $i \in \{j : |p_j| \leq 1\}$. Hence, the only relevant singular points are those such that $|p_i| > 1$.
Since T_P and T_S are stable, Ω, $\Phi \in l_1$ and $\Omega * (\Phi * x) \in l_\infty$, $\forall x \in l_p$, $1 \leq p \leq \infty$.
By Lemma 25 (i), since T_S is stable and causal, $\{n^j p_i^{-n} g_n\} \in l_1$, for $i = 1, ..., N$, $j = 0, ..., k_i$, and $\{n^j p_i^{-n} x_n\} \in l_1$, whenever $\{n^{\bar{k}}\bar{p}^{-n}x_n\} \in l_1$. Hence, by Theorem 18 (d),

$$T_G(T_S x) = T_P(T_S x) + T_R x$$

where

$$T_R x = \sum_{i}^{N} \sum_{j=0}^{k_i} c_{i,j} \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} ((z\frac{d}{dz})^r (1+G(z))^{-1}_{z=p_i})(\Psi_{i,j} * x)$$

By Theorem 23 (a)(i) and (b), all the singular points such that $|p_i| > 1$ are internal points of the domain of T_S, since T_S is causal and stable, and

$$(z\frac{d}{dz})^j (1+G(z))^{-1}_{z=p_i}) = 0$$

for $i = 1, ..., N$, $j = 1, ..., k_i$, since, by Lemma 26, the elements of the sequence $\{n^j a_i^n\}$, $i = 1, ..., N$, $j = 1, ..., k_i$ are zero eigenvectors of $(1+G(z))^{-1}$. It follows that $T_R(T_S x) = 0$ and $T_G(T_S x) = T_P(T_S x)$. Therefore, $\forall r \in \bar{p}l_p$

$$\begin{cases} y = T_G u = T_G(T_S r) = T_P(T_S r) \ \in l_p \\ u = r - y = r - T_P(T_S r) = r - (r - T_S r) = T_S r \ \in l_p \end{cases}$$

■

Theorem 29 implies that the response of a feedback system enclosing the unstable plant, T_G, to an input restricted to $\bar{p}l_p$, is the same as the response of the feedback system enclosing the stable plant, T_P, when the latter feedback is stable and causal. Therefore, when the class of inputs is restricted to $\bar{p}l_p$, the feedback system 4.1 is stable and causal if and only if the feedback system 4.2 is causal and stable.

To be more precise, by Young's Theorem and Theorem 27

$$\|y\|_p \le \|\Omega * \Phi\|_1 \|r\|_p, \ \forall r \in l^p, \ 1 \le p \le \infty$$

Hence, the feedback system enclosing T_P being stable in the sense that, for some $c > 0$, $\|y\|_p \le c\|r\|_p$, $\forall r \in l_p$, implies that the feedback system enclosing T_G is stable in the sense that $\|y\|_p \le c\|r\|_p$, $\forall r \in \bar{p}l_p$.

Note that $\bar{p}l_p$ is a subspace of l_p, but it is not closed in l_p. In fact, consider $r \in l_p$ such that $r \notin \bar{p}l_p$ and $\forall n \in \mathbb{Z}$, let s_n be the sequence with elements $s_{n,k}$ such that $s_{n,k} = 0$ for $n < -k$, $s_{n,k} = r_n$ otherwise. Then, $\forall n \in \mathbb{Z}$, $s_n \in l_p$, $1 \le p < \infty$, and $s_n \to r$ in l_p.

Example 3. Let T_G, T_P and T_S be the systems defined, respectively, by

$$T_G x = \Pi * x, \ \Pi = \{kba^{(n-1)}\Theta_n\}$$

$$T_P x = \Gamma * x, \ \Gamma = \{-kba^{(n-1)}(1-\Theta_n)\}$$

$$T_S x = \Phi * x,\ \Phi = \{\delta_{n0} - kb(a-kb)^{(n-1)}\Theta_n\}$$

where $\Theta_n = 1$, $n > 0$, zero otherwise. The system functions are, respectively, $\left\{\frac{kb}{z-a}, |z| > a\right\}$, $\left\{\frac{kb}{z-a}, |z| < a\right\}$ and $\left\{\frac{z-a}{z-(a-kb)}, |z| > |a-kb|\right\}$. The system function for T_G has a singular point at $z = a$ and $\{a^n\}$ is a zero eigenvector of T_S.
Let $x = \{a^{(n-1)}\Theta_n\}$, then, when $|a-kb| < |a|$, $T_S x = \{(a-kb)^{n-1}\Theta_n\}$ and

$$(I + T_P)(T_S x) = \{-a^{(n-1)(1-\Theta_n)}\}$$

Hence, $(I + T_P)(T_S x) \neq x$ and T_S is not the inverse of $(I + T_)$. When $a > 1$ and $|a - kb| < 1$, T_P, T_G and T_S are, respectively, stable, causal and stable and causal. Hence, the feedback system enclosing T_G is stable and causal with the same response as the feedback system enclosing T_P but only for the inputs $r \in a^{-n} l_p$.

4.3.2 Transform domain analysis

Consider a signal $x \in L^T$. The correspondent element in transform domain analysis is the doublet $\{X(z), D_X\}$. The doublet is the double sided z transform of the signal x. Hence, the doublets of the signals and of the systems of the Standard Formalism of Definition 22 are the elements for a transform domain analysis. Note that this Formalism is closely related to the conventional analysis in [1].

Similarly to Theorem 23 (i) and (ii), when

$$D_X = \{z \in \mathbb{C} :\ |z| > R\},\ x_n \neq 0, \forall n > 0$$

the signal is causal, and, when

$$D_X = \{z \in \mathbb{C} :\ |z| < R\},\ x_n \neq 0, \forall n \leq 0$$

the signal is acausal. Similarly to 23 (iii), when $D_G \supseteq \{z \in \mathbb{C} :\ |z| = 1\}$ the signal is stable. When the signals x and y have double sided z transforms, $\{X(z), D_X\}$ and $\{Y(z), D_Y\}$, respectively, and $D_X \cap D_Y \neq \emptyset$, $\{X(z)+Y(z), D_{X+Y} \supset D_X \cap D_Y\}$ corresponds to the signal $x + y$. Moreover, when the system T has system function $\{G(z), D_G\}$, $\{Y(z) = G(z)X(z), D_Y = D_{GX} \supset D_G \cap D_X\}$ is the double sided z transform of the the signal y, such that $y = Tx$. Hence, transform domain analysis can be applied to systems and signals with Definition 22.

Example 4. The contradiction between the time domain analysis and the transform analysis of 2.6 in [19] is further analysed using the doublet notation. With that the right-hand side in 2.8 becomes

$$\{G(z)W(z), (D_G \cap D_V) \cap D_D \supseteq D_G \cap D_W\}$$

where $W(z) = -kD(z) + V(z)$. In order to avoid a pole zero cancellation the condition $(D_G \cap D_V) \cap D_D \supseteq D_G \cap D_W$ becomes $(D_G \cap D_V) \cap D_D = D_G \cap D_W$. The algebraic manipulation above is made possible by requiring that there exists solutions in time domain, hence when $a^{-i}w[i] \to 0$ as $i \to -\infty$.
At the same time, supposed the algebraic result above, the right-hand side of 2.8 does not exists in a meaningful sense when $|a| > \bar{R}_W > 1$, since $D_G \cap D_W = \emptyset$. Hence this condition is sufficient for the existence of solutions in time and transform domain.

4.3.3 Standard Framework for double-sided signals and LTI systems

Note that Definition 22 is the definition of a Formalism, not of a Framework. In fact, the elements of the Standard Formalism do not meet three requirements for a consistent mathematical Framework:

1) The class of signals L^T is not a linear space. In fact, if $\{a_n\}$ and $\{b_n\}$ are in L^T the Laurent series of their sum might not exists if the intersection of the two domains is empty. This is demonstrated by the following example.

Example 5. Consider the Laurent series of the sequences $\{1^n\}_{n=1}^{\infty}$ and $\{1^n\}_{n=0}^{-\infty}$. The domain of the first if the region $|q| > 1$ in the complex plane, while the domain of the second is the region $|q| < 1$ in the complex plane. The sum of the two sequences is $\{1^n\}_{n=-\infty^\infty}$, which Laurent series does not exists since it has to satisfy the condition $1 < |q| < 1$.

2) The class of systems does not constitute an algebra. If $\{G_1, D_1\}$ and $\{G_2, D_2\}$ are two transfer functions such that $D_1 \cap D_{G_2} = \emptyset$, then their sum or composition might not exist.

3) The system with system function $\{(1+G(z))^{-1}, D_{(1+G(z))^{-1})}\}$ is not necessarily the inverse of the system with transfer function $\{\{(1+G(z)), D_{(1+G(z))}\}\}$.

Definition 30. Define $L_S^T \subset L^T$ by

$$L_S^T = \{\{a_n\} \in L^T : \ |q| = 1 \text{ implies } q \in D_{\mathcal{X}}\}$$

Note that $L_S^T \subset l_p$, $1 \leq p \leq \infty$.

Definition 31. L_S^T is the class of signals and the convolution sums on L_S^T, defined by $\Phi * x$, with $\Phi \in L_S^T$, are the systems for a Standard Framework in the time domain analysis.
The double-sided z transform of the signals and the convolution sums on L_S^T are the corresponding elements for a corresponding transform domain analysis.

For any system, T, with transfer function, $\{G(z), D_G\}$, and any signal, x, with double-sided z transform, $\{X(z), D_X\}$, $D_G \cap D_X$ is nonempty, open and contains the unit circle. Similarly, for any systems, T_1 and T_2, $D_{G_1} \cap D_{G_2}$ is also nonempty, open and contains the unit circle. Therefore the class of signals is a linear space and the class of systems constitutes an algebra. Moreover, the system with transfer function, $\{(1+G(z))^{-1}, D_{(1+G)^{-1})}\}$, is the inverse of the system with transfer function $\{(1+G(z)), D_{(1+G)})\}$. Since the requirements for a consistent Framework are satisfied Definition 31 is the definition of a consistent Mathematical Framework.

In the Standard Framework all signals and systems are stable but not necessarily causal. The analysis of the feedback system is no longer concerned with establishing the stability of the closed-loop system but with establishing its causality.
Consider T_G, a causal but unstable open loop system on L^T, together with its transfer function, $\{G(z), D_G\}$, such that $G(z)$ is analytic on the unit circle. Consider T_P, the associated acausal but stable open loop system on L_S^T, with transfer function $\{G(z), D_P\}$. By Theorem 29, the closed loop system for T_G is causal and stable provided the closed loop system for T_P is stable and causal, but only when the class of signals is restricted to $\bar{p}l_p$.

Example 6. Let T_G the unstable causal system

$$T_G x = \Pi * x, \ \Pi = \{kba^{(n-1)}\Theta_n\}, \ a > 1$$

Its system function is $\left\{\frac{kb}{z-a}, |z| > a\right\}$. The associated stable acausal system, T_P, has system function $\left\{\frac{kb}{z-a}, |z| < a\right\}$.
Since $(I + T_P)^{-1}$ has system function $\left\{\frac{z-a}{z-(a-kb)}, |z| > |a - kb|\right\}$, the closed loop system for T_P is causal as well as stable, provided $|a - kb| < 1$. Hence, by Theorem 29, the closed loop system for T_G is stable as well as causal, provided $|a - kb| < 1$. Note that the inputs to the latter closed loop are restricted to $a^{-n}l_p$. For these inputs, the responses of the two closed loop systems are the same.

4.4 Generalised Formalism for Stable and Unstable Systems

Define $C^\infty(\mathbb{R})$, the linear space of infinity differentiable complex valued functions on the real line. A subspace $\hat{C} \subseteq C^\infty(\mathbb{R})$ is shift-invariant, when $f(t) \in \hat{C}$ implies $f(t-a) \in \hat{C}$, $\forall a \in \mathbb{R}$. Define $\mathcal{T}$ the linear space of linear functionals with domain a shift-invariant subspace of $C^\infty(\mathbb{R})$ such that, when $x \in \mathcal{T}$,

$$f(t) \mapsto x[f(t)]$$

where $x[f(t)]$ is the value of the functional x assigned to each $f(t)$ in its domain.
$\mathcal{T}_D$ and $\mathcal{T}_S$ are subspaces of $\mathcal{T}$ with domain containing D and S, respectively.
Define x_a the shifted functional such that $x_a[f(t)] = x[f(t+a)]$ and $\delta_\tau \in \mathcal{T}_S$ the delta functional such that $\delta_\tau[f(t)] = f(\tau)$ for all functions in $C^\infty(\mathbb{R})$.
$\mathcal{T}_\Delta \subset \mathcal{T}_D$ is the subspace of functionals defined by

$$\sum_{m=-\infty}^{\infty} a_m \delta_{mT}$$

Definition 32. Two elements in $\mathcal{T}_\Delta$ are equivalent if, given x,y in $\mathcal{T}_\Delta$, $x[f(t)] = y[f(t)]$ for all $f \in D$.

Consider the operator T_A defined on a shift-invariant subspace of $C^\infty(\mathbb{R})$ containing D such that

$$T_A f(t) = \sum_{m=-\infty}^{\infty} c_{-m} f(t - mT) \tag{4.3}$$

Consider the operator T on $\mathcal{T}_\Delta$ such that

$$T\left(\sum_{m=-\infty}^{\infty} a_m \delta_{mT}\right) = \sum_{m=-\infty}^{\infty} b_m \delta_{mT} \tag{4.4}$$

where $b_m = \sum_{n=-\infty}^{\infty} a_n c_{m-n}$.

Lemma 33. *Consider the operators T on $\mathcal{T}_\Delta$ as above and T_A as above. If D_T and R_T are subspaces of $\mathcal{T}_\Delta$ and $\sum_{n=-\infty}^{\infty} a_n c_{(m-n)}$ exists, then*

$$Tx[f(t)] = x[T_A f(t)]$$

for all $f(t) \in D$.

Proof. It follows from Definition 32 that

$$\begin{aligned} &x[T_A f(t)] \\ &\quad = x[\sum_{m=-\infty}^{\infty} c_{-m} f(t - mT)] = \sum_{n=-\infty}^{\infty} a_n \sum_{m=-\infty}^{\infty} c_{-m} f(nT - mT) \\ &\quad = \sum_{n=-\infty}^{\infty} a_n \sum_{m=-\infty}^{\infty} c_{-(n-m)} f(mT) = \sum_{m=-\infty}^{\infty} (\sum_{n=-\infty}^{\infty} a_n c_{-(n-m)}) f(mT) \end{aligned}$$

Define $b_m = \sum_{n=-\infty}^{\infty} a_n c_{(m-n)}$ and the result follows immediately from Definition 32. ■

Define $\mathcal{Q}_T$ the class of operators T on $\mathcal{T}_\Delta$ as

$$\mathcal{Q}_T = \{T : \exists T_A \text{ as in 4.3 such that } Tx[f(t)] = x[T_A f(t)]\} \tag{4.5}$$

Lemma 34. *Let T_1 and T_2 operators on $\mathcal{T}_\Delta$ such that $T_1 x[f(t)] = x[T_{A1} f(t)]$ and $T_2 x[f(t)] = x[T_{A2} f(t)]$, where*

$$T_{A1} f(t) = \sum_{m=-\infty}^{\infty} c_{-m} f(t - mT)$$

and

$$T_{A2} f(t) = \sum_{m=-\infty}^{\infty} \bar{c}_{-m} f(t - mT)$$

Then $T_1 x + T_2 x = T_x$, when $T_1 x$ and $T_2 x$ exist, where T is an operator on $\mathcal{T}_\Delta$ such that

$$Tx[f(t)] = x[T_A f(t)]$$

with $T_A f(t) = T_{A1} f(t) + T_{A2} f(t)$

Proof. Since the operators are linear and by Lemma 33 the result follows immediately. ■

Lemma 35. *Suppose* $T_1x_1 = T_2x_2 \in \mathcal{T}_\Delta$ *and* $T(T_1x) \in \mathcal{T}_\Delta$*, with* T,T_1*,* $T_2 \in \mathcal{Q}_T$*, then* $T(T_1x_1) = T(T_2x_2)$*.*

Proof. Since $T_1x_1 = T_2x_2 \in \mathcal{T}_\Delta$, $T_1x_1[f(t)] = T_2x_2[f(t)]$ for all $f(t)$ in their domain. Let T be defined by T_A, then $T_1x_1[T_Af(t)]$ exists and $T_1x_1[T_Af(t)] = T_2x_2[T_Af(t)]$. Hence, $TT_1x_1[f(t)] = TT_2x_2[f(t)]$. ■

Consequently, the operators $T \in \mathcal{Q}_T$ can be extended to include all the pairs, $x \mapsto y$, such that $y = T(\hat{T}w)$ and $x = \hat{T}w$, for some $\hat{T} \in \mathcal{Q}_T$ and $w \in \mathcal{T}_\Delta$. All the operators in $\mathcal{Q}_T$ can be extended in this way and let $\mathcal{Q}_{T_1}$ be the class consisting of them. That can be repeated to construct the extension classes $\mathcal{Q}_{Tr}$, $r \geq 0$, with $\mathcal{Q}_{T0} = \mathcal{Q}_T$.

Definition 36. Let $\bar{\mathcal{Q}}_T$ be the class of maximal extensions of the operators in $\mathcal{Q}_T$, where the maximal extension of $T \in \mathcal{Q}_T$ is defined by all the pairs, $x \mapsto y$, such that $y = Tx$ is in $\mathcal{Q}_{Tr}$ for some $r \geq 0$.

Within $\bar{\mathcal{Q}}_T$, repeated cascading of operators is consistently defined.

Definition 37. $\mathcal{T}_\Delta$ is the class of signals and the operators in $\bar{\mathcal{Q}}_T$ are the systems for a Generalized Formalism in time domain analysis.

Let the signals

$$\sum_{m=-\infty}^{\infty} a^m \delta_{mT}$$

be in the domain of the system T. By Lemma 16 these are λ_a eigenvectors of T and $K(z)$, the analytic continuation of the eigenvalues λ_a is the system function of the operator T, as in Definition 17.

A system in $\bar{\mathcal{Q}}_T$, defined by the operator T_A

$$T_Af(t) = \sum_{m=-\infty}^{\infty} \Phi(-m)f(t - mT)$$

is causal if $\Phi_n = 0$ for $n > 0$, acausal if $\Phi_n = 0$ for $n < 0$. It is stable if there exists a $k > 0$ and a $0 < c < 1$ such that $|\Phi_n| < kc^{|n|}$.

Consider the the functional $v \in \mathcal{T}_\Delta$, an eigenvector of the system T, defined as in 4.4, on $\mathcal{T}_\Delta$ with eigenvalue λ. Define the operator V_A on a shift-invariant subspace of $C^\infty(\mathbb{R})$ containing D, such that

$$V_Af(t) = \sum_{m=-\infty}^{\infty} v_{-m}f(t - mT) \tag{4.6}$$

Lemma 38. *$V_A f(t)$ is an eigenvector of T_A with eigenvalue λ.*

Proof. Since v is a λ-eigenvector it follows

$$Tv[f(t)] = \lambda \sum_{m=-\infty}^{\infty} v_m f(mT) = \sum_{m=-\infty}^{\infty} (\sum_{n=-\infty}^{\infty} v_{-n} c_{m-n}) f(mT)$$

If $(\sum_{n=-\infty}^{\infty} v_{-n} c_{m-n})$ exists, then

$$\sum_{n=-\infty}^{\infty} v_{-n} c_{m-n} = \lambda v_{-m}$$

From the definition of $V_A f(t)$,

$$T_A V_A f(t) = \sum_{n=-\infty}^{\infty} c_{-n} \sum_{m=-\infty}^{\infty} v_{-m} f(t-(m+n)T)$$

$$= \sum_{m=-\infty}^{\infty} (\sum_{n=-\infty}^{\infty} c_{-n} v_{m-n}) f(t-mT)$$

hence, from above,

$$\sum_{m=-\infty}^{\infty} (\sum_{n=-\infty}^{\infty} c_{-n} v_{m-n}) f(t-mT) = \lambda \sum_{n=-\infty}^{\infty} v_{-n} f(t-nT) = \lambda V_A f(t)$$

■

Define the system V on $\mathcal{T}_\Delta$ as $Vx = x[V_A f(t)]$ where V_A is the same as in 4.6. Consider the systems T and T_S on $\mathcal{T}_\Delta$, such that

$$(I+T)T_S x[f(t)] = x[T_{SA}(I+T_A)f(t)] = x[f(t)]$$

Theorem 39. *Let T and T_S be two systems in $\bar{\mathcal{Q}}_T$ defined as above. Let V be a system in $\bar{\mathcal{Q}}_T$ defined as above, with v a zero eigenvector of T_S. Consider the two feedback systems*

$$\begin{cases} y = Tr \\ u = r - y \end{cases} \tag{4.7}$$

and

$$\begin{cases} y = (T+V)u \\ u = r - y \end{cases} \tag{4.8}$$

Then the feedback systems 4.7 and 4.8 have, $\forall r \in D_{TS}$, the same solution

$$\begin{cases} y = T(T_S r) \\ u = T_S r \end{cases}$$

Proof. The solution to 4.7 follows from the way in which T and T_S are defined. By Lemma 38 if v is a zero eigenvector of T_S then $V_A f(t)$ is a zero eigenvector of T_{SA}. Hence

$$VT_S x[f(t)] = x[T_{SA} V_A f(t)] = 0, \forall x \in \mathcal{T}_\Delta$$

Therefore, $\forall r \in D_{T_S}$

$$\begin{cases} y &= (T+V)u = (T+V)T_S r = r[T_{SA}(T_A + V_A)f(t)] \\ & \quad r[T_{SA}T_A f(t) + T_{AS}V_A f(t)] = T(T_S r) \qquad \in \mathcal{T}_\Delta \\ u &= r - y = r - T(T_S r) = r - (r - T_S r) = T_S r \qquad \in \mathcal{T}_\Delta \end{cases}$$

■

Theorem 39 implies that the response for a feedback system enclosing the plant T to an input in $\mathcal{T}_\Delta$ is the same as the response of the feedback system enclosing the plant $(T+V)$.

This Formalism is a generalization of the Standard Formalism. Consider $\{c_n\}$, an element of the class of signals L^T, and the functional $c \in \mathcal{T}_\Delta$ such that

$$c = \sum_{m=-\infty}^{\infty} c_m \delta_{mT}$$

The map $\{c_n\} \mapsto c$ is an isomorphism between L^T and a linear shift-invariant subspace of $\mathcal{T}_\Delta$.

Similarly, consider the system T_1 on L^T, such that $T_1 x = \Phi * c$, with $\Phi = \{g_n\}$, and the system $T_2 \in \bar{\mathcal{Q}}_T$ such that

$$T_2 x = \sum_{m=-\infty}^{\infty} (\sum_{n=-\infty}^{\infty} a_n g_{m-n}) \delta_{mT}$$

The map $T_1 \mapsto T_2$ is is isomorphism between the convolution sums on L^T and a subspace of $\bar{\mathcal{Q}}_T$. Clearly, when the response exists in both formulations, they are the same.

Consider T_1 on L^T as above. Let T_A be

$$T_A f(t) = \sum_{m=-\infty}^{\infty} \Phi(-m) f(t - mT)$$

where Φ is the same as in T_1. It follows that $T_2 \in \bar{\mathcal{Q}}_T$ as above is given by

$$T_2 x[f(t)] = x[T_A f(t)] = x[\sum_{m=-\infty}^{\infty} \Phi(-m) f(t - mT)]$$

Lemma 40. *If the double-sided sequence $\{v_n\} \in L^T$ is an eigenvector of a system T_1 on L^T, then the functional $\sum_{m=-\infty}^{\infty} v_m \delta_{mT} \in \mathcal{T}_\Delta$ is an eigenvector of the correspondent system T on $\mathcal{T}_\Delta$.*

Proof. The result is a consequence of the discussion above. ■

Because of the isomorphisms between L^T and a subspace of $\mathcal{T}_\Delta$ and the convolution sums on L^T and a subspace of $\bar{\mathcal{Q}}_T$, properties of the systems on L^T, such as causality and stability, are transferred to the equivalent systems on $\mathcal{T}_\Delta$. It follows that the feedback systems 4.1 and 4.2, when extended into their equivalent feedback systems in the Generalized Formalism, have the same response for all inputs in D_{T_S}. In fact, to the stable system T_P, the causal system T_G, the stable and causal system T_S, on L^T, correspond the system T, $(T+V)$ and T_S on $\mathcal{T}_\Delta$, of Theorem 39. Therefore, the response of a feedback system enclosing the unstable but causal plant $(T+V)$ is the same as the response of the feedback system enclosing the stable but acausal plant T_P, for any input in D_{T_S}.

Example 7. Consider the systems T_G, T_P and T_S, defined, respectively, by

$$T_G x[f(t)] = x[G_A f(t)],\ G_A f(t) = \sum_{i=-\infty}^{\infty} kba^{(1-i)} \Theta_{-i} f(t-iT)$$

$$T_P x[f(t)] = x[P_A f(t)],\ P_A f(t) = \sum_{i=-\infty}^{\infty} kba^{(1-i)} (1-\Theta_{-i}) f(t-iT)$$

$$T_S x[f(t)] = x[S_A f(t)],\ S_A f(t) = f(t) - \sum_{i=-\infty}^{\infty} kb(a-kb)^{(1-n)} \Theta_{-i} f(t-iT)$$

where $\Theta_i = 1$ for $i > 0$, zero otherwise. T_P, T_G and T_S are , respectively, stable, causal and stable and causal.

T_S is such that $(I+T_P)(T_S x) = x$. Hence, the solution to the feedback system

$$\begin{cases} y = T_P u \\ u = r - y \end{cases} \tag{4.9}$$

is given by

$$\begin{cases} y = T_P T_S r \\ u = T_S r \end{cases}$$

Consider the operator V, defined by

$$V x[f(t)] = x[V_A f(t)],\ V_A f(t) = \sum_{i=-\infty}^{\infty} kba^{(1-n)} f(t-iT)$$

Since $T_G = T_P + V$ and $VT_S x = 0$, by Theorem 39 the feedback system

$$\begin{cases} y = T_G u \\ u = r - y \end{cases} \tag{4.10}$$

has the same solutions as 4.9, for $\forall r \in D_{T_S}$. Hence, the feedback system enclosing T_G is causal and stable with the same response as the feedback system enclosing T_P, for all signals in D_{T_S}.

4.5 A Framework using Distributions

Consider the subclass of the distributions, $\mathcal{D}_E^T$, the subclass of $\mathcal{U}$, $\mathcal{U}_E^T$, and the class of multipliers $\mathcal{M}^T$.

Definition 41. $\mathcal{D}_E^T$ is the class of signals and the convolutes on $\mathcal{D}_S$, mapping $\mathcal{D}_{EN}^T$ into $\mathcal{D}_{EN}^T$, are the systems for a Framework in time domain analysis.
$\mathcal{U}_E^T$ is the class of signals and the multipliers in $\mathcal{M}^T$ on $\mathcal{U}_S$, mapping $\mathcal{U}_{EN}^T$ into $\mathcal{U}_{EN}^T$, are the systems for a Framework in transform domain analysis.

It is first proved that the the systems so defined are an algebra.

Theorem 42. $\mathcal{M}^T$ *constitutes an algebra.*

Proof. The multipliers in $\mathcal{M}^T$ are periodic linear operators on $\mathcal{U}_E^T$ mapping elements of $\mathcal{U}_{EN}^T$ into $\mathcal{U}_{EN}^T$. Hence, they constitute an algebra of periodic operators on $\mathcal{U}_E^T$. Moreover, the sum and product of two periodic multipliers are themselves periodic multipliers defined simply by the sum and product, respectively, of the functions defining the original multipliers. Since $\mathcal{M}$ is an algebra, the result follows. ∎

In what follows it is proved the existence of inverse of the return difference operator.

Lemma 43. *Let M be a regular functional defined by the infinitely differentiable function $M(\omega)$ and $M^{(r)}$, the regular functional defined by $M^{(r)}(\omega)$, be its r^{th} derivative. Then M is a periodic multiplier on $\mathcal{U}_E^T$ mapping $\mathcal{U}_{EN}^T$ into $\mathcal{U}_{EN}^T$ for all $N \geq 0$ provided M is periodic with period $2\pi/T$.*

Proof. Given a functional $x \in \mathcal{D}_{EN}^T$ defined by the sequence $\{x_n\}$, $x_n = ym_t^N$, where y is the functional defined by the sequence $\{y_n\} = \{x_n/(1 + jnT)^N\}$. It follows that

$$m * x = m * (ym_t^N) = \sum_{r=0}^{N} \left(\begin{array}{c} N \\ r \end{array} \right) \mathcal{F}^{-1}\{M^{(r)}Y\} m_t^{N-r}$$

where $Y = \mathcal{F}\{y\}$. By Theorem 15.24 of [4], since $\{y_n\}$ is square summable, the periodic function $Y(\omega) = \mathcal{P}\{y[k]\}(\omega)$ is square integrable over a single period and Y is the regular functional defined by $Y(\omega)$. In addition, $M^{(r)}(\omega)Y(\omega)$ is square integrable over a single period for all $r \geq 0$. Hence, $\mathcal{F}^{-1}\{M^{(r)}Y\}$ is the functional in $\mathcal{D}_{EN}^T$ defined by the square summable sequence $\mathcal{P}^{-1}\{M^{(r)}(\omega)Y(\omega)\}$. It follows that $m * x \in \mathcal{D}_{EN}^T$ and M is a multiplier on $\mathcal{U}T_E$ mapping $\mathcal{U}_{EN}^T$ into itself for all $N \geq 0$ as required. ■

Theorem 44. *Let $M \in \mathcal{M}^T$ be a multiplier on $\mathcal{U}_S$ defined by the function $M(\omega)$ such that $(I + M(\omega))$ has no finite zeros. Then, the functional $(I + M)^{-1}$ exists and is a multiplier and inverse on $\mathcal{U}_S$.*

Proof. Since $M(\omega)$ is periodic and everywhere infinitely differentiable and $(1 + M(\omega))$ has no finite zeros, $(1 + M(\omega))^{-1}$ is periodic and everywhere infinitely differentiable. It follows immediately from Theorem 16.22 of [4] and Lemma 43, that $(1 + M(\omega))^{-1}$ defines a periodic multiplier in $\mathcal{M}^T$. Furthermore, since $(1 + M(\omega))^{-1}(1 + M(\omega)) = 1$, the multiplier is an inverse as required. ■

Moreover, the class of signals $\mathcal{D}_E^T$ is a Banach space. In fact, consider $x \in \mathcal{D}_E^T$, $x = \sum_{k=-\infty}^{\infty} a_k \delta_{kT}$, then the norm associated to $\mathcal{D}_{EN}^T$ is

$$\|x\| = \left(\sum_{j=-\infty}^{\infty} \left| \frac{a_j}{1 + |jT|^N} \right|^2 \right)^{\frac{1}{2}}$$

It follows that all the 5 Requirements for a Framework to be consistent are satisfied. It remains to be proved that the system function and the multiplier in $\mathcal{M}^T$ are the same

Theorem 45. *The system function for T is the multiplier $K(j\omega) \in \mathcal{M}^T$.*

Proof. Consider v_ν the regular functionals defined by the function $e^{j\nu t} \sum_{k=-\infty}^{\infty} \delta_{kT}$. Then

$$\mathcal{F}\{v_\nu\} = \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta_{\nu + \frac{2\pi k}{T}}$$

and

$$\mathcal{F}\{\Phi * v_\nu\} = K(\omega)\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta_{\nu+\frac{2\pi k}{T}} = \frac{1}{T}\sum_{k=-\infty}^{\infty}K(\nu+\frac{2\pi}{T})\delta_{\nu+\frac{2\pi k}{T}}$$
$$= K(\nu)\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta_{\nu+\frac{2\pi k}{T}}$$

Hence $\Phi * v_\nu = K(j\nu)v_\nu$ Therefore v_ν is an eigenvector of T with eigenvalue $K(\nu)$. The result follows from the Definition of system function. ■

4.5.1 Standard Framework, Generalised Formalism and Distributions Framework

Consider the subspace of the distributions $\mathcal{D}_E^T$. Let Φ be a convolute on $\mathcal{D}_S$, the space of tempered distributions.

Theorem 46. *(i) the space $\mathcal{D}_E^T$ is isomorphic to the class of polynomially bounded signals.*

(ii) The systems in the Standard Framework are isomorphic to a subclass of the convolutes on $\mathcal{D}_S$.

Proof. (i) Obviously, the class of sequences is isomorphic to a subclass of $\mathcal{D}^T$. The result follows from the Definition of $\mathcal{D}_E^T$.

(ii) Consider $Tx = \{\Phi_n\}*\{x_n\}$ a system in the Standard Framework of Definition 31. Because the systems of the Standard Framework are stable, then

$$|\Phi_n| < kc^{|n|}$$

for k a constant and $0 < c < 1$. Hence the functional $\Phi = \sum_{m=-\infty}^{\infty}\Phi_m\delta_{mT}$ is a member of $\mathcal{D}_S$. ■

Let Φ be a convolute on $\mathcal{D}_S$ and let $K(j\omega)$ the corresponding multiplier in $\mathcal{M}^T$. Define the operator T on $\mathcal{D}_E^T$ as $Tx = \Phi * x$. Consider the same operator T as a system in the Generalised Formalism and let $\bar{K}(z)$ be the system function of T. The relation between $K(j\omega)$, the multiplier in $\mathcal{M}^T$ associated to the convolute Φ and $\bar{K}(z)$ is established in the next Theorem.

Theorem 47. *Let $K(\omega)$ be the regular functional defined by $K(\omega) = \bar{K}_{z=e^{j\omega T}}$. Then $K(j\omega)$ is the multiplier associated to the convolute Φ.*

Proof. Since v_ν, as defined in Lemma 45, is also an eigenvector of the Generalised Formalism version of T, and $\bar{K}(z)_{z=v_\nu} = K(v_\nu)$, the result follows from the Definition of system function and Theorem 45. ∎

Consider Φ in L^T, defining a causal but unstable convolution sum on L^T, together with its system function, the doublet $\{K(z), D_K\}$, with the further condition that $K(z)$ does not have finite zeros on the circle with unitary radius. The feedback system enclosing the unstable convolution sum

$$\begin{cases} y = \Phi * u \\ u = r - y \end{cases} \tag{4.11}$$

is extended in the Generalized Framework to

$$\begin{cases} y = T_G u \\ u = r - y \end{cases} \tag{4.12}$$

where $T_G : \mathcal{T}_\Delta \to \mathcal{T}_\Delta$ is given by

$$T_G x[f(t)] = x[T_G^A f(t)]$$

and $T_G^A = \sum_{m=-\infty}^{\infty} \Phi_{-m} f(t - mT)$. T_G is the operator of an unstable but causal system on $\mathcal{T}_\Delta$. By Theorem 39, 4.12 is equivalent to

$$\begin{cases} y = T_P u \\ u = r - y \end{cases} \tag{4.13}$$

where $T_P : \mathcal{T}_\Delta \to \mathcal{T}_\Delta$ is the operator of a stable systems such that $T_P = T_G - V$, $Vx = x[V_A f(t)]$, V_A the same as in 4.6, where the eigenvectors chosen are the zero eigenvectors of $(I + T_P)^{-1}$. Note that, for the two feedback systems to be equivalent, the inputs must be in $D_{(I+T_P)^{-1}}$. From the discussion above the stable feedback system enclosing T_P can be rewritten as a stable feedback system in which T_P is reconsidered as an operator acting on $\mathcal{D}_E^T$, with the advantage that the latter possesses a transfer function, $K(j\omega)$, such that $K(j\omega) = K(z)_{z=e^{j\omega T}}$.

Example 8. The feedback systems enclosing T_P and T_G of Example 5, in the Standard Formalism, are obviously extended to the feedback systems enclosing T_P and T_G of Example 7, in the Generalised Formalism. In both Formalisms it is proven that the feedback system enclosing T_G is stable and causal with the same response as the feedback system enclosing T_P. However, while in the Standard Formalism it is necessary to restrict the class of signals to $\bar{a}l_p$, in the Generalized Formalism the result holds for all the signals in $D_{T_S} \subseteq \mathcal{T}_\Delta$. Moreover, from the discussion above, the feedback system enclosing T_P in the Generalized Formalism is equivalent to a feedback system where T_P is a convolute on $\mathcal{D}_S$ mapping $\mathcal{D}^T_{EN}$ into itself. Not only a restriction of the class of signals is not necessary anymore, but the context used for the analysis, the Framework for Distributions, is a consistent mathematical framework, since $(I + T_P)^{-1}$ is well defined. Furthermore, the time domain analysis is coupled with a transform domain analysis. In fact to the convolutes T_P and $(I + T_P)^{-1}$ corresponds, as a transfer function, the multipliers in $\mathcal{M}^T$ on $\mathcal{U}_\mathcal{S}$, mapping $\mathcal{U}^T_{EN}$ into itself, $kb/(e^{j\omega T} - a)$ and $(e^{j\omega T} - a)/(e^{j\omega T} + kb - a)$, respectively. The analysis is not centred anymore on stability, but on causality.

4.6 A Framework for Singular Systems

In Distribution Theory a non regular functional in $\mathcal{D}^T_E$ can be always represented by the limits of a sequence of regular functionals in $\mathcal{D}^T_E$. A classical example ([4]) is the functional x^{-1}, that can be represented as the limit

$$\lim_{\epsilon \to 0} \frac{x}{x^2 + \epsilon^2}$$

However, the result does not apply to convolutes. Consider the sequence of convolutes $\{\Phi_n\}$, such that Φ_n is a convolute for any n. The convolute Φ, such that

$$\Phi = \lim_{n \to \infty} \Phi_n$$

exists as an operator on $\mathcal{D}^T_E$, but that is not necessarily a convolute anymore, hence it is not a system in the Distributions Framework. Moreover, even if Φ is a convolute, and $\Phi_n * x = y_n$ that is not sufficient to establish that

$$(\lim_{n \to \infty} \Phi_n) * x = \lim_{n \to \infty} y_n$$

In fact, it must be required that x is a convolute, as in [4].
Clearly, with this modification when singular, the operators Φ, limits of sequences of convolutes $\{\Phi_n\}$ form an algebra.

Definition 48. The member of $\mathcal{D}^T_E$ that are convolutes on $\mathcal{D}^T_E$ are the signals and the limits of sequences of convolutes on $\mathcal{D}^T_E$ are the systems for a Framework for singular systems.

The subspace of $\mathcal{U}_E^T$ that corresponds to the Fourier transforms of convolutes on $\mathcal{D}_E^T$ and the Fourier transform of the limits of convolutes on $\mathcal{D}_E^T$ are the corresponding elements for a corresponding transform domain analysis.

Consider the plant T such that $Tx = \lim_{n\to\infty}(T_n x)$, with $T_n x = \Phi_n * x$, Φ_n as above. The feedback system

$$\begin{cases} y = lim_{n\to\infty}(T_n u) \\ u = r - y \end{cases}$$

still has a solution. In fact if $\{M_n\}$ is the sequence of multipliers corresponding to the convolutes $\{\Phi_n\}$, then $(I + M_n)^{-1}$ is an infinite differentiable function for any n. Therefore, $\lim_{n\to\infty}(I+M_n)^{-1}$ is well defined and $(I+\lim_{n\to\infty}\Phi_n)^{-1}$ exists and is the system representing the inverse difference operator.

Example 9. Consider the singular system with system function $K(z) = 1/(z-1)$. Replace it with the limit

$$\lim_{n\to\infty} K_n(z) = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{z-1+\frac{1}{n}} + \frac{1}{z-1-\frac{1}{n}}\right)$$

$\{K_n(z)\}$ is a sequence of multipliers in $\mathcal{M}_E^T$. In time domain $\{\Phi_n\}$ is the sequence of corresponding convolutes. In the Framework for Singular Systems

$$(\lim_{n\to+\infty} \Phi_n) * x = \lim_{n\to\infty}(\Phi_n * x)$$

because x is a convolute on $\mathcal{D}_E^T$, but the operator $\lim_{n\to\infty}\Phi_n$ is not a convolute. However, the closed loop system

$$\begin{cases} y = lim_{n\to\infty}(\Phi_n * u) \\ u = r - y \end{cases}$$

admits solution for any input that is a convolute on $\mathcal{D}_E^T$ and its transfer function is the multiplier

$$\lim_{n\to+\infty} \frac{1}{1+K_n(z)} = \frac{z-1}{z}$$

The usual transform domain analysis can now be applied.

4.7 Comments on the previous literature

Consider the feedback system of Figure 4.2. When the plant is the stable operator

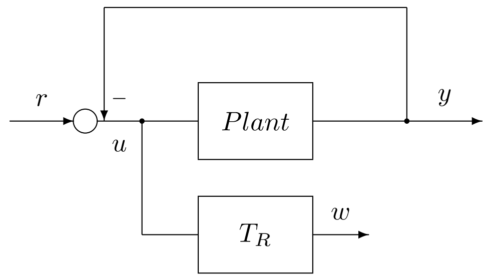

Figure 4.2: Feedback system.

T_P, the closed loop system is

$$\begin{cases} y = T_P u \\ u = r - y \end{cases}$$

Clearly, the solution is $y = T_P(T_S r)$, $u = T_S r$ when $(I + T_P)T_S x = x$, $x \in D_{T_S}$. When the plant is the unstable operator $T_G = T_P + T_R$, where T_R is constructed from the eigenvectors with eigenvalue zero of T_S, then the solutions are the same, provided $T_R(T_S x) = 0$. Hence, the requirement for the existence of the same solutions for the two systems is $r \in D_{T_S}$, $T_S r \in D_{T_R}$, $T_S r \in D_{T_P}$, regardless of the Framework chosen.

In [8] the feedback system of Figure 4.2 is treated as the input output model of Figure 4.3. Now the closed loop system is simply $y = T_C r$ and the requirement is

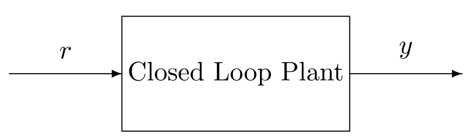

Figure 4.3: Input Output Box

$r \in D_{T_C}$. However, there exists some r that do not belong to D_{T_R}.

From above, regardless of the Framework chosen, when r satisfies the required conditions the two closed loop systems exist and have the same response. We denote by D_u the set of corresponding plant inputs, u. The graphs for the two plants can be constructed with the plant inputs restricted to D_u. Clearly they are the same. The closure of this common graph, when the plant is T_P, is the entire signal space of the Framework chosen, but when the plant is T_G there are signals that do not belong to D_{T_G}. Therefore this common graph is not closed, and the graph of T_S

has no closure. It follows that any conclusion regarding the properties of the plant that is deduced from the closure of the graph does not automatically apply to the unstable plant, but only to the stable one in the feedback situation.

In [11] the author states that the goals are "*to solve the problem concerning stabilisability for systems over the signal space* $l_2(\mathbb{Z})$" and "*show that the stabilisable systems and the stabilizing controllers are the same and thus it does not really matter wether we work with* $l_2(\mathbb{N}_0)$ *or* $l_2(\mathbb{Z})$ *as signal space* ". In order to do so an augmented definition of stabilisability is provided for systems with signal space $l_2(\mathbb{Z})$.

Definition 49. An $LTI(\mathbb{Z})$-system P is called stabilisable if P is closable and there exists an $LTI(\mathbb{Z})$-system C such that the feedback system $[\bar{P}, C]$ is stable and causal.

Let P an LTI system such that $[P, C]$ is stable. By Definition 4.1 in [11] it must be that $\bar{D}_P = l_2(\mathbb{Z})$. Moreover, because of Proposition 4.7 in [11], it must be that P is closed. It follows from the Definition of stability for an LTI system that P must be stable, but not necessarily causal.
Let P be a closable LTI system. Theorem 6.2 in [11] proves that $\bar{P}$ is stabilisable if and only if $\bar{P}_{\mathbb{N}}$ is stabilisable. Hence $\bar{P}$ must be stable, but not necessarily causal. $\bar{P}$ being stable, as defined in the paper, implies that P is stable in the conventional sense, mapping square summable signals onto square summable signals. Hence the main result is applicable to conventionally stable but possibly non-causal systems. Moreover, the above conclusion, is supported by the transform domain analysis in [11]. There is an extensive use of transfer functions and, from Definition 4.3 in [11], a transfer function for a closed system is a member of $R(L_\infty(T))$. Hence its domain is on the unit circle, the domain of the transfer function of a stable system, not necessarily causal.

In [23] the standard plant model $y = Pu$ is replaced by a $Ay = Bu$ when A and B are bounded operators. If the feedback system of the following example (the plant P is a convolution operator) has to be analysed by this method, then the convolution is converted in the two bounded operators A and B such that

$$(Ay)[k] = ay[k] - y[k-1]$$

$$(Bu)[k] = -u[k]$$

The convolution equation corresponding to P then becomes a difference equation corresponding to $Ay = Bu$. The following input is considered

$$u_a[k] = u[k] + \frac{1-a}{a} \sum_{i=0}^{+\infty} a^{-i} u[k+i]$$

for any $u \in l_\infty(\mathbb{Z})$ such that the series $\sum_{i=-\infty}^{\infty} |u[i]|$ converges absolutely to $\bar{u}$. Clearly

$$\sum_{i=0}^{\infty} a^{-i} u[k+i] \leq \bar{u}$$

if $a \geq 1$ and for all k. Therefore $u_a[k] \in l_\infty(\mathbb{Z})$ and $u_a \to u$ as $a \to 1$. It is further supposed that $u[k]$ is such that, if $a \to 1$

$$\sum_{i=0}^{\infty} a^{-(i+1)} u[i-k] \to \sum_{i=0}^{\infty} u[i-k]$$

uniformly over all k. It follows that $y_a = Pu_a$ and $y_a \to y^*$ in $l_\infty(\mathbb{Z})$ as $a \to 1$ where $y_a[k] = \sum_{i=0}^{\infty} a^{-(i+1)} u[k+i]$ and $y^*[k] = -\sum_{i=o}^{\infty} a^{-(i+1)} u[k+i]$. It is obvious that $y^* \neq Pu$, however $ay^*[k] - y^*[k-1] = -u[k]$. In fact a convolution equation admits uniqueness of solution, while a difference equation must be augmented by initial conditions in order to admit uniqueness. Therefore $Ay = Bu$ does not physically represent the same system as $y = Pu$, since it has many more solutions. In order to get the same physical meaning to $Ay = Bu$ must be added the extra condition that $u \in \mathcal{D}(P)$. However, this case is completely equivalent to the traditional input output framework. Hence there is no difference with the analysis made in [13].

Chapter 5

Continuous time feedback Systems

5.1 Bilateral Continuous Signals

In the previous chapter a consistent Mathematical Framework for feedback systems with double-sided discrete time polynomially bounded inputs was established. However, the seminal paper of Georgiou and Smith, describing the inconsistencies of the Mathematical Framework, was treating bilateral continuous time inputs. The later literature analysed the problem from a discrete time point of view, assuming that transition to the continuous time version would be obvious. In this chapter a consistent Mathematical Framework for feedback system with bilateral continuous time polynomially bounded input is established. Despite the similarities with the previous chapter the treatment of the problem needs the use of other technicalities.

5.2 Eigenvectors and System Function

Consider the operator

$$T : D_T \subset \mathcal{X} \to R_T \subset \mathcal{Y}$$

where D_T,R_T are the domain and the range of T, $\mathcal{X}$ and $\mathcal{Y}$ are classes of bilateral continuous time functions. Assume that

$$T(\lambda x) = \lambda T x, \ \forall \lambda \in \mathbb{C}$$

and, if $T(x(t)) = y(t)$, then

$$T(x(t-\tau)) = y(t-\tau)$$

for all $x \in D_T$. The operator T, the classes $\mathcal{X}$ and $\mathcal{Y}$ are the elements for the Mathematical Framework for a feedback system as in Figure 5.2.

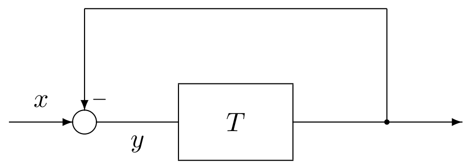

Figure 5.1: Feedback system.

The relation between exponentials, eigenvectors and eigenvalues of the operator T is investigated.

Lemma 50. *If $e^{at} \in D_T \subset \mathcal{X}$, then it is an eigenvector of T.*

Proof. If

$$T(e^{at}) = y(t)$$

$$y(t) = e^{at}(e^{-at}y(t))$$

define

$$h_a(t) = (e^{-at}y(t))$$

Hence,

$$T(e^{a(t-s)}) = y(t-s) = e^{a(t-s)}h_a(t-s)$$

$$e^{-as}T(e^{at}) = e^{-as}e^{at}h_a(t) = e^{-a(t-s)}h_a(t)$$

Then,

$$e^{a(t-s)}h_a(t-s) = e^{a(t-s)}h_a(t)$$

for any s. It follows that $h_a(t-s) = h_a(t)$, therefore $h_a(t) = \lambda_a$, a complex constant dependent only on a, and

$$y(t) = \lambda_a e^{at}$$

is what needed. ∎

Assume that λ_a exists for any a on some segment in the complex plane. The system function of the operator T is defined in what follows.

Definition 51. The meromorphic function $K(s)$, $s \in \mathbb{C}$, is the analytic continuation of λ_a, if

$$K(s)_{s=a} = \lambda_a$$

for any a on the segment of existence of λ_a. Furthermore, $K(s)$ is required to be bounded on some contour described by the semicircle encircling the origin and lying in the right half plane between the semicircles of radius k and $k+1$, for any $k \geq 0$. Define $K(s)$ the system function of the operator T.

As in the previous chapter the Definition of meromorphic function is taken from [25]. Consequently, $K(s)$ has only a countable number of poles and the set of meromorphic functions is a field.

5.3 Standard Formalism for bilateral signals and LTI systems

5.3.1 Time domain analysis

Consider the set of Lebesgue locally integrable functions. The operator T, mapping this set into itself, is defined by the convolution

$$x(t) \mapsto y(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

By Lemma 50, when the convolution exists, and there exists $K(s)$ as in Definition 51, the function e^{pt} is an eigenvector of the operator T with eigenvalue $K(p)$.

Theorem 52. *(a) Let T_r be the mapping*

$$T_r : f(t) \mapsto g(t) = \Phi_r(t) * f(t),\ \Phi_r(t) = t^r h(t),\ r \geq 0$$

Suppose p an interior point of the domain of $K(s)$, the system function for T_0 and $\Phi_{k,p} = t^k e^{-pt} h(t) \in L^1$, $k \geq 0$, then e^{pt} is an eigenvector of T_j, for $j = 0, ..., k$, with eigenvalue $K_j(p) = [(-\frac{d}{ds})^j K(s)]_{s=p}$.

(b) Let T_S and $T_{i,r}$, $i = 1, ..., N$ be the mappings

$$T_S : f(t) \mapsto g(t) = f + \Phi * f,\ \Phi(t) = \Phi_0(t) = h(t)$$

$$T_{i,r} : f(t) \mapsto g(t) = \Psi_{i,r} * f,\ \Psi_{i,r} = t^r e^{p_i t}$$

provided $\Phi * x \in L^1$ *and* $f_{k_i,p_i} \in L^1$, *for* $i = 1, ..., N$, *with* $f_{j,p} = \{t^j e^{pt} f(t)\}$, *then*

$$Tf = T_S f + \sum_{i=1}^{N} \sum_{j=0}^{k_1} c_{i,j} T_{i,j} f$$

where $Tx = (\Phi + \sum_{i=1}^{N} \sum_{j=0}^{k_i} c_{i,j} \Psi_{i,j}) * f$.

(c) Let $T_{S,r}$ *and* $T_{R,r}$, *respectively, be the mappings*

$$T_{S,r} : f(t) \mapsto g(t) = f + \Phi_r * f$$

$$T_{R,r} : f(t) \mapsto g(t) = \Psi_r * f, \ \Psi_r(t) = t^r e^{at}, \ r \geq 0$$

and $K_r(s)$ *the system function for* $T_{S,r}$. *Provided* $\Phi_{k,p} = t^k e^{-pt} h(t) \in L_1$ *and* $f_{k,p} = t^k e^{-pt} f(t) \in L^1$, *then*

(i) $\Psi_j * (\Phi_0 * f)$ *exists for* $j = 0, ..., k$;

(ii) $(\Psi_j * \Phi_0) * f$ *exists for* $j = 0, ..., k$;

(iii) $T_{R,k}(T_{S,0} f) = \sum_{r=0}^{k} (-1)^r \begin{pmatrix} k \\ r \end{pmatrix} K_r(p) T_{R,k-r} f$.

(d) Let T_S, T_P *and* T_G, *respectively be the mappings*

$$T_S : f(t) \mapsto g(t) = f + \Phi * f, \ \Phi(t) = h(t)$$

$$T_P : f(t) \mapsto g(t) = \Omega * f$$

$$T_G f = (\Phi + \sum_{i=1}^{N} \sum_{j=0}^{k_i} c_{i,j} \Psi_{i,j}) * f$$

where $\Psi_{i,r} = t^t e^{p_i t}$. *Suppose* $\Omega * (\Phi * x) \in L^1$, $t^j e^{-p_i t} h(t) \in L^1$ *and* $t^j e^{-p_i t} f(t) \in L_1$, *for* $i = 1, ..., N$, $j = 0, ..., k_i$, *and* p *is an interior point of the domain of* $K(s)$, *the system function for* T_P, *then*

$$(T_G(T_S x) - T_P(T_S x)) = \sum_{i=1}^{N} \sum_{j=1}^{N} c_{i,j} \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} ((-\frac{d}{ds})^j K(s))_{s=p_i} (T_{i,j-r} x)$$

where $T_{i,j} f = \Psi_{i,j} * f$.

Proof. (a) For $j = 0, ..., k$, $\Phi_{j,p} \in L^1$, since $\left|t^j e^{-pt} h(t)\right| \leq \left|t^k e^{-pt} h(t)\right|$. Hence,

$$\Phi_j * v = (t^j h(t)) * e^{pt} = \int_{-\infty}^{\infty} (t-s)^j h(t-s) e^{ps} ds = e^{pt} \int_{-\infty}^{\infty} (t-s)^j h(t-s) e^{p(s-t)} ds$$

when $v = e^{pt}$, exists and v is an eigenvector of Φ_j with eigenvalue

$$K_j(p) = \int_{-\infty}^{\infty} s^j h(s) e^{-ps} ds$$

In addition, for $j = 0, ..., k$,

$$\frac{d^j}{dp^j}K(a) = \int_{-\infty}^{\infty} \frac{d^j}{dp^j}(h(s)e^{-ps})ds = \int_{-\infty}^{\infty} (-1)^j s^j e^{-ps} h(s) ds$$

since p is an interior point of the domain of $K(s)$ and $\frac{d^j}{dp^j}(e^{-pt}h(t)) \in L^1$. It follows immediately that $K_j(p) = (-\frac{d}{dp})^j K(p)$, $j = 0, ..., k$.

(b) For $i = 1, ..., N$, $j = 0, ..., k_i$, $f_{j,p_i} \in L^1$, since $\left|t^j e^{-p_i t} f(t)\right| \leq \left|t^{k_i} e^{-p_i t} f(t)\right|$. Hence, $\Psi_{i,j} * f$ exists, since

$$|\Psi_{i,j} * f| < \left|e^{p_i t}\right| \int_{-\infty}^{\infty} \left| \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} (-1)^{j-r} t^r f_{j-r,p_i}(s) \right| ds$$

and $\sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} (-1)^{j-r} t^r f_{j-r,p_i} \in L^1$. It follows immediately that

$$Tx = T_S x + \sum_{i=1}^{N} \sum_{j=0}^{k_i} c_{i,j} T_{i,j} f$$

(c)(i) For $j = 0, ..., k$, $f_{j,p} \in L^1$, since $\left|t^j e^{-pt} f(t)\right| \leq \left|t^k e^{-pt} f(t)\right|$, and $\Phi_{j,p} \in L^1$, since $\left|t^j e^{-pt} f(t)\right| \leq \left|t^k e^{-pt} f(t)\right|$. Hence, by Young's Theorem, $\Phi_{i,p} * f_{j-r,p} \in L^1$ for $i, j = 0, ..., k$ and, since

$$\Omega_{j,p} = t^j v = \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} \Phi_{r,p} * f_{j-r,p}$$

where $v = \Phi_{0,p} * f_{0,p}$, $\Omega_{j,p} \in L^1$, for $j = 0, ..., k$ as required. It follows that $\Phi_0 * f = g$ exists, since $\Omega_{0,a} \in L^1$, $g = e^{at} v$ and $\Psi_j * (\Phi_0 * f)$ exists for $j = 0, ..., k$ as required, since it is bounded by

$$\left|e^{pt}\right| \int_{-\infty}^{\infty} \left| \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} t^r (-1)^{j-r} \Omega_{j-r,p}(s) \right| ds$$

and $\sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} t^r (-1)^{j-r} \Omega_{j-r,p} \in L^1$.

(ii) For $j = 0, ..., k$, $\Psi_j * \Phi_0$ exists, since

$$|\Psi_j * \Phi_0| < \left|e^{pt}\right| \int_{-\infty}^{\infty} \left| \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} t^r (-1)^{j-r} \Psi_{j-r,p}(s) \right| ds$$

and $\sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} t^r (-1)^{j-r} \Psi_{j-r,p} \in L^1$. In addition, for $j = 0, ..., k$, $\Phi_{j,1} * \Psi_0$ exists, since its absolute value is bounded by $\left|e^{pt}\right| \left\|\Phi_{j,p}\right\|_1$, and Ψ_0 is an eigenvector of $T_{S,j}$ with eigenvalue $K_j(p)$; that is, $\Phi_{j,p} * \Psi_0 = K_j(p)\Psi_0$. Hence

$$\int_{-\infty}^{\infty} e^{p(t-s)} t^j h(s) ds = K_j(p) e^{pt}$$

and

$$\begin{aligned}
&\Psi_j * \Phi_0 \\
&\quad = \int_{-\infty}^{\infty} (t-s)^j e^{p(t-s)} h(s) ds = \sum_{r=0}^{i} (-1)^r \left(\begin{array}{c} j \\ r \end{array} \right) \int_{-\infty}^{\infty} e^{p(t-s)} s^r h(s) ds \\
&\quad = \sum_{r=0}^{j} (-1)^r \left(\begin{array}{c} j \\ r \end{array} \right) \{K_r(p) t^{j-r} e^{pt} = \sum_{r=0}^{j} (-1)^r \left(\begin{array}{c} j \\ r \end{array} \right) K_r(p) \Psi_{j-r}
\end{aligned}$$

for $j = 0, ..., k$. Furthermore, for $j = 0, ..., k$, $\Psi_j * f$ exists, since

$$|\Psi_j * f| < \left|e^{pt}\right| \int_{-\infty}^{\infty} \left| \sum_{r=0}^{j} \left(\begin{array}{c} j \\ r \end{array} \right) t^r (-1)^{j-r} f_{j-r,p}(s) \right| ds$$

and $\sum_{r=0}^{j} \left(\begin{array}{c} j \\ r \end{array} \right) t^r (-1)^{j-r} f_{j-r,p} \in L^1$. It follows that $(\Psi_j * \Phi_0) * f$ exists for $j = 0, ..., k$.

(iii) By (i) and (ii),

$$\Psi_k * (\Phi_0 * f) = (\Psi_k * \Phi_0) * f = \sum_{r=0}^{k} (-1)^r \left(\begin{array}{c} k \\ r \end{array} \right) K_r(p) (\Psi_{k-r} * f)$$

It follows that

$$T_{R,k}(T_{S,0} f) = \sum_{r=0}^{k} (-1)^r \left(\begin{array}{c} k \\ r \end{array} \right) K_r(p) T_{R,k-r} f$$

(d) For $i = 1, ..., N$, $j = 0, ..., k_i$, $t^j e^{-p_i t} h(t)$, $t^j e^{-p_i t} f(t) \in L^1$, since $\left|t^j e^{-p_i t} h(t)\right| \leq \left|t_i^k e^{-p_i t} h(t)\right|$ and $\left|t^j e^{-p_i t} f(t)\right| \leq \left|t_i^k e^{-p_i t} f(t)\right|$ and $t^j e^{-p_i t} y \in L^1$ where $y = T_S x$, since

$$t^j e^{-p_i t} y(t) = \left(\begin{array}{c} j \\ r \end{array} \right) \sum_{r=0}^{j} (t^j e^{-a_i t} h(t)) * (t^{j-r} e^{-a_i t} f(t))$$

Hence, by part (b),

$$T_G(T_S x) - T_P(T_S x) = \sum_{i=1}^{N} \sum_{j=0}^{k_i} c_{i,j} \Psi_{i,j} * x$$

since $\Omega * x \in L^\infty$. The result follows immediately from parts (a) and (c). ■

Definition 53. Define the subclass L of of the locally Lebesgue integrable function

$$L = \{f(t) : \exists I_a \in I_0 \cup I_+ \cup I_\infty \text{ such that } \int_{-\infty}^{\infty} f(t)e^{qt}dt < \infty \text{ for } Re(q) \in I_a,\ q \in \mathbb{C}\}$$

where

$$I_0 = \{[l_1, l_2),\ l_1 = -\infty,\ \text{for some } l_2 > 0\}$$

$$I_+ = \{(l_1, l_2) \text{ for some } -\infty \le l_1 < l_2 \le \infty\}$$

$$I_\infty = \{(l_1, l_2],\ l_2 = \infty,\ \text{for some } l_1 \ge 0)\}$$

The class L is the class of locally Lebesgue integrable function for which the integral

$$\int_{-\infty}^{\infty} f(t)e^{qt}dt$$

exists for $Re(q) \in I_a$.

If $g(t) \in L$ consider the operator

$$T : D_T \subseteq L \to R_T \subseteq L$$

$$x(t) \mapsto y(t),\ \int_{-\infty}^{\infty} g(\tau)f(t-\tau)d\tau$$

The domain of T is the functions for which the convolution exists.

Lemma 54. *Consider the operator T, defined by the convolution above. Let $G(s)$ be the system function of T, as in Definition 51, then $G(s) = \int_{-\infty}^{\infty} g(t)e^{st}dt$.*

Proof. The result follows immediately from the Definitions of system function and L. ∎

Consider $X(s)$ the maximal analytic extension of $G(s) = \int_{-\infty}^{\infty} g(t)e^{st}dt$. For disjoint analytic domains, the algebraic function, $X(s)$, corresponds to different integrals. In order to recover a one-to-one relationship, the domain $D_X \subseteq \mathbb{C}$, on which the integral exists, must be specified. Therefore, the doublet notation, $\{X(s), D_X\}$, is preferred. Consequently, the doublet $\{G(s), D_G\}$ is the system function for T. The domain D_G is an open strip, parallel to the imaginary axis, with left boundary and right boundary the real part of singular points of $G(s)$.

Lemma 55. *Let T_1 and T_2 operators on L such that $T_1x = \Phi_1 * x$ and $T_2x = \Phi_2 * x$. Consider their system functions $\{G_1(s), D_{G_1}\}$ and $\{G_2(s), D_{G_2}\}$, respectively. Provided $D_{G_1} \cap D_{G_2} \neq \emptyset$, then*

(i) $\{G_1(s) + G_2(s), D_{G_1+G_2} \supset D_{G_1} \cap D_{G_2}\}$ is the system function for a system T, such that $Tx = T_1x + T_2x$, when T_1x and T_2x exist;

(ii) $\{G_1(s)G_2(s), D_{G_1G_2} \supset D_{G_1} \cap D_{G_2}\}$ is the system function for a system T, such that $Tx = T_1(T_2x)$, when T_1x and $T_1(T_2x)$ exist.

Proof. (i) Since,

$$\forall a \in D_{G_1}, \int_{-\infty}^{\infty} f(t)e^{at}dt = G_1(a) < \infty$$

and,

$$\forall a \in D_{G_2}, \int_{-\infty}^{\infty} f(t)e^{at}dt = G_2(a) < \infty$$

then,

$$\int_{-\infty}^{\infty} (f(t) + g(t))e^{at}dt = G_1(a) + G_2(a)$$

for all $a \in D_{G_1} \cap D_{G_2}$. Define $G_T(s) = (G_1(s) + G_2(s))$, with $D_{G_T} \supseteq D_{G_1} \cap D_{G_2}$ such that

$$G_T(s) = \int_{-\infty}^{\infty} (f(t) + g(t))e^{st}dt$$

It follows that G_T is the system function of the operator T on L, with $Tx = \Phi * x$, $\Phi(t) = (f(t) + g(t))$.

(ii) Similar to (i). ■

The domain in

$$\{G_1(s) + G_2(s), D_{G_1+G_2} \supset D_{G_1} \cap D_{G_2}\}$$

is greater than $D_{G_1} \cap D_{G_2}$ only when the removal of singular points through additive cancellations occurs. Similarly, the domain in

$$\{G_1(s)G_2(s), D_{G_1G_2} \supseteq D_{G_1} \cap D_{G_2}\}$$

is greater than $D_{G_1} \cap D_{G_2}$ only when the removal of singular points through multiplicative cancellation with zeros occurs.

Definition 56. L is the class of signals and the convolutions on L are the systems for a Standard Formalism in time domain analysis.

The doublet $\{G(s), D_G\}$ is the system function for the system.

A system on L, defined by the function $g(t)$ is causal if $g(t) = 0$ for $t < 0$, acausal if $g(t) = 0$ for $t \geq 0$. It is stable if there exists a $k > 0$ and a $0 < c < 1$ such that $|g(t)| < kc^{|t|}$.

Theorem 57. *(a)(i) The system $\{G(s), D_G\}$ is causal provided $\exists l \in \mathbb{R}$ such that*

$$D_G = \{s \in \mathbb{C} : \ Re(s) > l\}$$

(ii) the system $\{G(s), D_G\}$ is acausal provided $\exists l \in \mathbb{R}$ such that

$$D_G = \{s \in \mathbb{C} : \ Re(s) < l\}$$

(b) The system $\{G(s), D_G\}$ is stable provided

$$D_G = \{s \in \mathbb{C} : \ Re(s) = 0\}$$

Proof. (a)(i) The function $f(t) \in L$ is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\sigma + j\omega) e^{j\omega t} d\omega$$

when $\sigma \in D_G$.

From the Definition of system function, it follows that $G(s)$ is bounded on any open strip included in D_G. Hence, by Cauchy's Residue Theorem, $f(t) = 0$, for $t < 0$.

(ii) similar to (i).

(b) Since D_G is an open neighborhood of the imaginary axis, it follows immediately that the system is stable. ■

Properties of stable systems are proved in the next two Lemmas.

Lemma 58. *Let T_1 and T_2 be stable systems on L such that $T_1 x = \Phi_1 * x$ and $T_2 x = \Phi_2 * x$, where Φ_1 and $\Phi_2 \in L$. If $x \in L^p$, $1 \leq p \leq \infty$ then*

(i) $y = T_1 x + T_2 x = Tx \in L^p$, where T is a stable system;

(ii) $y = T_1(T_2 x) = Tx \in L^p$, where T is a stable system.

Proof. (i) Since T_1 and T_2 are stable, Φ_1 and $\Phi_2 \in L^1$ and so $\Psi = \Phi_1 + \Phi_2 \in L^1$. Hence, by Young's Theorem, $\forall x \in L^p$, $1 \leq p \leq \infty$, $\Phi_1 * x$, $\Phi_2 * x$, $\Psi * x \in L^p$ and it follows that $Tx = T_1 x + T_2 x$. There exist $k_1, k_2 > 0$ and $0 < c_1, c_2 < 1$ such that $|\Phi_1(t)| < k_1 c_1^{|t|}$ and $|\Phi_2(t)| < k_2 c_2^{|t|}$. Therefore, $|\Phi_1 + \Phi_2| < (k_1 + k_2)(max\{c_1, c_2\})^{|t|}$ and Tx is stable.

(ii) Similar to (ii). ■

Lemma 59. *(i) Let T be a stable and causal system such that $Tx = \Phi * x$. Then $t^k e^{-pt}\Phi \in L^1$, for $k \geq 0$, $|p| > 0$.*

(ii) Let T be a stable and acausal system such that $Tx = \Phi * x$. Then $t^k e^{-pt}\Phi \in L^1$, for $k \geq 0$, $|p| < 0$.

Proof. (i) Since $\Phi(t) = 0$, when $t < 0$, and $|\Phi| < kc^{|t|}$, for some $k > 0$ and $c < 1$, the result follows immediately.

(ii) Similar to (i). ∎

Since the analysis is centred on feedback systems the existence of the return difference operator must be investigated.

Lemma 60. *Let $G(s) \neq -1$ be meromorphic on $\mathbb{C}$ such that the singular points at p_i, $i = 1, ..., N$ of order $k_1, ..., k_N$ are in the domain of the doublet $\{(1 + G(s))^{-1}, D_{(1+G)^{-1}})\}$, then $\{(1 + G(s))^{-1}, D_{(1+G)^{-1}})\}$ is the system function for a system with zero eigenvectors, $\{t^j e^{p_i t}\}$, $i = 1, ..., N$, $j = 1, ..., k_i$.*

Proof. The proof is a consequence of the properties of the meromorphic functions. In fact, since they constitute a field, it follows immediately. ∎

Theorem 61. *Let T_P and T_S be stable systems with system functions, respectively, $\{G(s), D_G\}$ and $\{(1 + G(s))^{-1}, D_{(1+G)^{-1}}\}$. Then*

$$T_S x + T_P(T_S x) = (I + T_P)(T_S x) = T_S((I + T_P)x) = x, \ \forall x \in l^p, \ 1 \leq p \leq \infty$$

Proof. Since T_P and T_S are both stable, then $D_G \cap D_{(1+G)^{-1}} \neq \emptyset$. The proof is consequence of Theorem 55, Lemma 58 and Lemma 60. ∎

The system T_S, defined as above, is not necessarily the inverse of the system $T_{(1+G)}$, defined by $\{(1 + G(s)), D_{(1+G)}\}$.

The relation in time-domain between system functions is established when their domain of existence are disjoint.

Theorem 62. *Let Φ_1 and Φ_2 be two systems with system functions $\{G(s), D_1\}$ and $\{G(s), D_2\}$, respectively, such that $D_1 \cap D_2 = \emptyset$ and that $G(z)$ is the same meromorphic functions for both system functions. The singular points of $G(s)$*

are $p_1, ..., p_N$ *with order* $k_1, ..., k_N$*, respectively, and* $N \in [0, \infty]$*. Suppose that* $\sum_{i=1}^{N} \sum_{j=1}^{k_r} c_{i,j} t^{k_i - 1} e^{p_i t}$ *converges, then*

$$\Phi_1 - \Phi_2 = \sum_{i=1}^{N} \sum_{j=1}^{k_r} c_{i,j} t^{k_i - 1} e^{p_i t}$$

where $c_{i,j} = 1, ..., N,\ j = 1, ...k_i$ *are constants.*

Proof. Define he region $D_{1\backslash 2}$ as the region between D_1 and D_2. Since $G(s)$ is meromorphic, $D_{1\backslash 2}$ contains a countable number of poles. Moreover, from the Definition of system function, $G(s) \rightarrow 0$ when $Re(s) \rightarrow \infty$. The rest of the proof is an application of the Cauchy's Residue Theorem to the complex function $G(s)e^{st}$. In fact, since

$$\Phi_1(t) = \{\frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} G(s) e^{st} ds\}$$

$$\Phi_2(t) = \{\frac{1}{2\pi j} \int_{\sigma_2 - j\infty}^{\sigma_2 + j\infty} G(s) e^{st} ds\}$$

where σ_1 is inside the intersection between the real line and D_1, σ_2 is inside the intersection between the real line and D_2. Then

$$\{\frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} G(s) e^{st} ds\} - \{\frac{1}{2\pi j} \int_{\sigma_2 - j\infty}^{\sigma_2 + j\infty} G(s) e^{st} dz\} = \Lambda$$

with Λ the residues of the singular points of $G(s)$ in $D_{1\backslash 2}$. The rest follows immediately. ∎

Consider the systems T_G, T_P and T_S with system functions $\{G(s), D_G\}$, $\{G(s), G_P\}$ and $\{(1+G(s))^{-1}, D_{(1+G)^{-1}}\}$ respectively. The singular points of $G(s)$ are $p_1, ..., p_N$ with order $k_1, ..., k_N$, respectively, and $N \in [0, \infty]$. Require that

$$\sum_{i=1}^{N} \sum_{j=1}^{k_r} c_{i,j} t^{k_i - 1} e^{p_i t}$$

converges. Suppose that T_P is stable but not necessarily causal , that T_G is causal but not necessarily stable and that T_S is stable and causal. Let $T_G x = \Psi * x$, $T_P x = \Omega * x$ and $T_S x = \Phi * x$. Consider the feedback systems of T_G and T_P,

$$\begin{cases} y = T_G u \\ u = r - y \end{cases} \tag{5.1}$$

$$\begin{cases} y = T_P u \\ u = r - y \end{cases} \tag{5.2}$$

when $r, u, y \in L^p$, $1 \leq p \leq \infty$.

Define $\bar{p}$, a singular point of $G(s)$, such that

$$Re(\bar{p}) = max\{Re(p_i),\ i = 1, ..., N\}$$

and

$$\bar{k} = max\{k_i,\ i \in \{j:\ Re(p_i) = Re(\bar{p})\}\}$$

and define

$$e^{-\bar{p}}L^p = \{x \in L^p:\ \{t^{\bar{k}} e^{-\bar{p}t} x\} \in L^p\},\ 1 \leq p \leq \infty$$

The response of 5.1 is related to the response of 5.2 by the following Theorem.

Theorem 63. *Consider the systems T_G, T_P and T_S defined as above and the two feedback systems 5.1 and 5.2. Then*

(i) The feedback system 5.2 has, $\forall r \in D_{T_S}$, the solution

$$\begin{cases} y = T_P(T_S r) \\ u = T_S r \end{cases}$$

(ii) The feedback system 5.1 has, $\forall r \in e^{-\bar{p}}L^p$, the solution

$$\begin{cases} y = T_P(T_S r) \\ u = T_S r \end{cases}$$

Proof. (i) By Lemma 60 $T_P(T_S x) = x - T_S x$, $\forall x \in D_{T_S}$. Therefore, $\forall r \in D_{T_S}$

$$\begin{cases} y = T_P u = T_P(T_S r)\ \in L^p \\ u = r - y = r - T_P(T_S r) = r - (r - T_S r) = T_S r\ \in L^p \end{cases}$$

as required.

(ii) By Theorem 62

$$T_G x = (\Omega + \sum_{i=1}^{N} \sum_{j=0}^{k_i} c_{i,j} \Psi_{i,j}) * x$$

with $\Psi_{i,j} = t^j e^{pt}$ where p_i is a singular point of $G(s)$ with order k_i.

By Theorem 57 (a)(i) and (b), since T_P is stable and T_G is causal, $c_{i,j} = 0$, when $i \in \{j : Re(p_j) \leq 0\}$. Hence, the only relevant singular points are those such that $Re(p_i) > 0$.

Since T_P and T_S are stable, Ω, $\Phi \in L^1$ and $\Omega * (\Phi * x) \in L^1$, $\forall x \in L^p$, $1 \leq p \leq \infty$.

By Lemma 59 (i), since T_S is stable and causal, $te^{-p_i t} \in L^1$, for $i = 1, ..., N$, $j = 0, ..., k_i$, and $t^j e^{p_i t} x(t) \in L^1$, whenever $t^{\bar{k}} e^{-\bar{p}t} \in L^1$. Hence, by Theorem 52 (d),

$$T_G(T_S x) = T_P(T_S x) + T_R x$$

where

$$T_R x = \sum_{i=1}^{N} \sum_{j=0}^{k_i} c_{i,j} \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} ((-\frac{d}{dz})^r (1 + G(s))^{-1}_{s=p_i})(\Psi_{i,j} * x)$$

By Theorem 57 (a)(i) and (b) all the singular points such that $Re(p_i) > 0$ are internal points of the domain of T_S, since T_S is causal and stable, and

$$(z\frac{d}{dz})^j (1 + G(z))^{-1}_{z=p_i}) = 0$$

for $i = 1, ..., N$, $j = 1, ..., k_i$, since, by Lemma 60, the functions $t^j e^{p_i t}$, $i = 1, ..., N$, $j = 1, ..., k_i$ are zero eigenvectors of $(1 + G(s))^{-1}$. It follows that $T_R(T_S x) = 0$ and $T_G(T_S x) = T_P(T_S x)$.

Therefore, $\forall r \in e^{-\bar{p}} L^p$

$$\begin{cases} y = T_G u = T_G(T_S r) = T_P(T_S r) \ \in L^p \\ u = r - y = r - T_P(T_S r) = r - (r - T_S r) = T_S r \ \in L^p \end{cases}$$

■

Theorem 63 implies that the response of a feedback system enclosing the unstable plant, T_G, to an input restricted to $e^{-\bar{p}} L^p$, is the same as the response of the feedback system enclosing the stable plant, T_P, when the latter feedback is stable and causal. Therefore, when the class of inputs is restricted to $e^{-\bar{p}} L^p$, the feedback system 5.1 is stable and causal if and only if the feedback system 5.2 is causal and stable.

To be more precise, by Young's Theorem and Theorem 61

$$\|y\|_p \leq \|\Omega * \Phi\|_1 \|r\|_p, \ \forall r \in L^p, \ 1 \leq p \leq \infty$$

Hence, the feedback system enclosing T_P being stable in the sense that, for some $c > 0$, $\|y\|_p \leq c \|r\|_p$, $\forall r \in L^p$, implies that the feedback system enclosing T_G is stable in the sense that $\|y\|_p \leq c \|r\|_p$, $\forall r \in e^{-\bar{p}} L^p$.

Note that $e^{-\bar{p}} L^p$ is a subspace of L^p but is not closed in L^P. In fact, consider $r \in L^P$ such that $r \notin e^{-\bar{p}} L^p$ and let s_n, $n \in \mathbb{Z}$, be the function with values $s_n(t)$

such that $s_n(t) = 0$ for $t < -n$, $s_n(t) = r(t)$ otherwise. Then, $\forall n \in \mathbb{Z}$, $s_n \in L^p$, $1 \leq p < \infty$, and $s_n \to r$ in L^P.

Example 10. Let T_P, T_G and T_S be the systems defined, respectively, by

$$T_G x = \Omega * x,\ \ \Omega(t) = kbe^{at}\Theta(t)$$

$$T_P x = \Psi * x,\ \ \Psi(t) = -kbe^{at}\Theta(t)$$

$$T_S x = x + \Phi * x,\ \ \Phi(t) = -kbe^{(a-kb)t}\Theta(t)$$

where $x \in L^2(-\infty, \infty)$, $\Theta(t) = 1\ t > 0$, zero otherwise. The system functions are, respectively, $\{\frac{kb}{s-1}, Re(s) > a\}$, $\{\frac{kb}{s-a}, Re(s) < a\}$ and $\{\frac{s-a}{s-(a-kb)}, Re(s) > |a - kb|\}$. The system function for T_G has a singular point at $s = a$ and e^{at} is a zero eigenvector of T_S.

Let $x = e^{at}\Theta(t)$, then, when $|a - kb| < 1$, $T_S x = e^{(a-kb)t}\Theta(t)$ and $(I + T_P)T_S x = -e^t(a - \Theta(t))$. Hence $(I + T_P)(T_S x) \neq x$ and T_S is not the inverse of $(I + T_P)$. When $|a| > 0$ and $|a - kb| < 0$, T_P, T_G and T_S are, respectively, stable, causal and stable and causal. Hence, the feedback system enclosing T_G is stable and causal with the same response as the feedback system enclosing T_P but only for the inputs $r \in e^{-\bar{p}}L^p$, with $\bar{p} = a$.

5.3.2 Transform domain analysis

Consider a signal $x \in L$. The correspondent element in transform domain analysis is the doublet $\{X(s), D_X\}$. The doublet is the bilateral Laplace transform of the signal x. Hence, the doublets of the signals and of the systems of the Standard Formalism of Definition 56 are the elements for a transform domain analysis. Note that this Formalism is closely related to the conventional analysis in [1].

Similarly to Theorem 57 (i) and (ii), provided $\exists l \in \mathbb{R}$ such that

$$D_X = \{s \in \mathbb{C} :\ Re(s) > l\}$$

the signal is causal, and, provided $\exists l \in \mathbb{R}$ such that

$$D_X = \{s \in \mathbb{C} :\ Re(s) < l\}$$

the signal is acausal. Similarly to 57 (iii), when

$$D_G \supseteq \{s \in \mathbb{C} :\ Re(s) = 0\}$$

the signal is stable. When the signals x and y have bilateral Laplace transforms, $\{X(z), D_X\}$ and $\{Y(z), D_Y\}$, respectively, and $D_X \cap D_Y \neq \emptyset$, $\{X(s)+Y(s), D_{X+Y} \supset D_X \cap D_Y\}$ corresponds to the signal $x+y$. Moreover, when the system T has system function $\{G(s), D_G\}$, $\{Y(s) = G(s)X(s), D_Y = D_{GX} \supset D_G \cap D_X\}$ is the bilateral Laplace transform of the the signal y, such that $y = Tx$. Hence, transform domain analysis can be applied to systems and signals with Definition 56.

5.3.3 Standard Framework for bilateral signals and LTI systems

Note that Definition 56 is the definition of a Formalism, not of a Framework. In fact, the elements of the Standard Formalism do not meet three requirements for a consistent Mathematical Framework:

1) The class of signals L is not a linear space. In fact, if $g_1(t)$ and $g_2(t)$ are in L the bilateral Laplace transform of their sum might not exists if the intersection of the two domains is empty. This is demonstrated by the following example.

Example 11. Consider the bilateral Laplace transforms of

$$f_1(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

and

$$f_2(t) = \begin{cases} 0 & t \geq 0 \\ 1 & t < 0 \end{cases}$$

The domain of the first if the region $Re(s) > 0$ in the complex plane, while the domain of the second is the region $Re(s) < 0$ in the complex plane. The sum of the two sequences is $f(t) = 1$, whose bilateral Laplace transform does not exist since it has to satisfy the condition $0 < Re(s) < 0$.

2) The class of systems does not constitute an algebra. If $\{G_1, D_1\}$ and $\{G_2, D_2\}$ are two transfer functions such that $D_1 \cap D_{G_2} = \emptyset$, then their sum or composition might not exist.

3) The system with transfer function $\{(1+G(s))^{-1}, D_{(1+G(s))^{-1})}\}$ is not necessarily the inverse of the system with transfer function $\{\{(1+G(s)), D_{(1+G(s))}\}\}$.

Definition 64. Define $L_S \subset L$ by

$$L_S = \{f \in L : \ Re(s) = 0 \text{ implies } s \in D_{\mathcal{X}}\}$$

Note that $L_S \subset L^p$, $1 \leq p \leq \infty$.

Definition 65. L_S is the class of signals and the operators on L_S, defined by $\Phi * x$, with $\Phi \in L_S$, are the systems for a Standard Framework in the time domain analysis.
The bilateral Laplace transform of the signals and of the $\Phi \in L_S$ are the corresponding elements for a corresponding transform domain analysis.

For any system, T, with system function, $\{G(s), D_G\}$, and any signal, x, with bilateral Laplace transform, $\{X(s), D_X\}$, $D_G \cap D_X$ is nonempty, open and contains the imaginary axis. Similarly, for any systems, T_1 and T_2, $D_{G_1} \cup D_{G_2}$ is also nonempty, open and contains the imaginary axis. Therefore the class of signals is a linear space and the class of systems constitutes an algebra. Moreover, the system with transfer function, $\{(1 + G(s))^{-1}, D_{(1+G)^{-1}})\}$, is the inverse of the system with transfer function $\{(1 + G(s)), D_{(1+G)})\}$. Since the requirements for a consistent framework are satisfied Definition 65 is the definition of a consistent mathematical framework.

In the Standard Framework all signals and systems are stable but not necessarily causal. The analysis of the feedback system is no longer concerned with establishing the stability of the closed-loop system but with establishing its causality.
Consider T_G, a causal but unstable open loop system on L_S, together with its transfer function, $\{G(s), D_G\}$, such that $G(s)$ is analytic on the imaginary axis. Consider T_P, the associated acausal but stable open loop system on L_S, with transfer function $\{G(s), D_P\}$. By Theorem 63, the closed loop system for T_G is stable and causal provided the closed loop system for T_P is stable and causal.

Example 12. Let T_G the unstable causal system

$$T_G x = \Pi * x, \ \Pi(t) = kbe^{at}\Theta(t), \ a > 0$$

Its system function is $\left\{\frac{kb}{s-a}, Re(s) > a\right\}$. The associated stable acausal system, T_P, has system function $\left\{\frac{kb}{s-a}, Re(s) < a\right\}$. Since $(I + T_P)^{-1}$ has system function $\left\{\frac{s-a}{s-(a-kb)}, Re(s) > |a - kb|\right\}$, the closed loop system for T_P is causal as well as stable, provided $|a - kb| < 0$. Hence, by Theorem 29, the closed loop system for T_G is stable as well as causal, provided $|a - kb| < 0$. Note that the inputs to the latter closed loop are restricted to $e^{-\bar{p}}L^p$, for $\bar{p} = a$. For these inputs, the responses of the two closed loop systems are the same.

5.4 Generalized Formalism for Stable and Unstable Systems

Consider $C^\infty(\mathbb{R})$, the linear space of infinity differentiable complex valued functions on the real line. As in in the previous chapter a subspace $\hat{C} \subseteq C^\infty(\mathbb{R})$ is shift-invariant, when $f(t) \in \hat{C}$ implies $f(t-a) \in \hat{C}$, $\forall a \in \mathbb{R}$. Define $\mathcal{T}$ the linear space of linear functionals with domain a shift-invariant subspace of $C^\infty(\mathbb{R})$ such that, when $x \in \mathcal{T}$,

$$f(t) \mapsto x[f(t)]$$

where $x[f(t)]$ is the value of the functional x assigned to each $f(t)$ in its domain. $\mathcal{T}_D$ and $\mathcal{T}_S$ are subspaces of $\mathcal{T}$ with domain containing D and S, respectively. Define x_a the shifted functional such that $x_a[f(t)] = x[f(t+a)]$ and $\delta_\tau \in \mathcal{T}_S$ the delta functional such that $\delta_\tau[f(t)] = f(\tau)$ for all functions in $C^\infty(\mathbb{R})$.

Definition 66. Two elements x and y in $\mathcal{T}_D$ are equivalent if $x[f(t)] = y[f(t)]$ for all $f \in D$ implies that $x[f(t)] = y[f(t)]$ for all f in their domains.

Consider the operator T_A defined on a shift-invariant subspace of $C^\infty(\mathbb{R})$ containing D such that

$$T_A f(t) = \int_{-\infty}^{\infty} g(-\tau) f(t-\tau) d\tau \tag{5.3}$$

when $D \subseteq D_{T_A}$ and $D \subseteq R_{T_A}$. Define the linear shift-invariant operator T

$$T : D_T \subseteq \mathcal{T}_D \to R_T \subseteq \mathcal{T}_D$$

$$Tx[f(t)] = x[T_A f(t)]$$

provided $T_A f(t)$ is in the domain of x for all $f(t)$. When x is a regular functional then

$$Tx[f(t)] = (\int_{-\infty}^{\infty} g(s) x(t-s) ds)[f(t)]$$

provided $\int_{-\infty}^{\infty} g(t-s) x(s) ds$ exists.

Lemma 67. *Consider the operators T on T_D and T_A defined as above. If D_T and R_T are subspaces of T_D and $\int_{-\infty}^{\infty} g(s)x(t-s)ds$ exists, then*

$$Tx[f(t)] = x[T_A f(t)]$$

Proof. When x is a regular functional

$$\begin{aligned} & x[T_A f(t)] \\ &\quad = x[\int_{-\infty}^{\infty} g(-\tau)f(s-\tau)d\tau] = \int_{-\infty}^{\infty} x(s)\left(\int_{-\infty}^{\infty} g(-\tau)f(s-\tau)d\tau\right) ds \\ &\quad = \int_{-\infty}^{\infty} x(s)\left(\int_{-\infty}^{\infty} g(t-s)f(t)dt\right) ds = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(s)g(t-s)ds\right) f(t)dt \\ &\quad = (\int_{-\infty}^{\infty} x(t-s)g(s)ds)[f(t)] \end{aligned}$$

and the result follows immediately from Definition 66. ■

Define $\mathcal{Q}$ the class of operators T on $\mathcal{T}_D$ as

$$\mathcal{Q} = \{T: \ \exists T_A \text{ as in 5.3 such that } Tx[f(t)] = x[T_A f(t)]\} \tag{5.4}$$

Lemma 68. *Let T_1 and T_2 operators on $\mathcal{T}_D$ such that $T_1 x[f(t)] = x[T_{A1} f(t)]$ and $T_2 x[f(t)] = x[T_{A2} f(t)]$, where*

$$T_{A1} f(t) = \int_{-\infty}^{\infty} g(-\tau)f(t-\tau)d\tau$$

and

$$T_{A2} f(t) = \int_{-\infty}^{\infty} \bar{g}(-\tau)f(t-\tau)d\tau$$

Then $T_1 x + T_2 x = T_x$, when $T_1 x$ and $T_2 x$ exist, where T is an operator on $\mathcal{T}_D$ such that

$$Tx[f(t)] = x[T_A f(t)]$$

with $T_A f(t) = T_{A1} f(t) + T_{A2} f(t)$;

Proof. Since the operators are linear and by Lemma 67 the result follows immediately. ■

Lemma 69. *Suppose $T_1 x_1 = T_2 x_2 \in \mathcal{T}_D$ and $T(T_1 x) \in \mathcal{T}_D$, with T,T_1, $T_2 \in \mathcal{Q}$, then $T(T_1 x_1) = T(T_2 x_2)$.*

Proof. Since $T_1 x_1 = T_2 x_2 \in \mathcal{T}_D$, $T_1 x_1[f(t)] = T_2 x_2[f(t)]$ for all $f(t)$ in their domain. Let T be defined by T_A, then $T_1 x_1[T_A f(t)]$ exists and $T_1 x_1[T_A f(t)] = T_2 x_2[T_A f(t)]$. Hence, $TT_1 x_1[f(t)] = TT_2 x_2[f(t)]$. ■

Consequently, the operators $T \in \mathcal{Q}$, can be extended to include all the pairs, $x \mapsto y$, such that $y = T(\hat{T}w)$ and $x = \hat{T}w$, for some $\hat{T} \in \mathcal{Q}$ and $w \in \mathcal{T}_D$. All the operators in $\mathcal{Q}$ can be extended in this way and let $\mathcal{Q}_1$ be the class consisting of them. That can be repeated to construct the extension classes $\mathcal{Q}_r$, $r \geq 0$, with $\mathcal{Q}_0 = \mathcal{Q}$.

Definition 70. Let $\bar{\mathcal{Q}}$ be the class of maximal extensions of the operators in $\mathcal{Q}$, where the maximal extension of $T \in \mathcal{Q}$ is defined by all the pairs, $x \mapsto y$, such that $y = Tx$ is in $\mathcal{Q}_r$ for some $r \geq 0$.

Within $\bar{\mathcal{Q}}$, repeated cascading of operators is consistently defined.

Definition 71. $\mathcal{T}_D$ is the class of signals and the operators T in $\bar{\mathcal{Q}}$ are the system for a Generalized Formalism in time domain analysis.

Let the signals e^{at} be in the domain of the system T. By Lemma 50 those are λ_a eigenvectors of T and $K(s)$, the analytic continuation of the eigenvalues λ_a is the system function of the operator T, as in Definition 51.

A system on $\bar{\mathcal{Q}}$, defined by the operator T_A

$$T_A f(t) = \int_{-\infty}^{\infty} g(-\tau) f(t-\tau) d\tau$$

is causal if $g(t) = 0$ for $t < 0$, acausal if $g(t) = 0$ for $t > 0$ (note that if a system defined by T_A is causal, then T_A is acausal. Similarly, when the system is acausal, then T_A is causal). It is stable if there exists a $k > 0$ and a $0 < c < 1$ such that $|g(t)| < k\,|c|^t$.

Consider the functional $v \in \mathcal{T}_D$, an eigenvector of the system T on $\mathcal{T}_D$ with eigenvalue λ. Define the operator V_A on a shift-invariant subspace of $C^\infty(\mathbb{R})$ containing D, such that

$$V_A f(t) = \int_{-\infty}^{\infty} v(-\tau) f(t-\tau) d\tau \tag{5.5}$$

Lemma 72. *$V_A f(t)$ is an eigenvector of T_A with eigenvalue λ.*

Proof. Since v is a λ-eigenvector it follows

$$Tv[f(t)] = \lambda \int_{-\infty}^{\infty} v(\tau) f(\tau) d\tau = \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} v(\tau) g(s-\tau) d\tau) f(s) ds$$

If $(\int_{-\infty}^{\infty} v(\tau) g(t-\tau) d\tau)$ exists, then

$$\int_{-\infty}^{\infty} v(\tau) g(t-\tau) d\tau = \lambda v(t)$$

From the definition of $V_A f(t)$,

$$\begin{aligned} T_A V_A f(t) &= \int_{-\infty}^{\infty} g(-s)(\int_{-\infty}^{\infty} v(-\tau) f(t-(\tau+s))d\tau)ds \\ &= \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} g(-s)v(\tau-s)ds)f(t-\tau)d\tau \end{aligned}$$

hence, from above,

$$\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} g(-\tau)v(s-\tau)d\tau)f(t-s)ds = \lambda \int_{-\infty}^{\infty} v(-\tau)f(t-\tau)d\tau = \lambda V_A f(t)$$

■

Define the system V on $\mathcal{T}_D$ as $Vx = x[V_A f(t)]$ where V_A is the same as in 5.5. Consider the systems T and T_S on $\mathcal{T}_D$, such that

$$(I+T)T_S x[f(t)] = x[T_{SA}(I+T_A)f(t)] = x[f(t)]$$

Theorem 73. *Let T and T_S be two systems on $\mathcal{T}_D$ defined as above. Let V be a system on $\mathcal{T}_D$ defined as above, with v a zero eigenvector of T_S. Consider the two feedback systems*

$$\begin{cases} y = Tr \\ u = r - y \end{cases} \tag{5.6}$$

and

$$\begin{cases} y = (T+V)u \\ u = r - y \end{cases} \tag{5.7}$$

Then the feedback systems 5.6 and 5.7 have, $\forall r \in D_{T_S}$, the same solution

$$\begin{cases} y = T(T_S r) \\ u = T_S r \end{cases}$$

Proof. The solution to 5.6 follows from the way in which T and T_S are defined. By Lemma 72 if v is a zero eigenvector of T_S then $V_A f(t)$ is a zero eigenvector of T_{SA} when $f(t)$ has finite support. Hence

$$VT_S x[f(t)] = x[T_{SA} V_A f(t)] = 0, \forall x \in \mathcal{T}_D$$

Therefore, $\forall r \in D_{T_S}$

$$\begin{cases} y &= (T+V)u = (T+V)T_S r = r[T_{SA}(T_A + V_A)f(t)] \\ & \quad r[T_{SA}T_A f(t) + T_{AS} V_A f(t)] = T(T_S r) \qquad \in \mathcal{T}_D \\ u &= r - y = r - T(T_S r) = r - (r - T_S r) = T_S r \qquad \in \mathcal{T}_D \end{cases}$$

■

Theorem 73 implies that the response for a feedback system enclosing the plant T, to an input in $\mathcal{T}_D$, is the same as the response of the feedback system enclosing the plant $(T+V)$.

This Formalism is a generalization of the Standard Formalism. Consider g, an element of the class of signals L, and the regular functional $g \in \mathcal{T}_D$.
The map $g(t) \mapsto g$ is an isomorphism between L and and a linear shift-invariant subspace of $\mathcal{T}_D$.
Similarly, consider the system T_1 on L, such that $T_1 x = g * c$, and the operator T_A on a shift-invariant subspace of $C^\infty(\mathbb{R})$ such that

$$T_A f(t) = \int_{-\infty}^{\infty} g(-\tau) f(t-\tau) d\tau$$

Define the system T_2 on $\mathcal{T}_D$ such that

$$T_2 x[f(t)] = x[T_A f(t)] = x[\int_{-\infty}^{\infty} g(-\tau) f(t-\tau) d\tau]$$

The map $T_1 \mapsto T_2$ is is isomorphism between the convolutions on L and a subspace of the class of systems on $\mathcal{T}_D$. Clearly, when the response exists in both formulations, they are the same.

Lemma 74. *If the function $v(t) \in L$ is an eigenvector of a system T_1 on L, then the functional $v \in \mathcal{T}_D$ is an eigenvector of the correspondent system T on $\mathcal{T}_D$.*

Proof. The result is a consequence of the discussion above. ∎

Because of the isomorphisms between L and a subspace of $\mathcal{T}_D$ and the convolution on L and a subspace of $\bar{\mathcal{Q}}$, properties of the systems on L, such as causality and stability, are transferred to the equivalent systems on $\mathcal{T}_D$. It follows that the feedback systems 5.1 and 5.2, when extended into their equivalent feedback systems in the Generalized Formalism, have the same response for all inputs in D_{T_S}. In fact, to the stable system T_P, the causal system T_G, the stable and causal system T_S, on L, correspond the system T, $(T+V)$ and T_S on $\mathcal{T}_D$, of Theorem 73. Therefore, the response of a feedback system enclosing the unstable but causal plant $(T+V)$ is the same as the response of the feedback system enclosing the stable but acausal plant T_P, for any input in D_{T_S}.

Example 13. Consider the systems T_G, T_P and T_S, defined, respectively, by

$$T_G x[f(t)] = x[G_A f(t)],\ G_A f(t) = \int_{-\infty}^{\infty} kbe^{-at}\Theta(-t)f(s-t)dt$$

$$T_P x[f(t)] = x[P_A f(t)],\ P_A f(t) = \int_{-\infty}^{\infty} kbe^{-at}(1-\Theta(-t))f(s-t)dt$$

$$T_S x[f(t)] = x[S_A f(t)],\ S_A f(t) = f(t) - \int_{-\infty}^{\infty} kbe^{(a-kb)t}\Theta(-t)f(s-t)dt$$

where $\Theta(t) = 1$ for $t > 0$, zero otherwise, $a > 0$ and $(a - kb) > 0$. T_P, T_G and T_S are, respectively, stable, causal and stable and causal.

T_S is such that $(I + T_P)(T_S x) = x$. Hence, the solution to the feedback system

$$\begin{cases} y = T_P u \\ u = r - y \end{cases} \tag{5.8}$$

is given by

$$\begin{cases} y = T_P T_S r \\ u = T_S r \end{cases}$$

Consider the operator V, defined by

$$V x[f(t)] = x[V_A f(t)],\ V_A f(t) = \int_{-\infty}^{\infty} kbe^{-at} f(s-t)dt$$

Since $T_G = T_P + V$ and $VT_S x = 0$, by Theorem 73 the feedback system

$$\begin{cases} y = T_G u \\ u = r - y \end{cases} \tag{5.9}$$

has the same solutions as 5.8, for $\forall r \in D_{T_S}$. Hence, the feedback system enclosing T_G is causal and stable with the same response as the feedback system enclosing T_P, for all signals in D_{T_S}.

5.5 A Framework using Distributions

Consider the subclass of the Distributions $\mathcal{D}_E$, the subclass of $\mathcal{U}$, $\mathcal{U}_E$ and the class of multipliers $\mathcal{M}$.

Definition 75. $\mathcal{D}_E$ is the class of signals and the convolutes on $\mathcal{D}_S$, mapping $\mathcal{D}_{EN}$ into $\mathcal{D}_{EN}$, are the systems for a Framework in time domain analysis.

$\mathcal{U}_E$ is the class of signals and the multipliers in $\mathcal{M}$ on $\mathcal{U}_S$, mapping $\mathcal{U}_{EN}$ into $\mathcal{U}_{EN}$, are the systems for a Framework in transform domain analysis.

It is first proved that the systems so defined are an algebra.

Theorem 76. *$\mathcal{M}$ constitutes an algebra.*

Proof. The multipliers in $\mathcal{M}$ on $\mathcal{U}_E$ are linear periodic operators mapping elements of $\mathcal{U}_{EN}$ into $\mathcal{U}_{EN}$. As linear operators they define an algebra of operators on $\mathcal{U}_E$. However, the sum an product of two multipliers are themselves multipliers defined simply by the sum and product , respectively, of the functions defining the original multipliers. The result follows. ■

In what follows it is proved the existence of inverse of the return difference operator.

Lemma 77. *Let M be a regular functional defined by the infinitely differentiable function $M(\omega)$ and $M^{(r)}$, the regular functional defined by $M^{(\omega)}$, be its r^{th}-derivative. Then M belongs to the class $\mathcal{M}$ provided $M(\omega)$ and $M^{(r)}(\omega)$, for all $r > 0$, are bounded.*

Proof. Clearly M is a multiplier on $\mathcal{U}_S$ and $m = \mathcal{F}^{-1}\{M\}$ is a convolute on $\mathcal{D}_S$. The functional m_t, defined by the function $(1+jt)$ is a multiplier on $\mathcal{D}_S$ and, given a regular functional $x \in \mathcal{D}_{EN}$, defined by the function $x(t)$, $x = ym_t^N$, where y is the regular functional defined by the function $y(t) = x(t)/(1+jt)^N$. It follows that

$$m * x = m * (ym_t^N) = \sum_{r=0}^{N} \begin{pmatrix} N \\ r \end{pmatrix} \mathcal{F}^{-1}\{M^{(r)}Y\}m_t^{(N-r)}$$

where $Y = \mathcal{F}\{y\}$. Since $y(t)$ is square integrable, $Y(\omega) = \mathcal{F}\{y(t)\}(\omega)$ is also square integrable and Y is the regular functional defined by it. In addition, by Holder's Theorem ([4]), $M^{(r)}(\omega)Y(\omega)$ is square integrable for all $r \geq 0$. Hence, $\mathcal{F}^{-1}\{M^{(r)}Y\}$ is the regular functional defined by the square integrable function $\mathcal{F}^{-1}\{M^{(r)}(\omega)Y(\omega)\}(t)$. It follows that $m * x \in \mathcal{D}_{EN}$ and M belongs to $\mathcal{M}_E$ as required. ■

Theorem 78. *Let $M \in \mathcal{M}$ be a multiplier on $\mathcal{U}_E$ defined by the function $M(\omega)$such that $(I + M(\omega))$ has no finite zeros, $|I + M(\omega)|$ is bounded away from zero, and the modulus of all derivatives of $M(\omega)$ is bounded, then $(I + M(\omega))^{-1}$ exists and is a multiplier and inverse on $\mathcal{U}_S$, mapping $\mathcal{U}_{EN}$ into $\mathcal{U}_{EN}$.*

Proof. Since $(1 + M(\omega))$ has no finite zeros and the limit of $M(\omega)$ as $|\omega|$ tends to infinity is not -1, then $(1 + M(\omega))^{-1}$ is bounded. Furthermore, since $M(\omega)$

is infinitely differentiable with all its derivatives bounded, $(1 + M(\omega))^{-1}$ is also infinitely differentiable with all its derivatives bounded. Hence, $(1+M(\omega))^{-1}$ defines a regular functional in $\mathcal{U}_S$ that, by Lemma 77, is a multiplier on $\mathcal{U}_S$ mapping $\mathcal{U}_{EN}$ into $\mathcal{U}_{EN}$. Furthermore, since $(1 + M(\omega))^{-1}(1 + M(\omega)) = 1$, $(1 + M(\omega))^{-1}$ is an inverse on U_E as required. ∎

Moreover, the class of signals $\mathcal{D}_E$ is a Banach space. In fact, consider $x \in \mathcal{D}_E$, then the norm associated to $\mathcal{D}_{EN}$ is

$$\|x\| = \left(\int_{-\infty}^{\infty} \left| \frac{x(t)}{1 + |t|^N} \right|^2 dt \right)^{\frac{1}{2}}$$

It follows that all the 5 Requirements for a Framework to be consistent are satisfied. It remains to be proved that the system function and the multiplier in $\mathcal{M}^T$ are the same

Theorem 79. *The system function for T is the multiplier $K(j\omega) \in \mathcal{M}$.*

Proof. Consider v_ν the regular functionals defined by the function $e^{j\nu t}$. Then

$$\mathcal{F}\{v_\nu\} = \delta_\nu$$

and

$$\mathcal{F}\{\Phi * v_\nu\} = K(\omega)\delta_\nu = K(\nu)\delta_\nu$$

Hence $\Phi * v_\nu = K(j\nu)v_n$ Therefore v_ν is an eigenvector of T with eigenvalue $K(\nu)$. The result follows from the Definition of system function. ∎

5.5.1 Standard Framework, Generalised Formalism and Distributions Framework

Consider the subspace of the distributions $\mathcal{D}_E$. Let Φ be a convolute on $\mathcal{D}_S$, the space of tempered distributions.

Theorem 80. *(i) the space $\mathcal{D}_E$ is isomorphic to the class of polynomially bounded signals.*

(ii) The systems in the Standard Framework are isomorphic to a subclass of the convolutes on $\mathcal{D}_S$.

Proof. (i) Obviously, $C^\infty(\mathbb{R})$ is isomorphic to a subset of $\mathcal{D}$. The result follows from the Definition of $\mathcal{D}_E$.

(ii) Consider $Tx = \{\Phi(t)\} * \{x(t)\}$ a system in the Standard Framework of Definition 65. Because the systems of the Standard Framework are stable, then

$$|\Phi(t)| < k\,|c|^t$$

for k a constant and $0 < c < 1$. Hence, the functional $\Phi(t)$ is a member of $\mathcal{D}_S$. ■

Let Φ be a convolute on $\mathcal{D}_S$ and let $K(j\omega)$ be the correspondent multiplier in $\mathcal{M}$. Define the operator T on $\mathcal{D}_E$ as $Tx = \Phi * x$. Consider the same operator T as a system in the Generalised Formalism and let $\bar{K}(s)$ be the system function of T. The relation between $K(j\omega)$, the multiplier in $\mathcal{M}$ associated to the convolute Φ and $\bar{K}(s)$ is established in the next Theorem.

Theorem 81. *Let $K(\omega)$ be the regular functional defined by $K(\omega) = \bar{K}_{s=j\omega}$. Then $K(j\omega)$ is the multiplier associated to the convolute Φ.*

Proof. Since v_ν, as defined in Theorem 79, is also an eigenvector of the Generalised Formalism version of T, and $\bar{K}(s)_{s=j\nu} = K(j\nu)$, the result follows from the Definition of system function and Theorem 79. ■

Consider $\Phi(t)$ in L, defining a causal but unstable convolution on L, together with its system function, the doublet $\{K(s), D_K\}$, with the further condition that $K(s)$ does not have finite zeros on the imaginary axis. The feedback system enclosing the unstable convolution

$$\begin{cases} y = \Phi * u \\ u = r - y \end{cases} \tag{5.10}$$

is extended in the Generalized Framework to

$$\begin{cases} y = T_G u \\ u = r - y \end{cases} \tag{5.11}$$

where $T_G : \mathcal{T}_D \to \mathcal{T}_D$ is given by

$$T_G x[f(t)] = x[T_G^A f(t)]$$

and $T_G^A = \int_{-\infty}^{\infty} \Phi(-\tau) f(t-\tau) d\tau$. T_G is the operator of an unstable but causal system on $\mathcal{T}_D$. By Theorem 73, 5.11 is equivalent to

$$\begin{cases} y = T_P u \\ u = r - y \end{cases} \tag{5.12}$$

where $T_P : \mathcal{T}_D \to \mathcal{T}_D$ is the operator of a stable systems such that $T_P = T_G - V$, $Vx = x[V_A f(t)]$, V_A the same as in 5.5, where the eigenvectors chosen are the zero eigenvectors of $(I + T_P)^{-1}$. Note that, for the two feedback systems to be equivalent, the inputs must be in $D_{(I+T_P)^{-1}}$. From the discussion above the stable feedback system enclosing T_P can be rewritten as a stable feedback system in which T_P is reconsidered as an operator acting on $\mathcal{D}_E$, with the advantage that the latter possesses a transfer function, $K(j\omega)$, such that $K(s)_{s=j\omega} = K(j\omega)$.

Example 14. The feedback systems enclosing T_P and T_G of Example 10, in the Standard Formalism, are obviously extended to the feedback systems enclosing T_P and T_G of Example 13, in the Generalised Formalism. In both Formalisms it is proven that the feedback system enclosing T_G is stable and causal with the same response as the feedback system enclosing T_P. However, while in the Standard Formalism it is necessary to restrict the class of signals to $e^{-at}L^p$, in the Generalized Formalism the result holds for all the signals in $D_{T_S} \subseteq \mathcal{T}_\Delta$. Moreover, from the discussion above, the feedback system enclosing T_P in the Generalized Formalism is equivalent to a feedback system where T_P is a convolute on $\mathcal{D}_E$. Not only a restriction of the class of signals is not necessary anymore, but the context used for the analysis, the Framework for Distributions, is a consistent mathematical framework, since $(I+T_P)^{-1}$ is well defined. Furthermore, the time domain analysis is coupled with a transform domain analysis. In fact to the convolutes T_P and $(I+T_P)^{-1}$ corresponds, as a transfer function, the multipliers in $\mathcal{M}$ on $\mathcal{U}_\mathcal{S}$, mapping $\mathcal{U}_{EN}$ into itself, $kb/(j\omega - a)$ and $(j\omega - a)/(j\omega - (a - kb))$, respectively. The analysis is not centred anymore on stability, but on causality.

5.6 A Framework for Singular Systems

Similarly to the same Section of the previous Chapter a non regular functional in $\mathcal{D}_E$ can be always represented by the limits of a sequence of regular functionals in $\mathcal{D}_E$. However, as in the previous Chapter, the result does not apply to convolutes. Hence, considering the sequence of convolutes $\{\Phi_n\}$, the operator T, such that $Tu = (\lim_{n\to\infty} \Phi_n) * u$, is a suitable system for a Framework for singular systems if the signal space is restricted to elements of $\mathcal{D}_E$ that are themselves convolutes.

Definition 82. The members of $\mathcal{D}_E$ that are convolutes on $\mathcal{D}_E$ are the signals and

the limits of sequences of convolutes on $\mathcal{D}_E$ are the systems for a Framework for singular unstable systems.
The subspace of $\mathcal{U}_E$ that corresponds to the Fourier transforms of convolutes on $\mathcal{D}_E$ and the Fourier transform of the limits of convolutes on $\mathcal{D}_E$ are the correspondent elements for a correspondent transform domain analysis.

Consider the plant T such that $Tu = \lim_{n\to\infty}(\Phi_n * u)$, Φ_n as above. The feedback system

$$\begin{cases} y = lim_{n\to\infty}(T_n u) \\ u = r - y \end{cases}$$

still has a solution. In fact if $\{M_n\}$ is the sequence of multipliers corresponding to the convolutes $\{\Phi_n\}$, then $(I + M_n)^{-1}$ is an infinite differentiable function for any n. Therefore, $\lim_{n\to\infty}(I + M_n)^{-1}$ is well defined and $(I + \lim_{n\to\infty}\Phi_n)^{-1}$ exists and is the system representing the inverse difference operator.

Example 15. Consider the singular unstable system with system function $K(s) = 1/(s-1)$. We replace it with the limit

$$\lim_{n\to\infty} K_n(z) = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{s-1+\frac{1}{n}} + \frac{1}{s-1-\frac{1}{n}}\right)$$

$\{K_n(s)\}$ is a sequence of multipliers in $\mathcal{M}_E$. In time domain $\{\Phi_n\}$ is the sequence of corresponding convolutes. In the Framework for Singular Systems the limit $(\lim_{n\to+\infty} T_n)x = \lim_{n\to\infty}(T_n x)$ exists because x is a convolute on $\mathcal{D}_E$, but the operator $\lim_{n\to\infty}\Phi_n$ is not a convolute. However, the closed loop system

$$\begin{cases} y = lim_{n\to\infty}(\Phi_n * u) \\ u = r - y \end{cases}$$

admits solution for any input that is a convolute on $\mathcal{D}_E$ and its transfer function is the multiplier

$$\lim_{n\to+\infty} \frac{1}{1 + K_n(s)} = \frac{s-1}{s}$$

The usual transform domain analysis can now be applied.

Chapter 6

Hybrid feedback systems

6.1 Introduction

Consider the hybrid feedback system

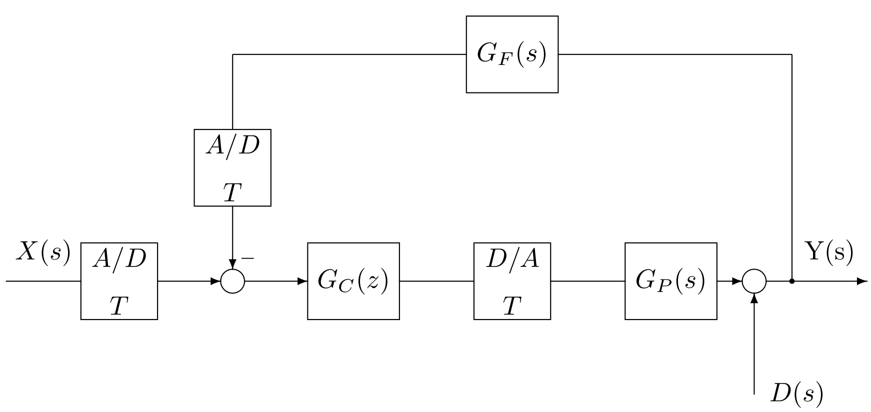

Figure 6.1: Hybrid Feedback System, s domain.

where $(D/A)T$ is an ideal D/A converter which acts, with a time constant T, on a discrete time signal in L^T, $\{x[k]\}$, to produce a piecewise constant continuous time signal, $y(t) \in L$, that is, it acts as an ideal zero-order-hold (ZOH). $(A/D)T$ is an ideal A/D converter which samples, with a sampling interval T, a continuous time signal, $x(t) \in L$, to produce a discrete time signal, $\{y[k]\} = \{x(kT)\} \in L^T$. $G_P(s)$ and $G_C(z)$ are the bilateral Laplace transform and the double sided z transform, respectively, of the stable, but not necessarily causal, plants of the feedback system and $G_F(s)$ is the bilateral Laplace transform of the controller. $X(s)$, $Y(s)$ and $D(s)$ are the bilateral Laplace transform of the signals $x(t)$, $y(t)$ and $d(t) \in L$, respectively. Define $\Phi(t)$ and $\Omega(t) \in L$, such that $G_P(s)$ and $G_F(s)$ are their bilateral Laplace transform, respectively. Define $\Psi[k] \in L^T$, such that $G_C(z)$ is its double sided z transform. Consider the operators

$$\begin{aligned}
T_P &: D_{T_P} \subseteq L \to R_{T_P} \subseteq L : \ T_P(x(t)) = \Phi * x \\
T_F &: D_{T_F} \subseteq L \to R_{T_F} \subseteq L : \ T_F(x(t)) = \Omega * x \\
T_C &: D_{T_C} \subseteq L^T \to R_{T_C} \subseteq L^T : \ T_C\{x[k]\} = \Psi * x \\
T_{(A/D)T} &: L \to L^T : \ T_{(A/D)T}(x(t)) = \{y[k]\} = \{x[kT]\} \\
T_{(D/A)T} &: L^T \to L : \ T_{(D/A)T}\{x[k]\} = y(t) = \textstyle\sum_{k=-\infty}^{\infty} x[k]h^T(t-kT)
\end{aligned}$$

where $h^T(t) = 1$ when $t \in [0, T)$, zero otherwise. When the sampling is well defined (that is discussed later), in time domain, the feedback system of Figure 6.1 becomes

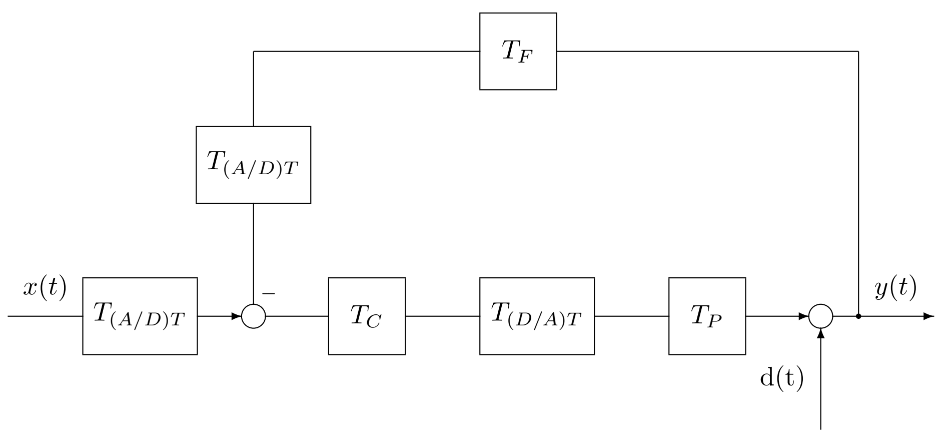

Figure 6.2: Hybrid Feedback System, time domain.

6.2 Hybrid feedback systems in the Generalised Formalism

Consider T_P, T_F, T_C, $T_{(A/D)T}$ and $T_{(D/A)T}$ defined in the previous section. Since T_G and T_F are continuous time systems, and T_C is a discrete time systems, in what follows they are redefined in their correspondent Generalised Formalism for continuous time domain or discrete time domain, respectively (see Definition 71 and

37).

$T_P : D_{T_P} \subseteq \mathcal{T}_D \to R_{T_P} \subseteq \mathcal{T}_D,\ (T_P x)[f(t)] = (\Phi{*}x)[f(t)] = x[\int_{-\infty}^{\infty} \Phi(-\tau)f(t-\tau)d\tau]$

$T_F : D_{T_F} \subseteq \mathcal{T}_D \to R_{T_F} \subseteq \mathcal{T}_D,\ (T_F x)[f(t)] = (\Omega{*}x)[f(t)] = x[\int_{-\infty}^{\infty} \Omega(-\tau)f(t-\tau)d\tau]$

$T_C : D_{T_C} \subseteq \mathcal{T}_\Delta \to R_{T_C} \subseteq \mathcal{T}_\Delta,\ (T_C x)[f(t)] = (\Psi * x)[f(t)] = x[\sum_{k=-\infty}^{\infty} \Psi_{-k} f(t-kT)]$

$T_{(A/D)T}$ and $T_{(D/A)T}$ are redefined as

$$T_{(A/D)T} : \mathcal{T}_D \to \mathcal{T}_\Delta,\ (T_{(A/D)T}x)[f(t)] = (\sum_{k=-\infty}^{\infty} x[k]\delta_{kT})[f(t)] \tag{6.1}$$

$$T_{(D/A)T} : \mathcal{T}_\Delta \to \mathcal{T}_D,\ (T_{(A/D)T}x)[f(t)] = (\sum_{k=-\infty}^{\infty} x[k]h^T(t-kT))[f(t)] \tag{6.2}$$

where $h^T(t) = 1$ when $t \in [0, T)$, zero otherwise.

Definition 83. The space of functionals $\mathcal{T}_D$ is the class of signals and the operators in $\bar{\mathcal{Q}}$ and in $\bar{\mathcal{Q}}_T$ represent the continuous time and the discrete time components of the hybrid feedback system for a Generalized Formalism in time domain analysis.

The notions of causality and stability are inherited from the notions of causality and stability of the correspondent discrete time or continuous time Generalised Formalism. Hence, an hybrid system is causal if all its components are causal, it is stable if all its components are stable.
The feedback system of Figure 6.2 can be written as

$$\begin{cases} y = d + T_P(T_{(D/A)T}T_C)u \\ u = T_{(A/D)T}(r - T_F y) \end{cases} \tag{6.3}$$

where y and $d \in \mathcal{T}_D$, $u \in \mathcal{T}_\Delta$. Define the system

$$\hat{T} : D_{\hat{T}} \subseteq \mathcal{T}_\Delta \to R_{\hat{T}} \subseteq \mathcal{T}_\Delta : \hat{T} = T_{(A/D)T}T_P T_{(D/A)T}T_C$$

Clearly $\hat{T}$ is equivalent to the system $T_{(A/D)T}T_P\hat{T}_C T_{(D/A)T}$ where $\hat{T}_C$ is defined as

$$\hat{T}_C : D_{\hat{T}_C} \subseteq \mathcal{T}_D \to R_{\hat{T}_C} \subseteq \mathcal{T}_D :\ x[f(t)] = (\sum_{k=-\infty}^{\infty} \Psi[k]h^T(t-kT) * x)[f(t)]$$

Consequently, the components of $\hat{T}$ given by the composition $T_P\hat{T}_C$ is a system that maps signals in $\mathcal{T}_D$ into signals in $\mathcal{T}_D$. Let $K_P(s)$ and $K_C(s)$ be the system functions of T_P and $\hat{T}_C$, respectively.

Reconsider $T_P\hat{T}_C$ as a system in the Standard Formalism. Clearly, that is a stable continuous time system. Moreover, that can be associated to an unstable continuous time system, defined as $T_{Pu}\hat{T}_{Cu}$, where T_{Fu} and $T_{\hat{C}u}$ are unstable systems with system functions $K_P(s)$ and $K_C(s)$, but with different domain of convergence. Denote with D_{PC} and D_{PCu} the region of convergence of the transfer function of $T_P\hat{T}_C$ and $T_{Pu}\hat{T}_{Cu}$, respectively. Define

$$v = \sum_{j=1}^{N}\sum_{j=1}^{k_r} c_{i,j} t^{k_i-1} e^{p_i t} \tag{6.4}$$

where p_i are the poles of $K_P(s)K_C(s)$ in the region $D_{PC} \backslash D_{PCu}$. Define the operator V on $\mathcal{T}_D$ such that

$$Vx[f(t)] = x[\int_{-\infty}^{\infty} v(-\tau)f(t-\tau)d\tau] \tag{6.5}$$

Lemma 84. *Reconsider $T_P\hat{T}_C$ and $T_{Pu}\hat{T}_{Cu}$ in the Generalised Formalism. Then*

$$T_P\hat{T}_C x[f(t)] - T_{Pu}\hat{T}_{Cu} x[f(t)] = Vx[f(t)]$$

where $\sum_{j=1}^{N}\sum_{j=1}^{k_r} c_{i,j} t^{k_i-1} e^{p_i t}$ is a regular functional.

Proof. The proof is an application of Theorem 62 when $T_P\hat{T}_C$ and $T_{Pu}\hat{T}_{Cu}$ are reconsidered in the Standard Formalism. The extension to the Generalised Formalism is done reconsidering $\sum_{j=1}^{k_r} c_{i,j} t^{k_i-1} e^{p_i t}$ as a regular functional. ∎

Define the system

$$T_S : D_{T_S} \subseteq \mathcal{T}_\Delta \to R_{T_S} \subseteq \mathcal{T}_\Delta$$

such that

$$(I + T_{(A/D)T} T_F T_P \hat{T}_C T_{(D/A)T}) T_S x[f(t)] = x[f(t)]$$

and

$$(VT_{(D/A)T)}) T_S x[f(t)] = 0$$

Theorem 85. *Consider the stable, but not necessarily causal feedback system 6.3. Suppose there exists an operator T_S defined as above. Consider the unstable feedback system*

$$\begin{cases} y = d + (T_{Pu}(T_{(D/A)T} T_{Cu}))u \\ u = T_{(A/D)T}(r - T_F y) \end{cases} \tag{6.6}$$

Then the feedback systems 6.3 and 6.6 have, $\forall r \in D_{T_S}$, the same solution

$$\begin{cases} y = (T_P T_{(D/A)T} T_C) T_S (T_{(A/D)T}) r + (I - (T_P T_{(D/A)T} T_C) T_S (T_{(A/D)T} T_F) d \\ u = T_S (T_{(A/D)T}) r - T_S (T_{(A/D)T} T_F) d \end{cases}$$

Proof. By Lemma 84

$$T_P \hat{T}_C x[f(t)] - T_{Pu} \hat{T}_{Cu} x[f(t)] = V$$

and from the Definition of T_S

$$T_{(A/D)T} V T_{(D/A)T} T_S x[f(t)] = x[f(t)]$$

Hence, $\forall r \in D_{T_S}$

$$\begin{cases} y = [(T_{Pu} \hat{T}_{Cu} + V)(T_{(D/A)T})] T_S (T_{(A/D)T} r) \\ +[I - (T_{Pu} \hat{T}_{Cu} + V) T_{(D/A)T}] T_S (T_{(A/D)T} T_F d) \\ = T_P \hat{T}_C (T_{(D/A)T}) T_S (T_{(A/D)T} r) + (I - T_P \hat{T}_C T_{(D/A)T}) T_S (T_{(A/D)T} T_F d) \ \in \mathcal{T}_D \\ u = T_{(A/D)T} r - T_{(A/D))T} T_F y = T_S (T_{(A/D)T}) r - T_S (T_{(A/D)T} T_F) d \in \mathcal{T}_D \end{cases}$$

■

Theorem 85 implies that the response of a stable hybrid feedback system enclosing the stable components T_P and T_C, for any input $r \in D_{T_S}$, is the same as the response of the unstable hybrid feedback system enclosing the unstable components T_{Pu} and T_{Cu}.

6.3 Hybrid feedback systems in the Framework using Distributions

In Chapter 4 a consistent Mathematical Framework for the discrete time analysis of feedback systems is introduced. The same happens in Chapter 5, for the continuous time analysis of feedback systems. There, exploiting the extensions from a Standard Formalism to a Generalized Formalism and finally to a Framework that uses Distributions, it is shown how the analysis of a feedback system can be performed in a Framework without any loss of meaning. In what follows the same procedure is repeated for the feedback system of Figure 6.1.

Consider the stable plants T_P and T_C and the controller T_F. In a Framework using

Distributions T_P and T_F are reconsidered as systems on $\mathcal{D}_E$ and T_C as a system on $\mathcal{D}_E^T$. However, since it is required that the idealised sampling of continuous time signals is well-defined, a more appropriate reformulation of continuous time signals is provided by the subclass of distributions $\mathcal{D}_B$. Consequently, the operators $T_{(D/A)T}$ and $T_{(A/D)T}$ are the restrictions of 6.2 and 6.1 to $\mathcal{D}_B^T$ and $\mathcal{D}_B$, respectively.
In transform domain the Fourier transforms of signals are represented by functionals in $\mathcal{U}_B$ and the transfer functions of systems are functionals in $\mathcal{M}_B$, the class of multipliers on $\mathcal{U}_B$ mapping $\mathcal{U}_{BN}$ into itself for all $N \geq 0$.

It remains to be established a correct formulation for the D/A and A/D converters.

6.4 Frequency Domain Analysis - D/A converter

Consider an ideal D/A converter which acts, with a time constant T, on a discrete time signal, $\{x[k]\}$ to produce a piecewise constant continuous time signal, $y(t)$; that is, it acts as an ideal zero-order-hold (ZOH). The linear relationship between $\{x[k]\}$ and $y(t)$ in the frequency domain is established by the following Theorem.

Theorem 86. *A discrete time signals $\{x[k]\}$ is acted on by a ZOH, with time constant T, to produce a piecewise constant time signal $y(t)$ such that*

$$y(t) = \sum_{k=-\infty}^{\infty} x[k]h^T(t-k)$$

where $h^T(t) = 1$ when $t \in [0, T)$, zero otherwise. Provided there exists a periodic functional $X \in \mathcal{U}_{BN}^T$ with Fourier coefficients $\{x[k]\}$, then $y(t)$ defines a regular functional, $y \in \mathcal{D}_{BN} \cap \mathcal{D}_{VN}$ such that $Y = H^T X$ where $Y = \mathcal{F}\{y\} \in \mathcal{U}_{BN} \cap \mathcal{U}_{VN}$ and $H^T = \mathcal{F}\{h^T\}$ with h^T the functional in $\mathcal{D}$ defined by $h^T(t)$.

Proof. $y(t)$ is of bounded variation on any finite interval, and, since $X \in \mathcal{U}_{BN}^T$ implies $|x[k]| \leq c(1+|k|)^N$ for some c, $|y(t)| < c^*(1+|t|)^N$ for some c^*. Hence $y = \sum_{k=-\infty}^{\infty} x[k]h_{kT}^T \in \mathcal{D}_{BN}$. Furthermore for all $b_i \in \{-1, 1\}$ and $\{\tau_1, \tau_2, ..., \tau_{n+1}\}$

satisfying $a \leq \tau_1 < \tau_2 < ... < \tau_{n+1} \leq b$

$$\begin{aligned}\sum_{i=1}^{n} b_i(y(t+\tau_{i+1}) - y(t+\tau_i)) &= \sum_{i=1}^{\bar{n}} b_i(y(t+\tau_{i+1}) - y(t+\tau_i)) \\ &\leq \sum_{i=1}^{\bar{n}} (|y(t+\tau_{i+1})| + |y(t+\tau_i)| \leq \sum_{i=1}^{\bar{n}} (c^*(1+|t+\tau_{i+1}|) + c^*(|t+\tau_i|) \\ &\leq 2c^*\bar{n}(1+|t+b|)^N\end{aligned}$$

where $\bar{n} = int(t/(kT))$. Hence, $Var_{[a+t,b+t]}\{y(t)\} \leq \bar{c}(1+|t|)^N$, for some $\bar{c} > 0$, and $y \in \mathcal{D}_{VN}$. In addition, since h^T is a convolute on $\mathcal{D}$,

$$y = \lim_{n\to\infty} h^T * \sum_{k=-n}^{n} x[n]h_{kT}^T = \lim_{n\to\infty} * \sum_{k=-n}^{n} x[k]\delta_{kT} = h^T * \lim_{n\to\infty} \sum_{k=-n}^{n} x[k]\delta_{kT} = h^T * x$$

with $x = \mathcal{F}^{-1}\{X\}$ and $Y = H^T X$ as required. ■

Therefore, a D/A converter is represented in the frequency domain by the multiplier H^T mapping $\mathcal{U}_{BN}^T$ into $\mathcal{U}_{BN} \cap \mathcal{U}_{VN}$. Moreover, as a consequence, a discrete time subsystem positioned before a D/A converter is equivalent to a continuous time subsystem positioned after the D/A converter, provided their frequency response functions are the same.

6.5 Frequency Domain Analysis - A/D converter

Consider an ideal A/D converter which samples, with a sampling interval T, a continuous time signal, $x(t)$, to produce a discrete time signal $\{y[k]\} = \{x[k]\}$. The linear relationship between $x(t)$ and $\{y[k]\}$ in the frequency domain is established by the following Theorem.

Theorem 87. *A continuous time signal, $x(t)$, is acted by a sampler with sampling interval T to produce a discrete time signal $\{y[k]\}$. Provided there exists a regular functional $x \in \mathcal{D}_{BN}$ defined by $x(t)$ then*

(i) $x(t)$ is equal almost everywhere to a function $x_D(t)$ such that, at all t,

$$x_D(t) = \frac{(x_D^-(t) + x_D^+(t))}{2}$$

and so sampling is well defined with $y[k] = x_D(kT)$.

(ii) the summation $\frac{1}{T}\sum_{k=-\infty}^{\infty} X_{2\pi k/T}$ converges in $\mathcal{U}$, where $X = \mathcal{F}\{x\} \in \mathcal{U}_{BN}$, and $\{y[k]\}$ are the Fourier coefficients for a periodic functional $Y \in \mathcal{U}_{BN}^T$ with period $2\pi/T$ such that $Y = \mathcal{O}^T[X] = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_{2\pi/T}$

Proof. Since $X \in \mathcal{D}_{BN}$, $x(t)$ is of bounded variation on each finite interval and part (i) follows from Theorem 89. In addition, there exists a periodic functional $Y \in \mathcal{U}$, with period $2\pi/T$ and Fourier coefficients $y_k[k] = x_D(kT)$ such that the summation $\frac{1}{T}\sum_{k=-\infty}^{\infty} X_{2\pi k/T}$ converges in $\mathcal{U}$ and $Y = \mathcal{O}^T[X] = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_{2\pi k/T}$. Furthermore, since $x \in \mathcal{D}_{BN}$, $y = \mathcal{F}^{-1}\{Y\} \in \mathcal{D}_{BN}^T$ as required by part (ii). ∎

Therefore, an A/D converter is represented in the frequency domain by the linear operator $\mathcal{O}^T$ on $\mathcal{U}_B$ mapping $\mathcal{U}_{BN}$ into $\mathcal{U}_{BN}^T$ for all $N \geq 0$. Further properties of the operator $\mathcal{O}^T$ are established in the following Theorem.

Theorem 88. *If X is a functional in $\mathcal{U}_B$ with n^{th} derivative $X^{(n)}$, Y is a functional in $\mathcal{U}_B$ and M^T is a periodic multiplier in $\mathcal{M}_B$ with period $2\pi/T$ then*

(i) $\mathcal{O}^T[X]$ is a periodic multiplier in $\mathcal{M}_B$ with period $2\pi/T$ provided $j^n X^{(n)} \in \mathcal{U}_{B0}$ for all $n \geq 0$;

(ii)$\mathcal{O}^T[M^T X] = M^T \mathcal{O}^T[X]$;

(iii) $\mathcal{O}^T[Y\mathcal{O}^T[X]] = \mathcal{O}^T[Y]\mathcal{O}^T[X]$ provided $j^n X^{(n)} \in \mathcal{U}_0$ for all $n \geq 0$.

Proof. (i)The regular functional $x = \mathcal{F}^{-1}\{X\} \in \mathcal{D}_B$ is defined by a function $x(t)$, which by Theorem 87 part (i) is equal almost everywhere to a function $x_D(t)$ such that, at all t,

$$x_D(t) = \frac{(x_D^-(t) + x_D^+(t))}{2}$$

For all $n \geq 0$, since $j^n X^{(n)} \in \mathcal{U}_{B0}$, $y \in \mathcal{D}_{B0}$, where y is the functional defined by $t^n x(t)$, and the series $\sum_{k=-\infty}^{\infty}(kT)^n x_D(kT)e^{-jk\omega T}$ converges for all ω. Hence, by Theorem 87 part (ii), $\mathcal{O}^T[X]$ is an infinitely differentiable regular functional. Furthermore, the n^{th} derivative of $\mathcal{O}^T[X]$ is continuous and periodic and so bounded. Consequently, $\mathcal{O}^T[X]$ is a multiplier in $\mathcal{M}_B$ with period $2\pi/T$.

(ii) For any $X \in \mathcal{U}_{BN}$, $M^T X \in \mathcal{U}_{BN}$ and by Theorem 87 both $\mathcal{O}^T[X] \in \mathcal{U}_{BN}^T$ and $\mathcal{O}^T[M^T X] \in \mathcal{U}_{BN}^T$ exist. Moreover, since M^T is a multiplier in $\mathcal{M}_B$ with period $2\pi/T$,

$$\begin{aligned}
&\frac{1}{T}\lim_{n\to\infty}\sum_{k=-n}^{n} M_{kT}^T X_{kT} \\
&\quad = \frac{1}{T}\lim_{n\to\infty}\sum_{k=-n}^{n} M^T X_{kT} = \frac{1}{T}\lim_{n\to\infty} M^T \sum_{k=-n}^{n} X_{kT} = M^T\frac{1}{T}\lim_{n\to\infty}\sum_{k=-n}^{n} X_{kT}
\end{aligned}$$

and $\mathcal{O}^T[M^T X] = M^T\mathcal{O}^T[X]$ as required.

(iii) It follows directly from part (i) and (ii). ∎

A consequence of Theorem 87 part (ii) is that, in frequency domain, a continuous time sub systems positioned before an A/D converter is equivalent to a discrete time subsystem positioned after the A/D provided their frequency response functions are the same.

6.6 Analysis in the Distributions Framework

Consider in the Standard Formalism the hybrid feedback system of Figure 6.1 and the associated hybrid feedback system with unstable components T_{Pu} and T_{Cu}, such that they have the same transfer functions $G_P(s)$ and $G_C(z)$, but with different region of convergence.

Both the hybrid feedback systems are reconsidered in the Generalised Framework. Similarly to the Standard Formalism, T_P and T_{Pu} have system function, $K_P(s)$, and T_C and T_{Cu} have the system function $K_C^T(z)$, but with different domain of existence. It is proved that, assumed there exists the operator T_S and that $VT_{(D/A)T}T_Sx[f(t)] = 0$, for all $x \in D_{T_S}$, with V as in 6.5, the stable feedback system and the unstable one have the same solutions, $\forall x \in D_{T_S}$.

Reconsider the stable hybrid feedback system in the Framework using Distributions. The system functions T_P and T_C are the multipliers $K_P(j\omega)$ and $K_C^T(j\omega)$. The stable hybrid feedback system has solution

$$\begin{cases} y = (T_PT_{(D/A)T}T_C)T_S(T_{(A/D)T})r + (I - (T_PT_{(D/A)T}T_C)T_ST_{(A/D)T}T_F)d \\ u = T_S(T_{(A/D)T})r - T_S(T_{(A/D)T}T_F)d \end{cases}$$

Note that T_S exists because it corresponds, in transform domain, to the operator

$$[I + \mathcal{O}^T[K_FK_PH^T]K_C^T]^{-1}$$

defined few lines below. Moreover, the condition $VT_{(D/A)T}T_Sx[f(t)] = 0$ is not anymore an assumption, but it is a consequence of Theorem 87 part (ii). Hence $VT_{(D/A)T}$ is an eigenvector with eigenvalue zero of T_S. Consequently, the solution to the stable hybrid feedback system exists for all inputs in $\mathcal{D}_B$.

Reconsider the stable feedback system in transform domain. That corresponds to the feedback system of Figure 6.3. The system functions K_P and K_F are the multipliers defined by $G_P(j\omega)$ and $G_F(j\omega)$, respectively, and K_C^T is the periodic multiplier defined by $G_C(e^{j\omega T})$. The class, to which each signal belongs, is explicitly

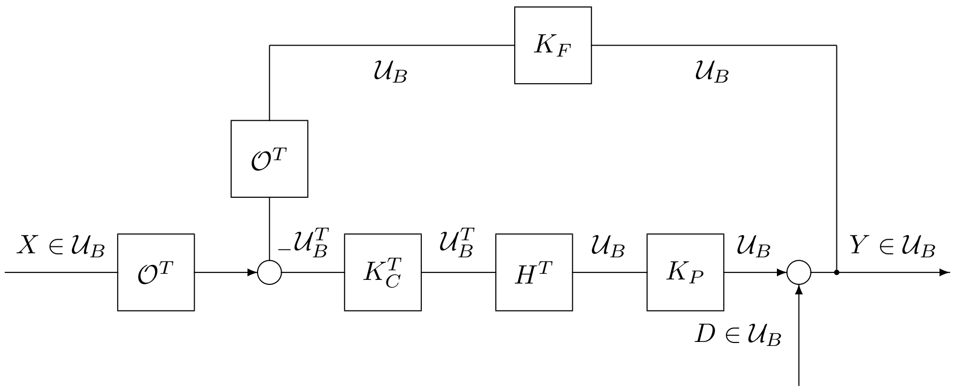

Figure 6.3: Hybrid Feedback System, frequency domain.

indicated in Figure 6.3. Formally,

$$Y = D + K_P H^T K_C^T \{\mathcal{O}^T[X] - \mathcal{O}^T[K_F Y]\} \tag{6.7}$$

It follows, assuming $[I + \mathcal{O}^T[K_F K_P H^T] K_C^T]^{-1}$ is non singular, that

$$\mathcal{O}^T[K_F Y] = [I + \mathcal{O}^T[K_F K_P H^T] K_C^T]^{-1} \{\mathcal{O}^T[K_F D] + \mathcal{O}^T[K_F K_P H^T] K_C^T \mathcal{O}^T[X]\}$$

and, hence,

$$Y = \mathcal{R}[X] + \mathcal{S}[D]$$

where

$$\mathcal{R}[X] = K_P H^T K_C^T [I + \mathcal{O}^T[K_F K_P H^T] K_C^T]^{-1} \mathcal{O}^T[X]$$

and

$$\mathcal{S}[D] = D - \mathcal{R}[K_F D]$$

Example 16. Consider a System defined as in Figure 6.1 with

$$G_P = \frac{1}{s-1}, \; G_C(z) = 1, \; G_F(s) = 1$$

and the corresponding system in the frequency domain with the regular functionals defined by

$$K_P(\omega) = \frac{1}{j\omega - 1}, \; K_c(\omega) = 1, \; K_F(\omega) = 1 \text{ and } H^T(\omega) = \frac{1}{j\omega}(1 - e^{-j\omega T})$$

It follows that

$$\mathcal{F}^{-1}\{K_F K_P H^T\}(t) = \begin{cases} -(1 - e^{-T})e^T & ; \quad t < 0 \\ -(1 - e^{-T} e^t) & ; \quad 0 < t < T \\ 0 & ; \quad T < t \end{cases}$$

and

$$\mathcal{O}^T[K_F K_P H^T](\omega) = -(1-e^{-T})\sum_{k=0}^{\infty} e^{-kT} e^{j\omega T} = -\frac{(1-e^{-T})}{(1-e^{-T}e^{j\omega T})}$$

Hence

$$\begin{aligned} &K_P H^T K_C^T [I + \mathcal{O}^T[K_F K_P H^T] K_C^T]^{-1}(\omega) \\ &\quad = \frac{c}{1-c(1-e^{-T})} \frac{(1-e^{-T}e^{j\omega T})}{(j\omega - 1)(1-e^{-T}e^{j\omega T}/(1-c(1-e^{-T})))} \frac{(1-e^{-j\omega T})}{j\omega} \end{aligned}$$

Chapter 7

Conclusion

The primary objective of this thesis is the investigation of three different Mathematical Formalisms for a System Theory approach to feedback systems, when the systems are either discrete time, or continuous time, or hybrid single rate. Central to the investigation is the idea of consistency of the Mathematical Formalism, in order to avoid paradoxes and difficulties like the Georgiou Smith paradox.

It is shown that, using a Standard Formalism, consistency be regained, but with the price of restricting the class of signals. With the Generalised Formalism, the use of transform domain analysis is not allowed. The disadvantages of those two Formalisms are avoided when employing the Framework using Distributions. In fact, it is shown that this Framework is consistent and that transform domain analysis still applies. Moreover, the class of signals is greatly enlarged to any polynomially bounded signal.

Using the three Formalisms it is shown how a conventional approach to feedback systems analysis can be replaced by an analysis in the Distributions Framework. In fact, a feedback system, enclosing a causal but unstable plant, is replaced by its equivalent feedback system enclosing a stable but noncausal plant. Therefore, in the Distributions Framework, the analysis of feedback systems is shifted to the property of causality.

Moreover, in the case of hybrid single rate feedback systems, the consistency of the Distributions Framework is done showing the well-posedness of sampling formulas in a distributions context.

Appendix A

Sampling Theorem

The following Theorem is quoted in [4] without a proof. Moreover, no proof could be found in the literature. That is provided in what follows.

Theorem 89. *Suppose $\tilde{f} \in \mathcal{U}$ has a transform $\tilde{F} \in \mathcal{D}$ that is regular and equal to a function F that is of bounded variation on each finite interval (though not necessarily on $(-\infty, \infty)$): then*

(i) $F(y)$ will be equal a.e. to a function $F_D(y)$ such that, at all y,

$$F_D(Y) = \frac{1}{2}[F_D(y^-) + F_D(y^+)]$$

(ii) also

$$X \sum_{-\infty}^{\infty} \tilde{f}(x - nX) \tag{A.1}$$

will converge in $\mathcal{U}$ to define a periodic functional $\tilde{g}$ whose Fourier coefficients G_n are given by

$$G_n = F_D(n/X), \; n = 0, \pm 1, \pm 2, ...$$

(iii) if in addition $\tilde{f} \in \mathcal{D}_S$ and $F(y)/(1 + |y|)^N$ is of bounded variation on $(-\infty, \infty)$, then A.1 will converge in $\mathcal{D}_S$.

Proof. (i) and (ii) Let $\tilde{f}_N \in \mathcal{D}$ be the regular functional defined by $f_N(x)$ where

$$f_N(x) = \sum_{n=-N}^{N} e^{jn(2\pi/X)x} = \frac{sin(\pi(2N+1)x/(2X))}{sin(\pi x/(2X))}$$

$\tilde{f}_N$ is a multiplier on $\mathcal{D}$ and $f_N(x)$ is periodic with period X such that

$$\int_{-X/2}^{X/2} f_N(x)dx = X$$

For any regular $\tilde{g} \in \mathcal{D}$, with $g(x)$ of bounded variation on any finite interval, and any $\psi(x) \in D$,

$$(\tilde{f}_N\tilde{g})[\psi(x)] = \tilde{g}[f_N(x)\psi(x)] = \int_{-\infty}^{\infty} g(x)f_N(x)\Psi(x)dx$$

Since $\psi(x)$ is of finite support, $\exists K$ such that $\psi(x) = 0$ for $|x| > (K + \frac{1}{2})X$. Hence,

$$\begin{aligned}
\tilde{f}_N\tilde{g}[\psi[x]] &= \int_{-(K+1/2)X}^{(K+1/2)X} g(x)f_N(x)\psi(x)dx \\
&= \int_{-X/2}^{X/2} \left\{ \sum_{k=-K}^{K} f_N(x)g(x+kX)\psi(x+kX) \right\} dx = \int_{-X/2}^{X/2} f_N(x)\phi_K(x)dx \\
&= \int_{-X/2}^{X/2} (sin(\pi(2N+1)x/(2X)/x)\{x\phi_K(x)/sin(\pi x/(2X))\}dx
\end{aligned}$$

where

$$\phi_K(x) = \sum_{k=-K}^{K} g(x+kX)\psi(x+kX)$$

For all k, $g(x)$ is of finite variation on $[(k-1/2)X, (k+1/2)X]$ and so $x\phi_K(x)/(sin(\pi x/(2X)))$ is of finite variation on $[(k-1/2)X, (k+1/2)X]$. Consequently, by Theorem 5.10 of [4], $x = 0$ is a Dirichlet point and

$$\lim_{N\to\infty} \int_{-X/2}^{X/2} (sin(\pi(2N+1)x/(2X))/x)\{x\phi_k(x)/sin(\pi x/(2X))\}dx = X(\phi_k(0^+)+\phi_k(0^-))/2$$

It follows that

$$\begin{aligned}
\lim_{N\to\infty}(\tilde{f}_N\tilde{g})[\psi(x)] &= X\sum_{k=-K}^{K} \frac{1}{2}(g(kX^-)+g(kX^+))\psi(kX) \\
&= X\sum_{k=-K}^{K} \frac{1}{2}(g(kX^-)+g(kX^+))\tilde{\delta}_{kX}[\psi(x)]
\end{aligned}$$

Hence, $\frac{1}{N}\tilde{f}_N\tilde{g}$ converges to

$$\tilde{h} = \sum_{k=-K}^{K} \frac{1}{2}(g(kX^-)+g(kX^+))\tilde{\delta}_{kX}$$

in $\mathcal{D}$. Furthermore,

$$\tilde{H} = \mathcal{F}\{\tilde{h}\} = \sum_{k=-K}^{K} \frac{1}{2}(g(kX^-)+g(kX^+))\tilde{e}_{k(2\pi/X)} \in \mathcal{U}$$

and by Theorem 16.3 of [4], $\tilde{H}$ is periodic with period $2\pi/X$ and Fourier coefficients $\left\{\frac{1}{2}(g(kX^-)+g(kX^+))\right\}$. However

$$\mathcal{F}\left\{\frac{1}{X}\tilde{f}_N\tilde{g}\right\} = \frac{1}{X}\mathcal{F}\{\tilde{f}_N\} * \mathcal{F}\{\tilde{g}\} = \frac{1}{X}\left(\sum_{n=-N}^{N}\tilde{\delta}_{n(2\pi/X)}\right) * \tilde{G} = \frac{1}{N}\sum_{n=-N}^{N}\tilde{G}_{n(2\pi/X)}$$

It immediately follows that $\frac{1}{X}\sum_{n=-\infty}^{\infty}\tilde{G}_{n(2\pi/X)} \in \mathcal{U}$ and is equal to $\tilde{H}$. Thus part (i) part and (ii) are established.

(iii) Let f_N as above. For any function $g(x)$, with $g(x)/(1+|x|)^M$ of bounded variation on $(-\infty,\infty)$ for some $M>0$, and any $\psi(x) \in S$

$$|g(x)\psi(x)| < c/(1+|x|)^2$$

for some $c>0$. Hence,

$$\int_{-\infty}^{\infty} g(x)f_N(x)\psi(x)dx = \lim_{K\to\infty}\left\{\int_{-(K+1/2)X}^{(K+1/2)X} f_N(x)g(x)\psi(x)dx\right\}$$
$$= \lim_{K\to\infty}\int_{-X/2}^{X/2} f_N(x)\left\{\sum_{k=-K}^{K} g(x+KX)\psi(x+KX)\right\}dx$$

In addition, for any x,

$$|g(x+k)\psi(x+kX)| < c/(1+|kX|)^2$$

for some $c>0$ and the series

$$\phi_K(x) = \sum_{k=-K}^{K} g(x+kX)\psi(x+kX)$$

is absolutely convergent. Hence, there exists a function, $\phi(x)$, such that $\phi_K(x)$ converges pointwise to $\phi(x)$ and there exists a constant, A, such that, for all $K>0$, $|\phi_K(x)| < A$, $\forall x \in [-X/2, X/2]$. Consequently, by Theorem 4.1 of [4],

$$\lim_{K\to\infty}\int_{-X/2}^{X/2} f_N(x)\left\{\sum_{k=-K}^{K} g(x+kX)\psi(x+kX)\right\}dx$$
$$= \int_{-X/2}^{X/2} f_N(x)\phi(x)dx = \int_{-X/2}^{X/2}(sin(\pi(2N+1)x/(2X))/x)\{x\phi(x)/(sin(\pi x/(2X)))\}dx$$

Furthermore, $x\phi(x)/(sin(\pi x/(2X))$ is of bounded variation on $[-X/2, X/2]$. By Theorem 5.10 of [4], $x=0$ is a Dirichlet point and

$$\lim_{N\to\infty}\int_{-X/2}^{X/2}(sin(\pi(2N+1)x/(2X))/x)\{x\phi_k(x)/sin(\pi x/(2X))\}dx = X(\phi_k(0^+)+\phi_k(0^-))/2$$

Since, for $|x| < X/2$,

$$|g(kX+x)\psi(kX+x)| < c/(1+|kX|)^2$$

for some $c > 0$

$$\phi(0^+) = \sum_{k=-\infty}^{\infty} g(kX^+)\psi(kX^+)$$

and

$$\phi(0^-) = \sum_{k=-\infty}^{\infty} g(kX^-)\psi(kX^-)$$

and it follows that

$$\begin{aligned}
&\lim_{N\to\infty}\int_{-\infty}^{\infty} f_N(x)g(x)\psi(x)dx \\
&= \frac{1}{2}X\sum_{k=-\infty}^{\infty}((g(kX^+)\psi(kX^+)) + (g(kX^-)\psi(kX^-))) \\
&= \frac{1}{2}X\sum_{k=-\infty}^{\infty}(g(kX^+) + g(kX^-)\psi(kX^-))\psi(kX)
\end{aligned}$$

Let $\tilde{f}_N \in \mathcal{D}_S$ be the regular functional defined by $f_N(x)$ then $\tilde{f}_N$ is a multiplier on $\mathcal{D}_S$. For the regular functional $\tilde{g} \in \mathcal{D}_S$ defined by $g(x)$

$$(\tilde{f}_N\tilde{g})[\psi(x)] = \tilde{g}[f_N(x)\psi(x)] = \int_{-\infty}^{\infty} g(x)f_N(x)\psi(x)dx$$

From the foregoing, it follows that

$$\begin{aligned}
\lim_{N\to\infty}(\tilde{f}_N\tilde{g})[\psi(x)] &= X\sum_{k=-\infty}^{\infty}\frac{1}{2}(g(kX^-) + g(kX^+))\psi(kX) \\
&= X\sum_{k=-\infty}^{\infty}\frac{1}{2}(g(kX^-) + g(kX^+))\tilde{\delta}_{kX}[\psi(x)]
\end{aligned}$$

Hence, $\frac{1}{N}\tilde{f}_N\tilde{g}$ converges to

$$\tilde{h} = \sum_{k=-\infty}^{\infty}\frac{1}{2}(g(kX^-) + g(kX^+))\tilde{\delta}_{kX}$$

in $\mathcal{D}_S$. Furthermore,

$$\tilde{H} = \mathcal{F}\{\tilde{h}\} = \sum_{k=-K}^{K}\frac{1}{2}(g(kX^-) + g(kX^+))\tilde{e}_{k(2\pi/X)} \in \mathcal{S}$$

and by Theorem 16.3 of [4], $\tilde{H}$ is periodic with period $2\pi/X$ and Fourier coefficients $\left\{\frac{1}{2}(g(kX^-) + g(kX^+))\right\}$. However

$$\mathcal{F}\left\{\frac{1}{X}\tilde{f}_N\tilde{g}\right\} = \frac{1}{X}\mathcal{F}\{\tilde{f}_N\} * \mathcal{F}\{\tilde{g}\} = \frac{1}{X}\left(\sum_{n=-N}^{N}\tilde{\delta}_{n(2\pi/X)}\right) * \tilde{G} = \frac{1}{N}\sum_{n=-N}^{N}\tilde{G}_{n(2\pi/X)}$$

It immediately follows that $\frac{1}{X}\sum_{n=-\infty}^{\infty}\tilde{G}_{n(2\pi/X)} \in \mathcal{D}_S$ and is equal to $\tilde{H}$. Thus part (iii) is established. $\blacksquare$

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