A Series of Surprises

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Riemann's theorem on conditionally–convergent series surprises a lot of people. Some people react to it by retreating to the view that only absolute convergence deserves to be taken seriously. I go the other way. What I take from it is that the ordinary notion of the convergence of a series is not so sacred, after all. That notion relates to one particular way of adding up a series, one of many. Depending on the circumstances, one of the other ways may be more appropriate or interesting. The idea that a series should add up in the usual way is just a prejudice. For instance, with the Fourier series of a continuous function, it is a fact that the series often fails to converge in the ordinary way, but we know that the Cesaro means always converge uniformly to the function.

Euler had a much healthier attitude to series than many people today. He wrote down:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots. \tag{1}$$

No–one has any problems with this for $|x| < 1$, but Euler was above such petty restrictions. He gave three examples of (1). First, taking $x = \frac{1}{2}$, gives

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots = 2.$$

No problem. Next, $x = -1$ gives

$$1 - 1 + 1 - 1 + \cdots = \tfrac{1}{2}.$$

Well, it's OK in the sense of Cesaro summability. Let it go. Finally, taking $x = 2$ gives

$$1 + 2 + 4 + 8 + \cdots = -1. \tag{2}$$

At this point, we realise we are in the presence of greatness.

The most amazing thing is, that Euler was right! Properly understood, (2) is true.

The way to understand it is in terms of another notion of convergence: *convergence with respect to the 2–adic metric.*

The 2–adic size of an integer is defined as follows. If $n = 2^r s$, with $(2, s)$ (the g.c.d. or h.c.f. of 2 and s) equal to 1, then

$$|n|_2 = \frac{1}{2^r}.$$

Thus, for instance,

$$|1|_2 = |3|_2 = |45|_2 = |1000001|_2 = 1,$$

$$|2|_2 = |42|_2 = \tfrac{1}{2},$$

$$|4|_2 = |100|_2 = \tfrac{1}{4}.$$

In other words, numbers that contain big powers of 2 are small, from the 2–adic point of view.

This size concept shares some properties with the ordinary absolute value, but in some ways is very different.

Proposition. *For* $a, b \in$, *we have*

(3) $|a + b|_2 \leq \max\{|a|_2, |b|_2\}$,

(4) $|ab|_2 = |a|_2|b|_2$.

Proof. (1) If 2^r divides a and b, then it also divides $a + b$, and this yields the result.

(2) Let $a = 2^r s$ and $b = 2^t u$, with $(2, r) = (2, t) = 1$. Then $ab = 2^{r+t} su$, and $(2, su) = 1$, so

$$|ab|_2 = 2^{-r-t} = 2^{-r} 2^{-t} = |a|_2 |b|_2.$$

■

We note that (3) implies the triangle inequality:

$$|a+b|_2 \leq |a|_2 + |b|_2.$$

It follows that becomes a metric space, when endowed with the metric

$$\rho(n,m) = |n-m|_2.$$

In terms of this metric (called *the 2–adic metric*), the series in (2) converges, and converges to -1. To see this, observe that

$$1+2+4+\cdots+2^n-(-1) = 2^{n+1}$$

which is very small, in terms of 2–adic size.

There are 2–adic Cauchy series of integers that do not converge to integers. For instance, consider

$$1-2+4-8+\cdots.$$

Denoting by S_n the n–th partial sum, we have

$$S_{n+r} - S_n = (-2)^{n+1} + \cdots + (-2)^r$$

which has 2–adic size at most 2^{-n-1}. Thus, given $\epsilon > 0$, we may choose $N = -\log_2 \epsilon$, and we then have

$$n > N \text{ and } r \geq 1 \Rightarrow |S_{n+r} - S_n|_2 \leq 2^{-n-1} < \epsilon.$$

Thus the series is 2–adically Cauchy. Euler's formula (1), with $x = -2$, suggests that it adds up to $\frac{1}{3}$. To justify this, we extend the 2–adic size concept to rational numbers, by defining

$$\left|\frac{2^r s}{t}\right|_2 = 2^{-r},$$

whenever $r, s \in$, $t \in$, and $(2, s) = (2, t) = 1$. We then get that

$$|S_n - \tfrac{1}{3}|_2 = \left|\frac{(-2)^{n+1}}{3}\right|_2 = 2^{-n-1} \to 0$$

as $n \uparrow \infty$.

Not all series of whole numbers will converge in the 2–adic metric. For instance,

$$1 + 1 + 1 + \cdots$$

is divergent. Essentially, the series that do converge are those obtained by grouping series of the form

$$\sum_{n=0}^{\infty} a_n 2^n,$$

where the a_n are bounded. In fact, these can be reduced to such series with $a_n = 0$ or 1.

A series or a sequence of whole numbers will never converge to $\frac{1}{2}$, or to any fraction having 2–adic size greater than 1. This is immediate from the triangle inequality:

$$|\tfrac{1}{2} - n|_2 \geq |\tfrac{1}{2}|_2 - |n|_2 \geq 2 - 1 = 1$$

for every integer n.

The sum of a 2–adically Cauchy series of integers does not have to be a rational number. In fact, it turns out that only those with eventually–periodic coefficients give rationals. The rest may be regarded as converging to other, non–rational numbers. These numbers are called *2–adic integers*. The rational that are sums of such series are also called 2–adic integers. For instance, $\frac{1}{3}$ is a 2–adic integer.

The set of 2–adic integers is denoted $_2$. It may be specified formally as the set of all sums

$$\sum_{n=0}^{\infty} a_n 2^n \tag{5}$$

with $a_n = 0$ or 1. It is a big set. The representations (5) are unique. Thus $_2$ has the same power as 2, so it has the power of the continuum.

The irrational 2–adic integers should not be thought of as real, or even complex, numbers. They are "out in a different direction" from the rationals.

There are p–adic integers for every prime number p, obtained as sums of power series

$$a_0 + a_1 p + a_2 p^2 + \cdots .$$

As an example, consider

$$s = 1 + 2 \cdot 5 + 3 \cdot 5^2 + 5^3 + 2 \cdot 5^4 + 3 \cdot 5^6 + 5^7 + \cdots .$$

We have

$$1 + 5^3 + 5^6 + \cdots = \frac{1}{1 - 125} = -\frac{1}{124},$$

$$2 \cdot 5 + 2 \cdot 5^4 + 2 \cdot 5^7 + \cdots = -\frac{10}{124}$$

$$3 \cdot 5^2 + 3 \cdot 5^5 + 3 \cdot 5^8 + \cdots = -\frac{75}{124}$$

so

$$s = -43/64.$$

Ratios of p–adic integers are called p–adic numbers. The set of such things is denoted $_p$. It is a complete metric field. It should be thought of as a kind of alternative to the set of real numbers. From a purely mathematical point of view, there is no particular reason to regard the reals as more important than any of the p–adic systems $_p$. There are some important differences between the reals and the p–adic systems. There is no sensible total ordering of $_p$, and $_p$ is disconnected (— in fact, totally–disconnected). On the other hand, the p–adic balls

$$\{x \in {}_p : |a - x|_p \leq r\}$$

are compact sets, and this means that analysis over the p–adics is quite similar to that over . For instance, you can use methods like the Newton–Raphson method to solve polynomial equations for p–adic roots (when they have such roots).

These alternative number systems are very useful in number theory, and in some other areas of Mathematics. Besides, they are a lot of fun.

Exercises

1. Write $\frac{1}{3}$ as a series

$$a_0 + a_1 2 + a_2 2^2 + \cdots$$

using only 0's and 1's for the a_n's.

2. Write $\frac{1}{5}$ as a series

$$a_0 + a_1 2 + a_2 2^2 + \cdots.$$

3. Write $\frac{1}{7}$ as a series

$$a_0 + a_1 2 + a_2 2^2 + \cdots.$$

4. Write $\frac{1}{9}$ as a series

$$a_0 + a_1 2 + a_2 2^2 + \cdots.$$

5. Write $\frac{1}{2}$ as a series

$$a_0 + a_1 3 + a_2 3^2 + \cdots.$$

6. Write $\frac{1}{5}$ as a series

$$a_0 + a_1 3 + a_2 3^2 + \cdots.$$

7. Work out the 7–adic sum

$$1 + 2 \cdot 7 + 7^2 + 2 \cdot 7^3 + 7^4 \cdots.$$

8. Work out the 2–adic sum

$$1 + 7 \cdot 2 + 2^2 + 7 \cdot 2^3 + 2^4 \cdots.$$