HENRY SMITH AND THE ARITHMETICAL THEORY OF QUADRATIC FORMS

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For my mother, Ellen Veronica Bennett (b. 1942),
to whom I owe everything.

ABSTRACT

Henry John Stephen Smith (1826–1883) FRS was Savilian Professor of Geometry at Oxford University from 1861 to 1883 (Figure 1). He distinguished himself as a superb lecturer and researcher who brought international recognition to Oxford mathematics. His unique and caring personality ensured he was held in widespread affection and admiration by his students and the University community. The two volumes of Henry Smith's collected mathematical papers, first published in 1894, are little read today. They include interesting biographical material, but it may not be immediately obvious why he was regarded, and not just at his *alma mater*, as being in the front rank of European mathematicians. To understand the reasons behind this it will be helpful to look more closely at Henry Smith's life in mathematics.

His mathematical writings, on which his reputation chiefly rests, were on the theory of numbers and elliptic functions, topics in which European mathematicians were pre-eminent. He prepared a series of *Reports on the Theory of Numbers*, commissioned by the British Association for the Advancement of Science, from 1859 to 1865. These were followed by his memoirs on the arithmetical theory of integral quadratic forms leading eventually to his crowning memoir to the *French Académie des Sciences* for its *Grand Prix des Sciences Mathématiques* of 1882. He advanced the theory of quadratic forms by returning to original sources and remaining true to arithmetic, confirming Henry Smith's reputation as a brilliant arithmetician for whom fine arithmetical details and presentation mattered. The principle theme of these memoirs was the classification of quadratic forms. In addition to considering his early education and life in mathematics, this thesis will include a careful distillation of some of the mathematical techniques contained in these memoirs. It will reveal that Henry Smith's mathematical techniques and presentation style was guided and influenced by the earlier writings of Gauss, Eisenstein, and Dirichlet. In any attempt to assess Henry Smith's reputation in his day, and since, such considerations will be important.

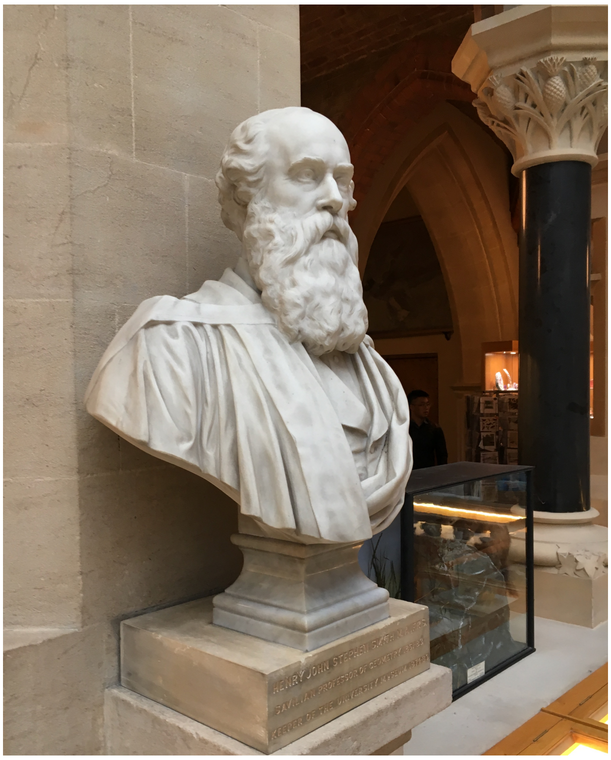

Figure 1: Marble Bust of Henry Smith (1826–1883) at the Oxford University Museum. Inscription: Henry John Stephen Smith MA FRS. Savilian Professor of Geometry 1861–83. Keeper of the University Museum 1874–83.

[Photograph the author's own, taken February, 2018].

I am sure that no subject loses more than mathematics
by any attempt to dissociate it from its history.

— James W.L. Glaisher (1848–1928)

ACKNOWLEDGEMENTS

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Getting to know Henry Smith through letters and recollections was a great pleasure. This could only be achieved with the gracious help of the librarians and archivists at: the Maynooth University Libraries, the Bodleian Libraries in Oxford, the Royal Society London, the Royal Astronomical Society London, Balliol College and Corpus Christi College, Oxford. I am very grateful to the British Society for the History of Mathematics (BSHM) for their kind invitation to speak at their annual 'Research in Progress' meeting at the Queen's College Oxford in 2019 and 2020. It was a wonderful experience. I would also like to thank the BSHM and the Department of Mathematics and Statistics, Maynooth University, for the financial support I received for a four week research visit to Oxford, in April 2022. I would like to thank my colleagues at the South East Technological University for their support throughout my studies. Finally I would like to acknowledge the support of the Irish History of Mathematics (IHoM) community.

INTRODUCTION

One hundred years after the death of the Oxford mathematician Henry Smith a commemorative article was published: *The Mathematician the World Forgot* (Hannabuss, 1983). It would seem that Henry Smith's name was unfamiliar even to many professional mathematicians who made regular use of the ideas he introduced. The reasons given for this range from his comparative isolation in a country that was only beginning to regain its mathematical confidence, to his own modest and caring disposition, utterly devoid of ambition. Fortunately, there has been renewed interest in the life and mathematics of Henry Smith. Biographical essays have, in recent years, been complemented by a number of excellent chapters on the subject of mathematics in Victorian Oxford. These chapters, written by Dr Keith Hannabuss of the Mathematical Institute, University of Oxford, may be found in the following publications.

Flood, Rice, and Wilson, 2011	pp. 35–50
Fauvel, Flood, and Wilson, 2013	pp. 239–255
Wilson, 2021	pp. 93–119

I first encountered Henry Smith's name in an essay by Professor Rod Gow, University College Dublin, in *Creators of mathematics: the Irish connection* (Houston, 2000, pp. 63–69). Smith, born in Dublin in 1826, was elected Savilian Professor of Geometry at Oxford University in 1860. Sixty years later the newly elected Savilian Professor was Godfrey Harold Hardy FRS (1877–1947), the foremost pure mathematician working in Britain during the first half of the 20th century. Upon reading G.H. Hardy's inaugural lecture as Savilian Professor my interest in Henry Smith's life and mathematics began. In his lecture G.H. Hardy described Henry Smith as 'a most brilliant arithmetician' and praised his memoirs on the arithmetical theory of quadratic forms, first published during the 1860's (Chapters 3, 4, 5). He reminded his audience that Henry Smith's final memoir on the theory of numbers was awarded a distinguished prize from the *French Académie des Sciences* in 1882. This tribute by G.H. Hardy, to a previous incumbent of the Savilian chair, would indicate that Henry Smith was a significant mathematician whose reputation then and now is a topic worthy of consideration.

I was initially attracted to the circumstances surrounding the *Grand Prix des Sciences Mathématiques* of 1882, as it represents an interesting episode in the history of mathematics (Chapter 6). On April 2nd, 1883, the French Academy announced the result of its competition to solve a problem in the theory of numbers. The *Grand Prix* was to be awarded jointly to Henry Smith, the recently deceased Oxford Professor, and to a young 18-year-old student at the University of Königsberg in East Prussia. The circumstances leading up to the announcement, along with the contrast between the ages of the honorands, meant that the award was soon at the centre of a public scandal. The French press criticized the Academy for incompetence and, most unfairly of all, falsely accused the young student of plagiarism. To understand the reasons behind some of these allegations it would helpful initially to look more closely at Henry Smith's early education and life in mathematics (Chapter 1, 2).

* Illustrative examples throughout this thesis are the author's own, unless otherwise stated.

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ACRONYMS

BAAS British Association for the Advancement of Science

LMS London Mathematical Society

1

BANTRY BAY TO BALLIOL COLLEGE OXFORD

Figure 2: Looking West along Bantry Bay, County Cork, Ireland.

Henry Smith was a brilliant Irish mathematician who was gifted with an endearing and jovial personality. One of his closest friends at Oxford recalled that 'he would at times break out into fits of laughter and joviality, which showed that the original Irish nature was not extinguished, but only kept under by him' (Smith, 1894a, p. lxiii). An interesting narrative for this thesis will be to consider Smith's contribution to the theory of numbers, along with aspects of his life at Oxford, told from the recollections of those who knew him best.

Despite the loss of his father and two of his siblings in young age, Henry Smith was part of a strong family unit; his mother, who sought the best education for her children, and his sister who was his constant support and companion. His childhood experiences and early education will give interesting insights on how it guided his early career and enriched his life in mathematics. In this chapter I will present newly discovered details of Smith's Irish ancestry, taking note of his family traditions, where formal education was important. His early education and association with Balliol College, Oxford will also be considered.

Figure 3: Some Towns and Parishes of County Cork, Ireland.

1.1 HENRY SMITH'S IRISH HERITAGE

Henry Smith's ancestors held important positions in the Church of Ireland community in the Parish of Kilmacomogue, Bantry, County Cork (Figure 3). They were landowners with brewing and milling interests. His parents' generation, particularly through marriages, continued in this strong tradition where high education attainment was important. However, in the early to mid 19th century, emigration from Ireland became commonplace.[1]

Henry John Stephen Smith was born on November 2nd, 1826 in Dublin.[2] He was the youngest of four children of John Smith (1792–1828) and his wife Mary Murphy (d. May 13th, 1857 aged 63) who were both from Bantry, County Cork (Figure 2). John Smith, barrister-at-law, was a graduate of Trinity College, Dublin and Brasenose College, Oxford. He was the eldest son of the Reverend Charles Smith (d. March 3rd, 1823, aged 65), Church of Ireland Vicar of Kilmacomogue Parish, Bantry. Charles Smith was ordained Deacon in 1784 at Cork and ordained Priest that same year at Cloyne. From 1784 until 1786 he was curate at Castlemartyr, in East Cork, and from 1786 until 1796 he was Vicar of Cannaway, in North Cork. From 1796 until his death in 1823 he was Vicar of Kilmacomogue. Charles Smith first married Elizabeth Forsayeth. In 1790 he married Jane Henah of Igtermurragh Parish, East Cork, with whom they had five children. Their eldest son John was born in 1792 followed by three daughters, Margaret, Ellen and Alicia and a youngest son Godfred (later the Reverend Godfred Smith, Vicar of Kinneigh, West Cork) (Brady, 1863, p. 166).[3] As Vicar of Kilmacomogue for almost 30 years, Charles Smith presided over the building of a new vicarage and new parish church, to replace the old church located on Church Road, Bantry. The vicarage was built in 1816 on land at Cappanaloha East, southwest of Bantry.[4] Building of the new parish Church on Wolfe Tone Square, Bantry, began in 1818 and was completed in 1828. It

1 Unless otherwise stated many of the biographical details contained in this chapter are taken from the *Biographical Sketch* of Henry Smith written by Charles Henry Pearson (1830–1894) on the request of Eleanor Smith (sister) who provided him with a complete memoir of details. This comprehensive account formed part of the introduction to Henry Smith's collected mathematical papers first published in two volumes in 1894 (Smith, 1894a, pp. x–xxxvi). Charles Henry Pearson was a British-born Australian historian, educationist, politician and journalist. His correspondence with Eleanor Smith, relating to the publication of the Henry Smith's collected mathematical papers, are considered in Section 2.7.

2 See entry for Henry John Stephen Smith in the *Dictionary of Irish Biography* (Byrne, 2009).

3 Records relating to the administration of the parish of Kilmacomogue during the 18th and 19th century may be found in (Brady, 1863).
https://archive.org/details/clericalandparo05bradgoog

4 In 1814 Lord Bantry, in return for an annual rent, leased to Reverend Charles Smith almost 20 acres of land at Cappanaloha East. The lease on this property was surrendered on October 11th, 1825 by John Smith and his mother Jane Smith, who in widowhood had moved to the city of Cork. See Bantry Estate Collection Descriptive List – Item 956.
https://libguides.ucc.ie/BantryEstateCollection/descriptivelist
The vicarage at Cappanaloha East, Bantry, is now a private residence.

was capable of accommodating 500 people and the old site on Church Road was abandoned.[5] Reverend Charles Smith died on March 3rd, 1823 and was interred at Garryvurcha graveyard which adjoins the old parish site on Church Road. The gravestone inscription reads:[6]

> Rev Charles Smith, Vicar of the Parish of Kilmacomogue, died 3rd Mar 1823, aged 65; Miss Elizabeth Smith died 24th Apr 1815, aged 50; Miss Anne Smith, died 20th Aug 1816, aged (?).[7]

Henry Smith's mother, Mary Murphy (d. May 13th, 1857 aged 63), was daughter of John Murphy (d. August 28th, 1806 aged 62) of Newtown House, Newtown in the Parish of Kilmacomogue, and Elizabeth Jervois (d. June 2nd, 1829) from Brade, Skibbereen, County Cork. They were married in August 1784 and they had a large family.[8] John Murphy was a brewery and mill owner with his mill located at Dunnamark, Bantry (now demolished). John Murphy was also interred at Garryvurcha Graveyard.

The marriage records for Kilmacomogue Parish show that the children of John Murphy and Elizabeth Jervois who married were Ellen/Elizabeth, Michael, Mary, Daniel, Catherine, John and Martha. It is likely that Mary Murphy had other siblings who remained unmarried.[9] Mary Murphy married John Smith on 14th April 1818 at Saint Fin Barre's Church of Ireland Cathedral, Cork City. Another connection between the Murphy and Smith families was through Mary Murphy's brother Reverend John Murphy (1795–1870). He was born at Newtown and entered Trinity College Dublin in 1813. He was ordained a Deacon at Cloyne on April 10th 1825 and Priest at Kildare on February 25th, 1827. He was Curate at Kilbrogan from April until November 1825 and from that time until 1842 was Curate at Murragh, both parishes in County Cork. From 1842 until 1861 he was Vicar of Kilmacomogue Parish. He resigned this ministry in 1861 for the Treasurership of the Cork Diocese. In 1826 he married Alicia, daughter of Reverend Charles Smith, the former Vicar of Kilmacomogue Parish. They had four daughters (Brady, 1863, p. 25).

5 In 1830, the Parish of Kilmacomogue was estimated at 14 miles long by 12 miles broad, with a gross population of 14,483. The Church of Ireland population of the parish was estimated to be 948 (Brady, 1863, p. 166).

6 https://www.ornaverum.org/family/bantry/garryvurcha-graveyard.html

7 Miss Elizabeth Smith and Miss Anne Smith may have been sisters of Reverend Charles Smith.

8 The details throughout this section that relate to Henry Smith's Irish ancestry, particularly the Murphy family, are located in the Church of Ireland records of births, marriages and deaths for Kilmacomogue Parish, Bantry, County Cork. https://durrushistory.com/2013/05/28/some-kilmocomogue-bantry-church-of-ireland-and-methodist-births-marriages-and-deaths-from-1629/

9 The entry for Henry John Stephen Smith in the *Oxford Dictionary of National Biography* suggests that 'Mary Murphy (d. 1857) was one of the fourteen children of John Murphy from near Bantry Bay' (Hannabuss, 2010).

The same parish records show that in 1818, at Ross Cathedral, Mary Murphy's brother Michael married Jane Besnard. The Besnard family were of Huguenot descent and, from the late 18th century, owned one of the largest sail factories in the world at Douglas, Cork, employing thousand of workers. The mills produced sail-cloth and supplied sails to the British Royal Navy, amongst other clients. Michael became the principle member of the family to remain in Bantry into the 19th century, to carry on the family business. He and his wife raised a family there, some of whom emigrated to Australia.

Following their marriage in April 1818 John and Mary Smith moved to Dublin. They spent the ten years of their married life in Dublin, living on Leeson Street, near St Stephen's Green. Located in the parish of Saint Peter's (Oxford, 1888, p. 97), it was the largest Church of Ireland parish in Dublin at that time. The parish Church of Saint Peter's was located on Aungier Street, Dublin 2.[10] Their first child, a daughter, was followed by Charles, Eleanor (b. 1822) and Henry (b. 1826).

In 1828 John Smith died and his last will and testament was signed on November 4th, 1828.[11] The document outlines that he had set up a trust fund following a Deed of Marriage settlement of £3900 to provide a payment of £200 per year to 'my beloved wife' and, upon his death, the sum of £3350 was to be paid to his wife and children, 'first and last alike'. This provision was to be made only when his children reach the age of twenty three years or date of marriage, whichever should first happen. He instructed the executors to permit his wife to occupy and enjoy their house on Leeson Street, as long as she wished to do so, and that upon her death, or in case she may prefer to reside elsewhere, that the 'house and premises' be sold and the proceeds form part of his custodian fund.[12] His law books were to be sold and the proceeds added to his estate. He finally nominated his wife as the sole guardian of their children. He expressed his unlimited and absolute confidence in his wife in the proper 'maintenance, clothing and education' of their 'dear children'. Just six months after her husbands death Mary Smith, along with her four children, left Ireland. Arriving first on the Isle of Man (1829), she then continued to England moving to Harborne near Birmingham (1829–30), then to Leamington (1830–31) and finally to Ryde on the Isle of Wight (1831), where they remained for ten years (Smith, 1894a, p. x).

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10 In the 1980s the church was demolished and an exhumation carried out in the churchyard under the supervision of Dublin Corporation and the Eastern Health Board. https://excavations.ie/report/2004/Dublin/0011702/

11 Public Record Office, The National Archives, Catalogue Reference: Prob 11/1750/418. https://discovery.nationalarchives.gov.uk/browse/r/h/D173428

12 The Solicitors' Journal & Reporter, Volume 1 of May 23rd, 1857 records that on May 13th, Mary Smith died of fever at Upper George Street, Bryanston Square, London, aged 63. It states that she was widow of the late John Smith, barrister-at-law, of Hatch Street, Dublin.

1.2 EARLY EDUCATION

Henry Smith's early education was guided and nurtured by his mother, who, after the death of her husband, devoted herself to her children. She developed in her young children a very broad range of interests, from nature to languages, and each day of their childhood was a mixture of lessons and leisure time. Given her knowledge of the classics she instructed her children, and Henry, from a Greek textbook, 'learned the alphabet, the nouns, the adjectives and the pronouns for his own pleasure' (Pearson, Glaisher, and Smith, 1894, p. xi). In 1838 she placed Henry, along with his brother and sisters, in charge of a private tutor, Mr Wheler Bush of Ryde. Henry was eleven when he read a large portion of the Greek and Latin authors commonly studied. His tutor recalled Henry's brilliant talents for such a young age. He wrote that 'he was a deeply interesting boy, singularly modest, lovable, and affectionate'.[13] Less than one year later Mr Bush was called away to a head–mastership and Henry's mother found it difficult to secure a new tutor for her children. However, on two days each week a tutor visited who was strong in mathematics, and with his aid Henry became acquainted with advanced arithmetic, and elements of algebra and geometry. In 1840 her eldest son Charles left to attend Addiscombe Military Seminary outside London and Mary Smith decided to move her family to Oxford, where she was certain that better tuition could be found than on the Isle of Wight.

Once settled in Oxford Mary Smith was fortunate in the tutor she secured for Henry. The Reverend Henry Highton (1816–1874), Fellow of Queen's College Oxford and then Curate of St. Ebbe's, was a scholar who had a strong interest in mathematics.[14] Within a classroom setting Henry could, for the first time, measure his ability with pupils of his own age. In the summer of 1841 Highton received the offer of a Mastership at Rugby, a public school north of Oxford, which at that time was a prestigious appointment as it came with a boarding-house. Highton accepted the offer and proposed that he should take Henry with him as his first boarder. His mother agreed to the suggestion and so began his formal school life at Rugby (Table 1).

Henry Smith was an exceptional student at Rugby. In July 1843 the headmaster, Dr Archibald Campbell Tait (1811–1882) wrote to Mary Smith to say 'there is no young man in the Sixth Form from whose abilities I am led to expect more than from him, and I have formed a very high opinion of his character and conduct generally' (*ibid*, p. xv). However Henry had only completed his second year at Rugby when, in September 1843, his brother Charles died of tuberculosis. Given that his elder sister had also died to the same disease some years earlier the difficult decision was made to remove Henry from Rugby and for the

13 Times, February 12th, 1883.
14 St Ebbe's is a Church of England parish church in central Oxford.

Table 1: Education at Rugby School and Oxford University

Year	
1841	Entrance to Rugby School – August.
1844	Matriculated – November 30th.
1844	Scholarship Balliol College, Oxford.
1845	Attends Balliol College, Oxford. (First term only)
1847	Resumed Studies at Balliol College, Oxford.
1848	Dean Ireland Scholarship, Oxford.
1849	BA, Oxford. (First class Mathematics and Classics)
1849	Lectureship in Mathematics, Balliol College, Oxford.
1849 – 74	Fellow of Balliol College, Oxford.
1851	Senior Mathematical Scholarship, Oxford.
1855	MA, Oxford.

family to move to a healthier climate.[15] Unfortunately it would not be the last time that his formal education was interrupted by concerns for his health. The winter of 1843 was spent with his mother and only surviving sibling Eleanor at Nice, France. The summer of 1844 was spent at Lake Lucerne, Switzerland which, according to his sister 'were months of steady reading, though his books were few, and he was even unprovided with a Greek Lexicon' (*ibid*, p. xv).

In the Autumn of 1844, just before his eighteenth birthday, Henry returned to England to visit Henry Highton who assisted him to prepare for a Balliol College scholarship. Oxford University was outstanding in the Classics, the branch of learning in which Henry excelled at as a boy. However his choice of Balliol College may have been influenced by his headmaster, Dr Tait, who himself entered Balliol College in 1830 and obtained a first in Classics in 1833. Before his Headmastership at Rugby School he was a Fellow and tutor of Balliol College. Henry won the scholarship and so began an association with Balliol College and Oxford which lasted for the rest of his life (Figure 4). As he was not to go into residence until the following Easter, he returned to his family in Rome.

15 The decision to remove Henry from Rugby was made with the help of his guardian, 'an Uncle', who was most likely his father's only brother, the Reverend Godfred Smith, Vicar of Kinneigh, West Cork, Ireland (*ibid*, p. xv).

1.3 BALLIOL COLLEGE OXFORD FROM 1845

In the summer holidays of 1845, having completed his first term at Oxford, Henry travelled to join his mother and Eleanor at Frascati, south east of Rome. There he contracted malaria and following medical advice, they travelled to the sea at Naples in the hope of helping his recovery. Eleanor, who helped to nurse her brother over the following months, recalled that 'he was too weak even to put up his glass that he might look at an eruption of Mount Vesuvius' (Smith, 1894a, p. xvi). His mother would read aloud to him from English newspapers and Latin and Greek classics. From Naples they moved to Wiesbaden in Germany where the waters restored his health. Rather than returning to England they remained on the Continent spending the winter of 1845–46 in Paris. Henry put his convalescence to good use by adding fluency in French, German and Italian to his other accomplishments, and attending lectures of the French physicist François Arago at the Sorbonne. A second visit to Wiesbaden followed during the summer of 1847 but by then Henry, having recovered his health fully, had resumed his studies at Oxford by Easter 1847. He never afterwards needed to suspend it.

On resuming his studies he managed to compress his delayed undergraduate career into just eighteen months. The University's most prestigious classical scholars prize, the Dean Ireland Scholarships, was founded in 1825 by the Dean of Westminster for the promotion of classical learning. The Oxford *University Calendar* of 1849 describes this scholarship and lists Henry Smith as the successful recipient for 1848. He was awarded first class honours in both classics and mathematics in the Easter Term of 1849. Faced with a choice of disciplines Smith's close friend Charles Henry Pearson (1830–1894) had the following recollection.

Figure 4: Balliol College Oxford (pre–1865), by Joseph Murray Ince (1806-1859).

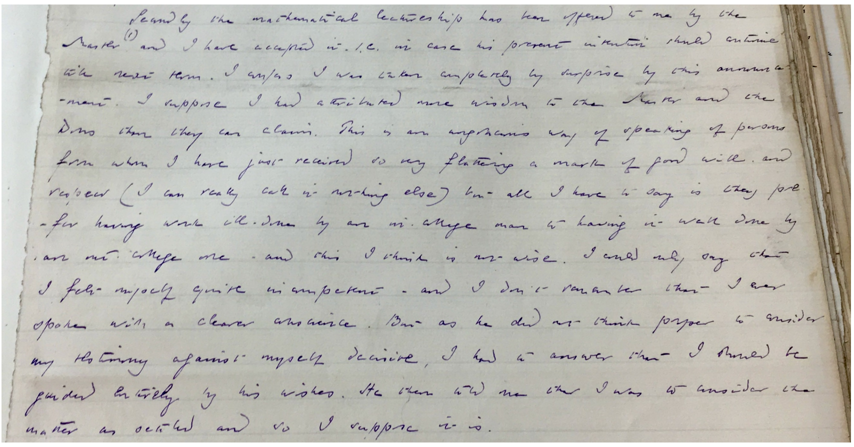
> It was a common story in Oxford at that time that Henry Smith, being uncertain after he had taken his degree whether he should devote himself to classics or mathematics, had solved the doubt by tossing up a halfpenny. His sister remembers how he actually expressed a wish that some one would do this for him. He was, in fact, the last man on earth to have committed any important decisions to chance, and he has himself told me that his choice was partly determined by the fact that having at that time weak sight he found he could do more work in thinking out problems than in any other way without using his eyes (Smith, 1894a, p. xviii).

Smith was elected a Fellow of Balliol College in November 1849 and later gained a Senior Mathematical Scholarship in 1851. Any doubt that remained as to his choice of discipline was soon settled when his old mathematics tutor, Fredrick Temple (1821–1902), resigned his fellowship in 1849.[16] The Master of Balliol College invited Smith to take over as Mathematical Lecturer. With modest pride, and with his self-deprecating sense of humour in full evidence, he wrote to his sister Eleanor (Figure 5):

> I confess I was taken completely by surprise by this appointment. I suppose I had attributed more wisdom to the Master and the Dons than they can claim. This is an ungracious way of speaking of persons from whom I have just received so very flattering a mark of good will and respect (I can really call it nothing else), but all I have to say is they prefer having work ill done by an in-college man to having it well done by an out-college one, and this I think is not wise (Pearson, Glaisher, and Smith, 1894, p. 39).

Smith's early teaching duties were not confined to mathematics. To understand the reasons for this it should be noted that from the early 19th century Oxford University had Honours Schools and Pass Schools where the former had a more rigorous examinations than the latter. Most colleges had both Honours and Pass students, but Balliol, Christchurch, and Oriel Colleges insisted that students study for Honours. Until around 1860 the number of student matriculating at Oxford University each year was approximately 400 with the numbers steadily increasing. Undergraduates could study for a Honours BA in Classics (*in Literis Humanioribus*) or Mathematics (*in Disciplinis Mathematicis et Physicis*) with some students choosing both subjects. However the vast majority of students took Honours in Classics with mathematics attracting only a minority of students. Oxford did not rank its students individually by

16 Fredrick Temple (1821–1902) became Chaplain to Queen Victoria and later Archbishop of Canterbury. He retained his interest in mathematics throughout his life. During an uninspiring episcopal meeting in the 1800's he attempted a proof of the Four Colour Problem (Biggs, Lloyd, and Wilson, 1986, p. 105).



Secondly the mathematical lectureship has been offered to me by the Master(1) and I have accepted it. i.e. in case his present intention should continue till next term. I confess I was taken completely by surprise by this announcement. I suppose I had attributed more wisdom to the Master and the Dons than they can claim. This is an ungracious way of speaking of persons from whom I have just received so very flattering a mark of good will and respect (I can really call it nothing else) but all I have to say is they prefer having work ill-done by an in-college man to having it well done by an out-college one – and this I think is not wise. I could only say that I felt myself quite incompetent – and I don't remember that I ever spoke with a clearer conscience. But as he did not think proper to consider my testimony against myself decisive, I had to answer that I should be guided entirely by his wishes. He then told me that I was to consider the matter as settled and so I suppose it is.

Figure 5: Letter (Extract) from Henry Smith to his sister Eleanor informing her of his appointment as Mathematics Lecturer at Balliol College, 1850. [Courtesy of the Bodleian Libraries, The University of Oxford, Charles Pearson Collection, MSS. Eng. misc. b. 74, fols. 58–59.]

order of merit in examinations, as was done at Cambridge University, so the competition was much less intense. Potential students of mathematics would naturally gravitate to Cambridge and consequently, Oxford had comparatively fewer mathematics students.[17]

In 1849 the University decided to develop an Honour School of *Natural Science* leading to the degree of BA and, as there was 'no initial practical work to be undertaken, Balliol College decided to build and equip a laboratory for this' (Fauvel, Flood, and Wilson, 2013, p. 240). Having attended chemistry lectures in Paris some years earlier Smith was requested to give lectures in chemistry from 1853 to 1855 and to run the first college science laboratory. To prepare for this he was sent to study with the German chemist August Hofmann (1818–1892) at the Royal College of Chemistry in London and with Nevil Story Maskelyne (1823–1911) in the basement of the Old Ashmolean, Oxford.[18] According to author Vanda Morton the two characters complemented each other for humour and vitality. Nevil Maskelyne, despite a brusque temperament, was a 'modest and humorous personality, well able to laugh at himself and at life in general'. Smith was 'full of Irish wit, plays on words and a readiness to appreciate the humour in any situation' (Morton, 1987, p. 95).

17 For an account of the role played by Henry Smith, and other Oxford mathematicians, in the university reform process and their objective to revitalise English mathematics after a century of stagnation, see (Hannabuss, 2000, pp. 443–455).

18 Charles Henry Pearson also joined Henry Smith as students of Nevil Maskelyne during the winter of 1852. All three would become lifelong friends. (Section 2.7)

> He and Nevil Maskelyne would sit up studying together at night, drinking more and more tea as they worked, and filling up the teapot with hot water until no tea was left in it. In the laboratory, Maskelyne trained Smith to approach mineral analysis in a reasoned and meticulous way, but they would break off for long and interesting talks over the bench, for Henry with his Irish blood loved nothing better than a good chat taken through to its conclusion, and would never stop a conversation for a clock. He had no problem in setting others at their ease, and would often break into fits of laughter as he talked, so the Old Ashmolean Museum cellar saw some lively fun as well as earnest discussion in those years (*ibid*, p. 95).

From 1853 until 1860 Smith managed the Balliol College laboratory except for a two-year interval from 1856 when the University Chair of Chemistry, Benjamin Collins Brodie FRS (1817–1880), conducted his research there. To further develop sciences at Oxford, the University Museum project began in 1853 with a committee appointed to work out the the precise buildings and the rooms required. Nevil Story Maskelyne was secretary of this first committee and he presented a comprehensive report on the kind of museum display areas, lecture rooms, and equipment that were necessary. The design included a new science laboratory and with the completion of University Museum in 1860, the Balliol College laboratory was either closed or put to very little use for the following twenty years (Bowen, 1970, p. 228). From 1904 until the late 1930's the laboratory, in a co-operative scheme with Trinity College, became the centre for university instruction in physical chemistry (Figure 6).

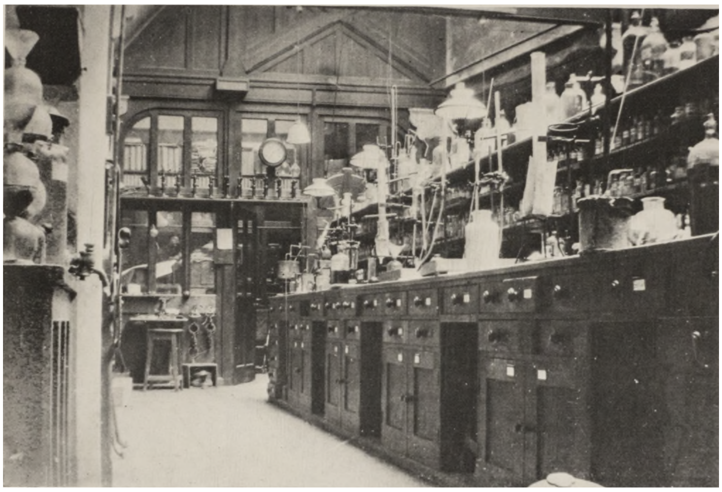


Figure 6: The Balliol-Trinity Laboratory about 1925 (Bowen, 1970, p. 234). In the background is the 'box-office' for conducting tutorials.

For Henry Smith, despite this digression into another discipline, it was mathematics that was never too far from his mind. In 1855 he published, in Latin, his first paper on the theory of numbers; *'De compositione numerorum primorum formae $4n + 1$ ex duobus quadratis'* (Smith, 1855, Smith, 1894a, pp. 33–34). It was an elegant new proof of an old result by Fermat, namely that every prime number of the form $4n + 1$ is the sum of two squares.[19] His interest in the theory of numbers was soon encouraged further by a commission, from the British Association for the Advancement of Science (BAAS), to write a series of *Reports on the Theory of Numbers*, which he prepared from 1859 to 1865 (Section 2.2). During this period, and up until 1868, his published papers related almost exclusively to the theory of numbers.

When the Savilian Professor of Geometry Baden Powell (1796–1860) died, the Oxford mathematicians petitioned the electors in support of Henry Smith. He was duly elected the Savilian Professor of Geometry in 1861 at just 34 years of age.[20] This appointment marked the beginning of a most productive decade for him, not just his mathematical output, but in his involvement with scientific societies and serving on University Boards and Committees. That same year he was elected a Fellow of the Royal Society and a Fellow of the Royal Astronomical Society. The stipend attached to the Savilian Professorship, at that time, was not sufficient to allow him relinquish his Balliol Fellowship and, as a consequence, he continued his Lectureship on topics such as Modern Geometry, Analytical Geometry, Theory of Numbers, Calculus of Variations, and Differential Equations.[21] Some of Smith's most distinguished students became notable scientists rather than mathematicians, a reflection perhaps of the standing of mathematics at Oxford at that time (Table 2).

Table 2: Distinguished Students of Henry Smith (Smith, 1894a, p. lxxvi)

	Profession
Augustus Vernon Harcourt FRS (1834–1919)	Chemist
Sir Lazarus Fletcher FRS (1854–1921)	Geologist
Sir William Thiselton-Dyer FRS (1843–1928)	Botanist
Arthur Buchheim (1859–1888)	Mathematician
Henry T. Gerrans (1858–1921)	Mathematician
John Wellesley Russell (1851–1922)	Mathematician

19 For a paper on Henry Smith's 1855 proof of Fermat's Two Square Theorem see (Clarke et al., 1999).

20 George Boole (1815–1864) of Queen's College Cork, Ireland, also entered his name as a candidate for the Savilian Professorship of Geometry in 1861. He did not submit any testimonials due to the religious controversies that existed in Oxford at that time. See (MacHale, 2014, p. 192) and (Fauvel, Flood, and Wilson, 2013, p. 242).

21 For further details on Henry Smith's University teaching see (*ibid*, pp. 250–252).

Smith had an immense personal charm, warmth and good humour which won him many friends and admirers. The Master of Balliol College, the Reverend Benjamin Jowett (1817–1893), had daily association with Henry Smith as a colleague and friend for over 35 years. In all matters regarding the University he looked to Smith more than anyone else for advice and help. Benjamin Jowett describes his jovial nature and Irish humour.

> But [Henry] Smith was not only the wise counsellor, he was the familiar friend who had been associated with Jowett in many common acts of hospitality. There was hardly any large gathering at the Master's Lodge at which he was not present, and whenever he was present he was the life of the party, charming in his conversation and possessing an inexhaustible fund of amusing stories, which he told with admirable grace and effect (Abbott and Campbell, 1897, p. 238).

Smith's endearing character, along with a strong sense of public duty, meant he was a natural choice for committees both in Oxford and more widely. 'He possessed in an extraordinary degree the gift of conciliation, and could say the happy word which quells the rising storm' (*ibid*, p. 238). In 1870 he was appointed a member of the Royal Commission (Devonshire Commission) on *Scientific Instruction and the Advancement of Science*, a role which he held for four years, and he drafted a large portion of its report. Also in 1870 he was appointed Mathematical Examiner at the University of London. In 1873 he was elected President of the BAAS (Section A) Mathematics and Physics, the week-long annual meeting of the BAAS being held that year in Bradford (Table 3). Despite the heavy burden of these extra duties his association with Balliol College continued until late 1873. It was then he accepted an offer of a Fellowship at Corpus Christi College, Oxford, one which did not require any teaching duties. Benjamin Jowett was not pleased with this news. On 20th October 1873 he wrote to Florence Nightingale:

> I have had great trouble the last few days – H. Smith, the most distinguished of our fellows has suddenly announced his intention to become a Fellow of Corpus, the wily President, without speaking to me, having offered him a fellowship. We have offered to do all that they would do, but he persists that he cannot, chiefly for conscientious! reasons, accept a sinecure Fellowship at a small College like ours. I cannot tell what his real motives are, but I suspect that he feels some kind of constraint in being here with me (Jowett, 1987, pp. 147–148).

Table 3: Distinctions and Offices – Henry Smith FRS

Year	
1849	BA, Oxford. (First class Mathematics and Classics)
1849	Lectureship in Mathematics, Balliol College, Oxford.
1849 – 74	Fellow of Balliol College, Oxford.
1851	Senior Mathematical Scholarship, Oxford.
1855	MA, Oxford.
1855	Membership of BAAS.
1858 – 62	Secretary of the BAAS (Section A) Mathematics and Physics.
1859	Junior Bursar and Senior Dean, Balliol College, Oxford.
1861	Savilian Professor of Geometry, Oxford.
1861	Fellow of the Royal Astronomical Society.
1861	Fellow of the Royal Society.
1865	Membership of the LMS.
1868	Steiner Prize of the Royal Prussian Academy of Sciences, Berlin.
1870	Mathematical Examiner at the University of London.
1870 – 74	Member of Royal Commission (Devonshire Commission).
1873	President of the BAAS (Section A) Mathematics and Physics.
1873 – 83	Fellow of Corpus Christi College, Oxford.
1874 – 82	Honorary Fellow of Balliol College, Oxford.
1874 – 83	Keeper of the University Museum, Oxford.
1874 – 76	President of the LMS
1876	Honorary Fellow of the Royal Society of Edinburgh.
1877	Member of the Oxford University Commission (Lord Salisbury).
1877	Chairman of the Meteorological Council.
1877 – 78	Vice–President of the Royal Society.
1878	Honorary Degree LLD, Trinity College Dublin.
1879	Honorary Degree LLD, University of Cambridge.
1882	Grand Prix des Sciences Mathématiques of the French Académie des Sciences.

It is reasonable to assume that Benjamin Jowett was disappointed with Smith's decision to leave Balliol College but, despite his decision, their mutual friendship continued (Section 2.6). On accepting the Fellowship at Corpus Christi College the number of offices and distinctions Smith held continued to increase. In 1874 alone he was appointed Keeper of the University Museum and elected President of the London Mathematical Society (LMS). Also that year Balliol College, who wished to keep its association with Smith, made him an Honorary Fellow. Henry Smith is commemorated on a memorial tablet within Balliol College Chapel (Figure 7).

Figure 7: Balliol College Chapel and Memorial Tablet (North Wall) commemorating late 19th Century Scholars of Balliol College. Among the names include Henry Smith, Savilian Professor of Geometry, William Spottiswoode, President of the Royal Society and Benjamin Collins Brodie, University Chair of Chemistry.

[Photograph the author's own, taken February, 2018].

1.4 CONCLUSION

Mary Smith, with her four children, emigrated from Ireland in 1829 following the death of her husband. Despite a turbulent start to his life Henry Smith's early years were happy ones where each day was filled with a mixture of lessons and leisure time. He was an exceptionally gifted child who was educated at home, firstly by his mother and then, by a succession of tutors. Before entering Oxford University in 1845 he had only spent two years at school. His early achievements as a student at Oxford may be attributed, not just to his academic talent, but to the dedication of his mother who sought, through education, to make a better life for her children. Their close family upbringing ensured Henry would have the companionship of his sister Eleanor throughout his life. They maintained contacts with their Irish relatives and Eleanor visited Ireland on a number of occasions. In 1878 Henry Smith visited Dublin to attend the annual meeting of the BAAS.[22] While there he received an Honorary Degree (LLD) from Trinity College Dublin, the *alma mater* of his father John Smith. Their Irish uncle, Reverend Godfred Smith, also visited them in Oxford.[23]

Smith's early studies were interrupted by poor health but his extended convalescence in Europe was crucial to his mathematical development. In the same year as his graduation he was elected to a Lectureship and to a Fellowship at Balliol College and quickly established himself as part of the Oxford University community. These appointments, at just 23 years of age, were important as they secured a path for him through his academic life. This was in contrast to other British mathematicians at that time, such as Arthur Cayley FRS (1821–1895) and James Joseph Sylvester FRS (1814–1897), who had somewhat fragmented careers in mathematics.[24] Smith's Irish personality, warmth and good humour

22 This is the only evidence uncovered of Henry Smith returning to the land of his birth.

23 In a letter to Charles Pearson, dated 1st September 1876, Eleanor Smith (Oxford) wrote that she had just been informed by telegraph of the death of her remaining uncle Godfred Smith, aged 73, in New Zealand. She confirmed that she had visited him at Kinneigh in West Cork and that he had only recently visited her, and her brother, in Oxford. Godfred Smith sailed from London to New Zealand in late June 1876. Eleanor travelled to London to bid him farewell. The letter also outlined the sad circumstances of Godfred Smith's death. Bodleian Library, Charles Pearson Collection, MSS. Eng. lett. d. 191, fols. 5–7.

24 Arthur Cayley FRS (1821–1895) was Senior Wrangler and Smith Prizeman at Cambridge. Early in his career he practiced as a barrister during which time he still continued his mathematical studies. He became the leading pure mathematician in Victorian Britain publishing almost 1,000 papers. He initiated the study of matrices and of n-dimensional geometry, gave the first definition of an abstract group, and made important contributions to the theory of invariants. He eventually became the first Sadlerian Professor at Cambridge. Many academic honours came his way. In addition to the Copley medal (1882) he was awarded the honorary degree of ScD at Cambridge (1887) and made a member of the French Légion d'honneur. For a complete account of Arthur Cayley's life in mathematics see (Crilly, 2006a). Also the entry for Arthur Cayley in the *Oxford Dictionary of National Biography* (Crilly, 2004).

James Joseph Sylvester FRS (1814–1897) was second wrangler at Cambridge, but being Jewish was unable to take his degree (he was later awarded a Bachelor's degree, on the basis of meant that he formed new friendships easily. He became actively involved in scientific societies and, through his sense of public duty and conciliatory nature, to various committees both in Oxford and more widely.

Questions that arise are, for example, as holder of Oxford's most prestigious mathematical chair, what were the significant contributions Smith made to Oxford mathematics? Did he sufferer from a lack of recognition in his day? Why did he remain at Oxford University for his entire career? What impact, for better or worse, was there on his mathematical career by doing so? Before answering these questions it will be helpful to look more closely at Henry Smith's life in mathematics.

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his work at Cambridge, as well as a Master's degree from Trinity College, Dublin). For part of his career he was a well-regarded actuary at the Institute of Actuaries (London) which he helped to establish in 1848. He was later appointed Professor of Mathematics at the Royal Military Academy in Woolwich, London. In his early sixties he became head of the mathematics department at the newly opened John Hopkins University in Baltimore, Maryland, USA. At the age of seventy he returned to England to succeed Henry Smith as Savilian Professor of Geometry at Oxford. His main mathematical work was in algebra and the theory of invariants, but he also had wide outside interests. For a complete account of James Joseph Sylvester's life in mathematics see (Parshall, 2006). Also see *James Joseph Sylvester* in (Wilson, 2021, pp. 120–143) and the entry for James Joseph Sylvester in the *Oxford Dictionary of National Biography* (Parshall, 2004).

From about 1848 Arthur Cayley and James Joseph Sylvester began a friendship that would characterise the rest of their lives. By the early 1850's almost daily mathematical exchanges between them would result in the formation of a new area of mathematics. 'The two men had very different personalities. Cayley was patient and equable, Sylvester fiery and passionate' (James, 2002, p. 168). Also see *Invariant Twins* in (Bell, 1986, pp. 378–405).

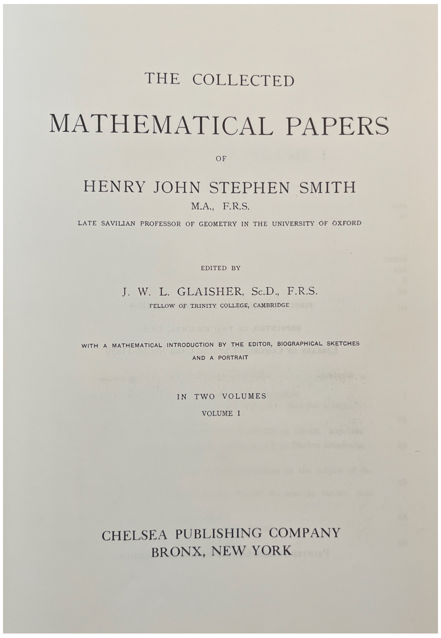
2

HENRY JOHN STEPHEN SMITH – A LIFE IN MATHEMATICS

Henry Smith's life in mathematics is best portrayed from the recollections of his mathematical friends, of which he had many, but particularly from those who knew him best – James W.L. Glaisher and William H. Spottiswoode. The mutual respect and admiration that developed between these three mathematicians was mainly the result of their active involvement in scientific societies. These recollections, and those of others, give interesting insights into Smith's life at Oxford University, his life outside of mathematics, and his affinity with Continental Europe. In this chapter I describe how his life in mathematics was enriched by his many friends and the important role played by his sister Eleanor. I begin the chapter with an overview of his contribution to the theory of numbers, during the 1860's, and later consider the factors that suggest he suffered from a relative lack of recognition in his day.

2.1 INTRODUCTION

James W.L. Glaisher FRS (1848–1928) wrote a number of articles of appreciation for his close friend Henry Smith, who died on February 9th, 1883. Glaisher edited the collected mathematical papers of Smith, first published in two volumes in 1894, eleven years after Smith's death (Smith, 1894a, vol. 1), (Smith, 1894b, vol. 2). He also wrote the introduction to this collection in which he details his early recollections of Smith and the mathematical friendship they shared for over ten years. It is a very personal account including extracts from the letters exchanged, between the Oxford and Cambridge mathematicians, from 1880 to 1883. Other contributions to this collection include an account of Smith's life at Oxford University, and in general society, by Lord Bowen of Colwood (1835–1894).[1] The other contributors to the collection are listed in its table of contents (Figure 8).


THE COLLECTED
MATHEMATICAL PAPERS
OF
HENRY JOHN STEPHEN SMITH
M.A., F.R.S.
LATE SAVILIAN PROFESSOR OF GEOMETRY IN THE UNIVERSITY OF OXFORD

EDITED BY
J. W. L. GLAISHER, Sc.D., F.R.S.
FELLOW OF TRINITY COLLEGE, CAMBRIDGE

WITH A MATHEMATICAL INTRODUCTION BY THE EDITOR, BIOGRAPHICAL SKETCHES AND A PORTRAIT

IN TWO VOLUMES
VOLUME I

CHELSEA PUBLISHING COMPANY
BRONX, NEW YORK

Figure 8: The Collected Mathematical Papers of Henry John Stephen Smith MA FRS – Volume I (Smith, 1894a).

In his introduction Glaisher describes Smith's personality and the characteristic features of his mathematical writing. He believed that there was a disconnect between his 'bright and winning gaiety of manner', when he introduced his research to a mathematical audience, with 'the severe style of his mathematical papers' (Smith, 1894a, p. lxxxi). He wrote two further articles on Smith's life,

1 Lord Bowen PC FRS was an English judge. He was educated at Rugby School and won a scholarship to Balliol College, Oxford in 1853. He was elected a fellow of Balliol in 1857. Lord Bowen's recollection of Henry Smith was first published in the *Spectator* on February 17th, 1883 and reprinted in Smith, 1894a, p. xlvi–li.
https://archive.spectator.co.uk

one of which was published in the *Fortnightly Review*, edited by Thomas H. Escott, in 1883 (Glaisher, 1883a, pp. 653–666). A similar article was published in the *Monthly Notices of the Royal Astronomical Society* in February, 1884 (Glaisher, 1884a, pp. 138–149). Both articles include a full biographical sketch of Smith, his contributions to mathematics and an account of his life at Oxford University. Furthermore, as president of the Cambridge Philosophical Society, James Glaisher's presidential address to the Society on 12th February 1883 was a tribute to his late friend. This address was published in the *Proceedings of Cambridge Philosophical Society* (Glaisher, 1883b, pp. 319–321).[2]

William H. Spottiswoode PRS (1825–1883) also wrote an appreciation of Henry Smith which was published in *Nature* on 22nd February 1883 (Spottiswoode, 1883, pp. 381–384). He wrote of Smith's contribution to mathematics, placing emphasis on his contribution to various societies, associations, and commissions, suggesting it was in this sphere that their friendship developed. However, it is possible that their friendship began during the late 1840's when they were both at Balliol College, Oxford. Reading Spottiswoode's article on Smith, like those of Glaisher, we encounter a personal recollection of the person he knew. Both Glaisher and Spottiswoode clearly held Smith in the highest regard. The deep sense of loss and appreciation that these men felt was undoubtedly as a result of their close working relationship. Scientific societies played an important role during this period in shaping both mathematical research and mathematical reputations. All three mathematicians were prominent members of these societies and, at various times, held the highest elected positions. All were elected Fellows of the Royal Society of London, Fellows of the Royal Astronomical Society and Honorary Fellows of the Royal Society of Edinburgh.[3] At different times, all three held the Presidency of the LMS and Presidency of the BAAS (Section A) Mathematics and Physics. Smith's involvement with these societies meant he could meet mathematicians who appreciated his unique knowledge and who would encouraged him to bring papers before their members. For these men, despite being based at different centres of England – at Oxford, Cambridge and London; their friendships, mutual respect and admiration developed as a result of their active involvement with these societies.

2 Arthur Cayley was in attendance at the *Cambridge Philosophical Society* meeting. He paid his own tribute to Henry Smith: 'It was impossible to speak in too high terms of the value of Professor Smith's work. His wonderful knowledge of the processes of the higher parts of the theory of numbers showed itself in everything he did. His work was of the very highest quality and excellence, and he could not too strongly express his sense of the great loss caused by his death' (Glaisher, 1883b, p. 321).

3 The certificates of election to the Royal Society for Henry Smith (1861), James W.L. Glaisher (1875) and William H. Spottiswoode (1853) may be viewed at https://catalogues.royalsociety.org/CalmView/PersonSearch.aspx?src=CalmView.Persons Henry Smith was one of the signatories to James Glaisher's election to the Royal Society.

I will begin by outlining how Smith's membership of the BAAS in 1855, and his subsequent *Reports on the Theory of Numbers*, marked the beginning of his contribution to the theory of numbers.

2.2 REPORTS ON THE THEORY OF NUMBERS – 1859 TO 1865

Henry Smith's memoirs on the arithmetical theory of integral quadratic forms were preceded by a series of *Reports on the Theory of Numbers*, commissioned by the BAAS, which he prepared from 1859 to 1865 (Table 4). These reports, presented in a condensed format, were the result of an immense amount of research. Each of the six reports were presented with a systematic arrangement, meticulous and were models of clear exposition. Smith's familiarity with the ideas and methods meant that the subject 'engaged his almost undivided attention for many years, and which was never afterwards quite absent from his thoughts' (Smith, 1894a, p. lxii).

Table 4: Reports on the Theory of Numbers 1859–1865

Year	Publication
1859	Report on the Theory of Numbers Part I, *Report of the British Association for 1859*, (Science, 1860, pp. 228–267).
1860	Report on the Theory of Numbers Part II, *Report of the British Association for 1860*, (Science, 1861, pp. 120–169).
1861	Report on the Theory of Numbers Part III, *Report of the British Association for 1861*, (Science, 1862 pp. 292–340).
1862	Report on the Theory of Numbers Part IV, *Report of the British Association for 1862*, (Science, 1863, pp. 503–526).
1863	Report on the Theory of Numbers Part V, *Report of the British Association for 1863*, (Science, 1864, pp. 768–786).
1865	Report on the Theory of Numbers Part VI, *Report of the British Association for 1865*, (Science, 1866, pp. 322–375).

Smith, like many Oxford scientists at the time, was an enthusiastic participant at meetings of the BAAS, and was a popular speaker, having first been elected a member of the association in 1855. For four consecutive years, starting in 1858, he was secretary of the sectional committee, Section A (Mathematics and Physics). The BAAS would regularly commission reports on various branches of science. In 1858 they selected Smith to report to them on the state of the theory of numbers. The annual report of the twenty-eighth meeting of the BAAS, held at Leeds in September 1858, records the following recommendation of the general committee: 'Mr HJS Smith MA of Balliol College (Oxon), be requested to draw up a Report on the Theory of Numbers' (Science, 1859, p. xlii).

Spottiswoode wrote that, prior to Smith's reports, research papers on the theory of numbers were 'scattered throughout the pages of various periodicals' and how there was a custom among mathematicians 'to publish results alone, without proof of their statements, and even without indication of the train of argument which led them to their conclusions' (Spottiswoode, 1883, p. 382). It seems that the decision of the BAAS to commission a report on the theory of numbers was a timely one. The selection of Smith for this task, as it would transpire, was an inspired choice. The reports would nurture Smith's own interest in the theory of numbers and in Continental mathematics. His published memoirs for the remainder of the decade being almost exclusively in this subject area (Section 2.3).

On the Continent, research on the theory of numbers was going through a transition during the 1860's with the death of many leading mathematicians during the 1850's. Carl Gustav Jacobi died in 1851, Gotthold Eisenstein in 1852 and Carl Friedrich Gauss in 1855. Peter Gustav Lejeune Dirichlet, who had become Gauss's successor in Göttingen, died in 1859. 'Thus the erstwhile proud and active leading group of European researchers in the domain opened up by the *Disquisitiones Arithmeticae* was decimated dramatically by the 1860's. Only Leopold Kronecker in Berlin represented a strong element of continuity across the years' (Goldstein, Schappacher, and Schwermer, 2007, p.67). Although the membership of the BAAS in 1858 included 60 corresponding members, mostly from Europe, the decision of the BAAS to commission these reports was more likely influenced by concerns about advancement in the theory of numbers at home rather than developments on the Continent.[4]

Smith's first *Report on the Theory of Numbers* was published in the annual report of the BAAS for the year 1859 (Science, 1860, pp. 228–267). The printed minutes of the Sectional Committee Meeting of Section A (Mathematics and Physics) of the BAAS of 1859 recorded that:

> The first paper in this section today was the report on the theory of numbers. It was purely abstract nature and Mr HJS Smith, who gave his report, contented himself by giving a mere outline of it. The president [William Parsons, 3rd Earl of Rosse], Sir WR Hamilton and Professor Stevelly, characterised the report as a very able one. It will be printed in the association transactions in full.[5]

In this first report Smith outlines from the outset that the classic works on the theory of numbers were still, at that time, the *Disquisitiones Arithmeticae* of Carl Friedrich Gauss and *Théorie des Nombres* of Adrien Marie Legendre (Legendre, 2011). He clarifies that the subsequent contributions of mathematicians such as Cauchy, Dirichlet, Eisenstein, Kronecker and Hermite had served

4 List of corresponding members of the BAAS for the year 1858 are located in the annual report of the association for that year (Science, 1859, p. xxviii).
5 Oxford, Bodleian Libraries, Dep. BAAS 20.

to simplify as well as extend the subject. 'From the labours of these and other eminent writers, the theory of numbers has acquired a great and increasing claim to the attention of mathematicians' (Science, 1860, p. 228). According to James Glaisher, Smith intended each report to exhibit an outline of the results of the most recent investigations and to trace their connections, as far as possible, with one another and to earlier research. His objective was to ensure that his reports be intelligible to a reader who had not occupied themselves especially. To achieve this he occasionally introduced a brief summary of principles and results from the earlier works of Gauss and Legendre.

> It is hardly necessary to add that we must confine ourselves to what we may term the great highways of the science; and that we must wholly pass by many outlying researches of great interest and importance, as we propose rather to exhibit in a clear light the most fundamental and indispensable theories, than to embarrass the treatment of a subject, already sufficiently complex, with a multitude of details, which, however important in themselves, are not essential to the comprehension of the whole (Science, 1860, p. 229).

The 'great highways of the science' he identifies to be the theory of congruences and the theory of homogeneous forms, hence the contents of his combined reports on the theory of numbers were arranged under these headings.

The second of his reports was published in 1860 for which he again received the following recommendation of the general committee: 'Mr. H.J.S. Smith, be requested to continue his Report on the Theory of Numbers' (Science, 1860, p. 47). The four subsequent reports were published in the annual report of the BAAS for the years 1861, 1862, 1863 and 1865. The combined reports occupied 250 pages and contain an almost complete account of the series of discoveries of Gauss and his successors.[6] James Glaisher described them as neither a history nor a full treatment of the ideas, but something intermediate. They presented, in condensed form, the result of Smith's extensive research of the subject and were described as 'models of clear exposition and systematic arrangement' (Smith, 1894a, p. lxii). We will see later in this thesis that this style of presentation was characteristic of Smith's writings. The reports were well received by Continental mathematicians, and Leopold Kronecker (1823–1891) praised it highly for its mastery and insight.[7] The young German mathematician Felix Klein (1849–1925) began a study of Smith's reports in 1871 which would form an important basis for his later work on the theory of numbers (Tobies, 2019, p. 149). The annual report of 1865 records the following recommendation of the general committee: 'That Professor Smith be requested to

6 Henry Smith did not appear to have access to all of Jacobi's material (Goldstein, Schappacher, and Schwermer, 2007, p. 28, n. 93).

7 Sitzungsberichte der Preussischen Akademie der Wissenschaften, Berlin (1875), p. 234

continue and conclude his *Report on the Theory of Numbers*' (Science, 1866, p. 43). Despite this recommendation, Smith produced no further reports. James Glaisher was quite certain that, in 1866, Smith had intended to write a seventh part of the report, which would relate to the frequency of primes and other asymptotic formulae in the theory of numbers (Smith, 1894a, p. lxxviii). Many years later James Glaisher would come into possession of Smith's mathematical notebooks and he confirmed from them that a seventh report was indeed in preparation. However, by 1864, Smith had already published his first of a series of memoirs on the arithmetical theory of quadratic forms.

2.3 MEMOIRS ON THE THEORY OF NUMBERS – 1864 TO 1868

The collected mathematical papers of Henry Smith were published in two volumes in 1894 containing all the memoirs he published in his lifetime, as well as those which were in press, or written out for printing at the time of his death. His first two papers, written in 1851 and 1852 were on geometry. From 1855 until 1868 the printed memoirs relate almost exclusively to the theory of numbers, followed by a number of further memoirs on geometry. In the last years of his life he was occupied principally with the subject of theta functions and elliptic transformations.[8]

This thesis will endeavour to illustrate that Smith's memoirs on the theory of numbers show an arithmetical style and presentation that was reminiscent of Gauss's *Disquisitiones Arithmeticae* (1801). In order to do this it will be important to initially draw attention to Smith's rare gift as an expositor of mathematics, and to his own declared admiration for the writings of Gauss. James Glaisher recalled that at various scientific society meetings Smith had an ability to engage an audience by presenting mathematics in an intelligible and interesting way (Smith, 1894a, p. lxxxi). In his published memoirs on the theory of numbers this ability shows itself as guiding and encouraging the reader, from fundamental definitions, with simple examples, to new developments while always acknowledging the contributions of earlier mathematicians. It will be important to explore the presentation, exacting style, coherence and beauty of these memoirs. It will show, not only the firm grasp he had on the subject but, that careful presentation of mathematics mattered to him. It appears that he took great care with notation and in examining all special cases of general theorems. According to James Glaisher, 'his natural love of precision in thought and expression was no doubt strengthened by his early study of the writings of Gauss, for whom he always felt the most unbounded admiration' (Smith, 1894a, p. lxxiv).

8 For a complete list of Henry Smith's published memoirs and reports see (Smith, 1894a, Vol 1, List of Papers, pp. v–viii)

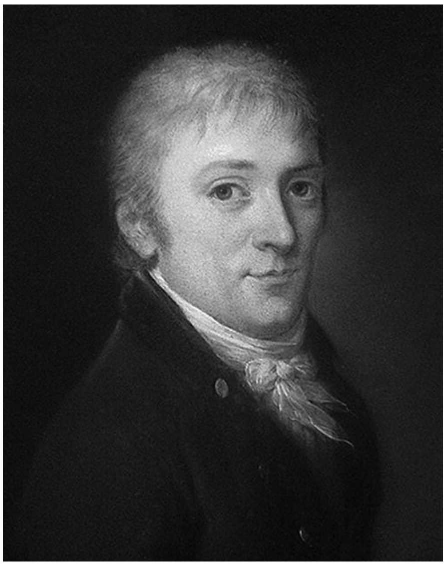


Figure 9: Carl Friedrich Gauss (1777–1855)

In 1877 Robert Tucker published an article titled *Carl Friedrich Gauss* to mark the occasion of the centenary of Gauss's birth (Tucker, 1877, p. 533–537).[9] In this article Robert Tucker acknowledges the assistance he received from Smith in writing an account of Gauss's arithmetical work and quotes directly from the notes he received. In his own words Smith pays the following tribute to Gauss.

> If we except the great name of Newton (and the exception is one which Gauss himself would have been delighted to make), it is probable that no mathematician of any age or country has ever surpassed Gauss in the combination of an abundant fertility of invention with an absolute rigorousness in demonstration, which the ancient Greeks themselves might have envied. It may be admitted, without any disparagement to the eminence of such great mathematicians as Euler and Cauchy, that they were so overwhelmed with the exuberant wealth of their own creations, and so fascinated by the interest attaching to the results at which they arrived, that they did not greatly care to expend their time in arranging their ideas in a strictly logical order, or even in establishing by irrefragable proof propositions which they instinctively felt and could almost see, to be true. With Gauss the case was otherwise (Tucker, 1877, p. 537).

9 Robert Tucker (1832–1905) was an English mathematician who served as Honorary Secretary of the LMS from 1867 until 1902. During this time he served under 19 Presidents of the Society, which included the Presidency of Henry Smith from 1874 until 1876. Robert Tucker edited the Mathematical Papers of William Kingdon Clifford (1845–1879) in 1882. Henry Smith wrote the introduction to the Mathematical Papers of William Kingdon Clifford (Clifford, 1882, pp. xxxii–lxx and Smith, 1894b, pp. 711–719).

Smith wrote of the extreme importance that Gauss attached to perfection of form in the presentation of mathematical results. He stated that the writings of Gauss had an ‘adamantine solidity and clear hard modelling which would keep his writings from being forgotten long after the chief results and methods contained in them have been incorporated in treatises more easily read’ (*ibid*, p. 537).

> It is not the greatest, but it is perhaps not the least, of Gauss’s claim to the admiration of mathematicians, that while fully penetrated with a sense of the vastness of the science, he exacted the utmost rigorousness in every part of it, never passed over a difficulty as if it did not exist, and never accepted a theorem as true beyond the limits within which it could actually be demonstrated (*ibid*, p. 537).

These words of Smith also capture the quality that he insisted upon his own work. As with Gauss, Smith did not publish his mathematical writing unless they were complete in all details and perfect in form and substance. These high standards of presentation, which he maintained throughout his life, would have an unfortunate consequence for him in 1882.

In 1861 Smith published a significant memoir titled, *On Systems of Linear Indeterminate Equations and Congruences* (Smith, 1862). The central theorem of this memoir established the *Smith normal form* for matrices with integer entries, an important concept which has best perpetuated Smith’s name in mathematics, especially in linear algebra.[10] This memoir is interesting in that it emphasises matrix methods, such as the formulation of the coefficient matrix and the augmented matrix of the system, for solving linear equations, which

10 Henry Smith showed, by employing simultaneously a premultiplying and postmultiplying unit matrix, that any matrix with integer entries can be put into a diagonal form in which each diagonal entry divides its successors, and that these diagonal entries are, up to sign, unique. This transformation of a square matrix became known as the *Smith normal form* of a matrix.
THEOREM (Smith, 1862, p. 89)
If $\|A\|$ represents any square matrix in integral numbers, ∇_n its determinant, and

$$\nabla_{n-1}, \nabla_{n-2}, \cdots\cdots\cdots, \nabla_1, \nabla_0$$

the greatest common divisors of its successive orders of minors, it is always possible to assign two unit matrices $\|\alpha\|$ and $\|\beta\|$, of the same dimension as $\|A\|$, and satisfying the equation

$$\|A\| = \|\alpha\| \times \begin{Vmatrix} \frac{\nabla_n}{\nabla_{n-1}} & 0 & 0 & \dots & 0 \\ 0 & \frac{\nabla_{n-1}}{\nabla_{n-2}} & 0 & \dots & 0 \\ 0 & 0 & \frac{\nabla_{n-2}}{\nabla_{n-3}} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{\nabla_1}{\nabla_0} \end{Vmatrix} \times \|\beta\|$$

were quite new at the time. Smith would later employ these methods of linear algebra in his final memoir to the *French Académie des Sciences* in 1882 (Chapter 6). His work on the order and genera of quadratic forms appeared in 1864, 1867 and 1868 (Table 5). These important contributions to the theory of numbers continued and completed the earlier work of Gotthold Eisenstein (1823–1852) whose treatment of ternary quadratic forms remained unfinished at the time of his death. These memoirs, and his final memoir of 1882, I collectively refer to as Smith's memoirs on the arithmetical theory of quadratic forms. With a presentation style reminiscent of Gauss's *Disquisitiones Arithmeticae* (1801), the most noticeable feature of Smith's writings being the strong arithmetical spirit that runs throughout them.

Table 5: Memoirs on the Theory of Numbers 1861–1868 and 1882

Year	Publication
1861	On Systems of Linear Indeterminate Equations and Congruences, *Philosophical Transactions of the Royal Society*, Vol. cli. pp. 293–326.
1864	On the Criterion of Resolubility in Integral Numbers of the Indeterminate Equation $f = ax^2 + a'x'^2 + a''x''^2 + 2bx'x'' + 2b'xx'' + 2b''x'x = 0$, *Proceedings of the Royal Society*, Vol. xiii. pp. 110–111.
1864	On the Orders and Genera of Quadratic Forms containing more than Three Indeterminates, *Proceedings of the Royal Society*, Vol. xiii. pp. 199–203.
1864	On Complex Binary Quadratic Forms, *Proceedings of the Royal Society*, Vol. xiii. pp. 278–298.
1867	On the Orders and Genera of Ternary Quadratic Forms, *Philosophical Transactions of the Royal Society*, Vol. clvii. pp. 255–298.
1868	On the Orders and Genera of Quadratic Forms containing more than Three Indeterminates, *Proceedings of the Royal Society*, Vol. xli. pp. 197–208.
1882	Mémoire sur la Représentation des Nombres par des Sommes de Cinq Carrés. *Mémoires présentés par divers savants à l'Académie des Sciences de l'Institut National de France*, Vol. xxix, pp. 1–72 (1887).

These memoirs received praise from notable 20th century mathematicians. The Cambridge mathematician Louis J. Mordell (1888–1972) described Smith's prize memoir of 1882 as 'an example of the most delicate and intricate demonstrations to be found in the whole range of analysis' (Dickson, 1930, p. 198).[11]

11 Louis J. Mordell (1888–1972) was an American-born British mathematician, known for pioneering research in number theory. He was the Sadleirian Professor of Pure Mathematics at the University of Cambridge from 1945 to 1953.

Godfrey Harold Hardy FRS (1847–1947) had particular praise for these memoirs, describing Smith as a 'most brilliant arithmetician'.[12] In his inaugural lecture as Savilian Professor of Geometry at Oxford University in 1920, G.H. Hardy acknowledged the significant contribution made by Smith in his memoirs on the arithmetical theory of quadratic forms:

> Henry Smith was very many things, but above all things a most brilliant arithmetician. Three-quarters of the first volume of his memoirs is occupied with the theory of numbers, and Dr. Glaisher, his mathematical biographer, has observed very justly that, even when he is primarily concerned with other matters, the most striking feature of his work is the strongly arithmetical spirit which pervades the whole. His most remarkable contributions to the theory are contained in his memoirs on the arithmetical theory of forms, and in particular in the famous memoir on the representation of numbers by sums of five squares, crowned by the Paris Academy and published only after his death. This memoir is peculiarly interesting to me, for the problem is precisely one of those of which I propose to speak to-day.[13]

Before considering aspects of Smith's memoirs on the arithmetical theory of quadratic forms (Chapter 5) it will be useful to present details of his life in mathematics as told from the recollections of those who knew him best. These recollections also give interesting insights into his life at Oxford, his many friends outside of mathematics, and his affinity with Continental Europe.

12 Godfrey Harold Hardy FRS (1847–1947) was the most important British pure mathematician of the first half of the 20th century. Although he is usually thought of as a Cambridge mathematician, his years from 1920 to 1931, as Savilian Professor of Geometry at Oxford University, were actually his happiest and most productive. While in Oxford Hardy distinguished himself as both a superb lecturer and an inspiring leader of research. He published over 100 papers there, including many of his most important investigations with his long-term Cambridge collaborator J.E. Littlewood FRS (1885–1977). G.H. Hardy was described 'as the consummate craftsman, a connoisseur of beautiful mathematical patterns and a master of stylish writing' (Harman and Mitton, 2002, p. 202). For a biographical essay on G.H. Hardy see (Wilson, 2021, pp. 146–172) and (James, 2002, pp. 299–306). Also see the entry for G.H. Hardy in the *Oxford Dictionary of National Biography* (Bollobás, 2004) and the posters of *Oxford Mathematicians through History* from the Mathematical Institute, University of Oxford. See https://www.maths.ox.ac.uk/outreach/posters

13 Inaugural Lecture, G.H. Hardy, MA, FRS, Savilian Professor of Geometry University of Oxford, 1920 (Hardy, 1920).

2.4 RECOLLECTIONS BY JAMES W.L. GLAISHER

James Whitbread Lee Glaisher FRS (1848–1928) was a Senior Fellow of Trinity College Cambridge.[14] He was ranked as one of the most recognised English pure mathematicians of his generation, who worked tirelessly to promote pure mathematics at Cambridge University. Throughout all his years he was devoted to astronomy, chiefly in its mathematical developments. He had a particular interest in the theory of elliptic functions. He published widely over many fields of mathematics and astronomy and served with distinction as an editor (eventually the sole editor) of the *Quarterly Journal of Mathematics* and the *Messenger of Mathematics*. At the time of his death he was the most senior member of the LMS, and was almost the most senior in standing among the Fellows of the Royal Society and among the Fellows of the Royal Astronomical Society (Table 6). James Glaisher was not awarded the Sadleirian Chair of Mathematics, following the death of Arthur Cayley, and it is probable that this disappointment was one of the reasons he immersed himself in a new interest of collecting pottery. He became a leading authority on the subject and built up a collection of great value. On his death the Fitzwilliam Museum at Cambridge acquired his extensive collection. James Glaisher resided within rooms at Trinity College Cambridge for more than 60 years. 'He was a keen cyclist, riding a 'penny-farthing', and also a keen walker, striding out at a speed which few could match'.[15]

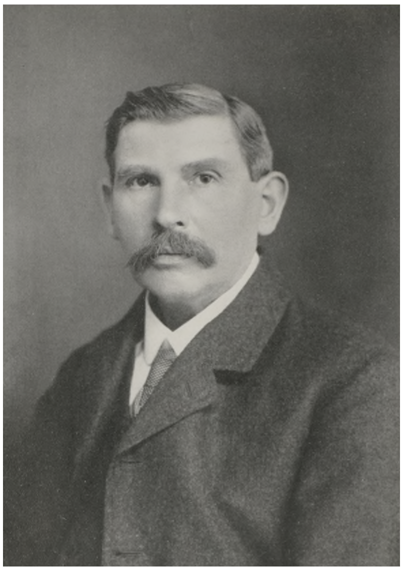


Figure 10: James W.L. Glaisher FRS (1848–1928)

14 For a biographical essay on James W.L. Glaisher see (Turner, 1929), (Hunt, 1996) and (Forsyth, 1929). Also see the entry for James W.L. Glaisher in the *Oxford Dictionary of National Biography* (Forsyth, 2010).

15 A brass memorial plaque to commemorate James Glaisher is located in the Trinity College Chapel, Cambridge. See http://trinitycollegechapel.com/about/memorials/brasses/glaisher/

James Glaisher was seventeen years of age when he was first introduced to Henry Smith. In 1866 Smith was one of the secretaries of Section A (Mathematics and Physics) of the BAAS. The annual meeting of the BAAS was held at Nottingham in August of that year and a young James Glaisher attended the meeting, accompanied by his father.[16] He remembers Smith reading aloud the papers of others as well as communicating his own papers. 'His tall handsome figure, his commanding presence, and the charm of his manners, stand out clearly before me, as I watched him then, and in no essential respect was there any change in him between the first time I saw him and the last' (Smith, 1894a, p. lxxvii). His first meeting with Smith was a clear memory for him.

> I was introduced to him in the committee room of the section by my father, and although I was not eighteen years of age, he welcomed me with as much cordiality as if I had been a fellow mathematician of equal standing with himself. I was a shy and retiring schoolboy, but, in spite of the respect with which his knowledge inspired me, his kind and friendly manner at once placed me at my ease. I mention so particularly this experience of my own because it was very characteristic of his gentle and considerate nature. I am sure that no one was ever treated by him with less courtesy or attention on account of youth or junior standing: on the contrary, I believe that in such cases he instinctively and unconsciously showed even more consideration, I may perhaps mention that on this occasion he gave me the first separate reprint of a mathematical paper which I ever possessed: it was not a paper of his own, but one which had been given to him, and seeing me interested in it he told me I might have it, as he could procure another copy from the author (Smith, 1894a, p. lxxviii).

In 1871 James Glaisher graduated at Cambridge as second Wrangler and, in the same year, was elected to a Fellowship of Trinity College Cambridge and to a Lectureship in Mathematics (Table 6). His second meeting with Smith was in 1872 at the annual meeting of the BAAS, held at Brighton that year. At this meeting he could, perhaps for the first time, fully appreciate Smith's great ability as an expositor of mathematics. 'His winning manners and graceful delivery charmed me as before, but I was even more struck with the skill with which he succeeded in giving, in the simplest language, a correct idea of complicated theories to those to whom they were entirely new' (Smith, 1894a, p. lxxviii).

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16 James Glaisher [*Senior*] FRS (1809–1903), was an astronomer, mathematician, a pioneer in meteorology, and an aeronaut of note. In 1862, with Henry Coxwell, they made a famous balloon ascent which reached the greatest height (about 7 miles) ever recorded, at that time, by survivors. The *Aeronauts* is a 2019 biographical film based on their adventures. See the official trailer at https://www.youtube.com/watch?v=vmsBuG1eY8w

Table 6: Distinctions and Offices – James W.L. Glaisher FRS

Year	
1867	Attended Trinity College, Cambridge.
1868	Scholarship, Cambridge.
1871	BA, Cambridge, 2^{nd} Wrangler.
	Fellow of Trinity College, Cambridge.
	Lectureship in Mathematics, Cambridge (until 1901).
	Fellow of the Royal Astronomical Society.
	Membership of BAAS.
1872 – 73	Secretary of the BAAS (Section A) Mathematics and Physics.
1872	Membership of the LMS.
1874	MA, Cambridge.
1874	Council of the Royal Astronomical Society (until 1927).
1875	Fellow of the Royal Society.
1877 – 83	Secretary of the Royal Astronomical Society.
1880's	Vice–President of the LMS (4 occ.'s).
1882 – 84	President of the Cambridge Philosophical Society.
1883 – 84	Council of the Royal Society.
1884 – 86	President of the LMS.
1886 – 88	President of the Royal Astronomical Society.
1890	President of the BAAS (Section A) Mathematics and Physics.
1890 – 92	Council of the Royal Society.
1892	Honorary Doctor of Science, Trinity College, Dublin.
1901 – 03	President of the Royal Astronomical Society.
1902	Honorary Doctor of Science, Victoria University of Manchester.
1908	DeMorgan Medal of the LMS.
1913	Sylvester Medal of the Royal Society.
1916	Honorary Keeper of Ceramics, Fitzwilliam Museum.
1916	Honorary Fellow of the Royal Society of Edinburgh.
1917 – 19	Council of the Royal Society.

In 1871, the year of his graduation, James Glaisher joined the Committee on Mathematical Tables (BAAS) where some of the committee members included Arthur Cayley, George Gabriel Stokes, William Thompson and Henry Smith. As part of his work on this committee Glaisher prepared *Tables on the Theta Functions* and, in late 1873 (or early 1874), he recalls asking Smith if he would contribute an introduction to the volume. So began his extended collaboration with Smith which gave rise to a number of other papers written and published during the memoir's progress. Smith's work on this memoir resulted in two further publications which were connected with it.

> The progress of this work naturally brought us into closer and more frequent contact. I used to meet him at the Mathematical and Astronomical Societies, often walking with him to the Athenaeum Club at the close of the meetings, and we had long mathematical conversations at Cambridge when he came to the dinners of the Ad Eundem Club. When the memoir on the Theta Functions in its final form was passing through the press, we both read the proof-sheets, and at the same time he was sending me the *Notes on Elliptic Transformation* for the *Messenger*. I also had occasion to consult him on several mathematical and other questions and all these causes combined to produce a rapid interchange of correspondence during the last two years of his life (Smith, 1894a, p. lxxxvi).[17]

This collaboration with Glaisher formed the principal new work upon which Smith was engaged, from the time of their commencement in 1874, until his death in 1883. It was during this time that James Glaisher appears to have became a close confidant of Smith, writing in 1883 that 'I believe I was the only person who ever really knew him as he was as a mathematician and to whom he opened his heart'.[18] He described learning of the 'intensity and earnestness of his devotion to mathematics'.

> Even among mathematicians he referred so gaily and with so light a heart to his own studies and pursuits that I have been almost startled to find, when alone with him, how engrossed he really was with mathematical researches, and how completely they possessed his mind and affections. He derived intense pleasure both from working at Mathematics and from the contemplation of its

17 *Notes on the Theory of Elliptic Transformation* was published in the *Messenger of Mathematics* in 1882 (Smith, 1894b, pp. 321–367). Extracts of letters from Henry Smith to James Glaisher, from 1880 to 1883, are included at the close of the *Introduction to the Collected Mathematical Papers of Henry J.S. Smith*. These extracts relate to their progress of the Introduction (Memoir) on the *Theta and Omega Functions and the Notes on Elliptic Functions* (Smith, 1894a, pp. lxxxviii–xcv).

18 Letter from James W.L. Glaisher to Thomas H. Escott, 17th April 1883: Escott Papers (British Library), Add. 58780, fols. 55.

> truths and processes; and although he was undoubtedly anxious in the latter part of his life that what he had accomplished should not perish in his notebooks, he seemed quite indifferent to the amount of recognition that was accorded to his published writings by his contemporaries (Smith, 1894a, p. lxxxvii).

In the 19th century Oxford University was renowned for education in the classics, thus, while mathematics was not unimportant at Oxford, it did not have the same dominance over the course of study as at its rival institution at Cambridge. At Cambridge the mathematical tradition was one of pursuing well-established subjects by methods that were uninfluenced by developments on the Continent. Andrew Russell Forsyth FRS (1858–1942), writing of James Glaisher in 1929, believed that his work as a researcher was blighted by the failure of mathematicians at Cambridge to follow many of the advances that had been made on the Continent (Forsyth, 1929, pp. 101, 110).[19] Glaisher's interest in elliptic functions, a topic of great contemporary interest, may be given as an example. He chose to 'work within the Cambridge tradition of real rather than complex analysis that soon rendered much of his work obsolete, and his complete treatises were never published' (Heard, 2019, p. 111). Throughout his long career at Cambridge University, he was an advocate for Continental practices of mathematical research but continued to work within the Cambridge tradition along with a heavy burden of teaching duties. At Oxford, Smith also had considerable teaching duties but had not been restricted in the same fashion and so was able to remain interested in Continental developments in mathematics. Despite Smith's comparative isolation at Oxford, James Glaisher acknowledged that 'the subjects in which [*Smith*] was a master, and to which his own contributions were of such high value, were quite beyond the range of Cambridge mathematical teaching' (Glaisher, 1926, p. 54). In *From Servant to Queen: A Journey through Victorian Mathematics*, the author presents an interesting assessment of how the status of pure mathematics at Cambridge affected the career of Glaisher (Heard, 2019, pp. 111–144). The author, comparing the career of Glaisher with that of his friend Smith, comes to the following conclusion.

> A comparison of the careers of the two friends Glaisher and Smith therefore leads to a paradoxical result. Glaisher, although the younger man, found himself institutionalised in Britain's mathematical powerhouse and enslaved by its traditional modes of thought. Consequently, burdened by teaching duties, he found himself unable to keep abreast of burgeoning developments on the Continent, and

19 Andrew Russell Forsyth FRS (1858–1942) was the Sadleirian Professor of Pure Mathematics at the University of Cambridge from 1895 to 1910 (having succeeded Arthur Cayley). He wrote important works on analysis which were responsible for introducing Continental methods into the subject in Britain.

> eventually collecting pottery displaced mathematics as his principal interest. Smith, who took a first in both mathematics and classics, was similarly burdened, but had not been trained in the same fashion and so was able to absorb, and stay in tune, with Continental mathematical advances to an extent that was unmatched in Britain. Although his output was less than Glaisher's, it was superior in originality, and it is a fair assessment of the stultifying power of Cambridge to say that Smith would not have been able to flourish there as he did in the mathematically insignificant confines of Balliol College Oxford (Heard, 2019, pp. 133–134).

Glaisher reminds us that Smith, as regards the higher branches of mathematics, was a 'self-made mathematician', whose mathematical powers developed over his lifetime into an almost passionate attachment in later years.

> Led on by pure fascination, under no pressure, but without either assistance or encouragement, he slowly and surely mastered everything that had been accomplished, and gained such an insight into the principles of the subject, and such a command over its methods, as could only have resulted from so long and complete a self-devotion. But one unfortunate result of his comparative isolation was that he allowed too much of his own work to accumulate in manuscript, and that, the 'note' of personal ambition (as Lord Bowen described it) being wanting in his character, and no external stimulus prompting him, he remained indifferent to the advantages of early publication, and was too little sensible of the difficulties that would stand in the way of preparing for the press any work which has been too long on hand (Smith, 1894a, pp. lxxxiii–lxxxiv).

These sentiments of concern were also expressed by Professor Thomas Henry Huxley FRS (1825–1895) in 1883. Huxley wrote that 'I think that he [*Smith*] would have been one of the greatest men of our time if he had added, to his wonderfully keen intellect and strangely varied and extensive knowledge, the power of caring very strongly about the attainment of any object' (Spottiswoode, 1883, p. 381). It seems that Smith's high standard of completeness, which he imposed on his published writings, added considerably to the effort required prior to publication of his work. His published mathematical papers occupy almost 1200 pages, however, according to Glaisher, this amount would have been tripled had he been less exacting in the quality of his work (Glaisher, 1883a, p. 664). Glaisher gives a fascinating insight into Smith's habits of mathematical investigations which may offer a partial explanation, at least among his closest friends, as to his mathematical anonymity.

> Except in vacations he seemed to have no time for mathematical investigation, and the amount that he accomplished was always a mystery to me until I learned that after a hard day's work, closing perhaps with a dinner party at which his lively wit and brilliant conversation had made him seem the gayest and the brightest of the circle, he would quietly settle down to work in his own room for some hours before going to bed. What he wrote related probably to matters that had been more or less in his mind all day, and to which at intervals he had actively turned his thoughts, making a few stray notes perhaps on slips of paper. The last thing of all at night he would enter the results of the day's work or thoughts in his note book. Most of his mathematical work he did in his head, by sheer mental effort, and he scarcely ever committed an investigation to paper in any detail except when writing it out for publication. The notes which he made while thinking out a subject were often written on scraps of paper or backs of envelopes, which were destroyed as soon as he had made a definite advance which would allow of an entry in his notes (Smith, 1894a, p. lxxxii).

Glaisher's recollections of Smith's mathematical writings and his mathematical style are important for the narrative of this thesis. He asserts that Smith had no sympathy for anyone who was contented to give imperfect demonstrations, or who regarded results proved merely because they had satisfied themselves of their truth. He believed that Smith did not publish his mathematics unless it was complete in every detail and perfect in form as in substance. He writes that:

> With respect to the character of Professor Smith's mathematical writings a very noticeable feature is the arithmetical spirit that runs through the whole of his work. The years of study which produced the Report upon the Theory of Numbers exercised a lasting influence upon his mode of thought; and his familiarity with the ideas and methods of the Higher Arithmetic continually shows itself in his treatment of Geometry and Elliptic Functions. In the latter subject the arithmetical tendency of his mind is especially evident in the point of view from which the theory of Transformation is always regarded. Another characteristic feature of his work is its completeness, both as regards attention to details and accuracy of demonstration. He had a very strong dislike to careless or slovenly work of any kind, and thought that it was nowhere so much out of place as in pure mathematics (Smith, 1894a, p. lxxiii).

In an article published in the *Times* on the day after Smith's death (February, 1883) it was written that: 'It is probable that of the thousands of Englishmen who knew Henry Smith, scarcely one in a hundred ever thought of him as a mathematician at all'.[20] He had a wide range of human interests and compassions which, along with his social skills, endeared him to his friends and to Oxford society. Glaisher was certain that very few people had any idea of the extent to which Smith was preoccupied with mathematical research.

> I am sure that very few even of his intimate friends were aware that in his own subjects he stood alone in England, and that his papers upon the Higher Arithmetic held a place among the most important productions of the century in abstract science. Even fewer still had any idea of the extent to which his heart and mind were engrossed by his mathematical researches. This want of recognition (if it may be so called) was no doubt partly due to his disinclination to speak of his own work except occasionally to those whom he knew to be interested in it, and his non-mathematical friends may be pardoned for not discovering an enthusiasm which showed itself so little; in fact it cannot be doubted that he would have been spared much of the voluntary work which he so unselfishly undertook at the solicitation of others, if the depth of his devotion to his own subject had been generally known (Smith, 1894a, p. lxxxii).

Glaisher suggests that another explanation may be found in the fact that so much of Smith's time and effort was devoted to other responsibilities, such as university work of various kinds, royal commissions and scientific societies. His friendship with Smith along with their mathematical collaboration during the 1870's meant that James Glaisher was, at that time, best placed to comment on Smith's published work. He was certain that Smith's published work could 'compare in extent and profundity with the researches of the ablest mathematicians, who have concentrated their whole lives upon their special subjects' (Smith, 1894a, p. lxxxii). He believed that Smith did not publish the amount of mathematical papers of which he was capable, but those that he did publish showed that he attempted to progress the bounds of knowledge on topics of real importance. James Glaisher found it remarkable that this could be achieved in the midst of so many varied pursuits all requiring constant care and attention. He believed that Smith's devotion to the subject he loved would not have endured without the balance of the life he lived.

> His victories were won by the hardest of intellectual conflicts, in which for the time his whole heart and soul and powers were entirely and absolutely absorbed. It was in his wide interests and sympathies, the pleasure of intercourse with others, and the love of

20 Times, 10th February 1883.

> all that was good and cultivated, that he found relief from these severe mental efforts. Had he not been gifted with a disposition that gave him the keenest sympathy with every human interest, that attracted him to society and endeared him to his friends, that gave him, in fact, his other noble life – the life the world knew – his fierce devotion to the subject he loved would have ended his days long since (Glaisher, 1883a, pp. 664–665).

James Glaisher was described as a man of rare humanity: 'To no one did warm friendship mean more' (Turner, 1929, p. 308). In 1883 James Glaisher, in service to his friend, started work to secure Smith's mathematical legacy. He began by completing the results of their collaboration, with the final articles of the memoir taken from a manuscript he found among Smith's papers: 'I believe that no more was written, even in draft' (Smith, 1894a, p. lxiv). He then set about the task of editing Smith's collected mathematical papers. He was in possession of Smith's notebooks and clarified that there were about forty notebooks in which Smith had written his early notes on geometry and the *Reports on the Theory of Numbers*, with others notebooks relating to his university lectures.[21]

> More than a dozen are devoted to the records of original work, a very large portion of which has never been published. I have repeatedly examined the notebooks relating to the subjects with which I was most familiar in hopes of being able to make extracts that could have been included in the present volumes. But in this I have been unsuccessful, for Professor Smith entered in these books not only the finished theorems which he had demonstrated, but also results which he had arrived at by rough explorations and inductions, as well as mere guesses sometimes; and it is certain that he would have published nothing himself from these books without submitting it

21 The location of these notebooks are unknown.
The author has continued this search. Bowes & Bowes Bookshop, No. 1. Trinity Street Cambridge prepared a sales catalogue of books purchased from the library of the late James W.L. Glaisher FRS (1848–1928). This sales catalogue has not been located however a pamphlet of the sale gives a table of contents. The pamphlet outlines that Bowes & Bowes have in the past 'purchased many libraries of eminent scholars, for instance, those of Professor Arthur Cayley, Dr. I. Todhunter, Professor John Couch Adams, Professor H.J. Stephen Smith'. [Archives of the Royal Astronomical Society, London. MSS Baily-Glaisher, GLAISHER Letters and papers of James Whitbread Lee Glaisher (1848–1928), F13].
See https://discovery.nationalarchives.gov.uk/details/c/F183295
On his death in 1928 James W.L. Glaisher (unmarried) had only one surviving sibling, his sister Cecilia Appelina Glaisher (1845–1932) who was residing in New Zealand having earlier emigrated there with her husband, Dr Frederick Everard Hunt (1840–1900), a practising doctor in Christchurch. They had one son Fredrick Knight Hunt (1868–1945).
Search also extended to Cecilia Glaisher's archives at the National Library of New Zealand.
See https://natlib.govt.nz/items?text=Cecilia+Appelina+Glaisher&commit=Search

> to the most careful examination and working out the demonstrations afresh. Under these circumstances it was decided, but with great reluctance, to confine the present work to the published writings, and make no attempt to give an account of the varied contents of the notebooks (Smith, 1894a, p. lxxv).

2.5 RECOLLECTIONS BY WILLIAM H. SPOTTISWOODE

William H. Spottiswoode PRS (1825–1883) was elected a Fellow of the Royal Society in 1853 and served as its president from 1878 until his death.[22] The presidency, the highest honour that fellows of the society could bestow, was testament to his exceptional attainments as a mathematician and physicist. Spottiswoode won a scholarship to Balliol College, Oxford, in 1842, and obtained a First Class Honours BA in Mathematics in 1845. He gained thereafter Junior (1846) and Senior (1847) University Mathematical Scholarships. After his degree he resided in Oxford for a short time, and gave a course of lectures on geometry in three dimensions at Balliol College. It is possible that Spottiswoode may have first encountered Smith as a student at that time.

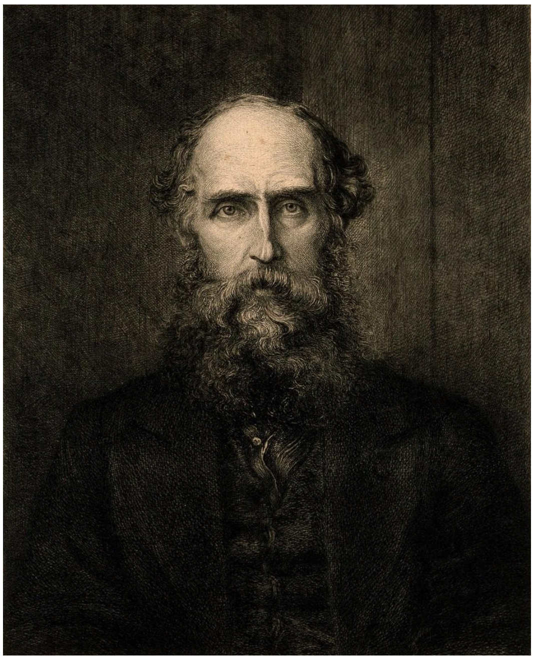

Figure 11: William H. Spottiswoode PRS (1825–1883)

22 For a biographical essay on William Spottiswoode see (Glaisher, 1884b, p. 150–153) and (Galton, 1883, pp. 489–491). Also see the entry for William Spottiswoode in the *Oxford Dictionary of National Biography* (Crilly, 2006b). For an account of Spottiswoode as 'Cayley's most prominent disciple, Sylvester apart,' see (Crilly, 2006a, p. 191).

Soon afterwards, William Spottiswoode left Oxford to take his father's place as a member of the firm of Eyre and Spottiswoode, the Queen's printers. He possessed remarkable gifts as an organiser and, with his financial acumen, the business developed under his guidance. He had an active interest in the well-being of his hundreds of employees 'by whom he was warmly beloved and generally looked upon as a personal friend' (Galton, 1883 p. 490). However, he always reserved time for his scientific and literary pursuits which were wide and varied. His high rank as a mathematician was mainly the result of his memoir *Elementary Theorems relating to Determinants*, published in *Crelle's Journal* in 1856 and, from 1862, a series of memoirs on the contact of curves and surfaces published in the *Philosophical Transactions of the Royal Society.*[23] In experimental physics he researched the polarisation of light and the electrical discharge in rarefied gases. On these subjects he gave popular lectures to crowded audiences at the Royal Institution, at the South Kensington College of Science, and the BAAS. He had a keen interest in geography and he contributed to the *Journal of Royal Asiatic Society* and to the *Proceedings of the Royal Geographical Society.* The distinctions and offices held by Spottiswoode within scientific societies show an outstanding list of accomplishments, including an international recognition from the French Académie des Science (Table 7).

> He was a man of the world, with a very wide circle of friends, chief among whom were the most earnest and most devoted labourers in different departments of knowledge. His house at Grosvenor Place was the centre of scientific society in London, and his garden parties at his country estate near Sevenoaks were brilliant gatherings of men eminent in various walks of life (Glaisher, 1884b, p. 152).

William Spottiswoode was a leading scientific figure in Victorian Britain but, perhaps similar to Henry Smith, his various commitments resulted in him becoming overburdened and tired. Towards the end of his life he went to Italy on a short holiday to recuperate from overwork. On returning home he contracted typhoid fever and, three weeks later, died on 27th June 1883 at his home; 41 Grosvenor Place, London. He was fifty-eight years of age and his death was regarded as a national loss. In recognition of his position as president of the Royal Society, and his contribution to science, he was buried in Westminster Abbey in the presence of civic dignitaries and the whole scientific establishment.

23 Henry Smith acted as a referee on a number of these memoirs. See Referees' Reports at https://catalogues.royalsociety.org/CalmView/

Table 7: Distinctions and Offices– William H. Spottiswoode PRS

Year	
1842	Attended Balliol College, Oxford.
1846	Junior Mathematical Scholarship, Oxford.
1846	BA, Oxford. (1^{st} class Mathematics).
1846	Membership of BAAS.
1847	Senior Mathematical Scholarship, Oxford.
1848	MA, Oxford.
1852	Fellow of the Royal Astronomical Society.
1853	Fellow of the Royal Society.
1855	Fellow of the Royal Geographical Society.
1857 – 58	Examiner in Mathematical Schools, Oxford University.
1861 – 74	Treasurer of the BAAS.
1865	President of the BAAS (Section A) Mathematics and Physics.
1865 – 73	Treasurer of the Royal Institution.
1865	Membership of the LMS.
1870 – 72	President of the LMS.
1871	Honorary Secretary of the Royal Institution.
1871 – 78	Treasurer of the Royal Society.
1876	Correspondent of the French Institut (Académie des Science).
1878	President of the BAAS.
1878 – 83	President of the Royal Society.
1883	Honorary Member of the French Institut (Académie des Science).
1883	Honorary Fellow of the Royal Society of Edinburgh.
	Honorary Degree of DCL at Oxford.
	Honorary Degree of LLD at Cambridge, Dublin and Edinburgh.

Just four months earlier William Spottiswoode wrote an appreciation of Henry Smith which was published in *Nature* on 22nd February 1883 (Spottiswoode, 1883, pp. 381–384). He wrote of Smith's contribution to mathematics and, in particular, to the various societies, associations and commissions on which they both served (Table 3, Table 7). He outlines Smith's early education and mathematical career at Oxford University. He describes his contributions to mathematics, in particular his work on the theory of numbers, as being work of the highest order.

The friendship between Spottiswoode and Smith developed through their prominent roles, including the highest elected positions, in societies such as the LMS, the BAAS and the Royal Society. These societies brought together the leading mathematicians and scientists of the time to cultivate their discipline through active engagement and collaboration. During the 1870's, council meetings of the LMS and the Royal Society would each convene nine to twelve times per year.[24] The week-long meeting of the BAAS would take place at locations throughout the British Isles in August/September each year. Further engagement and collaboration were facilitated by the formation of sub-committees. In 1865 Spottiswoode and Smith were both members of the BAAS sectional committee (Section A) Mathematics and Physics. In 1869 the BAAS appointed a sub-committee to report on the state of geometry teaching and on the adequacy of Euclid's Elements as a textbook. The committee members consisted of, among others, Arthur Cayley, James Joseph Sylvester, William Spottiswoode, Henry Smith and William Kingdon Clifford. The committee reported firstly at the Bradford meeting of the BAAS in 1873, and then at the Glasgow meeting in 1876. However the committee came to no significant conclusions, partly because of the conflicting opinions of its members.[25] Spottiswoode reminds us of the many instances in which Smith's name came uppermost in the minds of those looking for a leader, chairman, or president. His power of caring and sense of duty would mean that his time for mathematics was sacrificed.

> For those who knew him best were most fully aware of the effort which it cost him to postpone (as he often did, with apparent readiness) his beloved mathematics to other claims (*ibid*, p. 382).

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24 The Great Western Railway had reached Oxford in 1844 and provided fast access to London with a journey time of two hours and twenty minutes. Hence it was possible for Henry Smith to attend meetings in London and return to Oxford on the same day. Before then the quickest way to reach London from Oxford was by express coach which had a six hours journey time. 'One unexpected consequence of the new railway connection was the discovery that one express coachman had a wife and family at both ends of the journey' (Wilson, 2021, p. 94).

25 See (Flood, Rice, and Wilson, 2011, pp. 321–336).

From 1870 to 1875 Smith was appointed to the Royal Commission (Devonshire Commission) on Scientific Instruction and the Advancement of Science, and from 1877 to 1878 to the Selborne Commission on Oxford University (Wilson, 2021, p. 109). The recommendations of the Selborne Commission, all of which were achieved, affected reforms at the university in the decades that followed. The Meteorological Council was nominated by the Royal Society and appointed by the Government. In 1877 Smith was unanimously nominated as Chairman of the Meteorological Council and he represented the council at the International Meteorological Congress at Rome in 1879. In his 1883 appreciation of Smith, Spottiswoode outlined the important role of the Meteorological Council and praised Smith for his contribution.

However, contrary to the view of others, Spottiswoode believed that the time Smith devoted to scientific societies did not adversely affect the quality of his mathematical output.

> It was sometimes thought that his mind became diverted from mathematics by his many other distracting avocations; there are, however, reasons for doubting this. It is true that he did not pour out the amount of mathematical papers of which he was certainly capable, but those that he did publish showed that he cared little to add fringe work to the borders of our knowledge, and that he reserved himself for questions of real importance (Spottiswoode, 1883, p. 383).

Spottiswoode's 1883 appreciation continues with a description of Smith's lively enthusiasm for mathematics in the years up the end of his life, and quotes from his presidential address to the LMS in 1876 (Section 2.9). In this address Smith described the 1873 demonstration by Charles Hermite (1822–1901), that the base of the Napierian logarithms is a transcendental irrational, as 'singularly profound and beautiful analysis' (Smith, 1894b, p. 182). Smith lamented that similar research, at that time, had not been entered into to with regard to the number π. Spottiswoode reminds us of the accomplishment by Ferdinand von Lindemann (1852–1939) in 1882.

> Last year, Lindemann, starting from Hermite's researches, succeeded in supplying the proof required with reference to π. And while speaking of this achievement, with the satisfaction which his generous nature prompted, Henry Smith added that it was a problem on which he had long fixed his eye with a view to attacking it seriously so soon as he had leisure for the undertaking (*ibid*, p. 383).

Spottiswoode closes his appreciation of Smith by paying a personal tribute to his friend describing him as being as ‘young and vigorous in intellect at the age of fifty-six, the limit to which he attained, as he was when he gained the first of his many University honours’ (*ibid*, p. 382). He acknowledges Smith’s sister, Eleanor, whose ‘useful and sympathetic life worthily complemented his own’ (*ibid*, p. 384). He expressed the hope that the collected mathematical papers of Smith would be published as an appropriate memorial. This task was achieved by their mutual friend, James Glaisher, in 1894.

2.6 THE REMARK OF REVEREND BENJAMIN JOWETT

Henry Smith had mathematical anonymity among his closest friends at Oxford. A recollection from his lifelong friend, the Reverend Benjamin Jowett (1817–1893), is testament to this fact. Jowett was a renowned Oxford theologian and Master of Balliol College who had a daily association with Smith as a colleague and friend for over three decades.[26] Professor Jowett, then Vice-Chancellor of Oxford University, was the officiating clergyman at the funeral service for Henry Smith on Tuesday February 13th, 1883 at St. Paul’s Church, Oxford. Jowett contributed a short personal recollection of Smith which formed part of the introduction to Smith’s collected mathematical works, published in 1894. The following is a short extract:

Figure 12: Reverend Benjamin Jowett (1817–1893)

26 See the entry for Benjamin Jowett in the *Oxford Dictionary of National Biography* (Hinchliff and Prest, 2006).

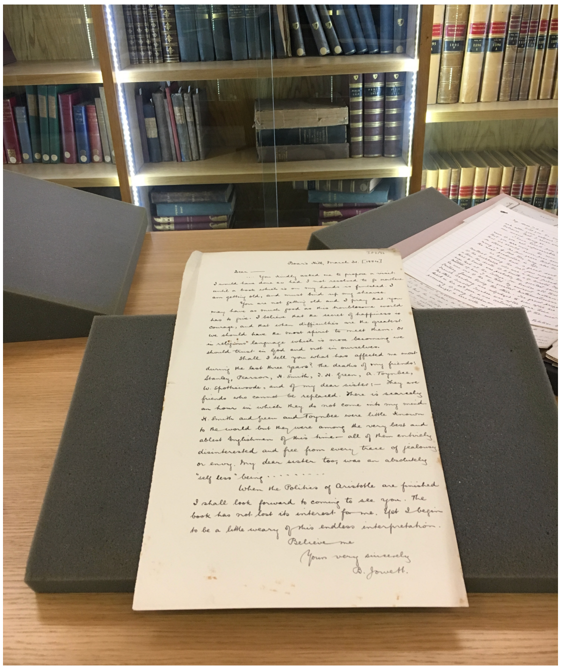
> I have endeavoured, in a few pages, to give a sketch of one with whom I was in daily intercourse during thirty-five years of his life, and who I think may be regarded, without exaggeration, as one of the most remarkable persons of his time. Yet he lived and died almost unknown to the world at large. I have sometimes asked myself what was the reason of this contrast between his reputation and his real merits. It has been said that 'the world knows nothing of its greatest men', but this familiar line, whether true or not, is not the whole account of the matter in his case. The explanation is partly to be sought in his own character. He had no ambition, he had not a strong will, and he had never made himself known to the public. He was once reproached by a friend for 'giving up to society what was meant for mankind', and the reproof, as far as it applied to his life at Oxford, was not without foundation. He was not the author of any considerable work. His mathematical writings, on which his fame chiefly rests, awaits the judgement of time (Smith, 1894a, pp. xxxvii–xlv).

His suggestion that Smith was 'not the author of any considerable work' may seem to some as a curious remark. However, as Regius Professor of Greek, Jowett could not be blamed for an oversight in commenting on Smith's life in mathematics. It seems that Smith's unassuming nature and the standing of mathematics at Oxford University, relative to the classics at that time, could provide an explanation.[27]

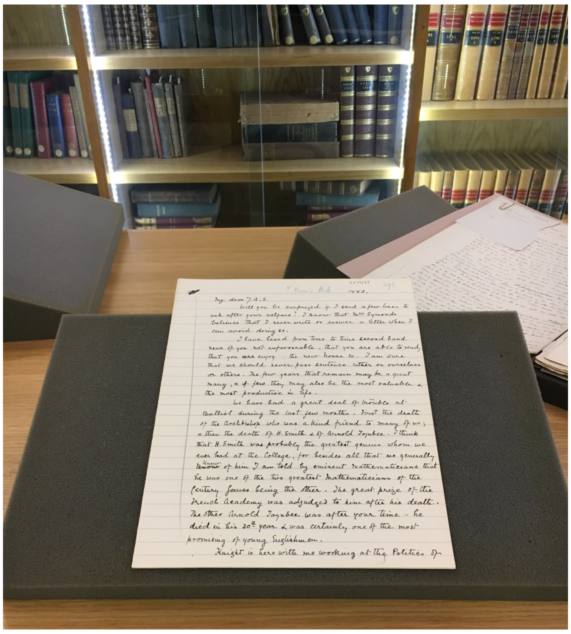
In the years 1881–84 Jowett lost many of his closest friends, all of whom were associated with Balliol College, and whose friendships he valued dearly. In a letter to Lady Abercromby (1840–1915), dated 31st March 1884, he expresses his sorrow at the loss of his friends, and of his sister (Figure 13a):

> Shall I tell you what has affected me most during the last three years? The deaths of my friends: Stanley, Pearson, H. Smith, T.H. Green, A. Toynbee, W. Spottiswoode, and of my dear sister. They are friends that cannot be replaced. There is scarcely an hour in which they do not come into my mind. H. Smith and Green and Toynbee were little known in the world, but they were among the very best and ablest Englishmen of this time – all of them entirely disinterested and free from every trace of jealousy or envy. My dear sister too, was an absolutely 'self less' being.

27 In 1930 G.H. Hardy published an article in *The Oxford Magazine* in which he makes reference to this comment by Benjamin Jowett. In the article G.H. Hardy was advocating for the establishment of a Mathematics Institute at Oxford and referenced this comment as an estimate of how undervalued mathematics was in Oxford at that time (Albers, Alexanderson, and Dunham, 2015, pp. 291–294).



(a) Letter to Lady Abercromby (1884)



(b) Letter to John Addington Symonds (1883)

Figure 13: Reverend Benjamin Jowett's letters to Lady Abercromby (1884) and to John Addington Symonds (1883) are courtesy of the Archives of Balliol College, University of Oxford [Balliol College Historic Collections, Jowett Papers, I F2/33 and I F13/55].

In 1883 Jowett wrote to his friend, the English poet John Addington Symonds (1840–1893). In this letter he writes that, despite knowing Smith, he has now been informed of his great achievements in mathematics and mentions the international recognition Smith achieved for his work (Figure 13b):

> I think that H. Smith was probably the greatest genius whom we ever had at the College, for besides all that we generally knew of him, I am told by eminent mathematicians, that he was one of the two greatest mathematicians of the century, Gauss being the other. The great prize of the French Academy was adjudged to him after his death.

This late awareness of Smith's mathematical achievements, by one of his closest friends, merits further investigation in this thesis. Jowett's initial recollection remained unamended and was included as part of the introduction to Henry Smith's collected mathematical papers. Further recollections were provided by Lord Bowen, Mr. J.L. Strachan-Davidson and Mr. Alfred Robinson. A formal introduction to the collection was provided by the editor, James W.L. Glaisher. In addition, a biographical sketch of Henry Smith was provided by his close friend Charles Henry Pearson (1830–1894). He was assisted in the task by Eleanor Smith, who provided him with a complete memoir of details. Eleanor Smith, with assistance from Charles Pearson, carefully reviewed the recollections provided by the various authors (Section 2.7). In a letter to Charles Pearson, dated October 26th 1888, Eleanor Smith wrote that:

> I have read the notice carefully once through with much satisfaction - seeing perhaps in some half dozen places where I doubted the appropriateness or accuracy of some or statement. Today Mr. Strachan-Davidson is to come to review the MSS and will go through it in the same way and then we must take measures to verify anything about which we both doubt.[28]

On 11th January 1889, Eleanor Smith continues her correspondence with Charles Pearson:

> I have only just received proofs of the first set of letters. Mr Robinson has sent me his statement about the election which I think excellently well done and the Master [Benjamin Jowett] has written his but, by some error, it has been sent to him at Malvern instead of coming to me so I have not yet seen it. I have shown your portion to some of the nearer friend: Albert Dicey and his wife, the Brodies, the Grant Duffs. The only one of these whom circumstances cannot express their impressions in writing was Dr

28 Oxford, Bodleian Libraries, Charles Pearson Collection, MSS. Eng. lett. d. 191, fols. 31–32.

> Acland, whose letter I sent you, but Albert Dicey and Sir Mountstuart both expressed the upmost satisfaction. It is still to go to Mr Maskelyne, Mr Glaisher, the Master and Mr Bradley of those whose opinion I specifically wish to conciliate but do not doubt that they will be equally pleased.[29]

Given other instances of Eleanor's attention to preserving her brother's memory, and righting any wrong, it is likely she never got to review Jowett's initial recollection of Smith. In reviewing the various notices, she expressed her distress at recalling the details of her brother's life. In concluding the above letter she wrote: 'I feel all this too bitterly and probably some of it unduly. I should not like this to come through in the notice except, in as far as it ought and, for the sake of those who come after. I cannot write more today'.

A footnote to the Jowett recollection of Smith reads: 'Professor Jowett's death has deprived these recollections of the author's final version'. Benjamin Jowett died October 1st, 1893. He left explicit directions, in writing, that he wished to be buried in the same cemetery near his friends Henry Smith and Thomas Hill Green, at St. Sepulchre's cemetery, Oxford.[30]

Henry Smith's friends in mathematics wrote of his intellectual power and academic achievements but other aspects of his life and nature were known only to his friends at Oxford. The friendships he formed during his student days at Balliol College were ones he maintained throughout his life. One of his closest friends was Charles Pearson who, despite spending many years in Australia, maintained a written correspondence with Eleanor Smith for almost 20 years. These letters give an interesting insight into Henry and Eleanor Smith's life at Oxford, of their many friends and travels throughout Europe. These letters range from celebrating the most joyful occasions to the most poignant expressions of loss.

2.7 CORRESPONDENCE OF ELEANOR SMITH AND CHARLES HENRY PEARSON

Throughout his life in mathematics Henry Smith enjoyed the lifelong support and companionship of his sister Eleanor (Figure 14). Eleanor Smith was four years older than Henry, but from an early age there was a strong bond between them, strengthened perhaps by the loss of their only other siblings in 1834 and 1843. Throughout her life she had pride in Henry's achievements and expressed concern for his health as a consequence of his university work and other commitments. These emotions she revealed in letters to their many friends in Oxford and beyond. Later in life Eleanor would defend her late brother in

29 Oxford, Bodleian Libraries, Charles Pearson Collection, MSS. Eng. lett. d. 191, fols. 33.
30 Balliol College Historic Collections, Jowett Papers, Group III Class MA – Letters to Robert Morier, MA38.

relation to an international prize, seeking to right a wrong she perceived had been done to him (Section 6).

Figure 14: Eleanor Smith (1822–1896) with Sir Henry Acland (1815–1900), the Regius Professor of Medicine, who was the inspiration behind the University Museum. [Courtesy of the Bodleian Libraries, The University of Oxford MS Minn. 202/6.]

On May 13th 1857, their mother Mary Smith died of fever at Upper George Street, Bryanston Square, London, aged 63. In that year Eleanor Smith moved to Oxford to set up home with her brother. From 1861 until 1874 they lived at 64 St Giles. They spent the university term together with each having complete freedom of movement during the vacations.[31]

> I cannot doubt that this arrangement contributed very much to Henry Smith's happiness. He was eminently domestic and hospitable, and having the cares of household life taken off his hands, and being supplemented by one who was almost another self, was able to fill his house with friends, who were certain of an Irish welcome, however unseasonably they might arrive to ask for a dinner or a bed (Smith, 1894a, p. xxiv).

During times of separation, vacations or otherwise, Henry and Eleanor wrote to one another at regular intervals. Sarah Angelina Acland (1849–1930), daughter of Sir Henry Acland, writing to Charles Pearson in 1891 wrote of them that

31 For an article on Eleanor Smith see *The Times*, 18 September 1896.
Also see https://www.stsepulchres.org.uk/burials/smith_henry_john.html
See the entry for Eleanor Smith (1822–1896) in the *Oxford Dictionary of National Biography* (Anonymous, 2004).
See the entry for Sir Henry Acland (1815–1900) in the *Oxford Dictionary of National Biography* (Cameron, 2004).

'I think that they are the most united family that I have ever seen; not a jar or a cross word was ever heard or a criticism of others'.[32]

Eleanor had travelled widely on the Continent and was a good linguist with an extensive knowledge of European literature. For her, life in Oxford was a one of active involvement. In the 1860s, before the foundation of the first women's colleges in Oxford, she persuaded various professors sympathetic to the education of women to give a series of women's lectures and organised the first course in 1866. She helped to found and was one of the original (1879) members of the council of Somerville, the women's college at Oxford, and served for many years as a trustee of Bedford College, London. In 1895 she supported the campaign to open Oxford degrees to women. In addition to her keen interest in women's education, Eleanor Smith gave generous support to schemes for improving the health of the poor. She served on the committees of both the Radcliffe Infirmary and the Sarah Acland Home for Nurses, and was a promoter and director of the Provident Dispensary.[33] Her life in Oxford was surrounded by many friends and confidants. Dr. Henry Acland and his wife Sarah, Albert Dicey and wife Eleanor, Reverend Benjamin Jowett, Reverend Mark Patterson and his wife Francis, Dr. George Rolleston and his wife Grace, Sir Benjamin Collins Brodie and his wife Philothea Margaret. She was described as an 'intelligent and compassionate woman' who maintained many friendships during her life (Green, Pattison, and Bradley, 1985, p. 18).

Charles Henry Pearson was one of Henry and Eleanor Smith's closest friends. He was a British-born Australian historian, educationist, politician and journalist. In 1850 Charles Pearson won a scholarship to attend Exeter College, Oxford. He obtained a First Class in *Literis Humanioribus* in 1852 and was elected to a Fellowship at Oriel College, Oxford in 1854 (Table 8). In his memoirs, published in 1900, Charles Pearson wrote of his time at Oxford:

> I have no doubt, in my own mind, that the only real advantage of Oxford as I knew it was in the opportunities it gave for social intercourse. Not only did I know about all the most distinguished of my contemporaries, but some of the most brilliant dons; in particular, Henry Smith (subsequently Savilian Professor of Geometry), and John Conington, the late Professor of Latin. Both were decided Liberals and brilliant talkers, and both stimulated the young men they mixed with to work for the nobler aims of life. The friends I made in Oxford have been staunch, with scarcely an exception, through life (Pearson and Stebbing, 1900, p. 74).

32 Oxford, Bodleian Libraries, Charles Pearson Collection, MSS. Eng. lett. d. 191, fols. 48–49.

33 The Provident Dispensary was set up in Oxford in 1892. Members paid a weekly subscription for medical attendance, advice and prescription drugs.

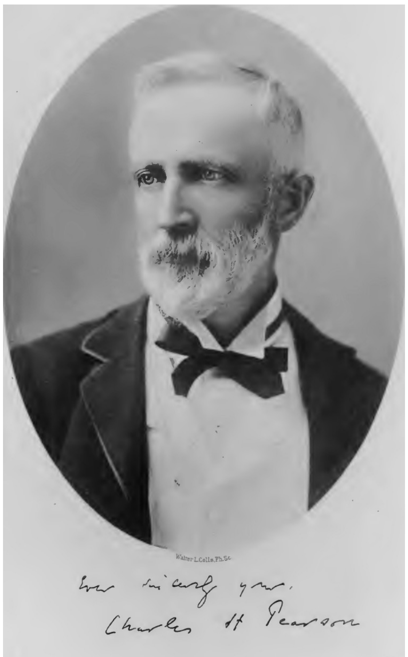


Figure 15: Charles Henry Pearson (1830–1894)

Henry Smith and Charles Pearson made each other's acquaintance about 1850.[34] During the winter of 1852 they were both students of Nevil Story Maskelyne, learning chemical analysis, in the basement of the Old Ashmolean Museum (Morton, 1987, p. 94). From this time on they would become lifelong friends. Pearson had the following early recollection of Smith.

> Henry Smith, the ablest, intellectually, of all my contemporaries, was living on at Oxford as he has continued to do, squandering his great powers upon hack work, and perfectly contented when he got the Savilian Professorship. He is the only one I have known whose superiority was so incontestable that it extinguished jealousy, and whose popularity was such that he had no personal enemies. He has passed pleasantly through life, working hard, giving proof now and again, by some isolated paper, of what his powers were, but doing nothing substantial to enlarge the boundaries of science (Pearson and Stebbing, 1900, p. 108).[35]

In 1864 Charles Pearson decided to emigrate to Australia. He was disappointed with Oxford University as they had not awarded him the Chichele Professorship of History. He decided to give up his Professorship at King's College London as they were not prepared to increase his modest salary. His

34 Mountstuart Grant Duff (1829–1906) recalls meeting Charles Pearson for the first time in June 1850 in the rooms of Henry Smith (Pearson and Stebbing, 1900, p. 172).

35 This extract gives another example of generous sentiments being expressed for Henry Smith from a close friend not fully aware of his mathematical prowess.

Table 8: Distinctions and Offices – Charles Henry Pearson LL.D

Year	
1843	Entrance to Rugby School – August.
1850	Scholarship, Exeter College, Oxford.
1852	First Class in *Literis Humanioribus*. B.A. Oxford.
1854	Fellowship, Oriel College. M.A. Oxford.
1855	Lectureship, English Language and Literature, King's College, London.
	Professorship, Modern History, King's College, London.
1857	Sacred Verse Prize, Oxford.
1862–63	Editorship, *National Review*.
1868	Lectureship, Liverpool and Manchester.
1869–70	Lectureship, History, Trinity College, Cambridge.
1873	Lectureship, History, University of Melbourne.
	M.A. University of Melbourne.
1875	Headmastership, Presbyterian Ladies' College, Melbourne.
1877–78	Commissioner to report on the best and most economical way to establish Free Education in Victoria.
1878	Member for Castlemaine, Legislative Assembly of Victoria.
1878–79	Commissioner to England.
1880–81	Minister in Mr. Berry's Administration, without Portfolio.
1883	Member for East Bourke Boroughs.
1886–90	Minister of Public Instruction in Gillies-Deakin Administration.
	Officier d'Instruction Publique, France.
1889	LL.D. St. Andrews University, Scotland.
1892 – 94	Secretary to Agent-General of Victoria.

decision to emigrate was also influenced by his failing health which he was confident would improve with a warmer climate. His closest friends in Oxford were concerned at his decision. Mountstuart Grant Duff (1829–1906) and Smith argued with him very strongly against taking a step which involved the surrender of all that he had laboriously achieved thus far. Eleanor Smith suggested that he 'was unfit for the rough life and coarse surroundings of a young country' (Pearson and Stebbing, 1900, p. 119). Despite these concerns Charles Pearson arrived in Adelaide, South Australia in 1864 following a ninety-three day sea voyage.

Charles Pearson would remain in Australia for over 20 years. He held a number of positions within education and politics, returning to England on a number of occasions, sometimes for extended periods (Table 8). In 1873 he was elected to a Lectureship at the University of Melbourne and, from 1875, the Headmastership at the Presbyterian Ladies' College, Melbourne. Although he was primarily a man of letters, he showed a practical ability in public affairs. In 1877 he was appointed by the minister of education to inquire into, and report on, the best and most economical way to establish free education in Victoria. He drew up an exhaustive report, issued in the spring of 1878, advocating several changes to the education system. In 1886 he was appointed minister of education of Victoria where he sought to implement his reforms. Charles Pearson was renowned for his gentle manner and for having scrupulous respect for the traditional rules and courtesies of public debate. He had strong convictions, which he stated courageously, but always sacrificed his personal interests to do what he believed to be right.[36]

Eleanor Smith maintained her written correspondence with Charles Pearson for almost 20 years. They shared a common interest in women's education and the education of the poor and underprivileged. Many of her letters begin with an update on the wellbeing of her brother – her sentiments always being that of contentment when Henry was quietly working on his mathematics and in good health. She wrote of the latest news in the lives of their mutual friends in Oxford, of which they had many, followed by the hope that he may make a return visit to England. Her letters range from describing her joy and excitement at moving to the Keeper's House of the University Museum, in October 1874, to her sadness of Henry Smith's final illness in February 1883.

In 1874 Henry Smith was appointed Keeper of the Oxford University Museum, a post he was to hold until his death, and he and his sister moved into the Keeper's house beside the museum (Figure 16). In a letter to Charles Pearson, dated 29th October 1874, Eleanor Smith wrote that:

36 See the entry for Charles Henry Pearson in the *Oxford Dictionary of National Biography* (Tregenza, 2004).

> Henry took the house and the office at the Museum in order to be near the croquet ground. The office was conferred on him about the end of June and at the end of July we began the preparations for getting the new house into order. Henry has established himself on the top floors which has three rooms – the attractions being the large comfortable study with a beautiful view towards Shotover. There is a nice yard and plenty of outhouses and a bit of garden which Henry will govern entirely and insists on having vegetables in it. I am happy to say that the cats bore the transition better than might have been expected and, were it not for the strange workmen who congregate about the house, they would be very happy. As it is they have found lofts and crannies to which they retreat whenever they hear a strange footstep. Henry has instituted a great reform in his habits. He gets up when he is called and goes out for a walk before breakfast and as a consequence goes to bed much earlier – all of which will I hope tell very beneficially on him.[37]

In this letter she describes the layout of the house and the renovations which were taking place.

> On the first floor the drawing room and spare bedroom fill the front of the house and my bedroom and dressing room, a spare dressing room and large room for the servants [*three*] fill the back of that floor. Below a dining room and morning room lie to the left and right of the hall and fill the front of the house. At the back a ground floor spare room which the Phillips used as a 'back study' and the kitchens and utilities fill the space.[38] The windows have to come out in succession and serval of the grates, but by the time all this is done, which will be by the end of April, I should think/I believe, we shall like the house very much. Even now, though my rooms are not papered and my grate has to be taken out, I feel I shall take to my new quarters kindly.

In this house they could entertain their many friends and Oxford visitors. Charles Pearson and his wife Edith were guests of Henry Smith and his sister, for a number of days in 1879, while on a visit to England from Australia (Pearson and Stebbing, 1900, p. 217). On a visit to Oxford in 1876 Ferdinand von Lindemann described their home as 'a country house in a large park – an

37 Oxford, Bodleian Libraries, Charles Pearson Collection, MSS. Eng. lett. d. 191, fols. 1–4.

38 John Phillips FRS (1800–1874) was the first Keeper of the Oxford University Museum from 1857 to 1874 and resided at the Keeper's house at the museum. In 1902, the office of Keeper was abolished, replaced by a newly established position of Secretary to the Delegates of the University Museum. From 1921, the Keeper's House was used to meet the extra needs of the science departments before being demolished in 1952 to provide space for an extension to the Inorganic Chemistry Laboratory. See https://web.prm.ox.ac.uk/sma/index.html

Figure 16: Keeper's House of the Oxford University Museum

official residence which he was not entitled to as a professor, but, as he told me, for the purely formal supervision of the scientific collections of the university' (Verholzer, 1971, pp. 64–65)[39]

Eleanor's letters to Charles Pearson give an indication of the many trips taken by her and her bother to the Continent to improve their general health. Her letters almost always included an update on her own health and a detailed account of the events in the lives of their mutual friends at Oxford. From the same letter, dated October 29th 1874, she wrote that:

> I got a great deal of benefit by my visit to Aix [*Aix-les-Bains, France*] and though far from as nimble as I wish – I hope by care to get through the winter without being crippled with rheumatism, as during the summer months, when I had to help myself up and downstairs with my hands. Your last letter which reached me when at Aix was very comforting as it told of your greatly improved health. I trust that this amendment was of a durable kind and that in other respects your position may not so improve as to induce you to wish to remain in it! We want you back again very much.

In a letter, dated 1st September 1876, Eleanor wrote that:

> Henry has been at home all summer having had his time at Nice and Naples in the winter. We have both very much enjoyed this house and its air and quite drawing the intense heat of this summer. With all it coolness we have not always been able to get food in an eatable condition but at least been able to breathe which was more than many could. Henry has got a good deal of quiet working time and I hope got forward with some of his various papers – but of

39 Courtesy of Professor Emeritus June Barrow-Green, Faculty of Science, Technology, Engineering & Mathematics, Open University.

> course his various governing bodies and the Museum have often broken in with claims – still as mathematics cannot be worked too continuously, it matters less.[40]

Eleanor was accurate in her assessment of her brother's progress on various mathematical papers for a quarter of all the papers in his *Collected Mathematical Papers* were published between June 1875 and February 1877. Although some of these papers were short works, perhaps written at the request of James Glaisher for his *Messenger of Mathematics*, there were also more substantial works, including his 1875 paper *On the integration of Discontinuous Functions* (Smith, 1874). The change of residence and habits did indeed appear to have a beneficial effect on Smith. From these letters a reader can identify that there is a sense of contentment and harmony in both their lives, with just the occasional concern expressed of the consequences of overworking on his health.

Smith's work for the university was tireless and his active involvement in the many offices and committees meant that he often had to sacrifice the time given to mathematics. In a letter, dated 16th July 1880, Eleanor begins to express to Charles Pearson her concern for her brother as a consequence of these commitments.

> Now that I have but few minutes to write and I must give you a brief summary of events, happily few, for the last 12 months. My brother has I think him in better health for the last year than the previous one – he has had his normal pressure of work. The Liberals, by withdrawing Lord Selbourne from all acting parts in the work of the Commission, he has incurred the responsibility and in many ways the labour of other members. He has not had a holiday and I didn't hear of any definite one in contemplation.[41]

In a letter, dated February 23rd 1883, Eleanor informed Charles Person of the death of her brother and, in a moving account to her dear friend, she gave the details of Henry's final illness. She wrote that 'there were some premonitory symptoms – some previous ailments not much regarded by him or me'. On February 1st Smith returned to Oxford from London complaining of abdominal pains during the two days he was there. The following day he had a busy day of lectures and meetings followed by a meeting at the Oxford Town Hall that same evening. Smith addressed this meeting to support a resolution by Joseph Arch (1826–1919) in favour of giving the franchise to the agricultural labourer. This was to be his last public appearance.[42] Eleanor recalls that:

40 Oxford, Bodleian Libraries, Charles Pearson Collection, MSS. Eng. lett. d. 191, fols. 5–7.
41 Oxford, Bodleian Libraries, Charles Pearson Collection, MSS. Eng. lett. d. 191, fols. 11–15.
42 Joseph Arch (1826–1919) was an English trade unionist and politician, who played a key role in unionising agricultural workers and in championing their welfare.

> When he returned from the meeting after 10 he came to my room. I was laid up with a cold, and he sat for an hour, happy and apparently well, talking a good deal about ? and enjoying his tea and toast. When he left me he said he must be made to get up in good time to be at Corpus Christi College by 9.[43]

The following morning he remained in his bedroom complaining of a migraine and was attended to by Dr. Acland that evening. In this letter Eleanor describes in detail the days that followed. On Monday and Tuesday a slight improvement in his health gave way to to a steady decline. The physician Sir William Gull (1816–1890), who had been telegraphed for and who arrived on Thursday evening, held out little hope.[44] On Friday morning (9th February) 'he passed quietly away without pomp or struggle'. Eleanor was later informed that her brother had died as a result of a 'large abscess at the back of his liver which could not be detected in life'. In concluding this letter she wrote that:

> The whole wonder was not that he died but that he had lived. He had taken so much out of himself and with such unfailing devotion and gaiety. If honours come to him [*it will not do him*] any good now – he has it all gained. I'll send you my Oxford papers, if you care for it.

The funeral service for Henry Smith took place on Tuesday February 13th at St. Paul's Church, Walton Street, Oxford. A newspaper article reported that at twelve o'clock friends and dignitaries assembled in the University Museum.[45] The article lists the names of those in attendance with the chief mourners reported as Miss Eleanor Smith, Dr. Acland, and the President of Corpus Christi College. In attendance from Cambridge University were Professor Arthur Cayley and Professor George Gabriel Stokes. The Royal Society was represented by its President, William H. Spottiswoode.[46] Shortly after twelve o'clock a procession of almost 300 people filed out of the museum and walked along Parks Street, Broad Street, St. Giles's, Little Clarendon Street, and Walton Street to St. Paul's Church, where the first portion of the funeral service was performed. Reverend Benjamin Jowett, then Vice-Chancellor of the University, officiated at the church service and burial at St. Sepulchre's cemetery (Figure 17).

43 Oxford, Bodleian Libraries, Charles Pearson Collection, MSS. Eng. lett. d. 191, fols. 16–20.
44 Sir William Gull (1816–1890) was one of the Physicians-in-Ordinary to Queen Victoria.
45 A detailed account of Henry Smith's funeral, including a list of the dignitaries present and the route of the funeral procession, is reported in the *The Oxford Weekly Supplement*, Saturday February 17th, 1883.
See https://www.stsepulchres.org.uk/burials/smith_henry_john.html.
46 *ibid.*

Figure 17: Inscription: MELIORA LATENT [*Better Things are waiting*] HENRY JOHN STEPHEN SMITH, Savilian Professor of Geometry, was born at Dublin Nov. 2nd 1826 & died at Oxford Feb. 9th 1883. O LORD GOD THOU KNOWEST. Also the sister of the above ELEANOR ELIZABETH SMITH, born at Dublin Sep. 30th 1822, died at Oxford Sep. 15th 1896. O LORD WHAT WAIT FOR, TRULY MY HOPE IS IN THEE.

[Photograph the author's own, taken November, 2019].

Soon afterwards Eleanor moved from the Keeper's house to a residence at 27 Banbury Road, Oxford. She continued to mourn the loss of her brother in the years that followed. Sarah Angelina Acland wrote to Charles Pearson in 1891 to report: 'I am afraid that Miss Smith shall quite break down - perhaps best to'.[47] Despite her obvious grief, Eleanor continued with her many interests in particular the establishment, from charitable donations, of the Sarah Acland Home for Nurses. The home was opened in 1882 in memory of Sarah Acland (1815–1878), wife of Sir Henry Acland, and initially located at 37 Wellington Square, Oxford. In 1897 the home moved location to 25 Banbury Road, Oxford, adjacent to Eleanor Smith's residence. Eleanor also continued her interest in travel. In 1888 she visited Ireland and in a letter to Charles Pearson, dated September 5th 1888, Eleanor wrote that:

> Galway, The Green Island – I am having a brief time in Ireland. I see little changes though till now (excepting Dublin) I have been over new ground – but the character of Donegal, Sligo, Mayo, Galway are much what I knew in the south. In the south 8 or 9 years

47 Oxford, Bodleian Libraries, Charles Pearson Collection, MSS. Eng. lett. d. 191, fols. 48–49.

> back I thought there was change for the better in cabins and clothing – now it seems status quo. The law commission are sitting here and I have spent some time in court today watching the process of extracting an accurate statement of fact from the Irish peasant – rival solicitors and valuators or surveyors adding to the confusion.[48]

In this letter she wrote of meeting James Fitzjames Stephen (1829–1894) at Westport, County Mayo, and 'taking a view of the West in [the] company of a bright young daughter Rosamond, his youngest'.[49]

Despite Eleanor's many friends in Oxford it was to Charles Pearson she turned to for help in preparing a biographical sketch of her brother, for inclusion as part his collected works. Eleanor provided him with a complete memoir of details and in 1888 Charles Pearson completed the task. He also assisted her in carefully reviewing the recollections provided by the various authors for inclusion in the collected works (Section 2.6). As a result of this collaboration many of Eleanor's letters may be found among the correspondence and papers of Charles Henry Pearson.[50] Mrs Edith Pearson (1852-1933) recalls her husband committing to the task of writing a biographical sketch of Henry Smith:

> Above all, he had lost, in February 1883, [*three years before the death of the Judge*], one who had been little less than a brother to him in soul, Professor Henry J. Smith, of Oxford.[51] At the request of Miss Smith, Pearson penned a sketch of her brother in 1888. Mrs. Pearson recollects how he devoted evening after evening to the composition, which he regarded as a labour of love. He would allow no call of pleasure to interfere with this sacred duty of friendship. The work was to have been printed within a twelvemonth at farthest. It did not appear till six years later, in 1894, eleven years after Professor Smith's death, and several months after that of its author, though in the same year (Pearson and Stebbing, 1900, pp. 297–298).

In the introduction to this biographical sketch Pearson suggested that the original plan for the memoir was that it would be supplemented by the publication of a large number of Smith's letters. This he stated 'was over-ruled in Oxford while I was in Australia, and cannot now be reverted to' (Smith, 1894a, p. ix). The first twelve of these letters, along with Pearson's biograph-

48 Oxford, Bodleian Libraries, Charles Pearson Collection, MSS. Eng. lett. d. 191, fols. 28–30.
49 James Fitzjames Stephen (1829–1894) was an English lawyer, judge, writer, and philosopher. He was a mutual friend of Eleanor Smith and Charles Pearson.
50 https://archives.bodleian.ox.ac.uk/repositories/2/resources/9223
51 Charles Pearson's brother Sir John Pearson (1819–1886) studied at Gonville and Caius College, Cambridge. He was a judge of the High Court of Justice, Queen's Bench Division, 1882.

ical sketch of Smith, were published in 1894 (Pearson, Glaisher, and Smith, 1894).[52] Charles Pearson died on May 29th 1894.

2.8 INTERNATIONAL RELATIONS

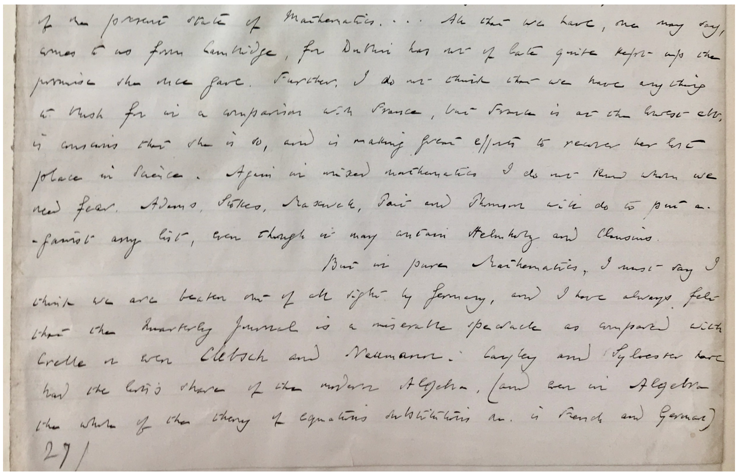
Henry Smith traveled extensively in Europe during his lifetime, maintaining an interest in languages and travel, following his extended convalesce in Europe as a young student. He visited, among other countries, Spain (Madrid & Valencia) in 1864, France in 1865 (Paris) & 1868 & 1870 (Paris), Italy in 1870 (Milan), Greece in 1871 (Athens) and Scandinavia in 1873 (Christiania). By visiting these countries he could readily form connections with leading European mathematicians and learn of their mathematical advances. By inviting European mathematicians to attend meetings in Britain, such as the annual meeting of the BAAS, new collaborations could be formed. He knew of the importance of such endeavours and gave an honest opinion on his views in a letter to his friend Isaac Todhunter FRS (1820–1884) of St John's College, Cambridge (Figure 18).[53]

> All that we have, one may say, comes to us from Cambridge; for Dublin has not of late quite kept up the promise she once gave. Further, I do not think that we have anything to blush for in comparison to France; but France is at a low ebb, is conscious that she is so, and is making great efforts to recover her lost place in Science.
>
> But in Pure Mathematics, I must say that I think we are beaten out of sight by Germany; and I have always felt that the *Quarterly Journal* is a miserable spectacle, as compared with Crelle, or even Clebsch and Newmann [the journal *Mathematische Annalen*]. Cayley and Sylvester have had the lion's share of the modern Algebra (but even in Algebra the whole of the modern theory of equations, substitutions, &c., is French and German). But what has England done in Pure Geometry, in the Theory of Numbers, in the Integral Calculus? What a trifle the symbolic methods, which have been developed in England are, compared with such work as that of Riemann and Weierstrass.[54]

52 In this collection there are a further thirty letters from Henry Smith to his sister from the years 1852 to 1874. These include letters from Henry Smith's visits to Italy and Spain during this time. See https://archives.bodleian.ox.ac.uk/repositories/2/resources/9223

53 Issac Todhunter FRS (1820–1884) was an English mathematician who is best known today for the books he wrote on mathematics and its history. He was a Fellow of St John's College, Cambridge.

54 Henry Smith's comments on British pure mathematics during the 19th century, when compared with European advancements, are reaffirmed in (Gray, 2011, pp. 178–185).



of the present state of Mathematics. . . All that we have, one may say, comes to us from Cambridge, for Dublin has not of late quite kept up the promise she once gave. Further, I do not think that we have anything to blush for in a comparison with France, but France is at the lowest ebb, is conscious that she is so, and is making great efforts to recover her lost place in Science. Again in mixed mathematics I do not know whom we need fear. Adams, Stokes, Maxwell, Tait and Thomson will do to put a-gainst any list, even though it may contain Helmholtz and Clausius.

But in pure Mathematics, I must say I think we are beaten out of all sight by Germany, and I have always felt that the Quarterly Journal is a miserable spectacle as compared with Crelle or even Clebsch and Neumann. Cayley and Sylvester have had the lion's share of the modern Algebra, (and even in Algebra the whole of the theory of equations substitutions &c. is French and German)

27/

Figure 18: Letter (Extract) from Henry Smith to Issac Todhunter, 1870's [Courtesy of the Bodleian Libraries, The University of Oxford, Charles Pearson Collection, MSS. Eng. misc. b. 74, fols. 96–97.]

Henry Smith was a foreign member of the Prussian Academy of Science and, as the recipient of major prizes awarded by the Berlin Academy (1868) and Paris Academy (1882), he brought an international recognition to Oxford mathematics.[55] He was active in establishing a wide association with European mathematicians. In 1865, during the election of the Lee's Professor of Experimental Philosophy, Smith tried to encourage the German physicist and physician Hermann von Helmholtz (1821–1894) to the Chair. Helmholtz mentioned learning of the plan from Smith when they met for lunch, with the French mathematician Charles Hermite, in Paris in April 1865 (Kurti, 1984). Henry Smith visited Sophus Lie (1842–1899) in [*Christiania*] Norway, most likely in 1873 while on a visit to Scandinavia with his friend Mountstuart Grant Duff FRS (Grant Duff, 1899, pp. 111-112).

Smith invited both Charles Hermite and the Felix Klein to participate in the 43rd annual meeting of the BAAS held in Bradford, England in September 1873. Charles Hermite presented a paper titled: *Sur I'irrationalité de la base des Logarithmes Hyperboliques* (Science, 1874, pp. 22–23). At this stage Smith was president of (Section A) Mathematics and Physics and gave a lecture *On Modular Equations*, a subject that would become a central research topic for Felix Klein. Two sectional reports were presented during the meeting: *Improving the Methods of Instruction in Elementary Geometry* and a comprehensive report *On Mathematical Tables* (Science, 1874, pp. 459–460 and pp. 1–174). The authors of these reports where the leading British and Irish mathematicians of

55 In 1866 the Academy of Sciences in Berlin posed a geometrical problem on the intersection of quartic curves. Henry Smith solved this, and in 1868 was the joint winner of the Academy's Steiner prize, with Hermann Kortum (1836–1904), a professor at Bonn. Henry Smith was elected a corresponding member of the Royal Prussian Academy of Sciences on April 15th, 1880. See https://www.bbaw.de/en/the-academy/history-of-the-academy/past-members

the time. Felix Klein wrote to Sophus Lie in November 1873 to report on his first visit to England.

> I have to tell you about England! Indeed, this is terribly difficult to do in a few words. Cayley is an extraordinary friendly man who takes an interest in everything that is presented to him. Sylvester is entirely different. When he has something on his mind, he tells everyone about it and is briefly but entirely absorbed by the topic. I wished that he worked more steadily. There is no doubt that he is more brilliant than Cayley, and everyone in London is generally of the same opinion. One of the finest, incidentally, is [*Henry*] Stephen Smith, who visited you in Christiania. I wished that I could have spent more time with him (Tobies, 2019, pp. 148).

While in Bradford Felix Klein was made a corresponding member of the BAAS (Science, 1874, p.81). In 1875 he, along with Leopold Kronecker and Hieronymus Georg Zeuthen (1839–1920) were appointed as Honorary Foreign Members of the LMS. The president of the LMS that year was Henry Smith.[56] In 1881 Henry Smith would send his student Arthur Buchheim to study with Felix Klein at Leipzig, Germany (Tobies, 2019, p. 150).

Ferdinand von Lindemann was a guest of Smith and his sister in 1876. An account of this visit is given by Irmgard Verholzer (neé Balser), a granddaughter of Lindemann, who edited his memoirs in 1971. Smith invited him, during his visit, to celebrate the Universities Commemoration Day. A formal procession of the professors in their gowns at Balliol College was followed by lunch in the hall of Christ Church College. Lindemann recalls that 'in the evening there was a dinner with ladies in the hall of Balliol College. After the soup, Professor Smith rose to a humorous address: 'We have a German among us today and he has, I've been told, been to a pub this afternoon to quench his thirst, we should therefore give him a glass of beer first' – and so it happened to general applause, but the beer was not good' (Verholzer, 1971, pp. 64–65).

2.9 PRESIDENTIAL ADDRESS TO THE LONDON MATHEMATICAL SOCIETY 1876

Henry Smith was president of the LMS from 1874 to 1876 and in November of that year he delivered his presidential address to the Society. The address titled *'On the present state and prospects of some branches of pure mathematics'* was published in the *Proceedings of the London Mathematical Society* (Smith, 1876) and (Smith, 1894b, pp. 166–190). In this address Smith discussed interesting questions and new ideas for mathematicians to study. The warm style of his

56 For an article on the early history of the LMS see (Rice and Wilson, 1998, pp. 185–217).

narration was matched with his gracious acknowledgment of the contributions of English mathematicians in their chosen fields of study.

> I have been led to believe that the Society may not be unwilling to allow a certain latitude in the scope of the remarks which they permit their Presidents to address to them upon retiring from the Chair. Relying upon this belief, I propose, on the present occasion, to invite your attention to some considerations relating to the present state of Mathematical Science, with especial reference to its cultivation in this country, and to our own position as representing a great number of those who are interested in its advancement. The subject is so extensive that I am sure you will excuse me if I endeavour to limit it in every way I can. I propose, therefore, to exclude from what I have to say all that relates to Applied Mathematics, and to ask you to confine your attention to questions of Pure Mathematics only (*ibid*, p. 166).

His address spoke of the vast increase in knowledge within the mathematical sciences in England during the 19th century. He highlights, by way of example, how these new developments increase the number and variety of new mathematical objects of interest to the 'rising generation of English mathematicians' which in turn increase the opportunity of discovering new truths. Smith's knowledge of mathematical developments on the Continent was very evident from his address, as he placed British pure mathematics within a European context (Section 2.8). However, rather than any critical comparison, he chose to acknowledge the strengths within British mathematics while gently encouraging young mathematicians into new endeavours. This encouragement was given throughout his address by his choice of examples taken from Continental mathematics. However, privately, Smith was concerned for the standing of British pure mathematics when compared to mathematical developments on the Continent. He wrote to his friend Issac Todhunter (1820–1884) in 1872 expressing the concern that 'in pure mathematics I must say that I think we are beaten out of sight by Germany' (Figure 18). In his opinion the published memoirs of English mathematicians did not present anything new and, instead of considering recognised difficult problems where real advances could be made, young British mathematicians were concerned with less important problems. He wrote that the German mathematician 'takes care to know what is known before he begins to work, and besides generally takes care to work at some really important problem'. These and other sentiments expressed in this letter suggest that this was a great concern for Smith. It is interesting to note that modern historians of mathematics express the same sentiments when reflecting on British pure mathematics during the 19th century.[57]

57 For an overview of the work of British pure mathematicians during the 19th century see (Gray, 2011, pp. 178–185).

Smith's presidential address was directed to what he considered to be the 'neglected regions' of mathematics and foremost among them he selected the theory of numbers.

> Of all branches of mathematical enquiry this is the most remote from practical applications; and yet, more perhaps than any other, it has kindled an extraordinary enthusiasm in the minds of some of the greatest mathematicians. We have the examples of Fermat, Euler, Lagrange, Legendre, and Gauss, of Cauchy, Jacobi, Lejeune Dirichlet, and Eisenstein, without mentioning the names of others who have passed away, and of a few who are still living. But, somehow, the practical genius of the English mathematician has in general given a different direction to his pursuits; and it would sometimes seem as if we in England measured the importance of the subject by what we find of it in our text–books of Algebra, or as if we regarded its enquiries as problems of mere curiosity, without a wider scope, and without direct bearing on other branches of mathematics. I might endeavour to remove this impression – if indeed it exists in the minds of any of those who hear me – by enumerating instances in which the advancement of Algebra and of the Integral Calculus appears to depend on the progress of the arithmetic of whole numbers. But, instead of wearying you with the details which would be necessary to make such an enumeration intelligible, I would rather ask you to listen to what is recorded of the most eminent master of this branch of science. Gauss, we are told by his biographer, 'held Mathematics to be the Queen of the Sciences, and Arithmetic to be the Queen of Mathematics' (*ibid*, pp. 168–169).

He continues to quote from Gauss and Jacobi as to the interesting truths, charms and attraction to be found within the theory of numbers, reflecting his own personal interest in the subject. He appealed to young English mathematicians, who may feel an instinctive liking for arithmetical enquiry, to be encouraged by an observation of Jacobi, and recorded in his brief notice of the life of Göpel, that 'many of those who have a natural turn for mathematical speculation find themselves, in the first instance, attracted to the theory of numbers' (Jacobi, 1847, p. 313). Smith was concerned that, in England, progress in theory of numbers had not maintained the same pace as progress in algebra. 'It is worthy of remembrance that some of the most fruitful conceptions of modern algebra had their origins in arithmetic, and not in geometry or even in the theory of equations' (Smith, 1894a, p. 170). He illustrates the interplay between these mathematical disciplines using a number of examples from the work of Continental mathematicians such as Gauss, Eisenstein, Dirichlet and Hermite.

In his combined *Reports on the Theory of Numbers* a decade earlier, Smith divided arithmetic into two sections: the theory of homogeneous forms (quadratic forms) and the theory of congruences. His presidential address maintained these headings while adding a further one: the determination of the mean or asymptotic values of arithmetical functions which he directed to 'some of the young mathematicians of this country'. In general remarks on the arithmetical theory of quadratic forms he expressed concern that progress in modern geometry and modern algebra had far outstripped progress in arithmetic. The 'great problem' as he saw it at that time was the need for arithmeticians to try and advance the subject with new results. He briefly mentions some instances which served to illustrate the actual position of arithmetical enquiry. He first spoke of problems which had been completely solved, at that time, such as the arithmetical formula which gives the automorphics of a binary quadratic form to the problem of equivalence of two definite or indefinite ternary quadratic forms. Acknowledging the contributions of Hermite, Cayley, Bachmann, Seeber, Selling and Eisenstein, Smith was satisfied that further developments by 'these distinguished authors' would advance the arithmetical theory of quadratic forms even further (*ibid*, p. 171).

> So far, then, as binary and ternary quadratic forms are concerned, we have not much reason to complain of the slowness of the advances made by arithmetic. But if we pass to quadratic forms of four or more indeterminates, we shall find that the limits within which our arithmetical knowledge is confined are indeed restricted (*ibid*, p. 171).

Smith reminded his audience that the characteristic properties of an invariant, and of a contravariant, appeared for the first time in the *Disquisitiones Arithmeticae* and, in doing so, Gauss had brought the study of quadratic forms, of any order and of any number of indeterminates, to the attention of mathematicians. He believed, at that time, that the criterion to decide the resolubility in the integral numbers of an indeterminate equation lay, not in geometry but, in the definition of the generic characters of ternary quadratic forms, first given by Eisenstein in 1847 (Eisenstein, 1847a, p. 147). In the published version of this presidential address a footnote refers to Smith's own publication on this problem of resolubility published in the Proceedings of the Royal Society of 1864 (Smith, 1864a). This short two page notice was Smith's first reference to a ternary quadratic form, its contravariant and their generic characters and the starting point of his further contributions to the arithmetical theory of quadratic forms. On the theory of elliptic functions, which became the focus of his own research in the years that followed, he was very encouraged.

> If I had had the honour of addressing the Mathematical Society ten years ago, I think I should have had to complain of the neglect in England of the study of elliptic functions. But I cannot

> do so now. The University of Cambridge has given this subject a place in its Mathematical Tripos, the University of London in its examination for the Doctorate of Science. The British Association has supplied the funds requisite to defray the cost of printing *Tables of the Theta Function*; tables of which the mathematicians of this country may justly be proud, and which will form an enduring memorial of the great ability and indefatigable industry of our colleague, Mr. Glaisher. We further owe to Professor Cayley an introductory treatise on elliptic functions, the first which has appeared in our language. I consider that the service which he has thus rendered to students is an important one, and one for which we ought to be very grateful (Smith, 1894a, p. 187).

His wide ranging presidential address to the society continued with further thoughts on other aspects of pure mathematics with supporting examples. He apologised for the 'fragmentary and disconnected' nature of his reflections claiming that 'over so wide a field I could only take a wandering course' (*ibid*, p. 189). However, he succeeded in making the transition from arithmetic of the whole numbers to some other branches of analysis in a seamless way. He admitted that most of his address was focused on arithmetical topics but he ventured to glance at some topics on which he wished he had more time to speak. These included integral calculus and the theory of ordinary differential equations. At every opportunity he celebrated the achievements within British mathematics, while rarely drawing any attention to his own achievements.

2.10 CONCLUSION

Henry Smith's life in mathematics was enriched by his many friends. They gave testimony of his intellectual power and unique personality which ensured that he was held in widespread affection at Oxford. The personal letters and recollections throughout this chapter reveal that Smith's life at Oxford was a very happy and fulfilled one. The role played by his sister Eleanor was a very important one in achieving this, as she maintain a family unit throughout their lives. Her personal letters reveal that she was proud of her brother and cared for his wellbeing. The home they shared together in Oxford, from 1874, ensured that they maintained a comfortable and social lifestyle, where they could entertain guests, and where Smith could work on his mathematics.

Henry Smith established himself early in his career as a superb lecturer and researcher. As holder of Oxford's most prestigious mathematical chair, he sought to make significant contributions to establish a mathematical culture as prevalent as that which existed at Cambridge. His most important contributions to mathematics was in the theory of numbers. His expertise in the subject developed during his writing of the six part *Report on the Theory of Numbers* between 1859 and 1865. His survey of the theory began with the investigations of Fermat, Legendre, and Gauss. He wrote on Jacobi's work on elliptic functions, Kummer's theory of ideal numbers, and the latest results of Hermite and Kronecker. This work gave Smith, early in his career, an exceptional wide knowledge of the subject. His subsequently published a number of memoirs in the theory of numbers with his final memoir on the subject being awarded the *Grand Prix des Sciences Mathématiques* in 1882 (Chapter 6). This award brought him, and Oxford mathematics, international recognition.

Recent historians of mathematics suggest that Smith suffered from a lack of recognition in his day. The first reason given for this was his long association, of almost forty years, with Oxford University. During the late 19th century, students of mathematics would naturally gravitate towards Cambridge and consequently Smith could not surround himself with close mathematical confidants. It would seem that his closest friend in mathematics was James Glaisher of Cambridge. An endeavour to form a research school in mathematics at Oxford would certainly have given greater exposure to Smith's mathematical achievements. Despite this, his active involvement in scientific societies would ensure that he could interact with the leading mathematicians of his generation. Given that mathematics at Oxford did not have the same dominance over a course of study, Smith found the freedom to absorb and stay in tune with Continental mathematical advances to an extend that was unmatched in Britain. He did so by maintaining relations with Continental mathematicians throughout his life. His comparative isolation at Oxford meant that he could flourish mathematically. If Smith suffered a lack of recognition in his day, by remaining at Oxford, the benefits of him doing so certainly enriched his life in mathematics.

Henry Smith's mathematical friends acknowledge that he did not publish the amount of mathematical papers of which he was certainly capable. Historians refer to this fact as a second reason why Smith was not as influential as he might have been. This, they suggest, was due to the amount of time he devoted to the administrative duties associated with the many positions and offices he held. Spottiswoode, however, believed that Smith's administrative duties did not adversely affect the quality of his mathematical output, and that the time he devoted to mathematics were on topics of real importance. James Glaisher also believed that Smith's devotion to the subject he loved would not have endured without the balance and variety of the life he lived. However, James Glaisher reminds us that Smith himself was anxious by the amount of unpublished work which remained in his notebooks, and how he sometimes conducted his mathematical research late at night, when the day's work would otherwise have been over. Personal letters also reveal that Eleanor Smith was, in later years, increasingly concerned about the consequences of administrative duties and overwork on her brother's health. Among his closest friends Smith had mathematical anonymity and perhaps, as a consequence, more demands were placed on his social gifts and powers of conciliation. His generosity of spirit suggest that he may have found it difficult to balance his mathematical ambition with his sense of public duty.

This chapter ends with Henry Smith's presidential address to the LMS in 1876. His address spoke of the vast increase in knowledge within the mathematical sciences in England during the 19th century. The address was an important one in the history of the Society because of the broad range of mathematical topics considered, and for his remarks on mathematical advances on the Continent. He was not publicly critical when making comparisons between mathematics in Britain and Europe, but chose to acknowledge the strengths within British mathematics while gently encouraging young mathematicians into new endeavours. He focused on what he termed the 'neglected regions' of mathematics and foremost among them he selected the theory of numbers. His characteristic modesty prevented him from outlining his own accomplishment in the subject.

An overview of Smith's contribution to the theory of numbers has been given. He advanced the theory of quadratic forms by a series of memoirs in which he returned to the earlier writings of Gauss, Eisenstein, and Dirichlet, and remained true to arithmetic. Consequently he confirmed his reputation as a brilliant arithmetician who nurtured fine arithmetical details and presentation. In order to consider these memoirs in detail some prerequisite details may are required. It will be helpful to set out clearly what was known of the arithmetical theory of ternary quadratic forms prior to Smith's publications on this subject. In particular the problem of reduction, equivalence and classification of ternary quadratic forms (Chapter 4). It may also be useful to begin with an overview of the same considerations within the theory of binary quadratic forms (Chapter 3).

3 THE CLASSIFICATION OF BINARY QUADRATIC FORMS

The *Disquisitiones Arithmeticae* contains a complete classification of binary quadratic forms (Gauss, 1986, Art. 223–232, pp. 108–291). Peter Gustav Lejeune Dirichlet (1805–1859) published three memoirs, from 1839 to 1840, with the title *Recherches sur diverses applications de l'Analyse infinitésimale à la Théorie des Nombres* (Dirichlet, 1839, 1840b,a). In the first of these memoirs Dirichlet complements Gauss's classification of binary quadratic forms using the symbols of quadratic reciprocity.[1] He also uses tables to display the complete generic characters necessary for classification. In this chapter I will consider the mathematical technique, presentation style, and the use of tables in the classification of binary quadratic forms. In view of Smith's later classification of ternary quadratic forms it will be interesting to comment on his writing on this subject with respect to these earlier contributions by Gauss and Dirichlet.

3.1 INTRODUCTION

In 1773 Joseph Louis Lagrange (1736–1813) laid the foundations for a study of binary quadratic forms by means of his general theory of reduction and equivalence.[2] He discovered that the theory of quadratic forms was complicated by the existence of inequivalent forms with the same determinant. In 1801, Gauss extended the theory of equivalence by introducing many new concepts. Instead of trying to find how many inequivalent forms existed, Gauss preferred to study how the equivalence classes of forms interact with each other algebraically. He

1 In these memoirs Dirichlet also establishes, in an original manner, the number of classes of binary quadratic forms for a given discriminant. Smith expressed his admiration for this particular mathematical achievement, writing that the 'originality of Dirichlet in this celebrated investigation is unquestionable' (Smith, 1894a, p. 208).

2 *Recherces d'Arithmétique*, Nouveaux Mémoires de l'Académie be Berlin, 1773. For an account of Lagrange's theory, placed within its historical context, see (Scharlau and Opolka, 1985, pp. 32–56).

defined an operation on the set of equivalence classes of forms with given determinant called the *composition of classes of forms*.

Henry Smith's *Report on the Theory of Numbers* Part III (1861) contains a brief systematic résumé of the general theory of reduction and equivalence of binary quadratic forms as they appear in the *Disquisitiones Arithmeticae* (Science, 1861, pp. 292–324, Smith, 1894a, pp. 163–207).[3] Part IV (1862) of his report was reserved for the general theory of composition of binary quadratic forms (Science, 1862, pp. 503–526, Smith, 1894a, pp. 229–262). Dirichlet's *Vorlesungen über Zahlentheorie*, published posthumously in 1863, includes simplifications of the *Disquisitiones* with an exceptionally clear synthesis (Dirichlet, 1999, pp. 91–145). These publications, almost sixty years after the *Disquisitiones*, rekindled interest among mathematicians in the theory of numbers. 'The four successive editions of the *Vorlesungen* alone testify to its success during the last decades of the 19th century'.[4]

Dirichlet's *Vorlesungen* and Smith's *Report on the Theory of Numbers* Part III both state clearly, what has since become, the standard steps for the study of quadratic forms. The following is a short extract, detailing these steps, from the *Report on the Theory of Numbers*:

> It remains to speak of the problem of equivalence. Of the three parts of which this problem consists, viz. [1.] to decide whether two given forms are equivalent or not, [2.] if they are, to obtain a single transformation of one form into the other, and [3.] from a single transformation to deduce all the transformations, the last only admits of being treated by a method equally applicable to forms of a positive and negative determinant. We shall therefore consider it first (Smith, 1894a, p. 176).[5]

The theory of the representation of numbers by quadratic forms reduces to these problems. A theory of reduction is important for the classification of binary quadratic forms. Two basic problems are based on the definition of a *reduced form*. Firstly, to construct a system of reduced forms for a given determinant D and, secondly, to decide if two given forms of the same determinant are equivalent. The theory of reduction is applicable to forms of a positive or negative determinant but when the determinant is positive, the reduced forms are not, in general, all non-equivalent. Consequently the problems are much harder to solve for positive determinants than negative.[6]

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3 For an account of binary quadratic forms in section V of the *Disquisitiones Arithmeticae* see (Goldstein, Schappacher, and Schwermer, 2007, pp. 8–13).

4 For an account of the simplifications introduced in the *Vorlesungen*, with respect to Gauss's *Disquisitiones*, and the influence of the *Vorlesungen* on late 19th century mathematicians, see (Goldstein, 2005, pp. 480–490).

5 For the corresponding remark in the *Vorlesungen* see (Dirichlet, 1999, p. 100).

6 Short extracts from the *Report on the Theory of Numbers* Part III, on the theory of reduction and equivalence for binary quadratic forms, are presented with illustrative examples as (Appendix A).

Article 85 (*ibid*, p. 170)
A binary quadratic form f may be represented as[7]

$$f = ax^2 + 2bxy + cy^2$$

where f is termed *primitive* (i.e. that the three integers a, b, c admit of no common divisor other than unity), and that its *discriminant* is different from zero. This *discriminant*, or the determinant of the matrix

$$-\begin{vmatrix} a & b \\ b & c \end{vmatrix}$$

is represented by D. A primitive form f is *properly primitive* when at least one of the coefficients a, c is odd; it is *improperly primitive* when those coefficients are both even. In an *improperly primitive* form b is odd or the form would not be primitive. The binary quadratic form f_1 becomes a new binary quadratic form f_2 when new variables are introduced. If x, y are the variables for the form f_1 and letting

$$x = \alpha X + \beta Y$$
$$y = \alpha' X + \beta' Y$$

where $\alpha, \beta, \alpha', \beta'$ are four particular integers and X, Y are the new variables. Forms f_1 and f_2 are said to be *equivalent* when one may be transformed into the other by a linear transformation of determinant unity i.e.

$$\begin{vmatrix} \alpha & \beta \\ \alpha' & \beta' \end{vmatrix} = \pm 1$$

Forms are *properly equivalent* if the determinant of this transformation is $+1$. Forms are *improperly equivalent* if the determinant is -1. We only consider *proper equivalence* of *positive definite forms*. A *positive definite form* is one in which the numbers represented by the form are positive, i.e., the first coefficient a of the form is positive. All equivalent forms are said to constitute a *class*. Equivalent forms represent the same integers and have the same discriminant. However, it is not true that forms of the same discriminant are necessarily equivalent. A *reduced form* is a form representing a class of equivalent forms. To decide whether two given forms of the same discriminant are equivalent, and hence members of the same class, we compare their *reduced forms*. All classes with the same discriminant D and the same greatest common divisor constitute an *order*.

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7 Symbolised by the formula $(a, b, c)(x, y)^2$ or, when it is not necessary to specify the indeterminates, by the simpler formula (a, b, c).

In article 98 of Smith's *Report on the Theory of Numbers* Part III he introduces the distribution of classes into orders and genera, i.e. the classification of quadratic forms.

Article 98 (*ibid*, p. 202)
The classes of forms of any given positive or negative determinant D are divided by Gauss into orders, and the classes belonging to each order into genera. Two classes, represented by the forms (a, b, c) and (a', b', c'), belong to the same order, when the greatest common divisor of a, b, c and $a, 2b, c$ are respectively equal to those of a', b', c' and $a', 2b', c'$. Thus the *properly primitive* classes form an order by themselves, and the *improperly primitive* classes form another order. To obtain the subdivision of orders into genera, it is only necessary to consider the primitive classes; because we can deduce the subdivision of a derived order of classes from the subdivision of the primitive order from which it is derived.[8]

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3.2 CLASSIFICATION OF FORMS BY CARL FRIEDRICH GAUSS

In this section I will review the mathematical technique and presentation style used by Gauss in his classification of binary quadratic forms. I will do so by considering direct transcriptions from article 229 of the *Disquisitiones Arithmeticae* (Gauss, 1986, pp. 221–222). The theorem and demonstrations of this article reveal an important arithmetical technique.[9] This arithmetical technique, along with Gauss's presentation style, supported throughout by simple examples, will be evident in Smith's 1867 classification of ternary quadratic forms (Section 5.2).

Theorem [Article 229] (Gauss, 1986, p. 221)
Let f be a primitive form with determinant D and p a prime number dividing D. The numbers not divisible by p, which can be represented by the form f, are all quadratic residues of p, or they are all quadratic non–residues of p.

8 Furthermore, in case there are two or more reduced, positive, primitive forms f_i of discriminant D, *arithmetical invariants* are required which serve to distinguish the numbers represented by f_1 from those represented by $f_2, f_3, \ldots\ldots\ldots$ Such invariants are called *generic characters*. They will differentiate the numbers represented by the separate f_i in case no two of the f_i belong to the same genus. Therefore forms that belong to the same *genus* represent the same numbers. Two forms belong to the same genus when their *generic characters values* are the same (Dickson, 1929, p. 82).

9 For an article focusing on Carl Friedrich Gauss's *Disquisitiones Arithmeticae* [and on the work of Charles Hermite], in the context of the classification of quadratic forms, see (Goldstein, 2016, pp. 106–112).

DEMONSTRATION Let $f = ax^2 + 2bxy + cy^2$, and n, m be any two numbers not divisible by p which can be represented by the form f, that is

$$n = ax^2 + 2bxy + cy^2$$
$$m = ax'^2 + 2bx'y' + cy'^2$$

Then we will have

$$nm = (ax^2 + 2bxy + cy^2)(ax'^2 + 2bx'y' + cy'^2)$$
$$\Rightarrow \quad nm = x^2 - Dy^2$$

where

$$x = axx' + bxy' + bx'y + cyy'$$
$$y = xy' - x'y.$$

Now nm will be congruent to a square relative to the modulus D and thus also relative to p, i.e. nm will be a quadratic residue of p. It follows therefore that both n, m are quadratic residues of p, or they are both non-residues.

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In the above extract Gauss demonstrates that all numbers, that are represented by a given primitive form (a, b, c), with determinant D, will have a fixed relationship to the individual prime divisors of D (by which they are not divisible). Gauss then suggests that odd numbers that can be represented by (a, b, c) will also have a fixed relationship to the numbers 4 and 8 in certain cases; to 4 whenever D is either $\equiv 0$ or $\equiv 3 \pmod 4$ and to 8 whenever D is $\equiv 0$ or $\equiv 2$ or $\equiv 6 \pmod 8$. If the determinant is divisible by 8 its relationship to the number 4 can be ignored since, in this case, it is already contained in the relationship to 8. Gauss establishes these results in the following extracts also taken from article 229. Gauss's use of simple examples is very evident here (*ibid*, p. 221).

A. When the determinant D of the primitive form f is $\equiv 3 \pmod 4$, all odd numbers representable by the form f will be $\equiv 1 \pmod 4$, or all $\equiv 3 \pmod 4$. For if n, m are two numbers representable by f, the product nm can be reduced to the form $p^2 - Dq^2$ just as we did above. When each of the numbers n, m is odd, one of the numbers p, q is necessarily even, the other odd, and therefore one of the squares p^2, q^2 will be $\equiv 0 \pmod 4$, the other $\equiv 1 \pmod 4$. Thus $nm = p^2 - Dq^2$ must certainly be $\equiv 1 \pmod 4$, and both n, m must be $\equiv 1 \pmod 4$, or both $\equiv 3 \pmod 4$. So, for example, no odd numbers other than those of the form $4n + 1$ can be represented by the form $(10, 3, 17)$.

B. When the determinant D of the primitive form (a, b, c) is $\equiv 2 \pmod 8$, all odd numbers representable by the form (a, b, c) will be either partly $\equiv 1 \pmod 8$ and partly $\equiv 7 \pmod 8$, or partly $\equiv 3 \pmod 8$ and partly $\equiv 5 \pmod 8$. For let us suppose that n, m are two numbers representable by (a, b, c), so the product nm can be reduced to the form $p^2 - Dq^2$.

When therefore both n, m are odd, p must be odd (because D is even) and so $p^2 \equiv 1 (\text{mod } 8)$; q^2 therefore will be $\equiv 0 (\text{mod } 8)$ or $\equiv 1 (\text{mod } 8)$ or $\equiv 4 (\text{mod } 8)$ and Dq^2 will be either $\equiv 0 (\text{mod } 8)$ or $\equiv 2 (\text{mod } 8)$. Thus $nm = p^2 - Dq^2$ will be either $\equiv 1 (\text{mod } 8)$ or $\equiv 7 (\text{mod } 8)$; if therefore n is either $\equiv 1 (\text{mod } 8)$ or $\equiv 7 (\text{mod } 8)$, m will also be either $\equiv 1 (\text{mod } 8)$ or $\equiv 7 (\text{mod } 8)$ and if n is either $\equiv 3 (\text{mod } 8)$ or $\equiv 5 (\text{mod } 8)$, m will also be either $\equiv 3 (\text{mod } 8)$ or $\equiv 5 (\text{mod } 8)$. For example, all odd numbers representable by the form $(3, 1, 5)$ are either $\equiv 3 (\text{mod } 8)$ or $\equiv 5 (\text{mod } 8)$, and no numbers of the form $8n + 1$ or $8n + 7$ can be represented by this form.

C. When the determinant D of the primitive form (a, b, c) is $\equiv 6 (\text{mod } 8)$, all odd numbers representable by the form (a, b, c) will be either only those that are $\equiv 1 (\text{mod } 8)$ and $\equiv 3 (\text{mod } 8)$, or only those that are $\equiv 5 (\text{mod } 8)$ and $\equiv 7 (\text{mod } 8)$. The reader can develop the argument without any trouble. It is exactly like the argument of the preceding part B. Thus, for example, for the form $(5, 1, 7)$, only those odd numbers can be represented which are either $\equiv 5 (\text{mod } 8)$ or $\equiv 7 (\text{mod } 8)$.

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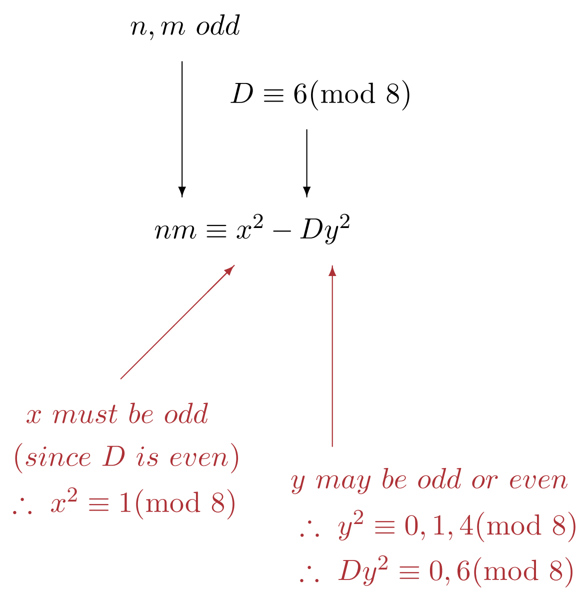
Gauss invites the reader to complete the details above suggesting that one 'can develop the argument without any trouble'. Accepting this invitation will provide an illustration of this common, yet simple, technique, also used by Smith throughout his memoir *On the Orders and Genera of Ternary Quadratic Forms* (Smith, 1868). If we suppose that n, m are two numbers representable by (a, b, c), so the product nm can be reduced to the form $x^2 - Dy^2$.



Thus $nm = x^2 - Dy^2$ will be either $\equiv 1 (\bmod 8)$ or $\equiv 3 (\bmod 8)$. As a consequence, all possible arrangements for n and m are as follows; if n is either $\equiv 1 (\bmod 8)$ or $\equiv 3 (\bmod 8)$, m will also be either $\equiv 1 (\bmod 8)$ or $\equiv 3 (\bmod 8)$ and if n is either $\equiv 5 (\bmod 8)$ or $\equiv 7 (\bmod 8)$, m will also be either $\equiv 5 (\bmod 8)$ or $\equiv 7 (\bmod 8)$. In all cases nm will be either $\equiv 1 (\bmod 8)$ or $\equiv 3 (\bmod 8)$, as established. The following is an example for this case, i.e. when $D \equiv 6 (\bmod 8)$.

EXAMPLE Let $(3, 3, 17)$ denote a primitive binary quadratic form with determinant $D = -42 \equiv 6 (\bmod 8)$. All odd numbers representable by the form (a, b, c) will be either only those that are $\equiv 1 (\bmod 8)$ and $\equiv 3 (\bmod 8)$, or only those that are $\equiv 5 (\bmod 8)$ and $\equiv 7 (\bmod 8)$. In this case, by observation, all odd numbers representable by f are either $\equiv 1 (\bmod 8)$ or $\equiv 3 (\bmod 8)$, and no numbers of the form $8n + 5$ or $8n + 7$ can be represented by this form.

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In article 230, Gauss makes the following points. Firstly he refers to these relationships between numbers as the *generic characters* of the form (a, b, c) and establishes a simple notation for each *particular character*.

ARTICLE 230 (*ibid*, p. 223)
If (a, b, c) is a primitive form with determinant D and p a prime number dividing D, the numbers not divisible by p, which can be represented by the form (a, b, c), agree in that they are all quadratic residues of p, or they are all non–residues. When only quadratic residues of p can be represented by the form (a, b, c), the character Rp is assigned. When only quadratic non–residues of p can be represented by the form (a, b, c), the character Np is assigned. The character $1, 4$ denotes that numbers, represented by the form (a, b, c), are only those that are $\equiv 1 (\bmod 4)$. The characters $3, 4$; $1, 8$; $3, 8$; $5, 8$; $7, 8$ are interpreted in a similar way. Finally, if the form f represents only those odd numbers that are either $\equiv 1 (\bmod 8)$ or $\equiv 7 (\bmod 8)$, then the characters are 1 and $7, 8$. It is immediately obvious what is meant by the character 3 and $5, 8$; 1 and $3, 8$; 5 and $7, 8$.

Secondly, the coefficients a, c of the form (a, b, c) are numbers represented by it, and if the form is properly primitive, one of the coefficients a, c is prime to 2 and to any prime divisor of the determinant. For whenever p is a prime divisor of D, certainly one of the numbers a, c will not be divisible by p. For if both were divisible by p, then p would also divide b^2 $(D + ac)$ and therefore also b, i.e. the form (a, b, c) would not be primitive. Similarly, in those cases where the form (a, b, c) has a fixed relationship to the number 4 or 8, at least one of the numbers a, c will be odd, and we can find the relationship from that number.

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Example (Gauss, 1986, p. 224)
$(7, 0, 23)$ with determinant $D = -161 = -7.23$ has characters

$$R7 \;\; ; \;\; N23 \;\; ; \;\; 3, 4$$

The particular character of the form $(7, 0, 23)$ with respect to the number 23 can be inferred from the number 7 to be $N23$, and the particular character of the same form with respect to the number 7 can be inferred from the number 23 to be $R7$. Finally the particular character of this form with respect to the number 4 can be found from either the number 7 or from the number 23 to be $3, 4$.

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The complex of all particular characters, of a given form or class, Gauss referred to as constituting its *complete* or *generic character*. Those classes which have the same complete character are considered to belong to the same *genus*. The complete character of a form is possessed, not only by every form of the same class, but by every form of any class belonging to the same genus. Hence a subdivision of the whole order of properly primitive classes of a given determinant may be achieved. This final example is also provided by Gauss.

Example (Gauss, 1986, p. 224)
There are 16 properly primitive, reduced, positive binary quadratic forms with determinant $D = -161 = -7.23$. They are distributed into 4 genera in the following way.

The determinant $D = -161$
Four genera; Four classes in each

R7 ; R23 ; 1,4	N7 ; R23 ; 3,4
$(1, 0, 161)$	$(3, 1, 54)$
$(9, 1, 18)$	$(6, -1, 27)$
$(2, 1, 81)$	$(6, 1, 27)$
$(9, -1, 18)$	$(3, -1, 54)$

R7 ; N23 ; 3,4	N7 ; N23 ; 1,4
$(7, 0, 23)$	$(10, 3, 17)$
$(11, -2, 15)$	$(5, 2, 33)$
$(14, 7, 15)$	$(5, -2, 33)$
$(11, 2, 15)$	$(10, -3, 17)$

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3.3 CLASSIFICATION OF FORMS BY GUSTAV LEJEUNE DIRICHLET

In preparing the first part of his *Report on the Theory of Numbers* Part I (1859) Henry Smith received the sad news of Dirichlet's death, and he could not help adding the following footnote to his text appreciating Dirichlet's great service to the theory of numbers:[10]

> The death of this eminent geometer in the present year (May 5, 1859) is an irreparable loss to the science of arithmetic. His original investigations have probably contributed more to its advancement than those of any other writer since the time of Gauss, if, at least, we estimate results rather by their importance than by their number. He has also applied himself (in several of his memoirs) to give an elementary character to arithmetical theories which, as they appear in the work of Gauss, are tedious and obscure; and he has done much to popularise the theory of numbers among mathematicians – a service which is impossible to appreciate too highly (Smith, 1894a, p. 72).

Figure 19: Peter Gustav Lejeune Dirichlet (1805–1859)

Dirichlet's *Vorlesungen über Zahlentheorie*, first published in 1863, was 'one of the most important mathematics books of the 19th century'.[11] It has been described as 'a bridge between Gauss's *Disquisitiones Arithmeticae* (1801) and the development of the theory of algebraic number fields as promoted by David Hilbert in his 1897 *Zahlbericht*' (Goldstein, 2005, p. 480). The German editions of the book were often called the 'Dirichlet–Dedekind' because Dedekind wrote

10 For a biographical essay on Peter Gustav Lejeune Dirichlet see (Elstrodt, 2007, pp. 1–37) and (James, 2002, pp. 103–109).

11 Translators Introduction by John Stillwell (Dirichlet, 1999, p. xi).

up Dirichlet's lecture notes and added supplements to the second and later editions.[12] It is a book of great historical interest as it documents Dirichlet's role as an expositor who made Gauss's *Disquisitiones Arithmeticae* more understandable to a wider audience. The book is an exceptionally clear synthesis of the number theory of his time, from 'absolute fundamentals to the threshold of research'.[13] Analysis and number theory were the two key features of Dirichlet's mathematical writings, and the *Vorlesungen* contains a lucid and thorough treatment of the class number formula for binary quadratic forms.[14] The legendary story is told of how Dirichlet kept a copy of Gauss's *Disquisitiones Arithmeticae* with him at all times and how Dirichlet strove to clarify and simplify Gauss's results. Dedekind, through his footnotes, documents what material Dirichlet took from Gauss, allowing insight into how Dirichlet transformed the ideas into a modern form.

The English translation of the 1st edition of Dirichlet's *Vorlesungen über Zahlentheorie* was published in 1999, and includes nine supplements, incorporating material both from Dirichlet and Dedekind (Dirichlet, 1999). The introduction by Professor John Stillwell, who translated the volume, also includes a historical essay which serves to assist the reading of Dirichlet's book. This historical account outlines, in advance, what the main problems within the theory of numbers were, at the time of Dirichlet's lectures, and what the basic principles were understood to be. Stillwell asserts that Dirichlet was an outstanding guide to the theory of numbers, describing him as 'a modern author who attempted to explain everything from basic arithmetic to L-functions in the same book'.[15]

The *Vorlesungen* would help educate, in the same way the *Disquisitiones* had done a generation earlier, young mathematicians such as Edouard Lucas (1842–1891), George Ballard Mathews (1861–1922) and Hermann Minkowski (1864–1909) suggesting its strong influence across the continent of Europe. For example, around 1880, the German physicist Heinrich Weber (1843–1912) made Dedekind aware of a promising young student, Hermann Minkowski, 'who leaves for the university next year and has worked his way completely on his own in analysis and number theory, which he had studied in the first edition of

12 Three further editions of the *Vorlesungen* were published between 1870 and 1894 along with an Italian (1881) and partial Russian translation (1899) (Goldstein, 2005, p. 480).

13 Translators Introduction by John Stillwell (Dirichlet, 1999, p. xi).

14 The method for the summation of the series

$$\sum \left(\frac{n}{m}\right)\frac{1}{m^2}$$

was first used in 1839 by Dirichlet to determined the number of properly primitive classes of binary quadratic forms for any given determinant (Dirichlet, 1839, 1840b,a). This method is central to Smith's demonstration of the formula for the weight of a genus of ternary quadratic forms first conjectured by Eisenstein in 1847 (Eisenstein, 1847a, pp. 128–129). This demonstration appears in the second half of Smith's 1867 memoir (Smith, 1867) (Section 5.8).

15 *ibid*, p. xi.

your Dirichlet *Vorlesungen*' (Strobl, 1985, p. 144). This involvement would lead Minkowski towards Gauss's *Disquisitiones* and a lifelong interest in quadratic forms and the development of the geometry of numbers.

Hermann Minkowski was also aware of Smith's *Reports on the Theory of Numbers*. In the last decade of the 19th century he and David Hilbert (1862–1943) were asked by the German Mathematical Society (*Deutsche Mathematiker-Vereinigung*) to prepare an updated German version of Smith's report, the *Zahlbericht*. They agreed that Minkowski would prepare subjects such as 'continued fractions, quadratic forms, and the geometry of numbers' while Hilbert would prepare algebraic number theory (Schappacher, 2005, p. 701). Minkowski never managed to complete his half, and in 1896 he wrote to ask Hilbert to publish the volume on his own. The reason he gave for his failure was the comprehensiveness of Smith's report and that there was no longer the same need for a volume of the kind he had been asked to prepare (Rüdenberg and Zassenhaus, 1973, p. 78).

As an interesting but related aside Professor Stillwell makes no reference, in his introduction, to Smith's combined *Reports*. He does however suggest that the 1892 publication, *Theory of Numbers* (Part I), by George Ballard Mathews could be considered a useful guide to the subject during the 19th century because it, follows the treatment by Dirichlet and, uses his notation (Mathews, 1892).[16] In the preface to his book, George Mathews wrote:

> It is hardly necessary to say that I have derived continual assistance from the works of Gauss and Dirichlet, and from HJS Smith's invaluable *Report on the Theory of Numbers*. I am also greatly indebted to Professor Dedekind for permission to make free use of his edition of Dirichlet's *Vorlesungen über Zahlentheorie* (Mathews, 1892, p. v).

Returning to the narrative on the classification of binary quadratic forms, Dirichlet recognised that the notation of quadratic residues, attributed to Adrien-Marie Legendre (1752–1833), was very suitable for simplifying Gauss's earlier demonstrations to establish the generic characters for binary quadratic forms. 'Voici maintenant les principles très faciles á éstablir, sur lesquels la division en genres repose (Disq. arithm. art. 229 et suiv.)' (Dirichlet, 1839, p. 335).

16 George Ballard Mathews (1861–1922) FRS was Senior Wrangler at Cambridge in 1883 and was elected a Fellow of St John's College, Cambridge. The 1884 the University College of North Wales was established and he was appointed its Professor of Mathematics. There he published his first textbook: *Theory of Numbers (Part I) (1892).* In 1896 he moved to Cambridge as a University Lecturer returning to the University College of North Wales in 1906. For an obituary article on George Ballard Mathews see (B., 1922).

I. Si l est un nombre premier impair qui divise D, les entiers m, non-divisibles par l, qui peuvent être représentés par une même forme ayant D pour déterminant, sont ou tous tels que $\left(\frac{m}{l}\right) = 1$, ou tous tels que $\left(\frac{m}{l}\right) = -1$.

II. Lorsqu'on a $D \equiv 3$ (mod. 4), les nombres impairs m, susceptibles d'être représentés par la même forme, sont ou tous tels que $(-1)^{\frac{m-1}{2}} = 1$, ou tous tels que $(-1)^{\frac{m-1}{2}} = -1$.

III. Lorsqu'on a $D \equiv 2$ (mod. 8), les nombres impairs m, susceptibles d'être représentés par la même forme, sont ou tous tels que $(-1)^{\frac{m^2-1}{8}} = 1$, ou tous tels que $(-1)^{\frac{m^2-1}{8}} = -1$.

IV. Lorsqu'on a $D \equiv 6$ (mod. 8), les nombres impairs m, susceptibles d'être représentés par la même forme, sont ou tous tels que $(-1)^{\frac{m-1}{2}+\frac{m^2-1}{8}} = 1$, ou tous tels que $(-1)^{\frac{m-1}{2}+\frac{m^2-1}{8}} = -1$.

V. Lorsqu'on a $D \equiv 4$ (mod. 8), les nombres impairs m, susceptibles d'être représentés par la même forme, sont ou tous tels que $(-1)^{\frac{m-1}{2}} = 1$, ou tous tels que $(-1)^{\frac{m-1}{2}} = -1$.

VI. Losqu'on a $D \equiv 0$ (mod. 8), les nombres impairs m, susceptibles d'êtres représentés par la même forme, sont tous exclusivement contenus dans l'une de ces quatre formes $8\mu+1$, 3, 5, 7, ou, ce qui revient au même, on a à la fois $(-1)^{\frac{m-1}{2}} = \pm 1$, $(-1)^{\frac{m^2-1}{8}} = \pm 1$, chacun des deux signes ambigus restant invariable pour la même forme.

Figure 20: Extract from *Recherches sur diverses applications de l'Analyse infinitésimale à la Théorie des Nombres* (Dirichlet, 1839, p. 335).

These principles follow from the theorem of Article 229 [*Disquisitiones Arithmeticae*] (Theorem 3.2). Dirichlet suggests that 'these principles are easy to establish', however, it may be helpful to consider case [ii] and case [iv] in order to give a clear interpretation of these symbolic formulae. Firstly, to restate the principles for prime p and n, m any two numbers, not divisible by p.

Let f be a primitive form with determinant D and p a prime number dividing D. Let n, m be any two numbers, not divisible by p, which can be represented by the form f.

i. $\left(\frac{m}{p}\right) = \left(\frac{n}{p}\right)$.

ii. If $D \equiv 3 (\text{mod } 4)$, then $(-1)^{\frac{1}{2}(n-1)} = (-1)^{\frac{1}{2}(m-1)}$.

iii. If $D \equiv 2 (\text{mod } 8)$, then $(-1)^{\frac{1}{8}(n^2-1)} = (-1)^{\frac{1}{8}(m^2-1)}$.

iv. If $D \equiv 6 (\text{mod } 8)$, then $(-1)^{\frac{1}{2}(n-1)+\frac{1}{8}(n^2-1)} = (-1)^{\frac{1}{2}(m-1)+\frac{1}{8}(m^2-1)}$.

v. If $D \equiv 4 (\text{mod } 8)$, then $(-1)^{\frac{1}{2}(n-1)} = (-1)^{\frac{1}{2}(n-1)}$.

vi. If $D \equiv 0 (\text{mod } 8)$, then $(-1)^{\frac{1}{2}(n-1)} = (-1)^{\frac{1}{2}(m-1)}$ and

$$(-1)^{\frac{1}{8}(n^2-1)} = (-1)^{\frac{1}{8}(m^2-1)}.$$

ii. If $D \equiv 3 (\text{mod } 4)$ then, for all odd numbers n representable by the form f, the expression $(-1)^{\frac{1}{2}(n-1)}$ has the same value. This is because, if n and m are two such numbers, not divisible by p, and representable by f, then

$$nm = x^2 - Dy^2$$
$$\Rightarrow \quad nm \equiv x^2 + y^2 (\text{mod } 4)$$

Since nm is odd, one of the numbers x, y must be even and the other odd. Hence

$$nm \equiv 1 (\text{mod } 4)$$
$$\Rightarrow \quad n \equiv m (\text{mod } 4)$$

Therefore $(-1)^{\frac{1}{2}(n-1)} = (-1)^{\frac{1}{2}(m-1)}$.
All odd numbers representable by f are all such that $(-1)^{\frac{1}{2}(n-1)} = +1$ or all such that $(-1)^{\frac{1}{2}(n-1)} = -1$, i.e. the odd numbers that can be represented by f are either all included in the form $4n+1$, or else in the form $4n-1$.

iv. If $D \equiv 6 (\text{mod } 8)$, for all odd numbers n representable by the form f, the expression $(-1)^{\frac{1}{2}(n-1)+\frac{1}{8}(n^2-1)}$ has the same value. Because, if n and m are two such numbers prime to p and representable by (a, b, c), then

$$nm = x^2 - Dy^2$$
$$\Rightarrow \quad nm \equiv x^2 + 2y^2 (\text{mod } 8)$$

Since nm is odd, then x is odd. According as y is even or odd, $nm \equiv 1 (\text{mod } 4)$ or $nm \equiv 3 (\text{mod } 8)$. Hence

$$n \equiv m (\text{mod } 4)$$
$$\text{or} \quad n \equiv 3m (\text{mod } 8)$$
$$\Rightarrow \quad n^2 \equiv m^2 (\text{mod } 8)$$

Therefore $(-1)^{\frac{1}{2}(n-1)+\frac{1}{8}(n^2-1)} = (-1)^{\frac{1}{2}(m-1)+\frac{1}{8}(m^2-1)}$.
All odd numbers representable by f are all such that $(-1)^{\frac{1}{2}(m-1)+\frac{1}{8}(m^2-1)} = +1$ or all such that $(-1)^{\frac{1}{2}(m-1)+\frac{1}{8}(m^2-1)} = -1$, i.e. the odd numbers that can be represented by f are either all included in one of the two forms $8n+1$, $8n+3$, or else in one of the two forms $8n-1$, $8n-3$.

☒

Finally, Dirichlet provides the following table representing the complete generic characters for binary quadratic forms (*ibid*, p. 338).

Table 9: The Complete Generic Characters for Binary Quadratic Forms

Let S^2 denotes the greatest square dividing D. The values P or $2P$ is the quotient D/S^2, according as the quotient is odd or even. Also let $p_1, p_2, ...$ denote the prime divisors of P and $r_1, r_2, ...$ are the odd primes dividing S, but not P.

$D = PS^2 \quad , \quad P \equiv 1 (\text{mod } 4)$

$S \equiv 1(\text{mod } 2)$	$\left(\frac{f}{p}\right), \left(\frac{f}{p'}\right), \ldots$	$\left(\frac{f}{r}\right), \left(\frac{f}{r'}\right), \ldots$
$S \equiv 2(\text{mod } 4)$	$\left(\frac{f}{p}\right), \left(\frac{f}{p'}\right), \ldots$	$(-1)^{\frac{1}{2}(f-1)}, \left(\frac{f}{r}\right), \left(\frac{f}{r'}\right), \ldots$
$S \equiv 0(\text{mod } 4)$	$\left(\frac{f}{p}\right), \left(\frac{f}{p'}\right), \ldots$	$(-1)^{\frac{1}{2}(f-1)}, (-1)^{\frac{1}{8}(f^2-1)}, \left(\frac{f}{r}\right), \left(\frac{f}{r'}\right), \ldots$

$D = PS^2 \quad , \quad P \equiv 3 (\text{mod } 4)$

$S \equiv 1(\text{mod } 2)$	$(-1)^{\frac{1}{2}(f-1)}, \left(\frac{f}{p}\right), \left(\frac{f}{p'}\right), \ldots$	$\left(\frac{f}{r}\right), \left(\frac{f}{r'}\right), \ldots$
$S \equiv 2(\text{mod } 4)$	$(-1)^{\frac{1}{2}(f-1)}, \left(\frac{f}{p}\right), \left(\frac{f}{p'}\right), \ldots$	$\left(\frac{f}{r}\right), \left(\frac{f}{r'}\right), \ldots$
$S \equiv 0(\text{mod } 4)$	$(-1)^{\frac{1}{2}(f-1)}, \left(\frac{f}{p}\right), \left(\frac{f}{p'}\right), \ldots$	$(-1)^{\frac{1}{8}(f^2-1)}, \left(\frac{f}{r}\right), \left(\frac{f}{r'}\right), \ldots$

$D = 2PS^2 \quad , \quad P \equiv 1 (\text{mod } 4)$

$S \equiv 1(\text{mod } 2)$	$(-1)^{\frac{1}{8}(f^2-1)}, \left(\frac{f}{p}\right), \left(\frac{f}{p'}\right), \ldots$	$\left(\frac{f}{r}\right), \left(\frac{f}{r'}\right), \ldots$
$S \equiv 0(\text{mod } 2)$	$(-1)^{\frac{1}{8}(f^2-1)}, \left(\frac{f}{p}\right), \left(\frac{f}{p'}\right), \ldots$	$(-1)^{\frac{1}{2}(f-1)}, \left(\frac{f}{r}\right), \left(\frac{f}{r'}\right), \ldots$

$D = 2PS^2 \quad , \quad P \equiv 3(\text{mod } 4)$

$S \equiv 1(\text{mod } 2)$	$(-1)^{\frac{1}{2}(f-1)+\frac{1}{8}(f^2-1)}, \left(\frac{f}{p}\right), \left(\frac{f}{p'}\right), \ldots$	$\left(\frac{f}{r}\right), \left(\frac{f}{r'}\right), \ldots$
$S \equiv 0(\text{mod } 2)$	$(-1)^{\frac{1}{2}(f-1)}, (-1)^{\frac{1}{8}(f^2-1)}, \left(\frac{f}{p}\right), \left(\frac{f}{p'}\right), \ldots$	$\left(\frac{f}{r}\right), \left(\frac{f}{r'}\right), \ldots$

We can see how each compartment is divided into two parts by a vertical line, and the generic characters placed to the left of this line are subject to the condition that their product is equal to $+1$. Using the law of quadratic reciprocity Dirichlet establishes this relationship between the characters from which he concluded that one-half of the complete set of characters are impossible, i.e. that no quadratic form characterised by them actually exists. Furthermore, the remaining half of the generic characters correspond to actual existing genera, and that each genus contains an equal number of classes. The genus which has every character value a positive unit, is called the *principal genus*. It contains the the principal class, and is therefore, in every case, an actual existing genus. Dirichlet tabulates this relationship between binary characters as follows (*ibid*, p. 337).

Table 10: Relationship between Binary Characters

$D = PS^2$	$P \equiv 1(\text{mod } 4)$	$\left(\frac{n}{p}\right)\left(\frac{n}{p'}\right) \ldots\ldots = 1$
	$P \equiv 3(\text{mod } 4)$	$(-1)^{\frac{1}{2}(n-1)}\left(\frac{n}{p}\right)\left(\frac{n}{p'}\right) \ldots\ldots = 1$
$D = 2PS^2$	$P \equiv 1(\text{mod } 4)$	$(-1)^{\frac{1}{8}(n^2-1)}\left(\frac{n}{p}\right)\left(\frac{n}{p'}\right) \ldots\ldots = 1$
	$P \equiv 3(\text{mod } 4)$	$(-1)^{\frac{1}{2}(n-1)+\frac{1}{8}(n^2-1)}\left(\frac{n}{p}\right).\left(\frac{n}{p'}\right) \ldots\ldots = 1$

⊠

These relationships between generic characters (Table 10) may be established using the generalised law of quadratic reciprocity. Dirichlet provides a statement of this law and a number of subsequent relationships that exist between Legendre symbols (Dirichlet, 1839, p. 337). He leaves these, and the relationships between generic characters, to be established by the reader. Appendix B provides some useful details on quadratic residues.

Let $f = ax^2 + 2bxy + cy^2$ and m a positive odd number not divisible by D. If D is capable of primitive representation by f, then D is a quadratic residue of m i.e. $x^2 \equiv D(\text{mod } m)$ and

$$\left(\frac{D}{m}\right) = +1$$

For $D = PS^2$ and $D = 2PS^2$ respectively

$$\left(\frac{D}{m}\right) = \left(\frac{PS^2}{m}\right) = \left(\frac{P}{m}\right)\left(\frac{S^2}{m}\right) = \left(\frac{P}{m}\right) = 1$$

Also

$$\left(\frac{D}{m}\right) = \left(\frac{2PS^2}{m}\right) = \left(\frac{2P}{m}\right)\left(\frac{S^2}{m}\right) = \left(\frac{2P}{m}\right) = 1$$

Let $p, p', \ldots\ldots\ldots\ldots$ be the prime divisors of P.
Applying the generalised law of quadratic reciprocity to both cases gives:

For $D = PS^2$

$$\begin{aligned} \left(\frac{m}{P}\right) &= \left(\frac{m}{P}\right)\left(\frac{P}{m}\right) \\ &= (-1)^{\frac{1}{4}(m-1)(P-1)} \\ \therefore \left(\frac{m}{p}\right)\left(\frac{m}{p'}\right)\ldots\ldots &= (-1)^{\frac{1}{4}(m-1)(P-1)} \qquad (*_A) \end{aligned}$$

For $D = 2PS^2$

$$\begin{aligned} \left(\frac{m}{P}\right) &= \left(\frac{m}{P}\right)\left(\frac{2P}{m}\right) \\ &= \left(\frac{m}{P}\right)\left(\frac{2}{m}\right)\left(\frac{P}{m}\right) \\ &= (-1)^{\frac{1}{4}(m-1)(P-1)+\frac{1}{8}(m^2-1)} \\ \therefore \left(\frac{m}{p}\right)\left(\frac{m}{p'}\right)\ldots\ldots &= (-1)^{\frac{1}{4}(m-1)(P-1)+\frac{1}{8}(m^2-1)} \qquad (*_B) \end{aligned}$$

If $D = PS^2$, $P \equiv 3 (\text{mod } 4)$, using $(*_A)$

$$\begin{aligned}\left(\frac{m}{p}\right)\left(\frac{m}{p'}\right)\ldots\ldots &= (-1)^{\frac{1}{4}(m-1)(P-1)} \\ (-1)^{\frac{1}{2}(m-1)}\left(\frac{m}{p}\right)\left(\frac{m}{p'}\right)\ldots\ldots &= (-1)^{\frac{1}{2}(m-1)}(-1)^{\frac{1}{4}(m-1)(P-1)} \\ &= (-1)^{\frac{1}{4}(m-1)(P-1)+\frac{1}{2}(m-1)} \\ &= +1\end{aligned}$$

If $D = 2PS^2$, $P \equiv 1 (\text{mod } 4)$, using $(*_B)$

$$\begin{aligned}\left(\frac{m}{p}\right)\left(\frac{m}{p'}\right)\ldots\ldots &= (-1)^{\frac{1}{4}(m-1).(P-1)+\frac{1}{8}(m^2-1)} \\ (-1)^{\frac{1}{8}(m^2-1)}\left(\frac{m}{p}\right)\left(\frac{m}{p'}\right)\ldots\ldots &= (-1)^{\frac{1}{8}(m^2-1)}(-1)^{\frac{1}{4}(m-1)(P-1)+\frac{1}{8}(m^2-1)} \\ &= (-1)^{\frac{1}{4}(m-1)(P-1)+\frac{1}{4}(m^2-1)} \\ &= +1\end{aligned}$$

⊠

We can see that the product of the particular characters which appear to the left of the line of division in the table is equal to $+1$ in the case of any really existing genus, i.e. that precisely one-half of the whole number of complete generic characters are impossible. The remaining half of the generic characters correspond to actual existing genera, and that each genus contains an equal number of classes. A final example will illustrate this point.

EXAMPLE There are six properly primitive, reduced, positive binary quadratic forms of determinant $D = -99 = -3^2.11$. They are distributed into two genera, in equal number, in the following way.

The determinant $D = -99$
Four genera; Three classes in each

$+1 \;;\; +1$	$-1 \;;\; +1$
$(1, 0, 99)$	$(5, 1, 20)$
$(4, 1, 25)$	$(5, -1, 20)$
$(4, -1, 25)$	$(9, 0, 11)$

where the particular character of each form is taken in the order

$$\left(\frac{n}{3}\right) \quad , \quad \left(\frac{n}{11}\right)$$

It has been established that two of these four complete characters are impossible i.e. that no quadratic form characterised by them can exist.

⊠

Finally, referring to the table of complete characters (Table 9), Dirichlet states the following. 'If μ denote the number of odd primes which divide D, the total number of complete characters that can be formed by combining the character values in every possible way is 2^{μ} when $D \equiv 1(\text{mod } 8)$ or $\equiv 5(\text{mod } 8)$, $2^{\mu+2}$ when $D \equiv 0(\text{mod } 8)$, and $2^{\mu+1}$ in every other case' (*ibid*, p. 339).

3.4 CONCLUSION

In this chapter I have considered the mathematical technique, presentation style, and the use of tables in the classification of binary quadratic forms. An important arithmetical technique is revealed in Gauss's *Disquisitiones.* This technique considers all possible arrangements of odd/even numbers modulo 4/modulo 8, for a given equation, and from basic implications the conclusions may be drawn. This arithmetical technique, and a presentation style supported throughout by simple examples, will be evident in Henry Smith's 1867 classification of ternary quadratic forms (Section 5.2). Dirichlet's use of symbols of quadratic reciprocity, and tables to display generic characters, will also be important in Smith's publications. Dirichlet observed a relationship between binary characters based on his table of complete generic characters (Table 10). In a similar way Smith also observed a relationship between ternary characters based on his table of complete generic characters and will define, for ternary quadratic forms, what he termed a 'condition of possibility' (Section 5.7).

Dirichlet's *Vorlesungen über Zahlentheorie*, first published in 1863, was one of the most important mathematics books of the 19th century. Dirichlet's *Vorlesungen* and Smith's *Reports on the Theory of Numbers* were both well received when first published, during the 1860's, but the influence of both publications over time differed. Smith's *Reports* gave an outline of the results of the most recent investigations and he traced their connections, as far as possible, with one another and to earlier research. His audience may have been the more experienced mathematician with its condensed form being described as 'models of clear exposition and systematic arrangement' (Smith, 1894a, p. lxii). Dirichlet's *Vorlesungen*, on the other hand, were lecture notes with illustrative examples which made Gauss's *Disquisitiones* more understandable to a much wider audience. Consequently, it would have a greater influence on a new generation of number theorists. There were four successive editions of the *Vorlesungen* published in last decades of the 19th century, with an English translation of the 1st edition published as recently as 1999. Smith's *Reports* have now been largely forgotten, being replaced by updated versions by 20th century mathematicians. This may provide another example of where, in a comparable treatment of aspects of Gauss's *Disquisitiones* during the 19th century, Continental mathematicians enjoyed greater recognition than their English counterparts. A comparative work from the early 20th century was Leonard Eugene Dickson's *History of the Theory of Numbers* (1919) which appeared in three volumes, totalling more that 1600 pages, and took 9 years to complete (Dickson, 1919, Dickson, 1920, Dickson, 1923). David Hilbert's *Zahlbericht*, published in 1897, gave a remarkable systematic and lucid treatment of algebraic number theory which established the subject as a major domain of pure mathematics. It became the principle reference book in the discipline for many decades.[17]

This chapter has given some interesting insight into the classification of binary quadratic forms in terms of mathematical technique and presentation style. The *Disquisitiones Arithmeticae* also presents a treatise on ternary quadratic forms which, for the purpose of this thesis, is certainly worth a closer look. Following the standard steps for the classification of quadratic forms, as outlined by Smith in 1861, I will first consider Gauss's initial work to develop a reduction theory for positive definite ternary forms and Eisenstein's contributions based on the definition of a *reduced form*.

17 For an account of the contents, impact, and reaction to David Hilbert's *Zahlbericht* see (Schappacher, 2005, pp. 700–709).

4

TERNARY QUADRATIC FORMS

Carl Friedrich Gauss made a preliminary study of ternary quadratic forms, with integer coefficients, as a short digression from his investigation of binary quadratic forms, for the purpose of determining the exact number of genera of the latter forms. Accordingly he studied especially the problem of representing binary forms by ternary forms, the details of which appear as articles 278–285 of his *Disquisitiones Arithmeticae* (Gauss, 1986, pp. 311–328). He acknowledges that this represents a 'brief digression into the theory' of ternary forms but that he wished to 'reserve a more exact treatment of this important subject for another occasion' (*ibid*, p. 292). In this chapter I will initially outline the historical and mathematical background leading to Henry Smith's 1867 memoir *On the Orders and Genera of Ternary Quadratic Forms* (Smith, 1867). This important memoir continued the earlier work of Gotthold Eisenstein whose treatment of ternary quadratic forms remained unfinished at the time of his death in 1852.

4.1 INTRODUCTION

The *Disquisitiones Arithmeticae* contains a theory of reduction for positive definite binary forms and, in a similar vein, Gauss began to develop a reduction theory for positive definite ternary forms.[1] This was completed by Ludwig August Seeber (1793–1855), while Professor at the University of Freiburg, in his 1831 mathematical treatise on positive ternary quadratic forms (Seeber, 1831). Dedicating this work to Gauss, Seeber presented complicated inequalities that were satisfied by one and only one reduced positive form of a class of ternary quadratic forms. In 1840 Gauss, acknowledging the results obtained by Seeber, supplied some simplified arguments and improvements (Gauss, 1840). Gauss

1 For a comprehensive account of the historical development of reduction theory of quadratic forms, see (Schwermer, 2007, pp. 483–504). For a description of the various theories of reduction before 1920 see (Dickson, 1923, pp. 206–224).

interpreted Seeber's results geometrically and extended his own geometrical interpretation of positive binary quadratic forms to positive ternary quadratic forms. Dirichlet subsequently gave a theory of reduction of positive ternary quadratic forms which was far simpler than Seeber. Dirichlet employed the notation and geometrical interpretation of Gauss and his own concept of a reduced parallelogram. His memoir was presented to the Prussian Academy of Sciences in July 1848 (Dirichlet, 1850). In 1851 Gotthold Eisenstein (1823–1852) published tables of primitive reduced positive ternary quadratic forms (Eisenstein, 1851, pp. 161–190). He calculated these tables by simplifying Seeber's inequalities for a reduced ternary quadratic form by replacing them with linear inequalities. Eisenstein's tables also gave the number δ of transformations of the form onto itself (Appendix C, Table 20). During this time Eisenstein also began his important study of genera and the weight of an order or genus of ternary quadratic forms. In 1847 he published an important memoir in which he defined the ordinal and generic characters of ternary quadratic forms of an odd discriminant only (Eisenstein, 1847a). In 1867 Smith presented a complete classification of ternary quadratic forms by extending Eisenstein's results to the more difficult and complicated case of the even discriminant. In his memoir Smith presented a table for the complete generic character of any ternary quadratic form (Chapter 5, Figure 24). In doing so he accomplished for the ternary theory that which had been already carried out for the binary theory.

In 1851 and succeeding years Charles Hermite (1822–1901) gave arithmetical theories of reduction of quadratic forms in n variables, both definite and indefinite, and in particular his theory of continual reduction (Hermite, 1851, Hermite, 1854). A new method of reduction was given by Eduard Selling (1834–1920) in 1874 (Selling, 1874).[2] In 1880, Léon Charve (1849–1937) would give a clearer exposition of the arithmetical part of Selling's theory of reduction (Charve, 1880). In 1898, Paul Bachmann (1837–1920) published *Volume IV: Die Arithmetik der Quadratischen Formen* (Part 1, 1898, Part 2, 1923) which contains a complete exposition of the arithmetical theory of quadratic forms in three or more variables (Bachmann, 1898).[3]

The *Disquisitiones Arithmeticae* presents a treatise on ternary quadratic forms, within articles 266–285, which provided Smith with key principles and methods to some of his demonstrations. In article 280, for example, Gauss's

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2 For an article on Selling's method of reduction for positive ternary quadratic forms see (Jones, 1932).

3 Paul Bachmann (1837–1920) was a German mathematician who studied under Dirichlet, Kummer, and Weierstrass. Bachmann became a professor at the University of Münster where he specialised in the theory of numbers. Around 1890 he resigned his professorship to focus on writing. In 1892 he published the first of a five-volume book series on the theory of numbers. In his introduction to *Volume IV: Die Arithmetik der Quadratischen Formen* (1898) he pays tribute to Henry Smith; 'this excellent mathematician who died prematurely is to be thanked for a number of advances in which he extends Eisenstein's investigations into ternary forms to those with any number of variables' (Bachmann, 1898, p. 4).

digression on ternary quadratic forms provides, when read backwards, several tools to reduce statements on ternary forms to those on binary forms.[4] Furthermore, article 282 states a principle from which a method may be derived to find all primitive representations of the binary form $\varphi = (a, b'', a')$, of discriminant d, by the ternary form f of discriminant D.

The following are the extracts from the *Disquisitiones Arithmeticae* which pertain to Smith's 1867 memoir on ternary quadratic forms. It will be convenient to briefly restate them in a form suited for this thesis.

4.2 CARL FRIEDRICH GAUSS AND TERNARY QUADRATIC FORMS

Gauss employed the following notation for ternary quadratic forms.

ARTICLE 266/267 (Gauss, 1986, pp. 292–294)
A ternary quadratic form f may be represented as

$$f = ax^2 + a'y^2 + a''z^2 + 2byz + 2b'xz + 2b''xy$$

where f is termed *primitive* (i.e. that the six integers a, a', a'', b, b', b'' admit of no common divisor other than unity) and its *determinant* is different from zero.[5] The *determinant* of f is the determinant of the matrix

$$-\begin{vmatrix} a & b'' & b' \\ b'' & a' & b \\ b' & b & a'' \end{vmatrix} \qquad (*_A)$$

and is represented by D. Gauss defined the minor determinants of the matrix $(*_A)$ as the *adjoint* of f, or the form

$$(b^2 - a'a'')x^2 + (b'^2 - a''a)y^2 + (b''^2 - aa')z^2$$
$$+2(ab - b'b'')yz + 2(a'b' - b''b)zx + 2(a''b'' - bb')xy$$

and is represented as

$$F = Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'xz + 2B''xy$$

The *determinant* of F is equal to D^2. Furthermore, the *adjoint* of F is

$$aDx^2 + a'Dy^2 + a''Dz^2 + 2bDyz + 2b'Dxz + 2b''Dxy$$

4 Articles 278–280 also provided Charles Hermite with a key technique to determine an upper bound for the minimal value at integers of a quadratic form. See (Goldstein, Schappacher, and Schwermer, 2007, pp. 377–410).

5 Smith would later use the terms *discriminant* and *contravariant* of f in preference to Gauss's use of the terms *determinant* and *adjoint* of f.

Example (*ibid*, p. 294)
Let $f = 29x^2 + 13y^2 + 9z^2 + 14yz - 2xz + 28xy$ denote a ternary form.
The *adjoint* of f is $F = -68x^2 - 260y^2 - 181z^2 + 434yz - 222xz + 266xy$.
The determinant of both f and F is 1.

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Article 268 (*ibid*, p. 294)
The ternary quadratic form f_1 becomes a new ternary quadratic form f_2 when new variables are introduced. Let f_1 be transformed into f_2 by

$$\begin{aligned} x &= \alpha X + \beta Y + \gamma Z \\ y &= \alpha' X + \beta' Y + \gamma' Z \\ z &= \alpha'' X + \beta'' Y + \gamma'' Z \end{aligned}$$

where $\alpha, \beta, \gamma, \alpha', \beta', \gamma', \alpha'', \beta'', \gamma''$ are nine particular integers and X, Y, Z are the new variables. Let this substitution be denoted by S i.e.

$$S = \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{pmatrix}$$

of determinant k. From this supposition will follow six equations for the six coefficients of f_2. From these the following conclusions result.

1. The determinant of f_2 is equal to $k^2 D$ where

$$k = \alpha\beta'\gamma''' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' - \gamma\beta'\alpha'' - \alpha\gamma'\beta'' - \beta\alpha'\gamma''$$

2. Let S' be the *adjoint* of S i.e.

$$S' = \begin{pmatrix} \beta'\gamma'' - \beta''\gamma' & \gamma'\alpha'' - \gamma''\alpha' & \alpha'\beta'' - \alpha''\beta' \\ \beta''\gamma - \beta\gamma'' & \gamma''\alpha - \gamma\alpha'' & \alpha''\beta - \alpha\beta'' \\ \beta\gamma' - \beta'\gamma & \gamma\alpha' - \gamma'\alpha & \alpha\beta' - \alpha'\beta \end{pmatrix}$$

   F_1 will be transformed into F_2 by S' where F_1, F_2 denote the *adjoints* of f_1, f_2 respectively.

3. By interchanging the rows and columns of the matrix for the substitution we obtain a transformation said to arise by *transposition*.
   Let S'' arises from S' by *transposition*.
   f_2 will be transformed into $k^2 f_1$ by this substitution.
   Let S''' arises from S by *transposition*.
   F_2 will be transformed into $k^2 F_1$ by this substitution.

EXAMPLE

Let

$$S = \begin{pmatrix} 3 & 5 & 33 \\ 3 & 4 & 29 \\ 1 & 1 & 8 \end{pmatrix}$$

denote a substitution of determinant $k = 1$.
Let $f_1 = -x^2 + 2y^2 - 16z^2 + 2yz$ and $F_1 = -33x^2 + 16y^2 - 2z^2 + 2yz$.
f_1 is transformed into f_2 by the substitution S.
F_1 is transformed into F_2 by the substitution S'.
Hence $f_2 = -x^2 - y^2 + 33z^2$ and $F_2 = -33x^2 - 33y^2 + z^2$. Furthermore
f_2 is transformed into f_1 by the substitution S''.
F_2 is transformed into F_1 by the substitution S'''.
The determinant of both f_1 and f_2 is 33.
The determinant of both F_1 and F_2 is 33^2.

⊠

ARTICLE 269 (*ibid*, p. 296)

Forms f_1 and f_2 are said to be *equivalent* when one may be transformed into the other by a substitution of determinant unity i.e.

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} = \pm 1$$

Their determinants are equal. Their adjoints are equivalent, and conversely. All equivalent forms of the same determinant constitute a *class*. All ternary quadratic forms of a given determinant can be distributed into a finite number of classes.

⊠

Having defined *equivalence* of forms f_1 and f_2 Gauss now shows how a ternary quadratic form can be reduced to a simpler form and, in doing so, establishes a method of reduction using his FIRST and SECOND REDUCTION. Employing the reduction of Lagrange the coefficients of a reduced binary quadratic form will satisfy two inequalities (Appendix A.2). By reducing a suitable ternary quadratic form, using special transformation matrices each of determinant ± 1, Gauss applies these inequalities to the resulting form to establish corresponding conditions for ternary forms. Analogous with the binary theory, the number of classes into which all ternary forms of a given determinant are distributed is always finite, will also be a consequence of this method of reduction.

Article 272 (*ibid*, pp. 300–303)

Theorem Any ternary form of determinant D can be reduced to an equivalent form with the property that its first coefficient is not greater than $\frac{4}{3}\sqrt[3]{D}$ and the third coefficient of the adjoint form is not greater than $\frac{4}{3}\sqrt[3]{D^2}$, disregarding sign, provided the proposed form does not already have these properties.

Demonstration

Let F_1, F_2 denote the *adjoints* of f_1, f_2 respectively i.e.
$f_1 = (a, a', a'', b, b', b'')$
$f_2 = (m, m', m'', n, n', n'')$
$F_1 = (A, A', A'', B, B', B'')$
$F_2 = (M, M', M'', N, N', N'')$
Let

$$S = \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{pmatrix}$$

of determinant ± 1 i.e.

$$\alpha\beta'\gamma''' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' - \gamma\beta'\alpha'' - \alpha\gamma'\beta'' - \beta\alpha'\gamma'' = \pm 1$$

First Reduction A first reduction of f is made by means of a substitution which leaves z unaltered and replaces x, y by linear functions of themselves of determinant ± 1. Thus the binary form (a, b'', a'), of determinant A'', goes into an equivalent binary form whose first coefficient may be made numerically less than or equal to $\sqrt{\frac{4}{3}|A''|}$. Let f_1 be transformed into f_2 by the substitution

$$S = \begin{pmatrix} \alpha & \beta & 0 \\ \alpha' & \beta' & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence

$$\begin{aligned} m &= a\alpha^2 + 2b''\alpha\alpha' + a'\alpha'\alpha' \\ m' &= \alpha\beta^2 + 2b''\beta\beta' + a'\beta'\beta' \\ m'' &= a'' \\ n &= b\beta' + b'\beta \\ n' &= b\alpha' + b'\alpha \\ n'' &= a\alpha\beta + \beta''(\alpha\beta' + \beta\alpha') + a'\alpha'\beta' \end{aligned}$$

Furthermore $\alpha\beta' - \beta\alpha' = \pm 1$.

It is clear that the binary quadratic form (a, b'', a') whose determinant is A'' will be transformed by the substitution $\alpha, \beta, \alpha', \beta'$ into the binary form (m, n'', m') whose determinant is $n''n'' - mm' = M''$ and, since $\alpha\beta' - \beta\alpha' = \pm 1$, they will be equivalent and hence $M'' = A''$. Unless (a, b'', a') is already the simplest form of its class, the coefficients $\alpha, \beta, \alpha', \beta'$ can be so determined that (m, n'', m') is simpler. From the theory of equivalence of binary quadratic forms this can be achieved so that m is not greater than $\sqrt{-\frac{4}{3}A''}$, if A'' is negative, or not greater than $\sqrt{A''}$ if A'' is positive. Thus the absolute value of m can be made either less than or at most equal to $\sqrt{\pm\frac{4}{3}A''}$. In this way the form f_1 is reduced to another with a smaller first coefficient (unless it is already the simplest form in its class). Furthermore the form which is adjoint to this has the same third coefficient as the form F_1 which is adjoint to f_1. This is the FIRST REDUCTION.

Second Reduction A second reduction of f is made by a substitution S which leaves x unaltered and replaces y, z by linear functions of themselves of determinant ± 1. The adjoint substitution to S is

$$\begin{aligned} x &= \pm X \\ y &= \gamma'' Y - \beta'' Z \\ z &= -\gamma' Y + \beta' Z \end{aligned}$$

and replaces F by G, and hence the binary form (A', B, A'') of determinant Da by an equivalent binary form whose last coefficient is numerically less than or equal to $\sqrt{\frac{4}{3}|Da|}$. Let F_1 be transformed into F_2 by the substitution

$$S = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \gamma'' & -\beta'' \\ 0 & -\gamma' & \beta' \end{pmatrix}$$

Hence

$$\begin{aligned} m &= a \\ m' &= a'\beta'\beta' + 2b\beta'\beta'' + a''\beta''\beta'' \\ m'' &= a'\gamma'\gamma' + 2b\gamma'\gamma'' + a''\beta''\beta'' \\ n &= a'\beta'\gamma' + b(\beta'\gamma'' + \gamma'\beta'') + a''\beta''\gamma'' \\ n' &= \beta'\gamma'' + b''\gamma' \\ n'' &= b'\beta'' + b''\beta' \\ N &= -A'\beta''\gamma'' + B(\beta'\gamma'' + \gamma'\beta'') - A''\beta'\gamma' \\ M' &= A'\gamma''\gamma'' - 2B\gamma'\gamma'' + A''\gamma'\gamma' \\ M'' &= A'\beta''\beta'' - 2B\beta'\beta'' + A''\beta'\beta' \end{aligned}$$

Furthermore $\beta'\gamma'' - \beta''\gamma' = \pm 1$.

It is clear that the binary quadratic form (A'', B, A') whose determinant is Da will be transformed by the substitution $\beta', -\gamma', -\beta'', \gamma''$ into the binary form (M'', N, M') whose determinant is $N^2 - M'M'' = Dm$ and, since $\beta'\gamma'' - \beta''\gamma' = \pm 1$, they will be equivalent and hence $Da = Dm$. Unless (A'', B, A') is already the simplest form of its class, the coefficients $\beta', \gamma', \beta'', \gamma''$ can be so determined that (M'', N, M') is simpler. From the theory of equivalence for binary quadratic forms this can be achieved so that, without respect to the sign, M'' is not greater than $\sqrt{\pm\frac{4}{3}Da}$. In this way the form f_1 is reduced to another with the same first coefficient. Furthermore the form which is adjoint to this will have, if possible, a smaller third coefficient than the form F_1 which is adjoint to f_1. This is the SECOND REDUCTION.

The necessary conditions from the FIRST and SECOND REDUCTIONS are restated for a ternary quadratic form f of determinant D. If a ternary quadratic form f cannot be transformed by the FIRST nor the SECOND REDUCTION into a simpler form then necessarily

$$\begin{aligned}
a &\ngtr \sqrt{-\frac{4}{3}A''}\\
a^2 &\ngtr -\frac{4}{3}A''\\
a^4 &\ngtr \frac{16}{9}A''^2 \ngtr \frac{64}{27}aD \quad \text{since } A''^2 \ngtr \frac{4}{3}aD\\
a^3 &\ngtr \frac{64}{27}D\\
a &\ngtr \frac{4}{3}\sqrt[3]{D}
\end{aligned}$$

Furthermore

$$\begin{aligned}
A'' &\ngtr \sqrt{-\frac{4}{3}aD}\\
A''^2 &\ngtr -\frac{4}{3}aD\\
A''^4 &\ngtr \frac{16}{9}a^2D^2 \ngtr \frac{64}{27}A''D^2 \quad \text{since } a^2 \ngtr \frac{4}{3}A''\\
A''^3 &\ngtr \frac{64}{27}D^2\\
A'' &\ngtr \frac{4}{3}\sqrt[3]{D^2}
\end{aligned}$$

Consequently, any ternary form of determinant D can be reduced to an equivalent form with the property that its first coefficient is not greater than $\frac{4}{3}\sqrt[3]{D}$ and the third coefficient of the adjoint form is not greater than $\frac{4}{3}\sqrt[3]{D^2}$, disregarding sign, provided the proposed form does not already have these properties.

⊠

In order to provide clarity to article 272, the following summary may be helpful:

Summary

Let F_1, F_2 denote the *adjoints* of f_1, f_2 respectively i.e.
$f_1 = (a, a', a'', b, b', b'')$
$f_2 = (m, m', m'', n, n', n'')$
$F_1 = (A, A', A'', B, B', B'')$
$F_2 = (M, M', M'', N, N', N'')$
Let S_1, S_2 denote special transformation matrices each of determinant ± 1.

FIRST REDUCTION

$$f_1 \xrightarrow{\quad S_1 \quad} f_2$$

$$(a, b'', a')$$

This transformation will ensure that the form (a, b'', a'), of determinant A'', goes into an equivalent form with $a \leq \sqrt{-\frac{4}{3}A''}$ when A'' is negative (or with $a \leq \sqrt{A''}$ when A'' is positive). The form f_1 is reduced to another with a smaller first coefficient (unless (a, b'', a') is already the simplest form in its class). The third coefficient A'' of the adjoint F_1 of f_1 remains the same.

SECOND REDUCTION

$$F_1 \xrightarrow{\quad S_2 \quad} F_2$$

$$(A'', B, A') \qquad\qquad (M'', N, M')$$

This transformation will ensure that the form (A'', B, A'), of determinant aD, goes into an equivalent form with $M'' \leq \sqrt{\pm\frac{4}{3}aD}$. The form f_1 is reduced to another with the same first coefficient (unless (A'', B, A') is already the simplest form in its class). Furthermore, if possible, there will be a smaller third coefficient M'' of the adjoint F_2 of f_2.

⊠

Gauss, in the following article, presents an example to illustrate the preceding principles.

Article 273 (*ibid*, pp. 303–304)

Example (*ibid*, p. 303)
Let F_1 denote the *adjoint* of f_1 where
$f_1 = (19, 21, 50, 15, 28, 1)$ with $D = -1$ and
$F_1 = (-825, -166, -398, 257, 573, -370)$.
First Reduction $(a, b'', a') = (19, 1, 21)$ is a reduced binary form hence the first reduction is not applicable here.

Second Reduction $(A'', B, A') = (-398, 257, -166)$, by the theory of equivalence of binary forms, can be transformed into a simpler equivalent $(-2, 1, -10)$ by the substitution $2, 7, 3, 11$. Applying the substitution

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -7 \\ 0 & -3 & 11 \end{pmatrix}$$

to the form f_1, it will be transformed into
$f_2 = (19, 354, 4769, -1299, 301, -82)$ with
$F_2 = (-825, -10, -2, 1, 36, -59)$.
The third coefficient of F_2 is 2 and in this respect f_2 is simpler than f_1.
First Reduction $(a, b'', a') = (19, -82, 354)$, by the theory of equivalence of binary forms, can be transformed into a simpler equivalent $(1, 0, 2)$ by the substitution $13, 4, 3, 1$. Applying the substitution

$$S = \begin{pmatrix} 13 & 4 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to the form f_2, it will be transformed into
$f_3 = (1, 2, 4769, -95, 16, 0)$ with
$F_3 = (-513, -4513, -2, -95, 32, 1520)$.
Second Reduction $(A'', B, A') = (-2, -95, -4513)$, by the theory of equivalence of binary forms, can be transformed into a simpler equivalent $(-1, 1, -2)$ by the substitution $47, 1, -1, 0$. Applying the substitution

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 47 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

to the form f_3, it will be transformed into
$f_4 = (1, 257, 2, 1, 0, 16)$ with
$F_4 = (-513, -2, -1, 1, -16, 32)$.
The first coefficient of f_4 cannot be any further reduced by the first reduction, nor can the third coefficient of F_4 be further reduced by the second reduction.

⊠

The methods of reduction outlined in article 272 will ensure that the first coefficient of a ternary form f_1, and the third coefficient of its adjoint F_1, can be reduced as far as possible. Gauss suggests how a further reduction of f_1 may be achieved by a similar method.

Article 274 (*ibid*, pp. 304–306)
Let f_1 be transformed into f_2 by the substitution

$$S = \begin{pmatrix} 1 & \beta & \gamma \\ 0 & 1 & \gamma' \\ 0 & 0 & 1 \end{pmatrix}$$

Hence

$$\begin{aligned}
m &= a \\
m' &= a' + 2b''\beta + a\beta^2 \\
m'' &= a'' + 2b\gamma' + 2b'\gamma + a\gamma^2 + 2b''\gamma\gamma' + a'\gamma'\gamma' \\
n &= b + a'\gamma' + b'\beta + b''(\gamma + \beta\gamma') + a\beta\gamma \\
n' &= b' + a\gamma + b''\gamma' \\
n'' &= b'' + a\beta \\
N &= B - A''\gamma' \\
N' &= B' - N\beta - A''\gamma \\
M'' &= A''
\end{aligned}$$

S has determinant 1 and will not change the coefficients a, A''. It remains to find a suitable determination of β, γ, γ' so that the remaining coefficients will be reduced. Setting n'', N, N' as not greater than $a/2, A''/2, A''/2$ respectively, and disregarding sign, the equations for n'', N and N' above will yield the appropriate values of β, γ, γ'.

Example (*ibid*, p. 305)
The previous example of article 273 produced the ternary form
$f_4 = (1, 257, 2, 1, 0, 16)$ with $D = 1$ and
$F_4 = (-513, -2, -1, 1, -16, 32)$.
The appropriate values of β, γ, γ' are found from the equations
$n'' = b'' + a\beta = 16 + \beta$ and
$N = B - A''\gamma' = 1 + \gamma'$ and
$N' = B' - N\beta - A''\gamma = -16 + \gamma$.
Setting $\beta = -16$ we will have $n'' = 0$.
Setting $\gamma' = -1$ we will have $N = 0$.
Setting $\gamma = 16$ we will have $N' = 0$.

Hence $\beta, \gamma, \gamma' = -16, 16, -1$.
Now applying the substitution

$$S = \begin{pmatrix} 1 & -16 & 16 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

to the form f_4, it will be transformed into $f_5 = (1, 1, 1, 0, 0, 0)$.

⊠

In article 278 Gauss gives a clear outline for future work to complete the theory of ternary quadratic forms. In doing so he begins the study of a new category of problem based on the representation of binary forms by ternary forms.

Article 278 (*ibid*, p. 311)
If the unknowns of a ternary form are x, y, z, the form will represent numbers by giving determined values to x, y, z and will represent binary forms by the substitutions

$$\begin{aligned} x &= \alpha_1 x_1 + \beta_1 y_1 \\ y &= \alpha_2 x_1 + \beta_2 y_1 \\ z &= \alpha_3 x_1 + \beta_3 y_1 \end{aligned}$$

where $\alpha_1, \beta_1, \alpha_2, ...$ etc. are determined numbers and x_1, y_1 the unknowns of the binary form. Thus for a complete theory of ternary forms we require a solution of the following problems.

1. To find all representations of a given number by a given ternary form.
2. To find all representations of a given binary form by a given ternary form.
3. To judge whether or not two given ternary forms of the same determinant are equivalent and, if they are, to find all transformations of one into the other.
4. To judge whether or not a given ternary form implies another given form of a greater determinant and, if it does, to assign all transformations of the first into the second.

Since these problems are more difficult than the analogous problems in binary forms we will treat them more in detail at another time.

⊠

Gauss begins to address some of these problems. In article 280, for example, he provides, when read backwards, several tools to reduce statements on ternary forms to those on binary forms (Gauss, 1986, pp. 312–315). Articles 282–283 relate to the representation of binary by ternary quadratic forms (Gauss, 1986, pp. 316–322). Smith considered a number of problems, separate from the main theme of the classification of quadratic forms, which rely on these important principles. He uses the terms *discriminant* and *contravariant* of f in preference to Gauss's use of the terms *determinant* and *adjoint* of f. Furthermore his definition of the *discriminant* of f is the negative of Gauss's determinant and his definition of the *contravariant* of f is the negative of Gauss's adjoint. Articles 282–284 are important for demonstrations within Smith's 1867 memoir: *On the Orders and Genera of Ternary Quadratic Forms*. It will be convenient to restate the details here, using this switch in notation, 'in a form suited for our present purpose' (Smith, 1867, p. 269).

ARTICLE 10 (Smith, 1867, p. 269).
For a ternary quadratic form $f(x, y, z)$, if we write

$$\begin{aligned} x &= \alpha_1 x_1 + \beta_1 y_1 \\ y &= \alpha_2 x_1 + \beta_2 y_1 \\ z &= \alpha_3 x_1 + \beta_3 y_1 \end{aligned}$$

we obtain a binary form $\varphi(x_1, y_1)$ which is said to be *represented* by f.
$f \longrightarrow \varphi$ is called a *primitive representation of* φ *by* f, when the determinants of the matrix of the transformation are relatively prime, i.e. $\gcd(C_1, C_2, C_3) = 1$

$$\begin{aligned} C_1 &= \alpha_2\beta_3 - \alpha_3\beta_2 \\ C_2 &= \alpha_3\beta_1 - \alpha_1\beta_3 \\ C_3 &= \alpha_1\beta_2 - \alpha_2\beta_1 \end{aligned}$$

If $\varphi = (a, b'', a')$ is primitively represented by f, then f is equivalent to a form containing φ as a part, i.e. to a form f' of the type

$$f' = ax^2 + a'y^2 + a''z^2 + 2byz + 2b'xz + 2b''xy$$

for f is transformed into such a form by a transformation of which the matrix for the transformation is

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

where $\gamma_1, \gamma_2, \gamma_3$ denoting any three numbers which render the determinant of the matrix equal to $+1$.

⊠

ARTICLE 10 (*ibid*, p. 270)

The investigation of the representations of a given binary form, with determinant not equal to zero, by a given ternary form depends on the following theorem.

THEOREM If φ is a binary quadratic form of discriminant $-\Omega A''$, and φ admits a primitive representation by a ternary quadratic form f of the invariants $[\Omega, \Delta]$, then $-\Delta\varphi$ is a quadratic residue of A''.

DEMONSTRATION[6]

Let f have discriminant $\Omega^2\Delta$.
Let φ have discriminant $b''^2 - aa' = -\Omega A''$.
Let $f \longrightarrow f'$ be a *primitive representation of f' by f*.
The discriminant of f' is $\Omega^2\Delta$.
The contravariant $F' = \Delta f'$, i.e.

$$(A'A'' - B^2)x^2 + (A''A - B'^2)y^2 + (AA' - B''^2)z^2 + \ldots\ldots\ldots$$
$$= \Delta\left[ax^2 + a'y^2 + a''z^2 + \ldots\ldots\ldots\right]$$

Multiplying the equations

$$\begin{aligned} A'A'' - B^2 &= \Delta a \\ BB' - A''B'' &= \Delta b'' \qquad (*_B) \\ AA'' - B'^2 &= \Delta a' \end{aligned}$$

by $x^2, 2xy, y^2$ respectively, we obtain

$$-\Delta \times (ax^2 + 2b''xy + a'y^2) = (B^2 - A'A'')x^2 - 2(BB' - A''B'')xy + (B'^2 - AA'')y^2$$

and this equation, considered as a congruence for the modulus A'', becomes

$$\Delta\varphi + (Bx - B'y)^2 \equiv 0 (\text{mod } A'')$$

the coefficients of $x^2, 2xy, y^2$ being all divisible by A''. If therefore $\varphi = (a, b'', a')$ is a binary quadratic form of discriminant $-\Omega A''$ and φ admits a primitive representation by a ternary quadratic form f of the invariants $[\Omega, \Delta]$, then $-\Delta\varphi$ is a quadratic residue of A''.

⊠

6 The notation $D = -\Omega^2\Delta$, where Ω denotes the gcd of the coefficients of the *adjoint* of f and Δ an integer, was introduced by Eisenstein in 1847 (Eisenstein, 1847a, p. 122). Henry Smith adopts this notation throughout his memoir: *On the Orders and Genera of Ternary Quadratic Forms* (Smith, 1867).

ARTICLE 10 (*ibid*, p. 270)
From the demonstration of the previous theorem a method may be derived to find all primitive representations of the binary form $\varphi = (a, b'', a')$, of discriminant d, by the ternary form f of discriminant D. Writing each equation $(*_B)$ as a congruence modulo A'' gives

$$\begin{aligned} B^2 &\equiv -\Delta a (\text{mod } A'') \\ BB' &\equiv \Delta b'' (\text{mod } A'') \\ B'^2 &\equiv -\Delta a' (\text{mod } A'') \end{aligned}$$

Solving this system of linear congruences yield integral solutions (B, B'). Any solution of this congruence supplies a system of five numbers A, A', B, B', B'' which satisfy $(*_B)$. Now $\gcd(A, A', B, B', B'') \mid \Delta$ since $\gcd(a, b'', a') = 1$. But $\gcd(A'', \Delta) = 1$, therefore $\gcd(A, A', A'', B, B', B'') = 1$, i.e., F' is *primitive*.

For each pair of solutions (B, B'), we seek a ternary form f', having determinant $\Omega^2\Delta$ with a, b'', a' known since f' contains φ as a part. Let b and b' be determined by the equations

$$\begin{aligned} -\Omega B &= ab - b'b'' \\ \Omega B' &= bb'' - a'b' \qquad (*_C) \end{aligned}$$

which are always resoluble because their discriminant $b''^2 - aa' = -\Omega A''$ is different from zero. Also let a'' be determined by the equation

$$bB + b'B' + a''A'' = \Omega\Delta \qquad (*_D)$$

If b, b', a'' are integers and if $f \longrightarrow f'$ is a primitive representation of f' by f, then every transformation which replaces f by f' yield a primitive representation of φ by f, and all representations are found by this method.

⊠

Smith, unusually, did not provide an example here for this important method. So, following Gauss's style of exposition, as Smith normally did, I have chosen to enumerate all primitive representations of the binary form $\varphi = (71, 101, 145)$ by the ternary form f of the invariants $[\Omega, \Delta] = [1, 13]$.

EXAMPLE If $\varphi = (71, 101, 145)$ is a binary quadratic form of determinant $-\Omega A'' = -94$ and φ admits a primitive representation by a ternary quadratic form f of the invariants $[\Omega, \Delta] = [1, 13]$, then

$$\begin{aligned} B^2 &\equiv 17 (\text{mod } 94) \\ BB' &\equiv 91 (\text{mod } 94) \\ B'^2 &\equiv 89 (\text{mod } 94) \end{aligned}$$

Solving we get $B \equiv 55 (\text{mod } 94)$ and $B' \equiv 29 (\text{mod } 94)$. So, for example:

Let $(B, B') = (55, 29)$.
From $(*_C)$, we find $b = -116$ and $b' = -81$.
From $(*_D)$, we find $a'' = 93$.
From the *contravariant of* F', we find $A = 29$, $A' = 42$ and $B'' = 3$.

Let $(B, B') = (149, 123)$.
From $(*_C)$, we find $b = -362$ and $b' = -253$.
From $(*_D)$, we find $a'' = 905$.
From the *contravariant of* F', we find $A = 181$, $A' = 246$ and $B'' = 181$.

Let $(B, B') = (243, 217)$.
From $(*_C)$, we find $b = -608$ and $b' = -425$.
From $(*_D)$, we find $a'' = 2553$.
From the *contravariant of* F', we find $A = 521$, $A' = 638$ and $B'' = 547$.

Let $(B, B') = (337, 311)$.
From $(*_C)$, we find $b = -854$ and $b' = -597$.
From $(*_D)$, we find $a'' = 5037$.
From the *contravariant of* F', we find $A = 1049$, $A' = 1218$ and $B'' = 1101$.

$f = (2, 2, 5, 1, 1, 1)$ $\longrightarrow$ $\varphi = (71, 101, 145)$

$[\Omega, \Delta] = [1, 13]$ $\qquad$ $-\Omega A'' = -94$

$$\begin{aligned} x &= 3x + 5y \\ y &= 3x + 4y \\ z &= x + y \end{aligned}$$

with $\gcd(-1, 2, -3) = 1$

$$\begin{aligned} x &= 3x + 5y + \gamma_1 z \\ y &= 3x + 4y + \gamma_2 z \\ z &= x + y + \gamma_3 z \end{aligned}$$

$\downarrow$

$f' = (71, 145, 93, -116, -81, 101)$, $\quad F' = (29, 42, 94, 55, 29, 3)$
$f' = (71, 145, 905, -362, -253, 101)$, $\quad F' = (181, 246, 94, 149, 123, 181)$
$f' = (71, 145, 2553, -608, -425, 101)$, $\quad F' = (521, 638, 94, 243, 217, 547)$
$f' = (71, 145, 5037, -854, -597, 101)$, $\quad F' = (1049, 1218, 94, 337, 311, 1101)$
....

4.3 GOTTHOLD EISENSTEIN AND TERNARY QUADRATIC FORMS

Gotthold Eisenstein belonged, together with Dirichlet, Jacobi, Hermite and Kummer, to the generation after Gauss that shaped the theory of numbers in the mid-19th century. Eisenstein's career in mathematics, like that of Dirichlet, was supported unfailingly throughout by Alexander von Humboldt (1769–1859).[7]

Figure 21: Alexander von Humboldt (1769–1859)

Von Humboldt was a German polymath, geographer, naturalist and explorer. In 1799 he embarked on a five year research expedition, with Aimé Bonpland (1773–1858), to South and Central America. This expedition earned him enormous world-wide fame and by 1807 he was living in Paris working on the 36 lavishly illustrated volumes on the scientific evaluation of his expedition. He became a corresponding member of the French Academy in 1804 and a foreign member in 1810. Von Humboldt took an exceedingly broad interest in the natural sciences and he made generous good use of his fame to support young talents in any kind of art or science, sometimes even out of his own pocket. In 1825 he was about to complete his great work and return to Berlin on the invitation of the Prussian King Friedrich Wilhelm III, who wanted to have such a luminary of science at his court.

In 1842 Eisenstein bought his own copy of Gauss's *Disquisitiones Arithmetice*, in French translation. The theory of numbers quickly became his favourite in-

7 For a biographical essay on Gotthold Eisenstein see (Schappacher, 1998) and (Schmitz, 2004). For an account of Alexander von Humboldt support of Gotthold Eisenstein see (Goldstein, Schappacher, and Schwermer, 2007, pp. 217–227). For an account of Alexander von Humboldt life long support of Dirichlet see (Elstrodt, 2007, pp. 1–37). For a biographical essay on Alexander von Humboldt see (Nassar, 2023).

terest in mathematics. In 1843 Eisenstein's mother Helene applied to the *Kultusministerium* for financial support for her son's studies, however, her application was rejected. At that time the young Eisenstein was already a student of mathematics at Berlin University, having already attended lectures by Dirichlet and Martin Ohm (1792–1872) as a high–school student and who had met William Rowan Hamilton (1805–1865) in Dublin.[8] Helene Eisenstein continued in her endeavour to secure financial support for her son. She wrote to the Queen who subsequently recommended Eisenstein to the *Kultusministerium* who in turn supported the request to King Friedrich Wilhelm III. Meanwhile Alexander von Humboldt was made aware of Eisenstein's precocious talent and predictably wrote to the King as well. These efforts succeeded and on May 1st, 1844 Eisenstein received an annual stipend of 250 Thaler. A few days later Humboldt explained to Eisenstein:

> Human life is a conditional equation, and the securing of some of its material needs is one of the most important conditions to satisfy. ... Do not be surprised by the small sum. There are limits which cannot easily be crossed, because the noble will of the monarch to help a young man with such rich gifts is counteracted by those cooling influences of the financial bureaucrats who calculate, which however, does not render them more positively inclined towards the theory of numbers (Biermann, 1959, pp. 122–123).

Humboldt's faith in Eisenstein was soon rewarded. Hamilton had earlier entrusted Eisenstein with a manuscript to deliver to the Berlin Academy on his behalf and, in doing so, Eisenstein also submitted a paper of his own on cubic quadratic forms.[9] This led to August Crelle's (1780–1855) interest in the prodigy. What happened next is not only a proof of Eisenstein's unusual talent, but also an amazing illustration of what the Berlin scientific establishment of the time was capable of doing for a young genius. Volume 27 of *Crelle's Journal* (1844) contains no less than fifteen papers (and one problem) by Eisenstein. Volume 28 (still 1844) had another eight (plus another problem), in German, French, and also Latin. Thus, within the first year, the young author had 'more publications to his credit than years of age' (Begehr et al., 1998, p. 57). Eisenstein's remarkable level of output continued over the remaining years of his short life, totalling some 40 mathematical articles and more than 700 printed pages.

On many occasions, and with his usual thoughtfulness and kindness, Humboldt continued to act in Eisenstein's favour, calling on the support of his

8 Eisenstein visited Dublin in 1843 where he met with William Rowan Hamilton in the same year in which Hamilton discovered quaternions. Eisenstein also met the then Lord Mayor of Dublin, Daniel O'Connell (1775–1847) (Lemmermeyer, 2020). Daniel O'Connell was the acknowledged political leader of Ireland's Roman Catholic majority in the first half of the 19th century.

9 $f = (a, b, c, d) = ax^3 + 3bx^2y + 3cxy^2 + dy^3$ with integral coefficients.

Table 11: Gotthold Eisenstein's Memoirs on Ternary Quadratic Forms

Year	Publication
1847	Neue Theoreme der höheren Arithmetik, *Journal für die reine und angewandte Mathematik* (Crelles Journal) Vol. 35, pp. 117–136.
1847	Note sur la représentation d'un nombre par la somme de cinq carrés, *Journal für die reine und angewandte Mathematik* (Crelles Journal) Vol. 35, p. 368.
1851	Tabelle der reducirten positiven ternären quadratischen Formen, nebst den Resultaten neuer Forschungen über diese Formen, in besonderer Rücksicht auf ihre tabellarische Berechnung, *Journal für die reine und angewandte Mathematik* (Crelles Journal) Vol. 41, pp. 141–190.

scientific friends in Berlin to advance the young student's career in mathematics. Eisenstein travelled to Göttingen in June 1844 to meet Gauss, a meeting arranged with the help of Humboldt and Johann Franz Encke (1791–1865), Humboldt providing Eisenstein with a letter of introduction. Eisenstein was awarded a doctorate *honoris causa* from the Philosophical Faculty of Breslau University in 1845, an award made possible by both Carl Gustav Jacobi (1804–1851) and Ernst Eduard Kummer (1810–1893), both acting in Eisenstein's favour. With this doctorate, Eisenstein became ostensibly a part of the network comprising of Humboldt, Gauss, Dirichlet, Jacobi and Kummer. A sequence of letters, set in action in 1850, resulted in Eisenstein being chosen as an ordinary member of the Berlin Academy of Sciences in March 1852. At 29 years of age, he was the youngest member.

Three major themes pervade Eisenstein's work. Firstly, the theory of quadratic forms, particularly cubic forms, where the goal was to generalise Gauss's arithmetic theory of quadratic forms contained in the *Disquisitiones Arithmeticae*. Second, the theory of higher reciprocity laws, i.e. the search for a generalisation of quadratic reciprocity which Gauss had also treated in his *Disquisitiones Arithmeticae*, and which he continued to come back to, giving new proofs of it. The final mathematical theme taken up by Eisenstein was the theory of elliptic functions, which Gauss had studied without publishing work on the subject.

Eisenstein died in 1852, before reaching 30 years of age, however, many of his publications formed a basis of further contributions to the arithmetical theory of quadratic forms, particularly for Smith (Table 11).

In 1847 Eisenstein's memoir *Neue Theoreme der Höheren Arithmetik* made a notable contribution to the arithmetical theory of ternary quadratic forms and, almost 20 years later, it became the starting point for Smith's own contribution to this theory (Eisenstein, 1847a). In this memoir Eisenstein considered the classification of ternary quadratic forms i.e. the separation of classes into orders and the separation of orders into genera. He stated that the positive greatest common divisor Ω of the coefficients of the form F has the same value for all forms f of a class. This he termed the *adjoint factor* of the class. All classes with the same determinant D and the same adjoint factor Ω constitute an order. Further to the theme of the classification of ternary quadratic forms Eisenstein, also in this memoir, defined a transformation which maps a form f onto itself as an *automorphic transformation*. He also stated, without demonstration, a formula to determine the weight of a genus of positive ternary quadratic forms (Eisenstein, 1847a, p. 128). Smith would later give a demonstration of this conjecture in 1867 (Section 5.8).

DEFINITION (*ibid*, p. 120)
A transformation which maps a form f onto itself is called an *automorphic transformation*. Let δ denote the number of *positive automorphics* of the form f. The *weight of a form* m is the reciprocal of the number of its positive automorphics, i.e.

$$m = \frac{1}{\delta}$$

⊠

DEFINITION (Smith, 1867, p. 278)
The *weight of a class* is the weight of any form contained in the class. The *weight of an order* is the sum of the weights of the reduced forms contained in the order. The *weight of a genus* is the sum of the weights of the reduced forms contained in that genus.

⊠

THEOREM (*ibid*, p. 291)
Let r any odd prime dividing both Ω and Δ, by δ any odd prime dividing Δ, but not Ω, and by ω any odd prime dividing Ω, but not Δ. The weight W of a proposed genus is

$$W = \frac{\Delta\Omega}{8} \times \zeta \times \Pi\frac{1}{4}\left[1 - \frac{1}{r^2}\right]\Pi\frac{1}{2}\left[1 + \left(\frac{-\Delta f}{\omega}\right)\frac{1}{\omega}\right]\Pi\frac{1}{2}\left[1 + \left(\frac{-\Omega F}{\delta}\right)\frac{1}{\delta}\right]$$

where the value of ζ is chosen from a distinction of cases (Section 5.8, Figure 25).

⊠

Eisenstein employed Gauss's notation for ternary quadratic forms and he outlines that since f is positive, F is negative and letting $F = -\Omega\mathfrak{I}$ will ensure that the ternary quadratic form $\mathfrak{I}$ is positive and primitive. The division of orders into genera does not depend (as in the case of binary forms) solely upon the quadratic characters of the numbers represented by the forms with respect to the various prime factors of D and to 4 and 8, but also upon the characters of their adjoint forms. The division of orders into genera is based on the following principals proposed, without demonstration, by Eisenstein.[10]

CONJECTURE (*ibid*, p. 126)
Let f be of odd determinant $D = -\Omega^2\Delta$, with Δ an integer. If ω represents any odd prime dividing Ω and δ represents any odd prime dividing Δ:

1. The numbers, prime to ω, which are represented by f, are either all quadratic residues of ω, or quadratic non-residues of ω. Consequently we attribute to f the *particular generic characters*

$$\left(\frac{f}{\omega}\right) = +1 \quad \text{or} \quad \left(\frac{f}{\omega}\right) = -1$$

2. The numbers, prime to δ, which are represented by f, are either all quadratic residues of δ, or quadratic non-residues of δ. Consequently we attribute to $\mathfrak{I}$ the *particular generic characters*

$$\left(\frac{\mathfrak{I}}{\delta}\right) = +1 \quad \text{or} \quad \left(\frac{\mathfrak{I}}{\delta}\right) = -1$$

Furthermore, if Ω and Δ are both divisible by any odd prime, f and $\mathfrak{I}$ will both have particular characters with respect to that prime.

⊠

For his classification Eisenstein considered ternary quadratic forms of an odd discriminant only. Such forms, and their adjoints, are always properly primitive and their particular generic characters are those with respect to the odd primes dividing the determinant. They have no supplementary characters i.e. characters with respect to 4 or 8. The case of forms of an even determinant would prove to be more complicated. Twenty years later, Smith, in his memoir *On the Orders and Genera of Ternary Quadratic Forms*, would supply these omissions and complete the work set out by Eisenstein.

In 1851, Eisenstein published tables of primitive reduced positive ternary quadratic forms (Eisenstein, 1851, pp. 161–190). He calculated these tables by simplifying Seeber's inequalities for a reduced ternary quadratic form by replacing them with linear inequalities. Eisenstein defined a form f as primitive when the coefficients a, a', a'', b, b', b'' have no common factor, as *properly primitive* when $a, a', a'', 2b, 2b', 2b''$ have no common factor and *improperly primitive*

10 Smith would give the demonstration of these principals in 1867 (Smith, 1867, p. 257).

when a, a', a'' are all even. A first table of properly primitive reduced positive ternary quadratic forms of discriminant -1 to -100 and -358 (Appendix C) was followed by a second table of improperly primitive reduced positive ternary quadratic forms of discriminant -2 to -100. He also included the number δ of transformations of the form onto itself. Eisenstein's conditions for a reduced form are as follows.

Case A (*ibid*, p. 143)
For

$$f = ax^2 + a'y^2 + a''z^2 + 2byz + 2b'xz + 2b''xy$$

with $b, b', b'' > 0$, the principal conditions are:

$$a \leq a' \leq a'' \ , \ 2b \leq a' \ , \ 2b' \leq a \ , \ 2b'' \leq a$$

The secondary conditions are:
If $a = a'$, then $b \leq b'$.
If $a' = a''$, then $b' \leq b''$.
If $2b = a'$, then $b'' \leq 2b'$.
If $2b' = a$, then $b'' \leq 2b$.
If $2b'' = a$, then $b' \leq 2b$.

Case B (*ibid*, p. 144)
For

$$f = ax^2 + a'y^2 + a''z^2 - 2byz - 2b'xz - 2b''xy$$

with $b, b', b'' \geq 0$, the principal conditions are:

$$a \leq a' \leq a'' \ , \ 2b \leq a' \ , \ 2b' \leq a \ , \ 2b'' \leq a \ , \ 2(b + b' + b'') \leq a + a'$$

The secondary conditions are:
If $2b = a'$, then $b'' = 0$.
If $2b' = a$, then $b'' = 0$.
If $2b'' = a$, then $b' = 0$.
If $2(b + b' + b'') = a + a'$, then $a \leq 2b' + b''$ i.e. $b'' \geq (a - 2b')$.

⊠

Eisenstein continues to outline how, from these conditions, tables of ternary quadratic forms may be generated (*ibid*, pp. 147–149).

The determinant $D = ab^2 + a'b'^2 + a''b''^2 - aa'a'' - 2bb'b''$.
Writing the determinant as $-D = 2bb'b'' - ab^2 - a'b'^2 + a''(aa' - b''^2)$.
The values of $-D$ compose an *arithmetic progression* for growing values of a''.
The *common difference* is $aa' - b''^2$.
The value $a = 1, 2, 3, 4, 5$ will be reduced by the condition

$$a \leq a' \leq \sqrt{\frac{200}{a}}$$

giving the inequalities $1 \leq a' \leq 14$, $2 \leq a' \leq 10$, $3 \leq a' \leq 8$, $4 \leq a' \leq 7$, $5 \leq a' \leq 6$. Consequently 35 combinations for a and a' may be considered.

$$\begin{array}{c}
(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)\ (1,7)\ (1,8)\ (1,9)\ (1,10)\ (1,11)\ (1,12)\ (1,13)\ (1,14) \\
(2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)\ (2,7)\ (2,8)\ (2,9)\ (2,10) \\
(3,3)\ (3,4)\ (3,5)\ (3,6)\ (3,7)\ (3,8) \\
(4,4)\ (4,5)\ (4,6)\ (4,7) \\
(5,5)\ (5,6)
\end{array}$$

He began by tabulating values of b, b', b'' for a chosen (a, a').
The principal conditions for CASE A are: $2b \leq a', 2b' \leq a, 2b'' \leq a$ i.e.

$$b \leq \frac{1}{2}a', \quad b' \leq \frac{1}{2}a, \quad b'' \leq \frac{1}{2}a$$

The principal conditions for CASE B are: $2b \leq a', 2b' \leq a, 2b'' \leq a, 2(b + b' + b'') \leq a + a'$ i.e.

$$b \leq \frac{1}{2}a', \quad b' \leq \frac{1}{2}a, \quad b'' \leq \frac{1}{2}a, \quad b + b' + b'' \leq \frac{1}{2}(a + a')$$

Eisenstein rejected those combinations which are incompatible with the the above conditions. With the remaining values of b, b', b'' in place for given a, a' he calculated $-D$ for $a'' = a', a' + 1, a' + 2, a' + 3, \ldots\ldots\ldots$ as is necessary. Observing the *arithmetic progression* for values of $-D$ helped with the calculations. Ensuring that the conditions outlined in CASE A and CASE B are satisfied will ensure that the remaining forms are primitive reduced ternary forms of determinant $-D$.[11]

⊠

In the following example I have determined the value of b, b', b'' in the case when $(a, a') = (4, 4)$.

11 For a table of properly primitive, and the improperly primitive, positive, reduced ternary quadratic forms of discriminant -1 to -50 (Appendix C). Furthermore, using Eisenstein's conditions, the author has tabulated the properly primitive, and the improperly primitive, positive, reduced ternary quadratic forms of discriminant -1 to -100 using a spreadsheet.

Example

Let $(a, a') = (4, 4)$. For b, b', b'' all positive, $b, b', b'' > 0$, we have

$$b = 1, 2 \qquad b' = 1, 2 \qquad b'' = 1, 2$$

For b, b', b'' all non-positive, $b, b', b'' \leq 0$, we have

$$b = 0, -1, -2 \qquad b' = 0, -1, -2 \qquad b'' = 0, -1, -2$$

Also $|b + b' + b''| \leq 4$.

⊠

4.4 CONCLUSION

In this chapter I have outlined the historical development of ternary quadratic forms from Gauss's digression into the topic in 1801 to Eisenstein's important study of genera and the weight of an order or genus of ternary forms in 1847. Eisenstein made further important contributions to the theory of ternary forms, in 1852, which are outside the scope of this thesis. It was not until 1867, almost twenty years after Eisenstein's work on the classification of ternary forms, that the topic was revisited and completed by Henry Smith. Conjectures made by Eisenstein on quadratic forms preoccupied Smith work throughout the 1860's. One of these conjectures became the subject of the *French Académie des Sciences* for its *Grand Prix des Sciences Mathématiques* of 1882.

Gauss began to develop a reduction theory for positive definite ternary forms. The mathematical technique involves a reduction of a suitable ternary form to one which allows the reduction theory of binary forms to be applied. The *Arithmeticae* also contains several tools to reduce statements on ternary forms to those on binary forms. The representation of binary forms by ternary forms would provide Smith with a mathematical technique to resolve more obscure results related to the classification of forms.

Smith's knowledge and expertise in the subject was extensive, following the publication of his *Reports on the Theory of Numbers*. In 1864 he had already set in place the notation for a classification of quadratic forms with more than three indeterminates (Smith, 1864b, pp. 199–203). Arthur Cayley published just three memoirs on the theory of binary quadratic forms between 1850 and 1862 and in 1871 he published a table of cubic forms. So it seems that Smith was best placed in Britain to advance the theory of quadratic forms. On the Continent, despite the loss of many leading mathematicians during the 1850's, Leopold Kronecker published extensively on binary quadratic forms during the 1860's and 1870's.

In 1867 Smith published the details for the classification of ternary quadratic forms for the more difficult case of the form having even determinant. The next chapter will show his skill as an expositor of mathematics with a systematic arrangement of details that was clearly the influence of Gauss.

5

THE CLASSIFICATION OF TERNARY QUADRATIC FORMS

Henry Smith advanced the theory of quadratic forms by returning to original sources and remaining true to arithmetic. The classification of quadratic forms by Gauss and Smith was separated by more than 65 years but, despite this, they employed the same mathematical technique and presentation style. In 1867, Smith published a memoir titled *On the Orders and Genera of Ternary Quadratic Forms* (Smith, 1867, pp. 255–277).[1] In this memoir he used identities to achieve a complete subdivision of an order of ternary forms into genera and, in doing so, complete the work set out by Eisenstein in 1847. Smith's complete classification of ternary forms required attention to a great many details and he took special care with atypical cases. He arranges the notation required for this task clearly and made extensive use of tables. In this chapter, I will highlight Smith's style of exposition by considering some of his demonstrations carefully. This will be supported with illustrative examples. Smith advanced the theory of classification further with new observations based on principles taken from the *Disquisitiones Arithmeticae*, articles 280–283.

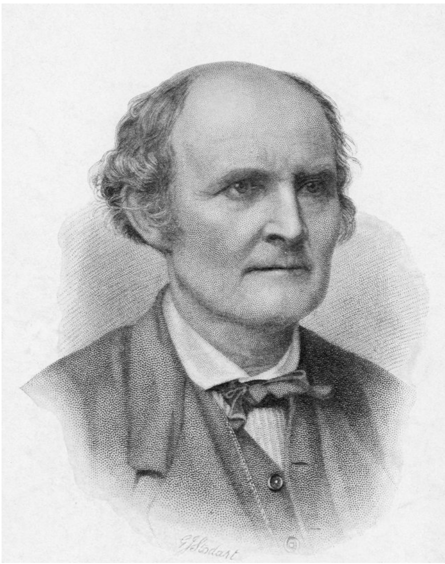
5.1 INTRODUCTION

Henry Smith's 1867 memoir was presented under two themes: the classification of ternary quadratic forms (Section 5.2) and the weight of a genus of ternary

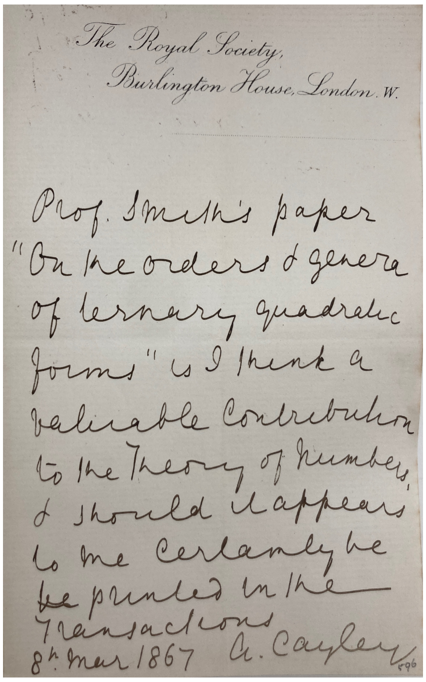
1 As part of his introduction to the *Report on the Theory of Numbers Part III* (1861) Henry Smith suggested some headings under which he would arrange the theory of quadratic forms. This included a reference to ternary quadratic forms suggesting he had planned to present the details at a future date. However, this work was not completed (Science, 1861, p. 169)

quadratic forms (Section 5.8). The presentation throughout has an exacting coherent style and structure showing the firm grasp he had on this subject. The opening paragraph of his memoir reads:[2]

> Eisenstein, in a memoir entitled *Neue Theoreme der Höheren Arithmetik*, has defined the ordinal and generic characters of ternary quadratic forms of an uneven determinant, and, in the case of definite forms, has assigned the weight of any given order or genus (Eisenstein, 1847a). But he has not considered forms of an even determinant, neither has he given any demonstrations of his results. To supply these omissions, and so far to complete the work of Eisenstein, is the object of the present memoir (Smith, 1867, p. 255).



(a) Arthur Cayley FRS (1821-1895)



The Royal Society,
Burlington House, London. W.

Prof. Smith's paper "On the orders & genera of ternary quadratic forms" is I think a valuable contribution to the Theory of Numbers, & should it appears to me certainly be be printed in the Transactions
8th Mar 1867 A. Cayley

(b) Referee's Report

Figure 22: Referee's Report by Arthur Cayley (1867) 'On the Orders and Genera of Ternary Quadratic Forms', *Philosophical Transactions of the Royal Society.* [Courtesy of the Royal Society, London Ref. RR/6/273]

2 Arthur Cayley also provided a referee's report for Henry Smith's memoir *On systems of linear indeterminate equations and congruences* (1861) published in the *Philosophical Transactions of the Royal Society* [Ref. RR/4/242] (Smith, 1862). Smith and Cayley were in communication by letter between June 1857 and January 1858 in relation to this memoir of 1861, the central theorem establishes the *Smith Normal Form* for matrices with integer entries (Crilly, 2006a, f.82, p. 520). Smith would later provide referee's reports for fifteen memoirs by Cayley, published in journals of the Royal Society, from 1863–1880.
See Referees' Reports at https://catalogues.royalsociety.org/CalmView/

5.2 CLASSIFICATION OF FORMS BY HENRY SMITH

In this memoir Smith makes use of Gauss's notation for ternary quadratic forms where the *discriminant* is the negative of Gauss's *determinant* and *contravariant* of f is the negative of Gauss's *adjoint* of f.

ARTICLE 1 (*ibid*, p. 270)
A ternary quadratic form f may now be represented as

$$f = ax^2 + a'y^2 + a''z^2 + 2byz + 2b'xz + 2b''xy$$

where f is termed *primitive* (i.e. that the six integers a, a', a'', b, b', b'' admit of no common divisor other than unity) and its *discriminant* is different from zero. The *discriminant*, or the determinant of the matrix

$$\begin{vmatrix} a & b'' & b' \\ b'' & a' & b \\ b' & b & a'' \end{vmatrix} \qquad (*_A)$$

is represented by D; by Ω is denoted the greatest common divisor of the minor determinants of the matrix $(*_A)$; by ΩF the *contravariant* of f, or the form

$$(a'a'' - b^2)x^2 + (a''a - b'^2)y^2 + (aa' - b''^2)z^2$$
$$+2(b'b'' - ab)yz + 2(b''b - a'b')zx + 2(bb' - a''b'')xy.$$

F is termed the *primitive contravariant* of f, and is represented as

$$F = Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'xz + 2B''xy$$

If $D = \Omega^2\Delta$, with Δ an integer, the discriminant, contravariant and primitive contravariant of F are respectively $\Delta^2\Omega$, Δf and f. The numbers Ω and Δ are the *arithmetical invariants* of f, i.e. they remain unaltered when f is transformed by any transformation whose determinant is unity and the coefficients integral numbers. The primitive form f, and the class of forms containing f, are described as a form, and class, of the invariants $[\Omega, \Delta]$. Similarly, F, and the class of forms containing F, are described as a form, and class, of the invariants $[\Delta, \Omega]$. The relation between the forms f and F is reciprocal and this reciprocity extends throughout the whole theory. The contravariant of f and F, and the invariants Ω and Δ are everywhere simultaneously interchangeable. The discriminant, contravariant and primitive contravariant of f are respectively $\Omega^2\Delta$, ΩF and F. The discriminant, contravariant and primitive contravariant of F are respectively $\Delta^2\Omega$, Δf and f. The contravariant of f and F, and the invariants Ω and Δ are everywhere simultaneously interchangeable.

⊠

Smith provides the following example which is followed by recalling Gauss's definition of equivalence of ternary quadratic forms [Article 268].

Example (*ibid*, p. 256).
Let $f = 2x^2 + 3y^2 + 3z^2 + 2yz + 2xz + 2xy$ denote a *primitive* ternary quadratic form with $\Omega = 1$. The discriminant D of f is the determinant of the matrix

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 12$$

The *primative contravariant* of f is $F = 8x^2 + 5y^2 + 5z^2 - 2yz - 4xz - 4xy$. If $D = \Omega^2\Delta$, then the discriminant of F is $\Delta^2\Omega = 12^2 1 = 144$.

⊠

Article 268 (Gauss, 1986, p. 294)
The ternary quadratic form f_1 becomes a new ternary quadratic form f_2 when new variables are introduced. Let f_1 be transformed into f_2 by

$$x = \alpha X + \beta Y + \gamma Z$$
$$y = \alpha' X + \beta' Y + \gamma' Z$$
$$z = \alpha'' X + \beta'' Y + \gamma'' Z$$

where $\alpha, \beta, \gamma, \alpha', \beta', \gamma', \alpha'', \beta'', \gamma''$ are nine particular integers and X, Y, Z are the new variables. Forms f_1 and f_2 are said to be *equivalent* when one may be transformed into the other by a linear transformation of determinant unity i.e.

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} = \pm 1$$

All equivalent forms are said to constitute a *class*.

⊠

To define an *order* of ternary quadratic forms Smith defines the following.

Article 2 (Smith, 1867, p. 270)
A primitive form f is *properly primitive* when at least one of its three principle coefficients a, a', a'' is odd; it is *improperly primitive* when those coefficients are all even. In an *improperly primitive* form, one at least of the three coefficients b, b', b'' is odd, or the form would not be primitive. In the case of definite forms, only those which are positive are considered. In the case of such forms Ω, as well as Δ, are positive, in order that F as well as f may be positive. In the case of indefinite forms, Ω and Δ will be attributed opposite signs, so that, in this case, the discriminant of f and F will be of opposite signs. Thus the

definiteness, or indefiniteness, of a form is indicated by the sign of its invariants. Let $D = \Omega^2\Delta$, with Δ an integer. Two primitive forms of the same invariants $[\Omega, \Delta]$ are said to belong to the same *order* when they and their primitive contravariants are alike properly or alike improperly primitive.

⊠

In his memoir of 1847 Eisenstein considers quadratic forms of an odd discriminant only. Such forms, and their contravariants, are always properly primitive and their particular generic characters are those with respect to the odd primes dividing the discriminant. They have no supplementary characters i.e. characters with respect to 4 or 8. The case of forms of an even discriminant is more complicated. Smith identified that apart from the properly primitive order, there may exist, in this case, an improperly primitive order in which the forms themselves are improperly primitive and their contravariants properly primitive. Similarly there may exist an improperly primitive order in which the forms themselves are properly primitive and their contravariants improperly primitive. Furthermore, forms of an even discriminant may have characters with respect to 4 or 8 and Smith provides a complete enumeration of these supplementary characters with careful distinction to the cases (Section 5.4). To facilitate this enumeration, the first table of this memoir identifies the supplementary characters of any proposed form (*ibid*, p. 258).

The second table of this memoir identifies the complete generic character of any proposed form (*ibid*, pp. 265–256). This table serves the same purposes for the ternary theory as that presented by Dirichlet for the binary theory (Dirichlet, 1839, p. 338). The table, like that of Dirichlet, distinguishes between the possible and impossible generic characters and Smith demonstrates the condition by which they are distinguished (Section 5.7). Besides the principal generic character, relating to odd primes dividing the discriminant, and in cases in which there is no supplementary character, Smith introduces a generic character which he calls the *simultaneous character* of the form and its contravariant (Section 5.5).

A synopsis of this memoir, under the theme of the classification of ternary forms, is given in the following extract from Leonard Eugene Dickson's *History of the Theory of Numbers Vol III* (1923) (Dickson, 1923, pp. 215–216).[3]

3 Leonard Eugene Dickson (1874–1954) greatly influenced the development of American mathematics at the turn of the century. One of the century's most prolific mathematicians, Dickson wrote 278 papers and 18 books covering a broad range of topics in his field. Dickson's *History of the Theory of Numbers* was published in three volumes from 1919, totalling more that 1600 pages, and took 9 years to complete. See (Fenster, 2005) and (Albert, 1955).
For an account of Dickson's research, in the context of how Gauss's *Disquisitiones Arithmeticae* penetrated American mathematics, see (Fenster, 2007).

Figure 23: Leonard Eugene Dickson (1874–1954)

The identities[4]

$$f(x_1, y_1, z_1)f(x_2, y_2, z_2) - \frac{1}{4}\left(x_1\frac{\partial f}{\partial x_2} + y_1\frac{\partial f}{\partial y_2} + z_1\frac{\partial f}{\partial z_2}\right)^2$$
$$= \Omega F(y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, x_1y_2 - y_1x_2) \quad (*_A)$$

$$F(x_1, y_1, z_1)F(x_2, y_2, z_2) - \frac{1}{4}\left(x_1\frac{\partial F}{\partial x_2} + y_1\frac{\partial F}{\partial y_2} + z_1\frac{\partial F}{\partial z_2}\right)^2$$
$$= \Delta f(y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, x_1y_2 - y_1x_2) \quad (*_B)$$

lead to the subdivision of the orders into genera. The first identity shows that the numbers, which are relatively prime to any odd prime factor ω of Ω and which are represented by f, are either all quadratic residues of ω or all non-residues of ω, hence f has the particular generic character (f/ω). The second identity shows that F has the character (F/δ), where δ is any odd prime factor of Δ. Also, as by Eisenstein, f and F have particular characters with respect to any odd prime dividing both Ω and Δ. These characters, which depend on the odd prime divisors of the invariants, are called the principal generic characters. These same identities led Smith to particular *supplementary characters* of each f and F with respect to 4 and 8, analogous to the case of binary forms. When f and F are both properly primitive and neither Ω nor Δ are multiples of 4, f and F taken separately have no particular characters with

4 Leonard Eugene Dickson refers to $(*_A)$ and $(*_B)$ as 'Smith's Identities' (Dickson, 1923, p. 216). Smith does not provide a demonstration for his identities, however, a demonstration may be found in (Bachmann, 1898, pp. 7–10) (Appendix D).

respect to 4 or 8, but have jointly a *simultaneously character* with respect to 4 or 8, defined by means of representations $m = f(x, y, z)$, $M = F(X, Y, Z)$ for which $xX + yY + zZ \equiv 0 (\text{mod } 2)$.

The aggregate of the particular characters of f and F gives the complete character. Two forms (or classes) with the same complete character (and the same Ω and Δ) are said to belong to the same genus. A two-page table serves to distinguish those complete characters which are possible (i.e., to which existing genera correspond) from those which are impossible, the distinction being expressed by a specified relation between the characters.

⊠

Smith's classification of ternary quadratic forms begins with the definition of the principle characters.

5.3 PRINCIPLE CHARACTERS

According to Smith a subdivision of an order of ternary quadratic forms into genera may be achieved from the identical equations $(*_A)$ and $(*_B)$. Firstly the principal generic characters, relating to odd primes dividing the discriminant, were first proposed in 1847 but without demonstration (Eisenstein, 1847a, p. 126). Smith states that the equations $(*_A)$ and $(*_B)$ imply the following theorem which he 'attributed to Eisenstein' (*ibid*, p. 257).

THEOREM (Smith, 1867, p. 257)
If ω represents any odd prime dividing Ω and δ represents any odd prime dividing Δ:

1. The numbers, prime to ω, which are represented by f, are either all quadratic residues of ω, or quadratic non-residues of ω. Consequently we attribute to f the *particular generic characters*

$$\left(\frac{f}{\omega}\right) = +1 \quad \text{or} \quad \left(\frac{f}{\omega}\right) = -1$$

2. The numbers, prime to δ, which are represented by f, are either all quadratic residues of δ, or quadratic non-residues of δ. Consequently we attribute to F the *particular generic characters*

$$\left(\frac{F}{\delta}\right) = +1 \quad \text{or} \quad \left(\frac{F}{\delta}\right) = -1$$

Furthermore, if Ω and Δ are both divisible by any odd prime, f and F will both have particular characters with respect to that prime.

⊠

5.4 SUPPLEMENTARY CHARACTERS

The complete enumeration of the supplementary characters for ternary quadratic forms requires a careful distinction of cases. Smith presents his table of supplementary characters, for each f and F with respect to 4 and 8, analogous to the case of binary forms (Table 9). He uses identities $(*_A)$ and $(*_B)$ to determine the form taken by all odd numbers representable by f for various arrangements of the discriminant D. The supplementary characters are assigned based on the following table of particular characters.

ARTICLE 4 (Smith, 1867, p. 257)
Let m be an odd number represented by f (or F).

Table 12: Particular Characters with respect to 4 and 8

$m \equiv 1 (\text{mod } 4)$	$(-1)^{\frac{1}{2}(f-1)} = +1$
$m \equiv 3 (\text{mod } 4)$	$(-1)^{\frac{1}{2}(f-1)} = -1$
$m \equiv 1 (\text{mod } 8)$ or $m \equiv 7 (\text{mod } 8)$	$(-1)^{\frac{1}{8}(f^2-1)} = +1$
$m \equiv 3 (\text{mod } 8)$ or $m \equiv 5 (\text{mod } 8)$	$(-1)^{\frac{1}{8}(f^2-1)} = -1$
$m \equiv 1 (\text{mod } 8)$ or $m \equiv 3 (\text{mod } 8)$	$(-1)^{\frac{1}{2}(f-1)+\frac{1}{8}(f^2-1)} = +1.$
$m \equiv 5 (mod\ 8)$ or $m \equiv 7 (\text{mod } 8)$	$(-1)^{\frac{1}{2}(f-1)+\frac{1}{8}(f^2-1)} = -1$

⊠

The following are his tables of supplementary characters for f and F.

ARTICLE 4 (*ibid*, p. 258)
f is a form of the invariants $[\Omega, \Delta]$,
F is a form of the invariants $[\Delta, \Omega]$.

Note that Ψ identifies the simultaneous characters. Ψ is defined in Section 5.5 along with Smith's demonstration of the existence of the simultaneous characters.

Table 13: Supplementary Characters: f and F Properly Primitive

$\Omega \equiv 1 \pmod 2$

$\Delta \equiv 1 \pmod 2$	Ψ	
$\Delta \equiv 2 \pmod 4$	$(-1)^{\frac{1}{8}(F^2-1)}\Psi$	
$\Delta \equiv 4 \pmod 8$	$*(-1)^{\frac{1}{2}(f-1)}$	$(-1)^{\frac{1}{2}(F-1)}$
$\Delta \equiv 0 \pmod 8$	$*(-1)^{\frac{1}{2}(f-1)}$	$(-1)^{\frac{1}{2}(F-1)}$ $(-1)^{\frac{1}{8}(F^2-1)}$

$\Omega \equiv 2 \pmod 4$

$\Delta \equiv 1 \pmod 2$	$(-1)^{\frac{1}{8}(f^2-1)}\Psi$	
$\Delta \equiv 2 \pmod 4$	$(-1)^{\frac{1}{8}(f^2-1)+\frac{1}{8}(F^2-1)}\Psi$	
$\Delta \equiv 4 \pmod 8$	$\dagger(-1)^{\frac{1}{8}(f^2-1)}$ $*(-1)^{\frac{1}{2}(f-1)+\frac{1}{8}(f^2-1)}$	$(-1)^{\frac{1}{2}(F-1)}$
$\Delta \equiv 0 \pmod 8$	$*(-1)^{\frac{1}{2}(f-1)+\frac{1}{8}(f^2-1)}$ $\dagger(-1)^{\frac{1}{8}(f^2-1)}$	$(-1)^{\frac{1}{2}(F-1)}$ $(-1)^{\frac{1}{8}(F^2-1)}$

$\Omega \equiv 4 \pmod 8$

$\Delta \equiv 1 \pmod 2$	$(-1)^{\frac{1}{2}(f-1)}$	$*(-1)^{\frac{1}{2}(F-1)}$
$\Delta \equiv 2 \pmod 4$	$(-1)^{\frac{1}{2}(f-1)}$ $*(-1)^{\frac{1}{2}(F-1)+\frac{1}{8}(F^2-1)}$	$\dagger(-1)^{\frac{1}{8}(F^2-1)}$
$\Delta \equiv 4 \pmod 8$	$(-1)^{\frac{1}{2}(f-1)}$	$(-1)^{\frac{1}{2}(F-1)}$
$\Delta \equiv 0 \pmod 8$	$(-1)^{\frac{1}{2}(f-1)}$	$(-1)^{\frac{1}{2}(F-1)}$ $(-1)^{\frac{1}{8}(F^2-1)}$

	$\Omega \equiv 0 \pmod 8$	
$\Delta \equiv 1 \pmod 2$	$(-1)^{\frac{1}{2}(f-1)}$	$*(-1)^{\frac{1}{2}(F-1)}$
	$(-1)^{\frac{1}{8}(f^2-1)}$	
$\Delta \equiv 2 \pmod 4$	$(-1)^{\frac{1}{2}(f-1)}$	$*(-1)^{\frac{1}{2}(F-1)+\frac{1}{8}(F^2-1)}$
	$(-1)^{\frac{1}{8}(f^2-1)}$	$\dagger(-1)^{\frac{1}{8}(F^2-1)}$
$\Delta \equiv 4 \pmod 8$	$(-1)^{\frac{1}{2}(f-1)}$	$(-1)^{\frac{1}{2}(F-1)}$
	$(-1)^{\frac{1}{8}(f^2-1)}$	
$\Delta \equiv 0 \pmod 8$	$(-1)^{\frac{1}{2}(f-1)}$	$(-1)^{\frac{1}{2}(F-1)}$
	$(-1)^{\frac{1}{8}(f^2-1)}$	$(-1)^{\frac{1}{8}(F^2-1)}$

$*$ prefix to f, character is attributed to f only when $(-1)^{\frac{1}{2}(F-1)} = (-1)^{\frac{1}{2}(\Omega'-1)}$.
$*$ prefix to F, character is attributed to F only when $(-1)^{\frac{1}{2}(f-1)} = (-1)^{\frac{1}{2}(\Delta'-1)}$.
$\dagger$ prefix to f, character is attributed to f only when $(-1)^{\frac{1}{2}(F-1)} = -(-1)^{\frac{1}{2}(\Omega'-1)}$.
$\dagger$ prefix to F, character is attributed to F only when $(-1)^{\frac{1}{2}(f-1)} = -(-1)^{\frac{1}{2}(\Delta'-1)}$
where Ω' and Δ' is greatest odd divisor of Ω and Δ, taken with the same sign as Ω and Δ.

Table 14: Supplementary Characters: f Improperly and F Properly Primitive

	$\Omega \equiv 1 \pmod 2$ and $(-1)^{\frac{1}{2}(f-1)} = -(-1)^{\frac{1}{2}(\Delta-1)}$	
$\Delta \equiv 2 \pmod 4$	$(-1)^{\frac{1}{2}(F-1)}$	
$\Delta \equiv 0 \pmod 4$	$(-1)^{\frac{1}{2}(F-1)}$	$(-1)^{\frac{1}{8}(F^2-1)}$

Table 15: Supplementary Characters: f Properly and F Improperly Primitive

	$\Delta \equiv 1 \pmod 2$ and $(-1)^{\frac{1}{2}(F-1)} = -(-1)^{\frac{1}{2}(\Omega-1)}$	
$\Omega \equiv 2 \pmod 4$	$(-1)^{\frac{1}{2}(f-1)}$	
$\Omega \equiv 0 \pmod 4$	$(-1)^{\frac{1}{2}(f-1)}$	$(-1)^{\frac{1}{8}(f^2-1)}$

Smith advocates that 'the use of the table is most easily explained by an example'. The following simple example, which is typical of Smith's mathematical style, gives confidence to the reader and provide a welcomed pause to reflect on the details.

EXAMPLE (*ibid*, p. 259)
Let $f = (2, 7, 7, -1, 0, 0)$ denote a form of the invariants $[\Omega, \Delta] = [2, 24]$.
Its primitive contravariant is $F = (24, 7, 7, 1, 0, 0)$. Since $\Omega \equiv 2 (\text{mod } 4)$ and $\Delta \equiv 0 (\text{mod } 8)$, F has the supplementary characters

$$(-1)^{\frac{1}{2}(F-1)} \quad \text{and} \quad (-1)^{\frac{1}{8}(F^2-1)}$$

The values of these characters are found by an inspection of the coefficients of F, and are -1 and $+1$ respectively. Since $\Omega' = 1$ and $(-1)^{\frac{1}{2}(F-1)} = -(-1)^{\frac{1}{2}(\Omega'-1)}$ the character

$$(-1)^{\frac{1}{8}(f^2-1)}$$

is therefore attributable to f. The value of this character is again found by an inspection of the coefficients of f, and is $+1$.

⊠

Smith writes that 'the demonstrations of the assertions in the table (in so far as they relate to the supplementary characters) are obtained without any difficulty from the equations $(*_A)$ and $(*_B)$. It will suffice to consider one case as an example of the rest' (*ibid*, p. 259). He presents the following demonstration of the supplementary characters in the case when $\Omega \equiv 2 (\text{mod } 4)$ and $\Delta \equiv 0 (\text{mod } 8)$. He is thoughtful in his choice here as it is not a straightforward case but one which requires attention to a great many details. For this demonstration I have added my own illustrative diagrams to draw attention to his arithmetical style, that is reminiscent of Gauss, as well as supporting the explanation. Illustrative examples will follow this demonstration.

ARTICLE 4 (*ibid*, pp. 259–260)
Let f and F be both properly primitive.
f is a form of the invariants $[\Omega, \Delta]$.
F is a form of the invariants $[\Delta, \Omega]$.
Let $\Omega \equiv 2 (\text{mod } 4)$ and $\Delta \equiv 0 (\text{mod } 8)$.
Let n, m are two odd numbers representable by f,
i.e. $n = f(x_1, y_1, z_1)$ and $m = f(x_2, y_2, z_2)$.
Consider $(*_A)$ as a congruence modulo 8.

n, m odd

must be odd (since Ω is even), $x^2 \equiv 1 \pmod 8$ when x is odd.

$$nm - \left(x_1 \frac{\partial f}{2\partial x_2} + y_1 \frac{\partial f}{2\partial y_2} + z_1 \frac{\partial f}{2\partial z_2} \right)^2$$

$$= \Omega F(y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, x_1y_2 - y_1y_2)$$

$\Omega \equiv 2 \pmod 4$

odd or evenly even

An *evenly even number* n is a multiple of 4, *unevenly even* otherwise. F cannot represent unevenly even numbers. If $F(x_1, y_1, z_1)$ is an unevenly even number and $F(x_2, y_2, z_2)$ is an odd number represented by F, then the equation $(*_B)$ will now have a square congruous, modulo 8, to an unevenly even number, which is impossible.

Consider $nm - 1 \equiv \Omega F(y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, x_1y_2 - y_1x_2) \pmod 8$ according as $F(\sim)$ is *evenly even* or *odd*.
If $F(\sim)$ is evenly even, then $\Omega F(\sim) \equiv 0 \pmod 8$.
Now $nm - 1 \equiv 0 \pmod 8$ implies $nm \equiv 1 \pmod 8$, hence $n \equiv m \pmod 8$.
If $F(\sim)$ is odd, then $\Omega F(\sim) \equiv 2 \pmod 8$ or $\Omega F(\sim) \equiv 6 \pmod 8$.
A partition of $\Omega F(\sim)$ is achieved by the following conditions:
Let Ω' be the largest odd divisor of Ω.

i If $(-1)^{\frac{1}{2}(F-1)} = (-1)^{\frac{1}{2}(\Omega'-1)}$, then $\Omega F(\sim) \equiv 2 \pmod 8$.
Furthermore, $\frac{1}{2}(F-1)$ and $\frac{1}{2}(\Omega'-1)$ are either both *even* or both *odd*.
Hence $\frac{1}{2}(\Omega'-1) + \frac{1}{2}(F-1)$ is *even*, and the congruence

$$nm \equiv 1 + 2 \times (-1)^{\frac{1}{2}(\Omega'-1) + \frac{1}{2}(F-1)} \pmod 8$$

becomes $nm \equiv 3 \pmod 8$.

ii If $(-1)^{\frac{1}{2}(F-1)} = -(-1)^{\frac{1}{2}(\Omega'-1)}$, then $\Omega F(\sim) \equiv 6 \pmod 8$.
Furthermore, $\frac{1}{2}(F-1)$ and $\frac{1}{2}(\Omega'-1)$ are *even* $\setminus$ *odd* or *odd* $\setminus$ *even*.
Hence $\frac{1}{2}(\Omega'-1) + \frac{1}{2}(F-1)$ is *odd*, and the congruence

$$nm \equiv 1 + 2 \times (-1)^{\frac{1}{2}(\Omega'-1) + \frac{1}{2}(F-1)} \pmod 8$$

becomes $nm \equiv -1 \pmod 8$.

If $(-1)^{\frac{1}{2}(F-1)} = (-1)^{\frac{1}{2}(\Omega'-1)}$, then $nm \equiv 1(\text{mod } 8)$ or $nm \equiv 3(\text{mod } 8)$, i.e. $n \equiv m(\text{mod } 8)$ or $n \equiv 3m(\text{mod } 8)$. All odd numbers represented by f are either of the forms $\equiv 1(\text{mod } 8)$ or $\equiv 3(\text{mod } 8)$ or else all of the forms $\equiv 5(\text{mod } 8)$ or $\equiv 7(\text{mod } 8)$. Hence f has the supplementary character $(-1)^{\frac{1}{2}(f-1)+\frac{1}{8}(f^2-1)}$.
If $(-1)^{\frac{1}{2}(F-1)} = -(-1)^{\frac{1}{2}(\Omega'-1)}$, then $nm \equiv 1(\text{mod } 8)$ or $nm \equiv 7(\text{mod } 8)$, i.e. $n \equiv m(\text{mod } 8)$ or $n \equiv 7m(\text{mod } 8)$. All odd numbers represented by f are either of the forms $\equiv 1(\text{mod } 8)$ or $\equiv 7(\text{mod } 8)$ or else all of the forms $\equiv 3(\text{mod } 8)$ or $\equiv 5(\text{mod } 8)$. Hence f has the supplementary character $(-1)^{\frac{1}{8}(f^2-1)}$.

Let N, M are two odd numbers representable by F,
i.e. $N = F(x_1, y_1, z_1)$ and $M = F(x_2, y_2, z_2)$.
Consider $(*_B)$ as a congruence modulo 8.

N, M odd

must be odd (since Δ is even),
$x^2 \equiv 1(\text{mod } 8)$ when x is odd.

$$NM - \left(x_1 \frac{\partial F}{2\partial x_2} + y_1 \frac{\partial F}{2\partial y_2} + z_1 \frac{\partial F}{2\partial z_2}\right)^2$$

$$= \Delta f(y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2)$$

$\Delta \equiv 0(\text{mod } 8)$

odd or even

Now $NM - 1 \equiv 0(\text{mod } 8)$ implies $NM \equiv 1(\text{mod } 8)$, hence $N \equiv M(\text{mod } 8)$. All odd numbers representable by the form F will be either all $\equiv 1(\text{mod } 8)$ or $\equiv 7(\text{mod } 8)$, or else all $\equiv 3(\text{mod } 8)$ or $\equiv 5(\text{mod } 8)$. Consider $(*_B)$ as a congruence modulo 4. Now $NM - 1 \equiv 0(\text{mod } 4)$ implies $NM \equiv 1(\text{mod } 4)$, hence $N \equiv M(mod\ 4)$. All odd numbers representable by the form F will be $\equiv 1(\text{mod } 4)$, or all $\equiv 3(\text{mod } 4)$. Hence F has the supplementary characters $(-1)^{\frac{1}{2}(F-1)}$ and $(-1)^{\frac{1}{8}(F^2-1)}$.

⊠

We have seen how Gauss would encourage the reader with the suggestion that 'one can develop the argument without any trouble' (Section 3.2, p. 67). In a similar way Smith suggests that all the supplementary characters may be 'obtained without any difficulty from the equations $(*_A)$ and $(*_B)$'. The following demonstrations I have chosen to support his assertion.

A. Let $\Omega \equiv 4 (\text{mod } 8)$ and $\Delta \equiv 4 (\text{mod } 8)$.
n, m are two odd numbers representable by f,
i.e. $n = f(x_1, y_1, z_1)$ and $m = f(x_2, y_2, z_2)$.
Consider $(*_A)$ as a congruence modulo 4.
Now $nm - 1 \equiv 0 (\text{mod } 4)$ implies $nm \equiv 1 (\text{mod } 4)$, hence $n \equiv m (\text{mod } 4)$. All odd numbers representable by the form f will be $\equiv 1 (\text{mod } 4)$, or all $\equiv 3 (\text{mod } 4)$. So, for example, all odd numbers representable by the form $(4, 4, 5, 0, -2, 0)$ are $\equiv 3 (\text{mod } 4)$ and no odd numbers of the form $4n + 1$ can be represented by the form. Its supplementary character is $(-1)^{\frac{1}{2}(f-1)}$ and its value is -1.

B. Let $\Omega \equiv 4 (\text{mod } 8)$ and $\Delta \equiv 4 (\text{mod } 8)$.
N, M are two odd numbers representable by F,
i.e. $N = F(x_1, y_1, z_1)$ and $M = F(x_2, y_2, z_2)$.
Consider $(*_B)$ as a congruence modulo 4.
Now $NM - 1 \equiv 0 (\text{mod } 4)$ implies $NM \equiv 1 (\text{mod } 4)$, hence $N \equiv M (\text{mod } 4)$. All odd numbers representable by the form F will be $\equiv 1 (\text{mod } 4)$, or all $\equiv 3 (\text{mod } 4)$. So, for example, all odd numbers representable by the form $(5, 4, 4, 0, 2, 0)$ are $\equiv 1 (\text{mod } 4)$ and no odd numbers of the form $4n + 3$ can be represented by the form. Its supplementary character is $(-1)^{\frac{1}{2}(F-1)}$ and its value is $+1$.

C. Let $\Omega \equiv 0 (\text{mod } 8)$ and $\Delta \equiv 0 (\text{mod } 8)$.
n, m are two odd numbers representable by f,
i.e. $n = f(x_1, y_1, z_1)$ and $m = f(x_2, y_2, z_2)$.
Consider $(*_A)$ as a congruence modulo 8.
Now $nm - 1 \equiv 0 (\text{mod } 8)$ implies $nm \equiv 1 (\text{mod } 8)$, hence $n \equiv m (\text{mod } 8)$. All odd numbers representable by the form f will be either all $\equiv 1 (\text{mod } 8)$ or $\equiv 7 (\text{mod } 8)$, or else all $\equiv 3 (\text{mod } 8)$ or $\equiv 5 (\text{mod } 8)$. So, for example, all odd numbers representable by the form $(1, 8, 64, 0, 0, 0)$ are either $\equiv 1 (\text{mod } 8)$ or $\equiv 7 (\text{mod } 8)$, and no numbers of the form $8n + 3$ or $8n + 5$ can be represented by this form. Its supplementary character is $(-1)^{\frac{1}{8}(f^2-1)}$ and its value is $+1$.

D. Let $\Omega \equiv 0 (\text{mod } 8)$ and $\Delta \equiv 0 (\text{mod } 8)$.
N, M are two odd numbers representable by f,
i.e. $N = F(x_1, y_1, z_1)$ and $M = F(x_2, y_2, z_2)$.
Consider $(*_B)$ as a congruence modulo 8.
Now $NM - 1 \equiv 0 (\text{mod } 8)$ implies $NM \equiv 1 (\text{mod } 8)$, hence $N \equiv M (\text{mod } 8)$.
All odd numbers representable by the form F will be either all $\equiv 1 (\text{mod } 8)$ or $\equiv 7 (\text{mod } 8)$, or else all $\equiv 3 (\text{mod } 8)$ or $\equiv 5 (\text{mod } 8)$. So, for example, all odd numbers representable by the form $(64, 8, 1, 0, 0, 0)$ are either $\equiv 1 (\text{mod } 8)$ or $\equiv 7 (\text{mod } 8)$, and no numbers of the form $8n + 3$ or $8n + 5$ can be represented by this form. Its supplementary character is $(-1)^{\frac{1}{8}(f^2-1)}$ and its value is $+1$.

Example $f = (2, 7, 7, -1, 0, 0)$ is a form of the invariants $[\Omega, \Delta] = [2, 24]$ and its contravariant $F = (24, 7, 7, 1, 0, 0)$ is a form of the invariants $[\Delta, \Omega] = [24, 2]$. All odd numbers representable by the form f will be either all $\equiv 1(\textit{mod } 8)$ or $\equiv 7 (\text{mod } 8)$, or else all $\equiv 3 (\text{mod } 8)$ or $\equiv 5 (\text{mod } 8)$. In this case, by observation, all odd numbers representable by f are either $\equiv 1 (\text{mod } 8)$ or $\equiv 7 (\text{mod } 8)$, and no numbers of the form $8n + 3$ or $8n + 5$ can be represented by this form. The supplementary character of f is $(-1)^{\frac{1}{8}(f^2-1)}$ and its value is $+1$.

All odd numbers representable by the form F will be either all $\equiv 1 (\text{mod } 4)$ or $\equiv 3 (\text{mod } 4)$. These numbers may also be $\equiv 1 (\text{mod } 8)$ or $\equiv 7(\textit{mod } 8)$, or else all $\equiv 3 (\text{mod } 8)$ or $\equiv 5 (\text{mod } 8)$. In this case, by observation, all odd numbers representable by F are either $\equiv 3 (\text{mod } 4)$ or $\equiv 1 (\text{mod } 8)$ or $\equiv 7 (\text{mod } 8)$, and no numbers of the form $4n + 1$ or $8n + 3$ or $8n + 5$ can be represented by this form. The supplementary characters of F are $(-1)^{\frac{1}{4}(F-1)}$ and $(-1)^{\frac{1}{8}(F^2-1)}$ and their values are -1 and $+1$ respectively.

⊠

5.5 SIMULTANEOUS CHARACTERS

When f and F are both properly primitive and neither Ω nor Δ are multiples of 4, f and F taken separately have no particular characters with respect to 4 or 8, but have jointly a *simultaneously character* with respect to 4 or 8.

Definition (*ibid*, p. 262)
If $m = f(x, y, z)$ and $M = F(X, Y, Z)$, then the representations of m by f and M by F are said to be *simultaneous* when x, y, z, X, Y, Z satisfy

$$xX + yY + zZ \equiv 0 (\text{mod } 2)$$

⊠

In article 6 of this memoir, with great detail and clarity, Smith showes that if f and F are both properly primitive and neither Ω nor Δ are multiples of 4,

f and F taken separately have no particular characters with respect to 4 or 8, but have jointly a *simultaneous character* with respect to 4 or 8. When f and F are properly primitive and neither Ω nor Δ is evenly even, the simultaneous characters of f and F are:

Table 16: Simultaneous Characters: f and F Properly Primitive

	$\Omega \equiv 1 (\text{mod } 2)$	$\Omega \equiv 2 (\text{mod } 4)$
$\Delta \equiv 1 (\text{mod } 2)$	Ψ	$(-1)^{\frac{1}{8}(f^2-1)}\Psi$
$\Delta \equiv 2 (\text{mod } 4)$	$(-1)^{\frac{1}{8}(F^2-1)}\Psi$	$(-1)^{\frac{1}{8}(f^2-1)+\frac{1}{8}(F^2-1)}\Psi$

The symbol Ψ is defined as

$$\Psi = (-1)^{\frac{1}{2}(\Delta' f+1)\frac{1}{2}(\Omega' F+1)}$$

where Ω' and Δ' is greatest odd divisor of Ω and Δ, taken with the same sign as Ω and Δ (f and F in the exponent of this unit denoting odd numbers simultaneously represented by f and F). The demonstration of these simultaneous characters requires a careful distinction of cases and Smith establishes each using the same mathematical technique. He firstly states and demonstrates a theorem which allow the forms f and F to be substituted by equivalent forms φ and Φ, which have no mixed terms and, which satisfy certain congruences for the modulus 4 or 8. In each of the four cases (using φ and Φ) he identifies the form taken by all pairs of odd numbers simultaneously represented by f and F. Finally, using the arithmetic of congruences, he shows that the proposed simultaneous character will always have the same value for every pair of simultaneously odd numbers.

THEOREM (*ibid*, p. 259).
There exists pairs of forms φ and Φ, equivalent to f and F, and satisfying the congruences

$$\begin{aligned} \varphi &\equiv \alpha x^2 + \beta\Omega y^2 + \gamma\Omega\Delta z^2 (\text{mod } \triangledown) \\ \Phi &\equiv \beta\gamma\Omega\Delta x^2 + \alpha\gamma\Delta y^2 + \alpha\beta z^2 (\text{mod } \triangledown) \\ \alpha\beta\gamma &\equiv 1 (\text{mod } \triangledown) \end{aligned}$$

for any proposed modulus $\triangledown$; but this modulus must be odd, if either f or F is improperly primitive.

⊠

The demonstration for this theorem, and an illustrative example, is presented as (Appendix E). Using this theorem Smith establishes the four simultaneous characters as follows:

ARTICLE 6 (*ibid*, pp. 262–264)
Let f and F be both properly primitive.
f is a form of the invariants $[\Omega, \Delta]$.
F is a form of the invariants $[\Delta, \Omega]$.
Let $m = \varphi(x, y, z)$ and $M = \Phi(X, Y, Z)$.

A. (*ibid*, p. 263)
Let $\Omega \equiv 1 (\text{mod } 2)$ and $\Delta \equiv 1 (\text{mod } 2)$.

There exists pairs of forms φ and Φ, equivalent to f and F, and satisfying the congruences

$$\begin{aligned} \Delta\varphi &\equiv \alpha x^2 + \beta y^2 + \gamma z^2 (\text{mod } 4) \\ \Omega\Phi &\equiv \beta\gamma X^2 + \alpha\gamma Y^2 + \alpha\beta Z^2 (\text{mod } 4) \\ \textit{or} \quad \Omega\Phi &\equiv \alpha X^2 + \beta Y^2 + \gamma Z^2 (\text{mod } 4) \end{aligned}$$

since $\alpha\beta\gamma \equiv 1 (\text{mod } 4)$.

Δm is odd. The possibilities for Δm are α, β and γ modulo 4. All possible combinations of the values x, y, z, X, Y, Z such that $xX + yY + zZ \equiv 0 (\text{mod } 2)$ are considered i.e. values which render m and M simultaneously odd. The following observations are made in this case:

If $\Delta m \equiv \alpha (\text{mod } 4)$ then $\Omega M \equiv \beta$ or $\gamma (\text{mod } 4)$.

If $\Delta m \equiv \beta (\text{mod } 4)$ then $\Omega M \equiv \alpha$ or $\gamma (\text{mod } 4)$.

If $\Delta m \equiv \gamma (\text{mod } 4)$ then $\Omega M \equiv \alpha$ or $\beta (\text{mod } 4)$.

The expression

$$\frac{1}{2}(\Delta m + 1)\frac{1}{2}(\Omega M + 1)$$

for all 6 cases will be congruent modulo 2 to one of the three numbers

$$\frac{1}{2}(\alpha + 1)\frac{1}{2}(\beta + 1) \quad , \quad \frac{1}{2}(\alpha + 1)\frac{1}{2}(\gamma + 1) \quad , \quad \frac{1}{2}(\beta + 1)\frac{1}{2}(\gamma + 1)$$

These numbers are all congruous to one another, modulo 2, by virtue of the congruence $\alpha\beta\gamma \equiv 1 (\text{mod } 4)$ implying $\alpha + \beta + \gamma + 1 \equiv 0 (\text{mod } 4)$.[5]

Therefore the unit $(-1)^{\frac{1}{2}(\Delta m+1)\frac{1}{2}(\Omega M+1)}$, i.e.

$$\Psi = (-1)^{\frac{1}{2}(\Delta' f+1)\frac{1}{2}(\Omega' F+1)}$$

always has the same value for every pair of odd numbers simultaneously represented by f and F.

⊠

5 $\alpha\beta\gamma \equiv 1 (\text{mod } 4) \Rightarrow \alpha, \beta, \gamma$ are odd numbers.
$\alpha\beta\gamma \equiv 1 (\text{mod } 4) \Rightarrow \alpha\beta \equiv \gamma (\text{mod } 4)$ or $\beta\gamma \equiv \alpha (\text{mod } 4)$ or $\alpha\gamma \equiv \beta (\text{mod } 4)$.
$\alpha\beta + \beta\gamma + \alpha\gamma \equiv \alpha + \beta + \gamma (\text{mod } 4)$,
$3 \equiv \alpha + \beta + \gamma (\text{mod } 4)$,
$-1 \equiv \alpha + \beta + \gamma (\text{mod } 4)$,
$\therefore \alpha + \beta + \gamma + 1 \equiv 0 (\text{mod } 4)$.

B. (*ibid*, p. 263)
Let $\Omega \equiv 2 (\text{mod } 4)$ and $\Delta \equiv 1 (\text{mod } 2)$.

There exists pairs of forms φ and Φ, equivalent to f and F, and satisfying the congruences

$$\begin{aligned} \Delta\varphi &\equiv \alpha x^2 + 2\beta y^2 + 2\gamma z^2 (\text{mod } 8) \\ \Omega'\Phi &\equiv 2\beta\gamma X^2 + \alpha\gamma Y^2 + \alpha\beta Z^2 (\text{mod } 4) \\ \textit{or} \quad \Omega'\Phi &\equiv 2\alpha X^2 + \beta Y^2 + \gamma Z^2 (\text{mod } 4) \end{aligned}$$

since $\alpha\beta\gamma \equiv 1 (\text{mod } 4)$.

Δm is odd. The possibilities for Δm are α, $\alpha + 2\beta$, $\alpha + 2\gamma$ and $\alpha + 2\beta + 2\gamma$ modulo 8. All possible combinations of the values x, y, z, X, Y, Z such that $xX + yY + zZ \equiv 0 (\text{mod } 2)$ are considered i.e. values which render m and M simultaneously odd. The following observations are made:

If $\Delta m \equiv \alpha (\text{mod } 8)$ then $\Omega' M \equiv \beta$ or $\gamma (\text{mod } 4)$.

If $\Delta m \equiv \alpha + 2\beta (\text{mod } 8)$ then $\Omega' M \equiv -\beta$ or $\gamma (\text{mod } 4)$.

If $\Delta m \equiv \alpha + 2\gamma (\text{mod } 8)$ then $\Omega' M \equiv \beta$ or $-\gamma (\text{mod } 4)$.

If $\Delta m \equiv \alpha + 2\beta + 2\gamma (\text{mod } 8)$ then $\Omega' M \equiv -\beta \textit{ or } -\gamma (\text{mod } 4)$.

The expression

$$\frac{1}{8}(\Delta^2 m^2 - 1) + \frac{1}{2}(\Delta m + 1)\frac{1}{2}(\Omega' M + 1)$$

for all 8 cases will be congruent to one of the four numbers:

$$\frac{1}{8}(\alpha + 1)(\alpha + 2\beta + 1) \quad , \quad \frac{1}{8}(\alpha + 1)(\alpha + 2\gamma + 1)$$

$$\frac{1}{8}(\alpha + 2\beta + 2\gamma + 1)(\alpha + 2\beta + 1) \quad , \quad \frac{1}{8}(\alpha + 2\beta + 2\gamma + 1)(\alpha + 2\gamma + 1)$$

These numbers are all congruous to one another, modulo 2, by virtue of the congruence $\alpha\beta\gamma \equiv 1 (\text{mod } 4)$ implying $\alpha + \beta + \gamma + 1 \equiv 0 (\text{mod } 4)$.

Therefore the unit $(-1)^{\frac{1}{8}(\Delta^2 m^2 - 1)}\Psi$ i.e.

$$(-1)^{\frac{1}{8}(f^2 - 1)}\Psi$$

always has the same value for every pair of odd numbers simultaneously represented by f and F.

$\boxtimes$

C. (*ibid*, p. 264)
Let $\Omega \equiv 1 (\text{mod } 2)$ and $\Delta \equiv 2 (\text{mod } 4)$.

The relation between the forms f and F is reciprocal and this reciprocity extends throughout the whole theory. The contravariant of f and F, and the invariants Ω and Δ are everywhere simultaneously interchangeable. In this case the simultaneous character of the forms f and F may be inferred (or demonstrated) from the result in case B.

D. (*ibid*, p. 264)
Let $\Omega \equiv 2 (\text{mod } 4)$ and $\Delta \equiv 2 (\text{mod } 4)$.

There exists pairs of forms φ and Φ, equivalent to f and F, and satisfying the congruences

$$\begin{array}{rrcl} & \Delta'\varphi & \equiv & \alpha x^2 + 2\beta y^2 + 4\gamma z^2 (\text{mod } 8) \\ & \Omega'\Phi & \equiv & 4\beta\gamma X^2 + 2\alpha\gamma Y^2 + \alpha\beta Z^2 (\text{mod } 8) \\ \textit{or} & \Omega'\Phi & \equiv & 4\alpha X^2 + 2\beta Y^2 + \gamma Z^2 (\text{mod } 8) \end{array}$$

since $\alpha\beta\gamma \equiv 1 (\text{mod } 4)$.

Δm is odd. The possibilities for Δm are α, $\alpha + 2\beta$, $\alpha + 4\gamma$ and $\alpha + 2\beta + 4\gamma$ modulo 8. All possible combinations of the values x, y, z, X, Y, Z such that $xX + yY + zZ \equiv 0 (\text{mod } 2)$ are considered i.e. values which render m and M simultaneously odd. The following observations are made:

If $\Delta' m \equiv \alpha (\text{mod } 8)$ then $\Omega' M \equiv \gamma$ or $\gamma + 2\beta (\text{mod } 8)$.

If $\Delta' m \equiv \alpha + 2\beta (\text{mod } 8)$ then $\Omega' M \equiv \gamma$ or $\gamma + 2\beta + 4 (\text{mod } 8)$.

If $\Delta' m \equiv \alpha + 4\gamma (\text{mod } 8)$ then $\Omega' M \equiv \gamma + 4$ or $\gamma + 2\beta + 4\gamma (\text{mod } 8)$. $(*_C)$

If $\Delta' m \equiv \alpha + 2\beta + 4\gamma (\text{mod } 8)$ then $\Omega' M \equiv \gamma + 4$ or $\gamma + 2\beta (\text{mod } 8)$.

The expression

$$\frac{1}{8}(\Delta'^2 m^2 - 1) + \frac{1}{8}(\Omega'^2 M^2 - 1) + \frac{1}{2}(\Delta' m + 1)\frac{1}{2}(\Omega' M + 1)$$

will be congruent to one of the two numbers

$$\frac{1}{8}(\alpha + \gamma)(\alpha + \gamma + 2) \quad , \quad \frac{1}{8}(\alpha + \gamma + 2\beta)(\alpha + \gamma + 2\beta + 2) \qquad (*_D)$$

These numbers are all congruous to one another, modulo 2 $(*_E)$, by virtue of the congruence $\alpha\beta\gamma \equiv 1 (\text{mod } 4)$ implying $\alpha + \beta + \gamma + 1 \equiv 0 (\text{mod } 4)$. Therefore the unit $(-1)^{\frac{1}{8}(\Delta'^2 m^2 - 1) + \frac{1}{8}(\Omega'^2 M^2 - 1)}\Psi$, i.e.

$$(-1)^{\frac{1}{8}(f^2-1)+\frac{1}{8}(F^2-1)}\Psi$$

always has the same value for every pair of odd numbers simultaneously represented by f and F.

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Smith presents the details here, in each of the four cases, using a consistent pattern of presentation. Similar point are sometimes phrased, from case to case, slightly differently to give the reader an improved perspective and understanding. The observations made may be easily established. I would like to present some further basic arithmetical details of $(*_C)$, $(*_D)$ and $(*_E)$ from demonstration D to confirm this point. These details are followed by an example.

$(*_C)$: If $\Delta' m \equiv \alpha + 4\gamma (\text{mod } 8)$ then $\Omega' M \equiv \gamma + 4$ or $\gamma + 2\beta + 4 (\text{mod } 8)$.

The condition $(*_C)$ will ensures that $m = \varphi(x, y, z)$ and $M = \Phi(X, Y, Z)$ are simultaneously odd i.e. $xX + yY + zZ \equiv 0(\text{mod } 2)$.
If $\Delta m \equiv \alpha + 4\gamma(\text{mod } 8) \equiv \alpha + 4(\text{mod } 8)$ then x must be odd, y even and z odd. Consequently:

1. X, Z can be odd and Y even i.e.

   odd *even* *odd*
   ↓ ↓ ↓
   $$xX + yY + zZ \equiv 0(\text{mod } 2)$$
   ↑ ↑ ↑
   odd *even* *odd*

   Now $\Omega' M \equiv 4\alpha + \gamma(\text{mod } 8) \equiv 4 + \gamma(\text{mod } 8)$ since α odd.

2. X, Y, Z are odd i.e.

   odd *even* *odd*
   ↓ ↓ ↓
   $$xX + yY + zZ \equiv 0(\text{mod } 2)$$
   ↑ ↑ ↑
   odd *odd* *odd*

   Now $\Omega' M \equiv 4\alpha + 2\beta + \gamma(\text{mod } 8) \equiv \gamma + 2\beta + 4(\text{mod } 8)$ since α odd.

Hence if $\Delta' m \equiv \alpha + 4(\text{mod } 8)$ then $\Omega' M \equiv \gamma + 4$ *or* $\gamma + 2\beta + 4(\text{mod } 8)$.

$(*_D)$: The expression

$$\frac{1}{8}(\Delta'^2 m^2 - 1) + \frac{1}{8}(\Omega'^2 M^2 - 1) + \frac{1}{2}(\Delta' m + 1)\frac{1}{2}(\Omega' M + 1)$$

evaluated in the case when $\Delta' m \equiv \alpha + 4(\text{mod } 8)$ and $\Omega' M \equiv \gamma + 4(\text{mod } 8)$ will be congruent to one of the two numbers

$$\frac{1}{8}(\alpha + \gamma)(\alpha + \gamma + 2) \quad , \quad \frac{1}{8}(\alpha + \gamma + 2\beta)(\alpha + \gamma + 2\beta + 2)$$

In this case

$$\frac{1}{8}((\alpha+4)^2-1)+\frac{1}{8}(\gamma+4^2-1)+\frac{1}{2}(\alpha+4+1)\frac{1}{2}(\gamma+4+1)$$
$$\equiv\frac{1}{8}(\alpha+\gamma)(\alpha+\gamma+2)(\text{mod } 8)$$

$(*_E)$: These numbers are all congruous to one another, modulo 2, by virtue of the congruence $\alpha\beta\gamma\equiv 1(\text{mod } 4)$ implying $\alpha+\beta+\gamma+1\equiv 0(\text{mod } 4)$.

$\alpha\beta\gamma\equiv 1(\text{mod } 4)\Rightarrow \alpha,\beta,\gamma$ are odd numbers.
$\alpha\beta\gamma\equiv 1(\text{mod } 4)\Rightarrow \alpha\beta\gamma\equiv 1(\text{mod } 2)$.
$\alpha\beta\gamma\equiv 1(\text{mod } 2)\Rightarrow \alpha\beta\equiv\gamma(\text{mod } 2)$ or $\beta\gamma\equiv\alpha(\text{mod } 2)$ or $\alpha\gamma\equiv\beta(\text{mod } 2)$.
Also $\alpha+\beta+\gamma+1\equiv 0(\text{mod } 2)$ and $\alpha^2\equiv\beta^2\equiv\gamma^2\equiv 1(\text{mod } 2)$.

Using the arithmetic of congruences it may be shown that

$$(\alpha+\gamma)(\alpha+\gamma+2)\equiv(\alpha+\gamma+2\beta)(\alpha+\gamma+2\beta+2)(\text{mod } 2)$$

Consequently the simultaneous character in this case will have an exponent of either 0 or 1 modulo 2. Hence the character will always has the same value for every pair of odd numbers simultaneously represented by f and F.

Example Let $\Omega\equiv 2(\text{mod } 4)$ and $\Delta\equiv 2(\text{mod } 4)$. There exists pairs of forms φ and Φ, equivalent to f and F, and satisfying the congruences

$$\begin{aligned}\Delta'\varphi &\equiv 7x^2+6y^2+4z^2(\text{mod } 8)\\ \Omega'\Phi &\equiv 4X^2+6Y^2+Z^2(\text{mod } 8)\end{aligned}$$

The simultaneous characters $(-1)^{\frac{1}{8}(f^2-1)+\frac{1}{8}(F^2-1)}\Psi$ always has the same value for every pair of odd numbers m, M simultaneously represented by φ and Φ. For example when $m=\varphi(3,2,2)=103$ and $M=\Phi(4,2,1)=89$

$$(-1)^{\frac{1}{8}(f^2-1)+\frac{1}{8}(F^2-1)}\Psi=+1.$$

⊠

In article 7 of this memoir Smith draws attention to the importance of these simultaneous characters by making some further observation on restrictions that apply in each case. He was always mindful of special cases and the necessity for complete details. William Spottiswoode wrote about this attribute of Smith's work: 'Of unfinished work, or of 'ragged ends' as he used to call them, he was as nearly intolerant as he could be of anything' (Spottiswoode, 1883, p. 382).

Article 7 (*ibid*, p. 265)
Let $m = f(x, y, z)$ and $M = F(X, Y, Z)$.

A. (Smith, 1867, p. 264)
Let $\Omega \equiv 1 \pmod 2$ and $\Delta \equiv 1 \pmod 2$.
The simultaneous character is $\Psi = (-1)^{\frac{1}{2}(\Delta' f+1)\frac{1}{2}(\Omega' F+1)}$.

Let $\Psi = -1$.
If m and M are any two odd numbers simultaneously represented by f and F, then $m \equiv \Delta \pmod 4$ and $M \equiv \Omega \pmod 4$. The restriction imposed on the numbers m and M by the simultaneous character are that f cannot represent numbers congruous to $7\Delta \pmod 8$ and F cannot represent numbers congruous to 7Ω i.e. it cannot represent them simultaneously with the representation of odd numbers. Hence f can only represent odd numbers congruous to $\Delta, 3\Delta, 5\Delta \pmod 8$ and F can only represent odd numbers congruous to $\Omega, 3\Omega, 5\Omega \pmod 8$. Numbers congruous to 3Δ are represented by f, and numbers congruous to 3Ω are represented by F; but these representations are not simultaneous with the representations of any odd number by F in the first case, and by f in the second.

Let $\Psi = +1$.
If m and M are any two odd numbers simultaneously represented by f and F, one at least of the two congruences $m \equiv -\Delta \pmod 4$ and $M \equiv -\Omega \pmod 4$ must be satisfied. Subject to this restriction f may represent odd numbers congruous to $\Delta, 3\Delta, 5\Delta, 7\Delta \pmod 8$ and F may represent odd numbers congruous to $\Omega, 3\Omega, 5\Omega, 7\Omega \pmod 8$.

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The restrictions which apply in the following cases

B. $\Omega \equiv 2 \pmod 4$ and $\Delta \equiv 1 \pmod 2$,

C. $\Omega \equiv 1 \pmod 2$ and $\Delta \equiv 2 \pmod 4$,

D. $\Omega \equiv 2 \pmod 4$ and $\Delta \equiv 2 \pmod 4$,

appear as Appendix F of this thesis.

Smith completes his classification by presenting a table of the complete generic characters for ternary quadratic forms (Table 24).

5.6 COMPLETE GENERIC CHARACTERS

The complex of all particular characters of a given form or class, and to its primitive contravariant, including their simultaneous character, if any, constitutes its *complete* or *generic character*. Those classes which have the same complete character are considered to belong to the same *genus*. The complete character of a form is possessed not only by every form of the same class but by every form of any class belonging to the same genus. Hence a subdivision of the whole order of properly primitive classes of a given determinant may be achieved. But not every complete generic character that can be assigned *à priori* is a character of any existing genus of forms. In article 8 of this memoir Smith presents a two–page table (Figure 24) which serves to distinguish those complete characters which are possible, i.e. to which existing genera correspond, from those which are impossible (Smith, 1867, pp. 267–268). In article 9 he makes this distinction by establishing a specific relation between the characters which he termed a 'condition of possibility'. In presenting these tables of characters Smith successfully completed the classification of ternary quadratic forms.

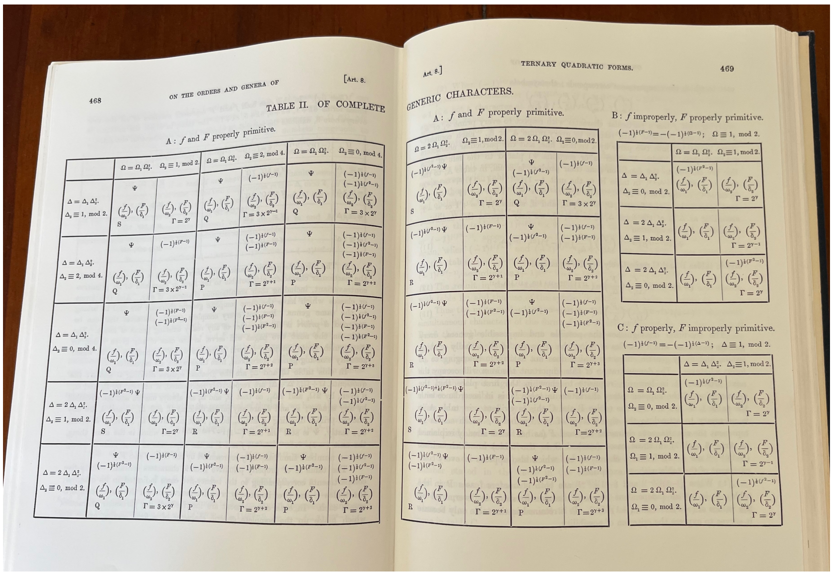


Figure 24: The Complete Generic Characters for Ternary Quadratic Forms taken from the *The collected mathematical papers of Henry John Stephen Smith–Volume I* (Smith, 1894a, pp. 468–469).

Article 8 (*ibid*, p. 266)
In this table Ω_2^2 and Δ_2^2 denote the greatest square dividing Ω and Δ. The quotients $\Omega \div \Omega_2^2$ and $\Delta \div \Delta_2^2$ are respectively represented, if odd, by Ω_1 and Δ_1, if even by $2\Omega_1$ and $2\Delta_1$, so that Ω_1 and Δ_1 are always odd and not divisible by any square. ω_1 and δ_1 are any primes dividing Ω_1 and Δ_1, ω_2 and δ_2 are any odd primes dividing Ω_2 and Δ_2, but ω_2 must not divide Ω_1, nor must δ_2 divide Δ_1. Lastly Ψ is still the unit

$$(-1)^{\frac{1}{2}(\Delta' f+1)\frac{1}{2}(\Omega' F+1)}$$

or, which is the same thing as the unit

$$(-1)^{\frac{1}{2}(\Delta_1 f+1)\frac{1}{2}(\Omega_1 F+1)}$$

where f and F in the exponents of these units denoting odd numbers simultaneously represented by the forms f and F.

There are three tables included in Table II. Table A of properly primitive genera contain twenty–five compartments corresponding to the twenty–five cases indicated in its margin; the Tables B and C of improperly primitive genera contain three such compartments each. In each compartment are inscribed all the particular characters which make up the complete generic character of a form coming under the case to which the compartment corresponds; the symbols

$$\left(\frac{f}{\omega_1}\right), \left(\frac{f}{\omega_1}\right), \left(\frac{F}{\delta_1}\right), \left(\frac{F}{\delta_2}\right)$$

implying that f has a particular character with respect to every prime ω_1 or ω_2, and F a particular character with respect to every prime δ_1 or δ_2.

⊠

Initially the use of this table, continuing Smith's style of exposition, is best illustrated by an example. In this example I have chosen the properly primitive, reduced, positive ternary quadratic forms of discriminant $D = 84$ and classified them into their order and genera (Table 17).

Example There are twenty-six properly primitive, reduced, positive ternary quadratic forms of discriminant $D = 84$. All reduced forms (classes) with the same discriminant D and the same value Ω constitute an order. There are two properly primitive orders of discriminant $D = 84$ and their subdivision into genera is based on the following observations. The first properly primitive order is of the invariants $[\Omega, \Delta] = [1, 84]$. Both f and F properly primitive. Their generic characters are located in the compartment of Table A labelled:

$\Omega = \Omega_1\Omega_2^2$, $\Omega_2 \equiv 1 (\text{mod } 2)$ and
$\Delta = \Delta_1\Delta_2^2$, $\Delta_2 \equiv 2 (\text{mod } 4)$,
since $\Omega_1 = 1$, $\Delta_1 = 21$, $\Omega_2 = 1$, $\Delta_2 = 2$.
Also $\Delta_1 = 21 = 3.7 = \delta_1$.

Table 17: Properly primitive, reduced, positive ternary quadratic forms f of discriminant 84 and their primitive contravariants F. The contravariant of f is ΩF. Also listed are the number δ of their automorphs.

f	F	Ω	δ
$(1,1,84,0,0,0)$	$(84,84,1,0,0,0)$	1	8
$(1,2,42,0,0,0)$	$(42,21,1,0,0,0)$	2	4
$(1,3,28,0,0,0)$	$(84,28,3,0,0,0)$	1	4
$(1,4,21,0,0,0)$	$(84,21,4,0,0,0)$	1	4
$(1,4,22,-2,0,0)$	$(42,11,2,1,0,0)$	2	4
$(1,5,17,-1,0,0)$	$(84,17,5,1,0,0)$	1	2
$(1,6,14,0,0,0)$	$(42,7,3,0,0,0)$	2	4
$(1,7,12,0,0,0)$	$(84,12,7,0,0,0)$	1	4
$(1,8,11,-2,0,0)$	$(84,11,8,2,0,0)$	1	2
$(1,10,10,-4,0,0)$	$(42,5,5,2,0,0)$	2	4
$(2,2,21,0,0,0)$	$(21,21,2,0,0,0)$	2	8
$(2,3,14,0,0,0)$	$(21,14,3,0,0,0)$	2	4
$(2,4,11,0,-1,0)$	$(44,21,8,0,4,0)$	1	4
$(2,6,7,0,0,0)$	$(21,7,6,0,0,0)$	2	4
$(2,6,9,-3,-1,0)$	$(45,17,12,6,6,3)$	1	4
$(2,7,7,1,1,1)$	$(48,13,13,-1,-6,-6)$	1	4
$(3,3,10,-1,-1,0)$	$(29,29,9,3,3,1)$	1	3
$(3,3,11,1,1,1)$	$(16,16,4,-1,-1,-5)$	2	2
$(3,4,7,0,0,0)$	$(28,21,12,0,0,0)$	1	4
$(3,4,8,-2,0,0)$	$(14,12,6,3,0,0)$	2	4
$(3,4,8,0,-1,-1)$	$(32,23,11,1,4,8)$	1	1
$(3,5,6,0,0,-1)$	$(15,9,7,0,0,3)$	2	2
$(4,4,7,0,0,-2)$	$(14,14,6,0,0,7)$	2	12
$(4,5,5,-2,0,0)$	$(21,20,20,8,0,0)$	1	4
$(4,5,5,-1,-1,-1)$	$(24,19,19,5,6,6)$	1	2
$(4,5,6,-2,-2,0)$	$(13,10,10,4,5,2)$	2	2

The second properly primitive order is of the invariants $[\Omega, \Delta] = [2, 21]$. There are two arrangements within this order. Firstly both f and F properly primitive and secondly f properly and F improperly primitive. Their generic characters are located in the compartment of Table A labelled:

$\Omega = 2\Omega_1\Omega_2^2$, $\Omega_2 \equiv 1(\text{mod } 2)$ and
$\Delta = \Delta_1\Delta_2^2$, $\Delta_2 \equiv 1(\text{mod } 2)$,
since $\Omega_1 = 1$, $\Delta_1 = 21$, $\Omega_2 = 1$, $\Delta_2 = 1$.
Also $\Delta_1 = 21 = 3.7 = \delta_1$.

The first properly primitive order of the invariants $[\Omega, \Delta] = [1, 84]$, when classified into their order and genera, may be presented as follows.

Invariants $[\Omega, \Delta] = [1, 84]$
f and F properly primitive; Six genera;

$-1 \;;\; +1 \;;\; +1 \;;\; +1$	$+1 \;;\; -1 \quad +1 \;;\; -1$
$(1, 1, 84, 0, 0, 0)$	$(1, 8, 11, -2, 0, 0)$
$(1, 4, 21, 0, 0, 0)$	$(3, 4, 8, 0, -1, -1)$
$+1 \;;\; +1 \;;\; -1 \;;\; +1$	$+1 \;;\; -1 \quad +1 \;;\; +1$
$(2, 7, 7, 1, 1, 1)$	$(2, 4, 11, 0, -1, 0)$
$(3, 4, 7, 0, 0, 0)$	$(3, 3, 10, -1. -1, 0)$
$-1 \;;\; -1 \;;\; -1 \;;\; +1$	$+1 \;;\; +1 \quad -1 \;;\; -1$
$(1, 5, 17, -1, 0, 0)$	$(1, 3, 28, 0, 0, 0)$
$(2, 6, 9, -3, -1, 0)$	$(1, 7, 12, -1, 0, 0)$
$(4, 5, 5, -2, 0, 0)$	$(4, 5, 5, -1, -1, -1)$

where the particular character of each form is taken in the order

$$\Psi \;,\; \left(\frac{F}{3}\right) \;,\; \left(\frac{F}{7}\right) \;,\; (-1)^{\frac{1}{2}(F-1)}$$

The second properly primitive order of the invariants $[\Omega, \Delta] = [2, 21]$, when classified into their order and genera, may be presented in a similar way.

☒

To complete the classification of ternary quadratic forms, Smith makes a number of final observations on Table II A of complete generic characters. Firstly, on the lower right–hand corner of each compartment he identifies the number of possible genera Γ contained in the order to which the compartment refers. γ denotes the number of odd primes dividing Ω, together with the number of odd primes dividing Δ, so that if the same prime divides both Ω and Δ, it is to be counted twice. Secondly he clarified how Table I of supplementary characters (Table 13) are now included within Table II A, and in particular the enumeration of the simultaneous character Ψ which appears in all twenty–five compartments of Table II A. These observations are another example of the delicate and intricate nature of his analysis, presenting all the fine details (Appendix G).

The table of complete of generic characters served to distinguish those complete characters which are possible, i.e. to which existing genera correspond, from those which are impossible. Continuing on the theme of the classification of ternary quadratic forms Smith defines what he termed a 'condition of possibility'. Every generic character which satisfies it being the character of an actually existing genus, and every generic character which does not satisfy it belongs to no form whatsoever.

5.7 CONDITION OF POSSIBILITY

To advance the theory of classification Smith relies on principles and methods found in the *Disquisitiones Arithmeticae*, articles 280–283. Within these articles Gauss defines a *primitive representation* of a binary quadratic form φ by a ternary quadratic form f, followed by a method of determining all such primitive representations (Gauss, 1986, pp. 312–322).[6] An example of an application of these principles is Smith's work on, what he termed, a 'condition of possibility' for ternary quadratic forms. Each condition distinguishes the possible and impossible genera, every generic character which satisfies it being the character of an actually existing genus, and every generic character which does not satisfy it belonging to no form whatsoever. In 1839 Dirichlet observed a relationship between binary characters (Table 10) based on his table of complete generic characters (Dirichlet, 1839, p. 337). Each compartment of the table was divided into two parts by a vertical line and the particular characters placed to the left of this line are subject to the condition that their product is equal to $+1$. Smith's corresponding relationship between ternary characters is also based on his table of complete generic characters (Figure 24).

6 These articles have been presented in some detail in (Section 4.2).

Smith states, in article 8 of his memoir, the conditions of possibility for ternary quadratic forms, one for each Table A, B and C, of his table of complete generic characters, taking particular care with atypical cases (Smith, 1867, pp. 266–268). In article 9, he demonstrates these conditions (*ibid*, pp. 268–269). In article 11, he applies these principles to consider the problem of the possibility of assigning a properly primitive ternary quadratic form, of any given invariants $[\Omega, \Delta]$, and of any given generic character, satisfying the condition of possibility (*ibid*, pp. 272–275).

Each compartment of his table, in a similar arrangement to Dirichlet, is divided into two parts by a vertical line and the particular character placed to the left of this line is subject to the condition that their product is equal in Table A (both f and F are properly primitive) to the unit

$$(-1)^{\frac{1}{2}(\Omega_1+1)\frac{1}{2}(\Delta_1+1)},$$

in Table B (f improperly and F properly primitive) to the unit

$$(-1)^{\frac{1}{8}(\Omega_1^2+1)} \times (-1)^{\frac{1}{2}(\Omega_1+1)\frac{1}{2}(\Delta_1+1)},$$

in Table C (f properly and F improperly primitive) to the unit

$$(-1)^{\frac{1}{8}(\Delta_1^2+1)} \times (-1)^{\frac{1}{2}(\Omega_1+1)\frac{1}{2}(\Delta_1+1)}.$$

From these observations Smith states the conditions of possibility for ternary quadratic forms, one for each of Table A, B and C. In the case of Table A (f and F properly primitive) the condition is as follows:

ARTICLE 9 (*ibid*, p. 268)
Let f and F be both properly primitive.
f is a form of the invariants $[\Omega, \Delta]$.
F is a form of the invariants $[\Delta, \Omega]$.
Let $\alpha = +1$ or -1 according as Ω is of the form $\Omega_1\Omega_2^2$ or $2\Omega_1\Omega_2^2$.
Let $\beta = +1$ or -1 according as Δ is of the form $\Delta_1\Delta_2^2$ or $2\Delta_1\Delta_2^2$.
The condition of possibility is

$$\Psi \times \alpha^{\frac{1}{8}(f^2-1)} \times \beta^{\frac{1}{8}(F^2-1)} \times \left(\frac{f}{\Omega_1}\right) \times \left(\frac{F}{\Delta_1}\right) = (-1)^{\frac{1}{2}(\Omega_1+1)\frac{1}{2}(\Delta_1+1)}$$

⊠

His demonstration of this condition when both f and F are properly primitive is included as Appendix H. The demonstration is achieved using the law of quadratic reciprocity and the arithmetic of Legendre symbols. Smith also states the condition of possibility in the case of Table B (f improperly and F properly primitive) and for Table C (f properly and F improperly primitive) and provides a demonstration of each of these cases (*ibid*, pp. 268–269).

The condition of possibility for Table B is

$$(-\beta)^{\frac{1}{8}(F^2-1)} \times \left(\frac{f}{\Omega_1}\right) \times \left(\frac{F}{\Delta_1}\right) = (-1)^{\frac{1}{8}(\Omega_1^2+1)} \times (-1)^{\frac{1}{2}(\Omega_1+1)\frac{1}{2}(\Delta_1+1)},$$

and for Table C is

$$(-\alpha)^{\frac{1}{8}(f^2-1)} \times \left(\frac{f}{\Omega_1}\right) \times \left(\frac{F}{\Delta_1}\right) = (-1)^{\frac{1}{8}(\Delta_1^2+1)} \times (-1)^{\frac{1}{2}(\Omega_1+1)\frac{1}{2}(\Delta_1+1)}.$$

⊠

With these mathematical details in place Smith poses the question – Can we assign a properly primitive ternary quadratic form of any given invariants $[\Omega, \Delta]$, and of any given generic character satisfying the condition of possibility? His treatment of this problem is another example of Smith's ability as an expositor of mathematics. Beginning with basic definitions he gives, in the simplest language, a clear concise overview of the mathematical technique. The mathematical demonstration is presented without impeding the reader from completing the technique. This clear exposition encourages the reader to try a simple illustration. The following details are taken from article 11 of this memoir.

ARTICLE 11 (*ibid*, p. 272)
Let f and F be both properly primitive.
f is a form of the invariants $[\Omega, \Delta]$.
F is a form of the invariants $[\Delta, \Omega]$.
Let M have the same sign as Δ, prime to 2Δ and satisfying the generic characters of F, except the simultaneous character, if any.
If Ω is odd and Δ odd or unevenly even, we shall suppose $M \equiv \Omega (\text{mod } 4)$.
Let φ be properly primitive of discriminant $-\Omega M$.
Let m be any number prime to $2\Omega M$ which is represented by φ.
By the theory of binary quadratic forms, the generic characters which are attributable to φ are its characters with respect to the primes dividing M, its characters with respect to the primes dividing Ω, and its supplementary characters. These supplementary characters are exhibited in the following table:

Table 18: Supplementary Characters attributable to φ

$-\Omega M \equiv 1 (\text{mod } 4)$	None
$-\Omega M \equiv 3 (\text{mod } 4)$	$(-1)^{\frac{1}{2}(\varphi-1)}$
$-\Omega M \equiv 2 (\text{mod } 8)$	$(-1)^{\frac{1}{8}(\varphi^2-1)}$
$-\Omega M \equiv 6 (\text{mod } 8)$	$(-1)^{\frac{1}{2}(\varphi-1)+\frac{1}{8}(\varphi^2-1)}$
$-\Omega M \equiv 4 (\text{mod } 8)$	$(-1)^{\frac{1}{2}(\varphi-1)}$
$-\Omega M \equiv 0 (\text{mod } 8)$	$(-1)^{\frac{1}{2}(\varphi-1)}, \ (-1)^{\frac{1}{8}(\varphi^2-1)}$

There is always one genus of properly primitive binary quadratic forms of determinant $-\Omega M$ capable of primitive representation by a given genus of ternary quadratic forms of the properly primitive order $[\Omega, \Delta]$, of which the contravariant characters coincide with the characters of M.

⊠

To isolate a single genus of a ternary quadratic form f, of the properly primitive order $[\Omega, \Delta]$, Smith chose generic characters for f and to each one assigned a value. It can be confirmed that this complete character (character and character value) for f satisfies the condition of possibility for ternary quadratic forms, i.e. that f, possessing such a complete character, actually exists. Smith now gives careful attention to assigning a value to each of the generic characters of the binary quadratic form φ of determinant $-\Omega M$. This will identify a genus of binary quadratic form capable of a primitive representation by f. Firstly he presents a rule that assigns to each principle character of φ a value based on the principle character value chosen for f.

Article 11 (*ibid*, p. 273)
Let μ be any prime dividing M and ω be any prime dividing Ω. The first set of characters are determined by the equation

$$\left(\frac{\varphi}{\mu}\right) = \left(\frac{-\Delta}{\mu}\right) \quad (*_E)$$

the second set by the equation

$$\left(\frac{\varphi}{\omega}\right) = \left(\frac{f}{\omega}\right) \quad (*_F)$$

$\left(\frac{f}{\omega}\right)$ being a particular character of f, of which the value is assigned in the proposed generic character.

With respect to the supplementary of φ, it will be found that on comparison of the above table with Table II A (Table 24), that, when the proposed generic character includes no simultaneous character, the supplementary characters attributable to φ are the same as those attributable to f. We then assign to the supplementary characters of φ the same values which were assigned to the supplementary of f in the proposed generic character. But when the proposed generic character includes a simultaneous character, there is always a supplementary character (and one only) attributable to φ, and not to f. This character of φ we determine so that the value of the simultaneous character of f and F, and the value of the unit similarly formed with m and M, may be coincident. This determination is always possible, as will be seen on a comparison of the cases (S) of Table II A, with the above table of supplementary characters of binary forms (*ibid*, p. 273).

⊠

From this extract of Smith's work, we can see that a value can now be assigned to every particular character attributable to φ. All primitive representations of φ by f may be enumerate using the method outlined in article 283 of the *Disquisitiones Arithmeticae*. Before doing so it must be confirmed that the complete character assigned to φ satisfies the condition of possibility for a binary quadratic forms, i.e. that φ, possessing such a complete character, actually exists. Smith states the following condition of possibility for binary quadratic forms. Every generic character which satisfies this condition is a character of an actual existing genus.

(*ibid*, p. 273)
Let φ be properly primitive binary quadratic form of discriminant $-\Omega M$.
Let $\alpha = +1$ or -1 according as Ω is of the form $\Omega_1\Omega_2^2$ or $2\Omega_1\Omega_2^2$.
The condition of possibility is

$$(-1)^{\frac{1}{2}(\Omega_1 M+1)\frac{1}{2}(\varphi-1)} \times \alpha^{\frac{1}{8}(\varphi^2-1)} \times \left(\frac{\varphi}{\Omega_1 M}\right) = +1 \qquad (*_G)$$

⊠

If this condition of possibility is true, Smith demonstrates that, using the rules that assigns to each principle character of φ a value based on the principle character value of f i.e. $(*_E)$ and $(*_F)$, that the condition of possibility for ternary quadratic forms must, as a consequence, be true. But this condition is satisfied by the proposed generic characters of f, therefore the equation $(*_G)$ is also satisfied (*ibid*, p. 273).

Again, following Smith's style of exposition, I have chosen the following example to illustrate the mathematical method just outlined. This example will enumerate properly primitive ternary quadratic forms of the invariants $[\Omega, \Delta] = [1, 84]$ with a specified complete generic character.

EXAMPLE

Let f and F be both properly primitive.
f is a form of the invariants $[\Omega, \Delta] = [1, 84]$.
F is a form of the invariants $[\Delta, \Omega] = [84, 1]$.
The generic characters are chosen and a value assigned to each character.

$$\Psi = -1 \quad , \quad \left(\frac{F}{3}\right) = +1 \quad , \quad \left(\frac{F}{7}\right) = +1 \quad , \quad (-1)^{\frac{1}{2}(F-1)} = +1$$

This complete character identifies a single genus of forms and satisfies the condition of possibility of f.
Let $\varphi = (1, 0, 85)$ be of discriminant $-\Omega M = -85$.
From $(*_E)$ we have that the generic characters of φ are

$$\left(\frac{\varphi}{5}\right) = +1 \quad , \quad \left(\frac{\varphi}{17}\right) = +1$$

This complete character satisfies the condition of possibility for φ.
If $\varphi = (1, 0, 85)$ is a binary quadratic form of determinant $-\Omega A'' = -85$, and φ admits a primitive representation by a ternary quadratic form f of the invariants $[\Omega, \Delta] = [1, 84]$, then

$$\begin{aligned} B^2 &\equiv 1 (\text{mod } 85) \\ BB' &\equiv 0 (\text{mod } 85) \\ B'^2 &\equiv 0 (\text{mod } 85) \end{aligned}$$

Solving we get $B \equiv 1 (\text{mod } 85)$ and $B' \equiv 0 (\text{mod } 85)$.
Enumerating f' using the method outlined in article 283 of the *Disquisitiones Arithmeticae* gives:

$f' = (1, 85, 1, -1, 0, 0)$
$f' = (1, 85, 89, -86, -1, 0)$
$f' = (1, 85, 349, -171, -2, 0)$
$f' = (1, 85, 349, -171, -2, 0)$
....

These forms are of the invariants $[1, 84]$. They have the preassigned generic characters values and satisfy the condition of possibility of f.

⊠

Further evidence of Smith's reliance on articles from the *Arithmeticae* appears as article 12 of this memoir. In 1852, Eisenstein defined a genus of forms as consisting of all forms which can be transformed into one another by transformations with rational coefficients of determinant unity (Eisenstein, 1975, pp. 713–722). In article 12 Smith remarked that, in the case of quadratic forms, it is desirable to add the limitation that the denominators of the fractional coefficients should be prime to $2\Omega\Delta$. Smith suggests that 'this may be proved nearly in the same way in which it is proved that equivalent forms have the same invariants and are of the same order and genus; it is only necessary to observe that F_1 and F_2, as well as f_1 and f_2, are transformable into one another by rational transformation, prime to $2\Omega\Delta$' (*ibid*, p. 275). Details of this demonstration are not included in this thesis however article 280 from the *Disquisitiones Arithmeticae* is a central part of his demonstration (Gauss, 1986, pp. 312–322).

5.8 THE WEIGHT OF A GENUS OF TERNARY QUADRATIC FORMS

In the second half of Henry Smith's 1867 memoir he demonstrates the formula for the weight of a genus of ternary quadratic forms, first conjectured by Eisenstein in 1847 (Eisenstein, 1847a, pp. 128–129). The method for the summation of the series

$$\sum \left(\frac{n}{m}\right) \frac{1}{m^2}$$

was first used by Dirichlet, from 1839, to determined the number of properly primitive classes of binary quadratic forms for any given determinant (Dirichlet, 1839, 1840b,a).[7] This method represents an important application of infinite series to the theory of numbers and is central to Smith's demonstration of Eisenstein's formula. The method is also important in Smith's final memoir to the *Académie des Sciences* in 1882 (Chapter 6). Due to the importance of this demonstration, and as a basis for further work, the following is a brief overview.

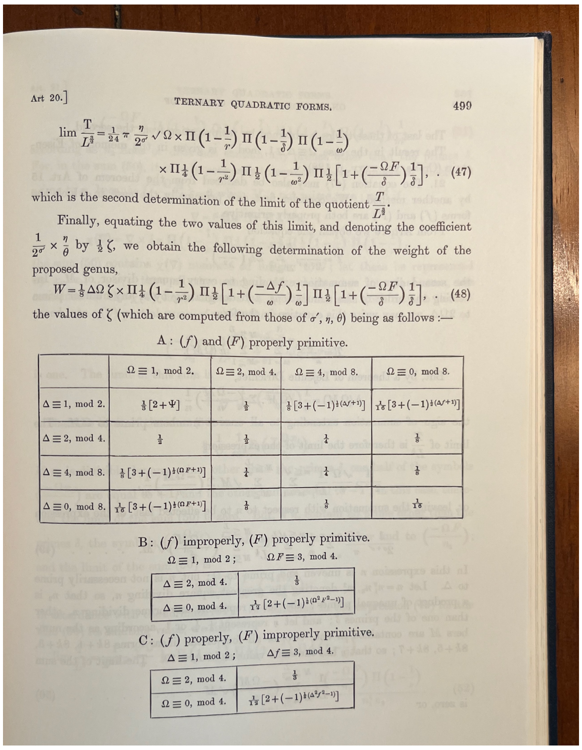
THEOREM (Smith, 1867, p. 291) Let r any odd prime dividing both Ω and Δ, by δ any odd prime dividing Δ, but not Ω, and by ω any odd prime dividing Ω, but not Δ. The weight W of a proposed genus is

$$W = \frac{\Delta\Omega}{8} \times \zeta \times \Pi \frac{1}{4}\left[1 - \frac{1}{r^2}\right] \Pi \frac{1}{2}\left[1 + \left(\frac{-\Delta f}{\omega}\right)\frac{1}{\omega}\right] \Pi \frac{1}{2}\left[1 + \left(\frac{-\Omega F}{\delta}\right)\frac{1}{\delta}\right]$$

where the value of ζ is chosen from a distinction of cases (Figure 25).

⊠

7 The principal parts of Dirichlet's researches is reproduced in Henry Smith's *Report on the Theory of Numbers* Part III (1861) (Science, 1861, pp. 325–340, Smith, 1894a, pp. 208–228).



Art 20.] TERNARY QUADRATIC FORMS. 499

$$\lim \frac{T}{L^{\frac{3}{2}}} = \tfrac{1}{24}\pi \frac{\eta}{2^{\sigma'}} \sqrt{\Omega} \times \Pi\left(1-\frac{1}{r}\right) \Pi\left(1-\frac{1}{\delta}\right) \Pi\left(1-\frac{1}{\omega}\right)$$
$$\times \Pi\tfrac{1}{4}\left(1-\frac{1}{r^2}\right) \Pi\tfrac{1}{2}\left(1-\frac{1}{\omega^2}\right) \Pi\tfrac{1}{2}\left[1+\left(\frac{-\Omega F}{\delta}\right)\frac{1}{\delta}\right], \quad . \quad (47)$$

which is the second determination of the limit of the quotient $\frac{T}{L^{\frac{3}{2}}}$.

Finally, equating the two values of this limit, and denoting the coefficient $\frac{1}{2^{\sigma'}} \times \frac{\eta}{\theta}$ by $\frac{1}{2}\zeta$, we obtain the following determination of the weight of the proposed genus,

$$W = \tfrac{1}{8}\Delta\Omega\,\zeta \times \Pi\tfrac{1}{4}\left(1-\frac{1}{r^2}\right) \Pi\tfrac{1}{2}\left[1+\left(\frac{-\Delta f}{\omega}\right)\frac{1}{\omega}\right] \Pi\tfrac{1}{2}\left[1+\left(\frac{-\Omega F}{\delta}\right)\frac{1}{\delta}\right], \quad . \quad (48)$$

the values of ζ (which are computed from those of σ', η, θ) being as follows :—

A : (f) and (F) properly primitive.

	$\Omega \equiv 1$, mod 2.	$\Omega \equiv 2$, mod 4.	$\Omega \equiv 4$, mod 8.	$\Omega \equiv 0$, mod 8.
$\Delta \equiv 1$, mod 2.	$\frac{1}{3}[2+\Psi]$	$\frac{1}{2}$	$\frac{1}{8}[3+(-1)^{\frac{1}{2}(\Delta f+1)}]$	$\frac{1}{16}[3+(-1)^{\frac{1}{2}(\Delta f+1)}]$
$\Delta \equiv 2$, mod 4.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$\Delta \equiv 4$, mod 8.	$\frac{1}{8}[3+(-1)^{\frac{1}{2}(\Omega F+1)}]$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$
$\Delta \equiv 0$, mod 8.	$\frac{1}{16}[3+(-1)^{\frac{1}{2}(\Omega F+1)}]$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$

B : (f) improperly, (F) properly primitive.

$\Omega \equiv 1$, mod 2 ; $\Omega F \equiv 3$, mod 4.

$\Delta \equiv 2$, mod 4.	$\frac{1}{3}$
$\Delta \equiv 0$, mod 4.	$\frac{1}{12}[2+(-1)^{\frac{1}{8}(\Omega^2F^2-1)}]$

C : (f) properly, (F) improperly primitive.

$\Delta \equiv 1$, mod 2 ; $\Delta f \equiv 3$, mod 4.

$\Omega \equiv 2$, mod 4.	$\frac{1}{3}$
$\Omega \equiv 0$, mod 4.	$\frac{1}{12}[2+(-1)^{\frac{1}{8}(\Delta^2f^2-1)}]$

Figure 25: The Weight of a Genus of Ternary Quadratic Forms taken from the *The collected mathematical papers of Henry John Stephen Smith–Volume I* (Smith, 1894a, p. 499).

The following is a simple illustrative example.

EXAMPLE Let W denote the weight of the properly primitive order of the invariants $[\Omega, \Delta] = [1, 84]$. The weight of an order is the sum of the weights of the reduced forms contained in the order. Therefore $W = 41/8$ (Table 17). Alternatively, using Eisenstein's formula: $\Omega \equiv 1 (\text{mod } 2)$ and $\Delta \equiv 4 (\text{mod } 4)$, hence

$$\zeta = \frac{1}{8}\left[3 + (-1)^{\frac{\Omega F+1}{2}}\right]$$

The odd prime divisors of Δ are $\delta = 3, 7$.

$$W = \frac{\Delta\Omega}{8} \times \zeta \times \Pi\frac{1}{2}\left[1 + \left(\frac{-\Omega F}{\delta}\right)\frac{1}{\delta}\right]$$

Evaluating for each genus of this properly primitive order:

$$W = \frac{3}{8} + \frac{3}{2} + \frac{1}{2} + \frac{3}{4} + \frac{1}{1} + \frac{1}{1} = \frac{41}{8}$$

⊠

In 1847, Eisenstein defined an automorphic transformation as follows:

DEFINITION (Eisenstein, 1847a, p. 120) A transformation which maps a form f onto itself is called an *automorphic transformation*. Let δ denote the number of *positive automorphics* of the form f. The *weight of a form m* is the reciprocal of the number of its positive automorphics, i.e.

$$m = \frac{1}{\delta}$$

⊠

In addition Smith defined the following:

ARTICLE 13 (Smith, 1867, p. 277)
Furthermore, the *weight of a class* is the weight of any form contained in the class. The *weight of an order* is the sum of the weights of the reduced forms contained in the order. The *weight of a genus* is the sum of the weights of the reduced forms contained in the genus. The weight of a ternary quadratic form and its contravariant have the same value. A ternary quadratic form has always $1, 2, 4, 6, 8, 12$ or 24 positive automorphics, i.e. an automorphic transformation of which the determinant is $+1$. When a number is represented by a ternary form, the weight of the representation is the weight of the ternary form. When a binary form is represented by a ternary form, the weight of the representation is the product of the weights of the two forms.

> To determine the weight of a given genus of ternary forms, we avail ourselves of the principles introduced into arithmetic by Gauss and Dirichlet, and employed by them to determine the number of binary forms of any given determinant (*ibid*, p. 278).

Smith outlines the following technique for this demonstration:

ARTICLE 13 (*ibid*, p. 278)
Let (f, F) represent a given genus of ternary forms of the invariants $[\Omega, \Delta]$, and either of the properly primitive order, or of that improperly primitive order in which f is improperly primitive and F properly primitive. Let $f_1, f_2, \ldots\ldots$ or (f) denote a system of forms representing the classes of the given genus. Let $F_1, F_2, \ldots\ldots$ or (F) denote the primitive contravariants of those forms. Let M represent any positive number, prime to $2\Omega\Delta$ and satisfying the generic characters of F. When (f, F) is a properly primitive genus with Ω odd and Δ odd or unevenly even, we suppose M satisfies the congruence $\Omega M \equiv 1 (\text{mod } 4)$. The numbers designated to M will be subject to the these restrictions throughout the whole of the investigation. Finally, let L be a positive quantity which we shall afterwards suppose to increase without limit. Let T be the sum of the weights of the representations of the forms (F) of all the numbers M which do

not surpass L. The quotient $T \div L^{\frac{3}{2}}$ approximates to a finite limit, when L is increased without limit. We obtain two distinct expressions for this limit. Firstly, one containing as a factor the weight W of the genus (f, F). Secondly, one depending on the arithmetical relation which exists between the sum of the weights of the representations of a given number M by the forms (F), and the sum of the weights of the properly or improperly primitive binary classes of determinant $-\Omega M$. A comparison of the two expressions will then give the required weight of the genus (f, F).

⊠

A full investigation of Smith's work on the weight of a genus of a ternary quadratic form is outside the scope of this thesis, however preliminary analysis reveal further examples of Smith's clear exposition style and his attention to detail.[8]

5.9 CONCLUSION

'During the nineteenth century, the classification of algebraic forms plays a key role in the fields of algebra and number theory in particular. It often borrows its vocabulary (class, genus, types) from the natural sciences' (Goldstein, 2016, p. 103). The classification of binary quadratic forms by Gauss and of ternary quadratic forms by Smith employed the same mathematical technique and style of exposition. Their common technique was to consider all possible arrangements of odd/even numbers modulo 4/modulo 8 for a given equation, and from basic implications the conclusions may be drawn. Smith's identities were central to his classification of ternary quadratic forms. These identities may be stated in more than three indeterminates, which lead to a subdivision of an order of quadratic forms containing more than three indeterminates into genera (Chapter 6). As regards their presentation style, it may be easily observed, that both Gauss and Smith present their work by a sequence of articles and that both use illustrative examples to support their demonstrations. Smith continues this presentation style in his further publications on quadratic forms. James Glaisher's recollection of Smith's mathematical writings was that Smith would always insist on perfect demonstrations and how he would only publish his mathematics when it was complete and perfect in every detail (Section 2.4).

This is in contrast to the style of, for example, Arthur Cayley's memoirs.[9] Cayley had been appointed Sadleirian Professor of Pure Mathematics at Cam-

8 See article 13–22 of Smith's memoir *On the Orders and Genera of Ternary Quadratic Forms* (Smith, 1867, pp. 277–298).

9 Smith used Cayley's matrix notation in his work on quadratic forms however Cayley, having recognised the importance of Smith's work on linear equations and congruences, did not appear to publish in this area (Crilly, 2006a, f.82, p. 520).

bridge in 1863. As holders of Victorian Britains most prodigious chairs of mathematics both Smith and Cayley would regularly meet at various Scientific Society meetings during the 1860's and 1870's. They also exchanged visits between their respective universities.[10] In his biography of Arthur Cayley, Tony Crilly writes of Cayley's interest in calculation and compiling mathematical tables (Crilly, 2006a, p. 68). His modus operandi was that 'he was up-to-date with the latest results, contributed to his own findings immediately, and did not wait to polish his work. The prospect of being in error did not seem to worry him unduly' (Crilly, 2006a, p. 281).

> The style of Cayley's papers served this purpose of collecting and classifying. They are typically discursive, contains little formal proof, and many of them simply assemble the specimens. Perhaps there is no other alternative in the early days of a new theory, even a mathematical one. Ponderous formulae can only be safely jettisoned when the underlying concepts are understood and the theory refined and initial clumsiness forgotten (Crilly, 2006a, p. 195).

To advance the theory of classification Smith relies on principles and methods found in the *Disquisitiones Arithmeticae*. His application of infinite series to the theory of numbers provided a demonstration of the formula for the weight of a genus of ternary quadratic forms, first conjectured by Eisenstein in 1847. His method was similar to that used by Dirichlet (1839) to determined the number of properly primitive classes of binary quadratic forms for any given determinant.[11] Smith's classification of forms, his method of exposition, his presentation, and his advancement in the subject were all the strong influence Gauss and the *Disquisitiones Arithmeticae*.

> To summarise the role of the DA [*Disquisitiones Arithmeticae*] in the constitution of number theory fifty years after its publication was two-fold. On one hand, the DA provided number theory with the features of it self-organisation as an academic discipline. It shaped what number theory was and ought to be: congruences and forms with integer coefficients (with their possible generalisation). This image informed advanced textbooks and helped to structure them, and it would, for an even longer time, structure the classification of mathematics. On the other hand the DA had launched an

10 In June of 1864 Cayley was at Oxford to receive a DCL, the degree of Doctor of Common Law (and in the company of Henry Smith breakfasted with Charles Dodgson) (Crilly, 2006a, p. 281). In 1883 Cayley wrote to Sylvester 'you will have heard of the grievous loss of we have had by the death of Prof. Smith: he was here at Cambridge not three weeks ago for the election of the Plumian Professorship & the last accounts, [of] the evening before his death were quite favourable' (Crilly, 2006a, p. 375).

11 For an account of the introduction of infinite series to number theory proofs see (Goldstein, Schappacher, and Schwermer, 2007, pp. 28–32).

> active research field, with a firm grasp on number theory, algebra and analysis, supported by close and varied readings of the book. It provided the field with concrete tools, and a stock of proofs to scrutinise and adapt (Goldstein, Schappacher, and Schwermer, 2007, pp. 57–58).

Smith makes extensive use of tables to present the generic characters required for his classification. Although Dirichlet used tables to present generic characters for binary quadratic forms, it more likely that Smith was influenced by the considerable work on compiling mathematical tables in England at that time. Smith was an active member of the Committee on Mathematical Tables of the BAAS. The other committee members included Arthur Cayley, George Gabriel Stokes, William Thompson, and James Glaisher (Secretary). After a number of years a comprehensive 175 page report *On Mathematical Tables* was presented at the annual meeting of the BAAS held in Bradford, England in September 1873 (Science, 1874, pp. 459–460 and pp. 1–174).

Smith's ability as an expositor of mathematics was supported by his systematic and well ordered notation. The principles he established for the classification of ternary quadratic forms could be extended to quadratic forms with more than three indeterminates. He was already thinking about this problem as far back as 1864 in his memoir *On the orders and genera of quadratic forms containing more than three indeterminates* (Smith, 1864b, pp. 199–203). In this memoir his notation for the classification of quadratic forms in n indeterminates would, almost 20 years later, be the basis of his memoir to the *French Académie des Sciences* for its *Grand Prix des Sciences Mathématiques* of 1882. In the final chapter, I will introduce Smith's notation for the classification of forms of n indeterminates, along with an interesting episode in the history of mathematics in which Smith was an unwilling participant.

6

GRAND PRIX DES SCIENCES MATHÉMATIQUES 1882

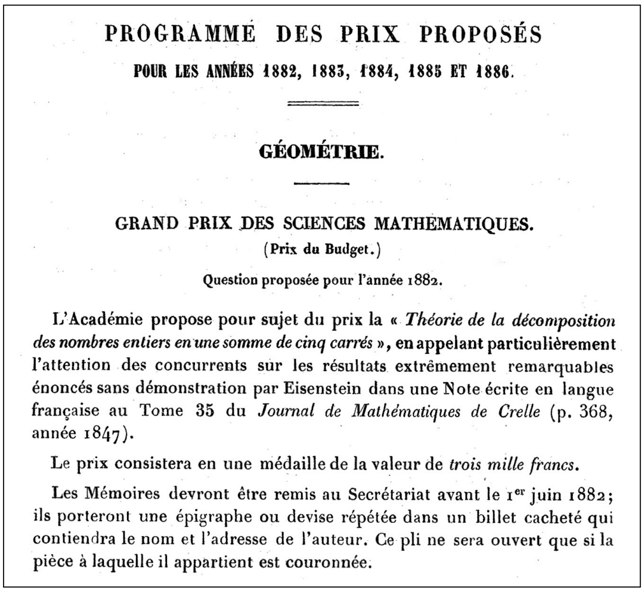
The subject proposed by the *French Académie des Sciences* for its *Grand Prix des Sciences Mathématiques* of 1882 was a special case of the general question of the classification of quadratic forms. This prestigious prize was awarded posthumously to Henry Smith, in 1883, bringing an international recognition to him and to Oxford mathematics. However, the awarding of the prize that year became the subject of an international controversy. During his lifetime Smith was renowned for his ability to reconcile opposing factions and for avoiding any kind of malicious gossip. To be at the centre of a controversy, after his death, would certainly have been abhorrent to him. In this chapter, I will outline the circumstances of the *Grand Prix* of 1882. I will also introduce Smith's prize memoir, *Mémoire sur la Représentation des Nombres par des Sommes de Cinq Carrés*, by indicating how the principles he established for the classification of ternary quadratic forms were extended to quadratic forms with more than three indeterminates.

Figure 26: Colbert Presenting Members of the Royal Academy of Sciences to Louis XIV in 1667. Artist: Henri Testelin (1616–1695) © Musée National du Château, Versailles.

6.1 GRAND PRIX DES SCIENCES MATHÉMATIQUES 1882

The *Académie des Sciences* was founded in 1666 by Louis XIV (1638–1715), at the suggestion of Jean-Baptiste Colbert (1619–1683), to encourage and protect the spirit of French scientific research and to evaluate scientific discoveries (Figure 26). The *Grand Prix des Sciences Mathématiques* was awarded by the *Académie des Sciences* between the end of the 18th century and the beginning of the 20th century at various intervals and arrangements.[1] The Academy, through a special academic Commission, would formulate a question for competitors, examine the memoirs submitted and report its judgement to the president of the Academy. The Academy president made the formal announcement, revealing the name of the prize author for the first time on the opening of a sealed envelope. All public announcements of the French Science Academy were published in the *Comptes rendus hebdomadaires des séances de l'Académie des sciences.*

In February 1882, Henry Smith was surprised to see, in the *Comptes Rendus*, that the subject proposed for the *Grand Prix* was the theory of the decomposition of integer numbers into a sum of five squares (Figure 27).[2] The first announcement had been the year previously, but the notice had escaped Smith's attention (Sciences, 1881, p. 622).



PROGRAMME DES PRIX PROPOSÉS

POUR LES ANNÉES 1882, 1883, 1884, 1885 ET 1886.

GÉOMÉTRIE.

GRAND PRIX DES SCIENCES MATHEMATIQUES.

(Prix du Budget.)

Question proposée pour l'année 1882.

L'Académie propose pour sujet du prix la « *Théorie de la décomposition des nombres entiers en une somme de cinq carrés* », en appelant particulièrement l'attention des concurrents sur les résultats extrêmement remarquables énoncés sans démonstration par Eisenstein dans une Note écrite en langue française au Tome 35 du *Journal de Mathématiques de Crelle* (p. 368, année 1847).

Le prix consistera en une médaille de la valeur de *trois mille francs*.

Les Mémoires devront être remis au Secrétariat avant le 1er juin 1882; ils porteront une épigraphe ou devise répétée dans un billet cacheté qui contiendra le nom et l'adresse de l'auteur. Ce pli ne sera ouvert que si la pièce à laquelle il appartient est couronnée.

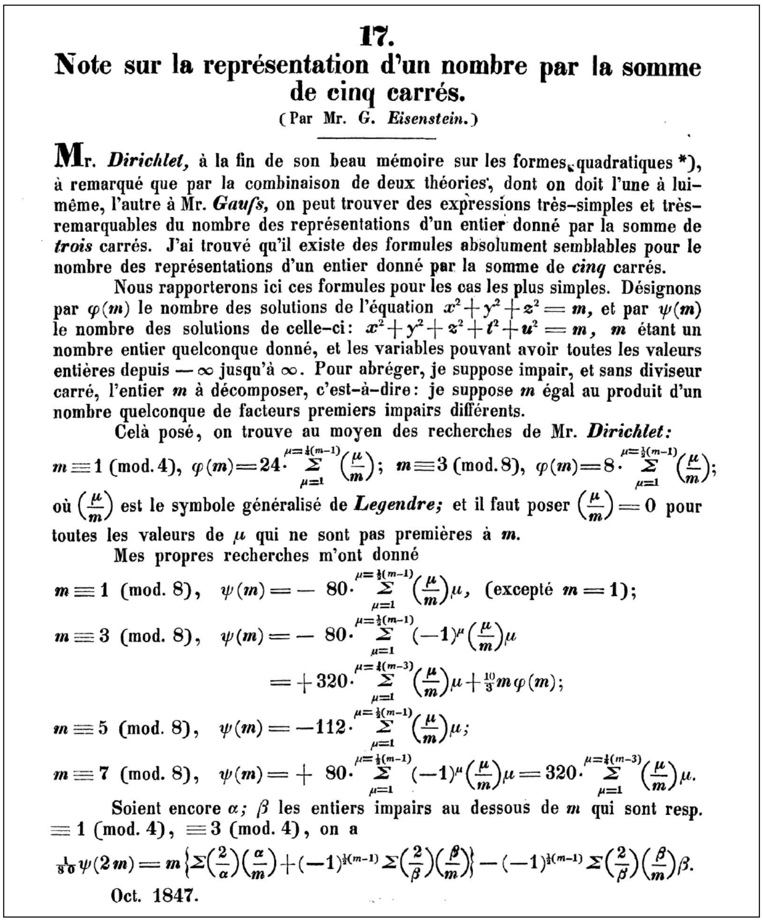
Figure 27: Grand Prix des Sciences Mathématiques (Sciences, 1882, p. 330–331).

The competitors were directed to results announced, without demonstration, by Eisenstein (1847) who stated formulae for the number of representations of

1 For an essay on mathematical prizes see (Carlson, Jaffe, and Wiles, 2006, pp. 3–27).

2 For a concise history on the representation of numbers as sums of squares see (Dickson, 1920, pp. viii–xi).

an integer when expressed as a sum of five squares (Figure 28). The terms of the announcement required competitors to submit their memoir, in French, to the *Secrétariat* before June 1st, 1882. It was to be accompanied by a signature or motto repeated on a sealed envelope which was to contain the name and address of the author. This envelope would only be opened if the signature or motto, to which it belonged, matched. The winning memoir would receive a prize of 3,000 francs.



17.

Note sur la représentation d'un nombre par la somme de cinq carrés.

(Par Mr. G. *Eisenstein.*)

Mr. *Dirichlet,* à la fin de son beau mémoire sur les formes quadratiques *), à remarqué que par la combinaison de deux théories, dont on doit l'une à lui-même, l'autre à Mr. *Gaufs,* on peut trouver des expressions très-simples et très-remarquables du nombre des représentations d'un entier donné par la somme de *trois* carrés. J'ai trouvé qu'il existe des formules absolument semblables pour le nombre des représentations d'un entier donné par la somme de *cinq* carrés.

Nous rapporterons ici ces formules pour les cas les plus simples. Désignons par $\varphi(m)$ le nombre des solutions de l'équation $x^2+y^2+z^2=m$, et par $\psi(m)$ le nombre des solutions de celle-ci: $x^2+y^2+z^2+t^2+u^2=m$, m étant un nombre entier quelconque donné, et les variables pouvant avoir toutes les valeurs entières depuis $-\infty$ jusqu'à ∞. Pour abréger, je suppose impair, et sans diviseur carré, l'entier m à décomposer, c'est-à-dire: je suppose m égal au produit d'un nombre quelconque de facteurs premiers impairs différents.

Celà posé, on trouve au moyen des recherches de Mr. *Dirichlet:*

$$m\equiv 1 \text{ (mod. 4)},\ \varphi(m)=24\cdot\sum_{\mu=1}^{\mu=\frac{1}{4}(m-1)}\left(\frac{\mu}{m}\right);\quad m\equiv 3 \text{ (mod. 8)},\ \varphi(m)=8\cdot\sum_{\mu=1}^{\mu=\frac{1}{2}(m-1)}\left(\frac{\mu}{m}\right);$$

où $\left(\frac{\mu}{m}\right)$ est le symbole généralisé de *Legendre;* et il faut poser $\left(\frac{\mu}{m}\right)=0$ pour toutes les valeurs de μ qui ne sont pas premières à m.

Mes propres recherches m'ont donné

$$m\equiv 1 \text{ (mod. 8)},\quad \psi(m)=-\ 80\cdot\sum_{\mu=1}^{\mu=\frac{1}{2}(m-1)}\left(\frac{\mu}{m}\right)\mu,\quad \text{(excepté } m=1);$$

$$m\equiv 3 \text{ (mod. 8)},\quad \psi(m)=-\ 80\cdot\sum_{\mu=1}^{\mu=\frac{1}{2}(m-1)}(-1)^\mu\left(\frac{\mu}{m}\right)\mu$$

$$=+320\cdot\sum_{\mu=1}^{\mu=\frac{1}{4}(m-3)}\left(\frac{\mu}{m}\right)\mu+\tfrac{10}{3}m\,\varphi(m);$$

$$m\equiv 5 \text{ (mod. 8)},\quad \psi(m)=-112\cdot\sum_{\mu=1}^{\mu=\frac{1}{2}(m-1)}\left(\frac{\mu}{m}\right)\mu;$$

$$m\equiv 7 \text{ (mod. 8)},\quad \psi(m)=+\ 80\cdot\sum_{\mu=1}^{\mu=\frac{1}{2}(m-1)}(-1)^\mu\left(\frac{\mu}{m}\right)\mu=320\cdot\sum_{\mu=1}^{\mu=\frac{1}{4}(m-3)}\left(\frac{\mu}{m}\right)\mu.$$

Soient encore $\alpha;\ \beta$ les entiers impairs au dessous de m qui sont resp. $\equiv 1$ (mod. 4), $\equiv 3$ (mod. 4), on a

$$\tfrac{1}{80}\psi(2m)=m\left\{\Sigma\left(\frac{2}{\alpha}\right)\left(\frac{\alpha}{m}\right)+(-1)^{\frac{1}{2}(m-1)}\Sigma\left(\frac{2}{\beta}\right)\left(\frac{\beta}{m}\right)\right\}-(-1)^{\frac{1}{2}(m-1)}\Sigma\left(\frac{2}{\beta}\right)\left(\frac{\beta}{m}\right)\beta.$$

Oct. 1847.

Figure 28: Single Page Memoir of Eisenstein (1847) (Eisenstein, 1847b).

The announcement makes no mention of Smith's own memoir dealing with the same subject in the *Proceedings of the Royal Society* in 1868, some fourteen years earlier (Smith, 1868, pp. 197–208). In the first part of this memoir Smith states further results on the classification of quadratic forms, containing more than three indeterminates, into their order and genera. This perfected his earlier investigation in his memoir, of the same title, in the *Proceedings of the Royal Society* in 1864 (Smith, 1864b, pp. 199–203). In the second part of this memoir (1868) he presents formulae for the determination of the weight of a given genus of definite quadratic forms containing more than three indeterminates. He makes the following remark as testament to the high standards of presentation he wished to maintain.

> It is easy to apply these general formulae to particular examples; but our imperfect knowledge of quadratic forms containing many indeterminates renders it practically impossible to test the results by any independent process. The demonstrations are simple in principle, but require attention to a great number of details with respect to which it is very easy to fall into error. As soon as they can be put into a convenient form, they shall be submitted to the Royal Society (Smith, 1868, p. 206).

Smith finally claims that the theorems given by Jacobi, Eisenstein, and Liouville, relating to the representation of numbers by four, six and eight squares, 'appear to be deducible by a uniform method from the principles indicated in this paper' (*ibid*, p. 207). He clarifies that the only cases which have not been fully considered are those of five and seven squares.

> The principal theorems relating to the case of five squares have indeed been given by Eisenstein (1847) but he has considered only those numbers which are not divisible by any square. We shall here complete his enunciation of those theorems, and shall add the corresponding theorems for the case of seven squares. We attend only to primitive representations (*ibid*, p. 207).

The following is Smith's enunciation of the theorems of Eisenstein.

THEOREM (*ibid*, p. 207).
Let Δ represent a number not divisible by any square, Ω^2 an odd square, a any exponent. By $\Phi_5(4^a\Omega^2\Delta)$ and $\Phi_7(4^a\Omega^2\Delta)$ we denote the number of representations of $4^a\Omega^2\Delta$ by five and seven squares respectively. By $Q_5(4^a\Omega^2\Delta)$ and $Q_7(4^a\Omega^2\Delta)$ we represent the products

$$5 \times 2^{3a} \times \Omega^3 \times \prod_q \left[1 - \left(\frac{\Delta}{q}\right)\frac{1}{q^2}\right]$$

$$5 \times 2^{5a} \times \Omega^5 \times \prod_q \left[1 - \left(\frac{-\Delta}{q}\right)\frac{1}{q^3}\right]$$

the sign of multiplication $\prod$ extending to every prime dividing Ω but not dividing Δ. We then have the following formulae.

1. For five squares.
   a) If $\Delta \equiv 1 (\bmod 4)$,

$$\Phi_5(4^a\Omega^2\Delta) = Q_5(4^a\Omega^2\Delta) \times \eta \times \sum_{s=1}^{\Delta} \left(\frac{s}{\Delta}\right) s(s-\Delta)$$

If $\Delta \equiv 1 (\bmod 8)$ then $\eta = 12$. If $\Delta \equiv 5 (\bmod 8)$, then $\eta = 28$ or $\eta = 20$ according as $a = 0$ or $a > 0$. If $\Delta = 1$, replace $\eta \times \sum$ by 2.

b) In every other case

$$\Phi_5(4^a\Omega^2\Delta) = Q_5(4^a\Omega^2\Delta) \times \eta \times \sum_{s=1}^{4\Delta} \left(\frac{s}{\Delta}\right) s(s-4\Delta)$$

where $\eta = 1$ or $\eta = 1/2$ according as $a = 0$ or $a > 0$.

2. For seven squares.

a) If $\Delta \equiv 3 (\text{mod } 4)$,

$$\Phi_7(4^a\Omega^2\Delta) = Q_7(4^a\Omega^2\Delta) \times \eta \times \sum_{s=1}^{\Delta} \left(\frac{s}{\Delta}\right) s(s-\Delta)(2s-\Delta)$$

If $a = 0$, $\Delta \equiv 3(\text{mod } 8)$, then $\eta = 30$. If $\Delta \equiv 7(\text{mod } 8)$, then $\eta = 2/3 \times 37$ or $\eta = 1/3 \times 140$ according as $a = 0$ or $a > 0$.

b) In every other case

$$\Phi_7(4^a\Omega^2\Delta) = Q_7(4^a\Omega^2\Delta) \times \eta \times \sum_{s=1}^{4\Delta} \left(\frac{s}{\Delta}\right) s(s-2\Delta)(s-4\Delta)$$

where $\eta = 1/3$ or $\eta = 5/12$ according as $a = 0$ or $a > 0$.

⊠

Smith clarifies that these formulae may be further simplified, to those of Eisenstein (1847), but he 'preferred to retain them in the form in which they first present themselves' (*ibid*, p. 207). The following example is a simple illustration of $\Phi_5(45)$.

Example To evaluate $\Phi_5(45)$ note that the primitive representations of 45 are

$$0^2 + 0^2 + 2^2 + 4^2 + 5^2$$
$$0^2 + 1^2 + 2^2 + 2^2 + 6^2$$
$$0^2 + 2^2 + 3^2 + 4^2 + 4^2$$
$$1^2 + 1^2 + 3^2 + 3^2 + 5^2$$

The convention for the number of representations is that all re-orderings and sign changes are to be counted as different i.e. we allow zeros and distinguish order and signs. For example, the representation $0^2 + 2^2 + 3^2 + 4^2 + 4^2$ may be arranged in $5! \div 2! = 60$ ways. Each of these arrangement has a further $2 \times 2 \times 2 \times 2 = 16$ possible arrangement of sign. Hence the total number of representations of $0^2 + 2^2 + 3^2 + 4^2 + 4^2$ is $60 \times 16 = 960$.

Let $4^a\Omega^2\Delta = 4^0 3^2 5 = 45$.
Now $\Delta \equiv 5(\text{mod } 8)$ and $a = 0$, hence $\eta = 28$.

$$\Phi_5(45) = 5 \times \frac{3^3}{5} \times \left[1 - \left(\frac{5}{3}\right)\frac{1}{3^2}\right] \times 28 \times \sum_{s=1}^{5}\left(\frac{s}{5}\right)s(s-5)$$

Now

$$\sum_{s=1}^{5}\left(\frac{s}{5}\right)s(s-5) = 4$$

Finally

$$\Phi_5(45) = 5 \times \frac{3^3}{5} \times \frac{10}{9} \times 28 \times 4 = 3,360.$$

Hence 45 may be represented by a sum of five squares in $3,360$ ways (allowing zeros and distinguish order and signs).

⊠

During the early 1880's Henry Smith and James Glaisher collaborated on their shared interest of Theta Functions and Elliptic Transformations. On February 17th, 1882, Smith wrote to James Glaisher.

> The Paris Academy have set for their Grand Prix for this year the theory of the decomposition of numbers into five squares, referring to a note of Eisenstein, Crelle, vol. xxxv, in which he gives without demonstration the formulae for the case in which the number to be decomposed has no square divisor. In the Royal Society's Proceedings, vol. xvi, pp. 207–208, I have given the complete theorems, not only for five, but also for seven squares: and though I have not given my demonstration, I have described the general theory from which these theorems are corollaries with some fullness of detail. Ought I to do anything in the matter? My first impression is that I ought to write to Hermite, and call his attention to it. A line or two of advice would really oblige me, as I am somewhat troubled and a little annoyed (Smith, 1894a, p. lxvi).

On February 22nd, he wrote again, in reply, to James Glaisher.

> You see I take your advice entirely upon the point that he ought to be written to. The worst of it is that it would take me a year, and a hundred pages, to work out the demonstrations of the paper in the Royal Society's Proceedings (*ibid*, p. lxvi).

Smith wrote immediately to Charles Hermite, a member of the Commission, drawing attention to this oversight. Smith had already known Hermite since at least April 1865, when they had lunch together in Paris (Kurti, 1984). He had invited Hermite to participate in the 43rd annual meeting of the BAAS held in Bradford, England in September 1873 and it is likely that he played a part in Hermite's election to a Fellowship of the Royal Society on 27th November, 1873.[3]

On February 26th, Hermite responded to Smith with a profuse apology (Appendix I). Recognising the potential embarrassment that could be caused to the Academy he advised Smith to rewrite his earlier memoir in French, with complete demonstrations, and to submit by the deadline in accordance with the rules of the competition. He reassured him that 'the commission will have my knowledge of your work if it has to make a decision and report to the Academy on memoirs submitted for its consideration' (Smith, 1894a, p. lxvi). He offered Smith a a partial explanation for the oversight.

> Until now, I do not know of any paper submitted. This can be explained by the direction of the mathematical trend which is no longer directed towards arithmetic. You are the only one in England to follow the path opened by Eisenstein. M. Kronecker is the only one in Germany; among us, M. Poincaré, after putting forward some good ideas on what he calls the arithmetic invariants, now seems to think only about Fuchsian functions and differential equations (*ibid*, p. lxvi).

Smith thought it his duty to accede to the suggestion by Charles Hermite and submit his demonstrations in the form of a memoir to the French Academy. In the weeks that followed he divided his time, according to James Glaisher, between working on Theta Functions, Elliptic Transformations and the prize subject. This was all the more remarkable given that he was suffering poor health during February and March 1882, confined to his sofa and unable to climb the stairs to reach his study. In April he wrote again to James Glaisher.

> I fear I cannot let you have the Transformation papers before the end of June. As I foresaw, getting the quadratic forms of n indeterminates into my mind again, putting my proofs into a rigorous form, and writing them out, will take up every moment till the end of May (the paper has to be in Paris by June 1st). My sole reason for taking this trouble is that sooner or later I should have had to do it unless I was to allow my demonstrations to perish (*ibid*, p. lxvii).

3 Charles Hermite's Certificate of Election to the Royal Society may be viewed at https://catalogues.royalsociety.org/CalmView/PersonSearch.aspx?src=CalmView.Persons Henry Smith admired the mathematical accomplishments of both Charles Hermite and Ferdinand von Lindemann. On 30th July, 1882, Smith wrote to James Glaisher: 'Do you see that Lindemann has covered himself with immortal renown by proving the transcendentality of π. Of course, nine-tenths of the discovery is really Hermite's, but then Lindemann has the immense glory of having seen that Hermite's method could be applied to prove the transcendentality of π, when Hermite himself despaired of it. I have never examined Hermite's method closely, but taking his results for granted, Lindemann's reasoning seems all right. It is difficult not to envy, as well as admire, people who do such beautiful things' (Smith, 1894a, p. xci).

Table 19: Grand Prix des Sciences Mathématiques 1882.

1881	
March 14th	1st Announcement *Comptes Rendus* (Sciences, 1881, p. 662).
1882	
February 6th	2nd Announcement *Comptes Rendus* (Sciences, 1882, p. 330).
17th & 22nd	Letter – Henry Smith to James W.L. Glaisher.
26th	Letter – Charles Hermite to Henry Smith.
June 1st	Deadline for submission to Académie des Sciences, Paris.
1883	
February 9th	Henry Smith dies.
12th	'Notice of Professor Henry Smith by the President'. *Proceedings of the Cambridge Philosophical Society.* (Society, 1880, pp. 319–321).
April 2nd	Prix Décernés *Comptes Rendus* (Sciences, 1883, pp. 879–882).
	Letter – Eleanor Smith to Charles Hermite.
	Letter – Charles Hermite to Eleanor Smith.
9th	Meeting of the Académie des Sciences. *Comptes Rendus* (Sciences, 1883, pp. 879–883).
	Journal Officiel de la République Française.
12th	'The French Academy Hoaxed' – The Times.
16th	Meeting of the Académie des Sciences. *Comptes Rendus* (Sciences, 1883, pp. 1095–1097).
	Journal Officiel de la République Française.
May 21st	Letter – Charles Hermite to Leopold Kronecker.

Smith submitted his complete memoir, *Mémoire sur la Représentation des Nombres par des Sommes de Cinq Carrés*, by the required deadline of June 1st, 1882 (Smith, 1894b, pp. 623–680) and (Sciences, 1887). Smith died on February 9th, 1883, two months before the adjudicating Commission of the Académie des Sciences were due to convene to make the *Prix Décernés*.

On February 12th, James Glaisher used his presidential address to the *Cambridge Philosophical Society* to pay tribute to his late friend, an honorary member of the society. Given that he had first hand knowledge of the circumstances of the priority issue, he relayed to the assembled audience Smith's earlier enunciation of the theorems of Eisenstein.

> These yielded to Professor Smith's powerful analytical methods, and he gave the enunciation of the theorems for the case of five and seven squares in the *Proceedings of the Royal Society* for 1867. In ignorance that the problem had been solved fifteen years before, the question of the resolution into sums of five squares was proposed as the subject for the Mathematical Prize by the French Academy last year (Society, 1880, pp. 319–321).

Figure 29: Hermann Minkowski (1864–1909)

The *Prix Décernés* was presented at the meeting of the Academy on April 2nd (Sciences, 1883, p. 879–882). The adjudicating Commission consisted of MM. Hermite, Bonnet, Bertrand, Bouquet and Jordan (rapporteur). They recommended that the prize be awarded equally to Henry Smith and Herman Minkowski (1864–1909), a young student of mathematics at the University of Königsberg, East Prussia.[4]

4 For articles relating to Minkowski's early scientific career see (Strobl, 1985) and (Schwermer, 2007, pp. 484–488).

A short analysis of both prize memoirs was presented by the rapporteur, Camille Jordan (1838–1922). The following is a short extract from his report:

> The two authors then deal with the representation of numbers by a quadratic form with n variables. They show, by generalising a Gaussian method, that this research returns to that of the representation of a quadratic form with $n-1$ variables. Then, for this latter problem, they show how the order and genera of the form is represented. The preceding results allow them to reduce the search for the weight of the representations of a given number by the set of forms of the same genus to that of the weight of a given genus. The application of Dirichlet's methods has provided the solution of this problem for quaternary forms. Pressed by time, they both give the proof of his results only as long as it was necessary to solve the problem posed by the Academy. It brings them back to the summation of an infinite series
>
> $$\sum \left(\frac{M}{m}\right) \frac{1}{m^2}$$
>
> very analogous to that which Dirichlet's had encountered in his famous memoir on the *Applications of the Infinitesimal Calculus to the Theory of Numbers* (Sciences, 1883, p. 882).

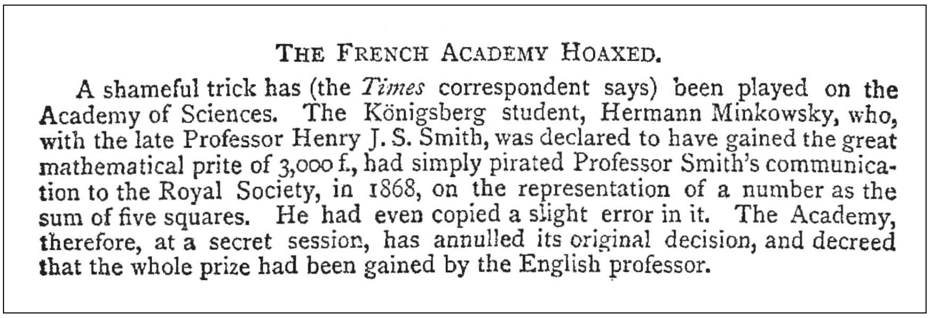
The report of the Commission shows that Smith's memoir was regarded as perfectly new work as no reference was made to his earlier memoir of 1868. Camille Jordan was not a member of the French Academy when the problem was originally posed and it seems, from the official report, that he was not appraised of Smith's priority. This is confirmed in a response to Eleanor Smith, who as the representative of her brother, wrote to Charles Hermite to bring to his attention his earlier assurance in his letter of February 26th, 1882. Charles Hermite replied that the omission of which she complained was due to 'absolutely involuntary forgetfulness' ['*ce tort ne consiste que dans un oubli, qui a été absolument involontaire*'] and he made no further statement of any kind (Smith, 1894a, p. lxx). Unfortunately, the report of the Commission contained the following remark on the similarities between both prize memoirs.

> It would be difficult to point out in one of them an important notion or theme that is not found in the other, and that, in order to avoid repetition and bring out the nuances that separate them, we had to analyse them simultaneously (Sciences, 1883, p. 880).

Henri De Parville (1838–1909) was scientific editor of the *Journal Officiel de la République Française*, the government gazette of the French Republic. On April 7th he reported on the meeting of the Académie des Sciences which took place on April 2nd.

> At the end of the last public meeting, the Academy met in secret committee, to repair an error which was made in the distribution of the prizes at the annual meeting of Monday, April 2nd. The subject proposed for the mathematical prize was known and had been dealt with ten or twelve years ago by M. Smith, an English scholar. A Königsberg student M. Hermann Minkowski had sent the exact same memoir as M. Smith; a calculation error committed by the latter had been faithfully reproduced by the German student. The prize had been shared between M. Smith and the student from Königsberg. The Academy awarded it to M. Smith.[5]

These reports, along with circumstances leading up to the announcement, now in the public domain meant the award was soon at the centre of a public controversy. Nationalistic feelings, following the Franco-Prussian war of 1871, were still heightened and could be easily inflamed. The continued French resentment towards Prussians resulted in the French press accusing the Academy of incompetence and blaming Minkowski for simply plagiarising Smith's earlier work. German mathematicians saw the French accusations against Minkowski as pure chauvinism.[6]



THE FRENCH ACADEMY HOAXED.

A shameful trick has (the *Times* correspondent says) been played on the Academy of Sciences. The Königsberg student, Hermann Minkowsky, who, with the late Professor Henry J. S. Smith, was declared to have gained the great mathematical prite of 3,000 f., had simply pirated Professor Smith's communication to the Royal Society, in 1868, on the representation of a number as the sum of five squares. He had even copied a slight error in it. The Academy, therefore, at a secret session, has annulled its original decision, and decreed that the whole prize had been gained by the English professor.

Figure 30: The Times, London (Paris Correspondent) – April 12th, 1883

As a consequence of criticisms in the French and English press the President of the *Académie des Sciences*, Joseph Louis François Bertrand (1822–1900), provided immediate clarity on the situation at the next meeting of the Academy on April 16th, 1883 (Sciences, 1883, pp. 1095–1097). He stated that the reporting of the previous meeting had been misinterpreted and that the

5 Journal Officiel de la République Française, 1883/04/07 (A15, N95), p. 1744 and 1883/04/13 (A15, N101), p. 1844.
https://gallica.bnf.fr

6 Following Hermann Minkowski's premature death in 1909 his *Gesammelte Abhandlungen von Hermann Minkowski* was published in 1911. This publication includes David Hilbert's tribute to Minkowski in which there is a sense of the German reaction to these accusations made against his close friend. Hilbert castigated the 'chauvinistic French press' for starting 'the most baseless attacks and suspicions' (Minkowski, 1911, p. vii). For an account of the prize competition, seen from the German perspective, see (Reid, 1970, pp. 9–14).

accusation made against Hermann Minkowski were false and needed to be refuted. Two formal notices, published in its *Comptes Rendus*, were read at the meeting to hopefully address the matter. The first notice, by Camille Jordan, was an appreciative obituary of Smith in which special reference was made to his contribution to the theory of numbers.[7] The second notice was from Joseph Bertrand. Both notices acknowledge, for the first time, the existence of Smith's 1868 paper in the *Proceedings of the Royal Society*, however both agreed that the demonstrations required complete details.

> The principal results of these vast researches are found set out in the *Proceedings of the Royal Society* of London, 1868, but these demonstrations were still hidden, and perhaps it was necessary for another great mathematician to come along, twenty years later, to awaken it for the third time, if the Academy had not the inspiration, by asking the question in the competition, to oblige M. Smith to give up part of his secret (Sciences, 1883, p. 1096).

This clarification was made without admitting that the subject was proposed in ignorance of Smith's work, or that the reporter was unaware of the existence of the 1868 paper, until after the publication of the report. According to James Glaisher 'the notices render justice to Hermann Minkowski, and offered a carefully framed defence of the Academy' (Smith, 1894a, p. lxxi). James Glaisher lamented that Charles Hermite had not communicated the existence of Smith's 1868 paper to the other members of the Commission. He believed that Smith would not have been willing to submit a memoir for the competition 'except under the special circumstances of the case and in response to M. Hermite's suggestion' (Smith, 1894a, p. lxxi). Henri De Parville again reported from the minutes of this meeting of the Academy in the *Journal Officiel de la République Française*.[8] Despite the clarifications provided by the Academy, there were further critical press reports.[9]

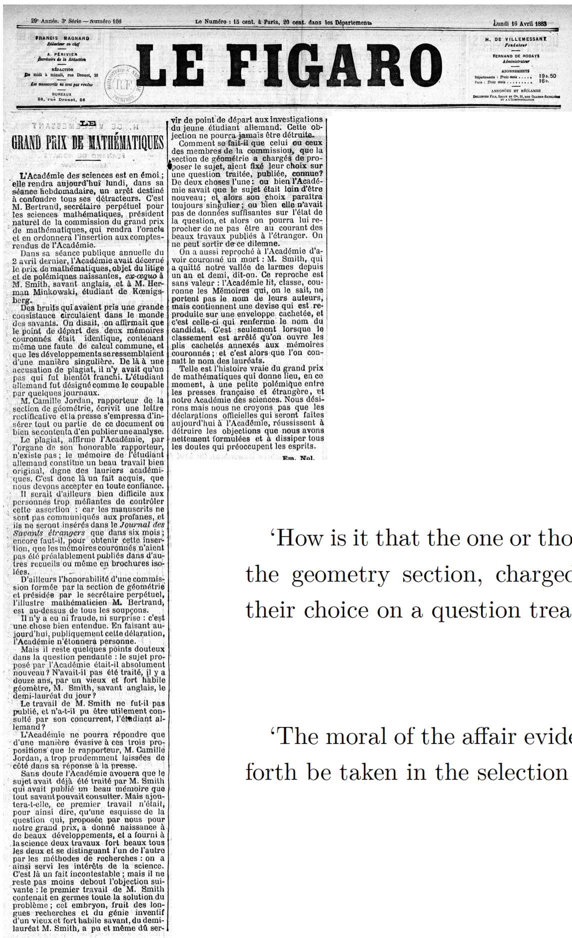
7 Camille Jordan would later write to Hermann Minkowski. In the style of the time he wrote 'work, sir, to become an eminent geometer' (Serre, 1993, pp. 3–9.)

8 Journal Officiel de la République Française, 1883/04/23 (A15, N111), p. 2015.
https://gallica.bnf.fr

9 Stamboul Journal Quotidien – Jeudi Mai 10, 1883 – 'Et voilà comme quoi l'un des prix les plus enviés de l'Académie des sciences, a été décerné à deux savants étrangers, dont l'un avait vu déjà son mémoire plusieurs fois récompensé, et dont l'autre n'avait pas craint de présenter un travail qui ne lui appartenait pas. Faut-il rire? Non, c'est triste. C'est à croire que l'Institut de France ne sera bientôt plus que la Sainte-Périne de la science!' Courtesy of Professor Catherine Goldstein, CNRS, Institut de mathématiques de Jussieu-Paris Rive Gauche, Paris.
https://gallica.bnf.fr

Figure 30:
The Times, London – April 12th, 1883.
https://www.britishnewspaperarchive.co.uk

Charles Hermite outlined his distress on the matter in a letter to Leopold Kronecker (1823–1891) dated May 21st, 1883. He wrote that his concern was that the academy award the two memoirs that seemed worthy to him, rather than to report thoroughly on them. He believed he was only doing his duty but that the resulting accusations had made him all too bitter and that there was no merit in it for him. He thanks Kronocker for his earlier letter which he said had 'consoled him from the wickedness of Miss Smith and her friends that a little negligence has attracted me'[10]



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LE
GRAND PRIX DE MATHÉMATIQUES

L'Académie des sciences est en émoi ; elle rendra aujourd'hui lundi, dans sa séance hebdomadaire, un arrêt destiné à confondre tous ses détracteurs. C'est M. Bertrand, secrétaire perpétuel pour les sciences mathématiques, président naturel de la commission du grand prix de mathématiques, qui rendra l'oracle et en ordonnera l'insertion aux comptes-rendus de l'Académie.

Dans sa séance publique annuelle du 2 avril dernier, l'Académie avait décerné le prix de mathématiques, objet du litige et de polémiques naissantes, *ex-æquo* à M. Smith, savant anglais, et à M. Herman Minkowski, étudiant de Kœnigsberg.

Des bruits qui avaient pris une grande consistance circulaient dans le monde des savants. On disait, on affirmait que le point de départ des deux mémoires couronnés était identique, contenant même une faute de calcul commune, et que les développements se ressemblaient d'une manière singulière. De là à une accusation de plagiat, il n'y avait qu'un pas qui fut bientôt franchi. L'étudiant allemand fut désigné comme le coupable par quelques journaux.

M. Camille Jordan, rapporteur de la section de géométrie, écrivit une lettre rectificative et la presse s'empressa d'insérer tout ou partie de ce document ou bien se contenta d'en publier une analyse.

Le plagiat, affirme l'Académie, par l'organe de son honorable rapporteur, n'existe pas ; le mémoire de l'étudiant allemand constitue un beau travail bien original, digne des lauriers académiques. C'est donc là un fait acquis, que nous devons accepter en toute confiance.

Il serait d'ailleurs bien difficile aux personnes trop méfiantes de contrôler cette assertion : car les manuscrits ne sont pas communiqués aux profanes, et ils ne seront insérés dans le *Journal des Savants étrangers* que dans six mois ; encore faut-il, pour obtenir cette insertion, que les mémoires couronnés n'aient pas été préalablement publiés dans d'autres recueils ou même en brochures isolées.

D'ailleurs l'honorabilité d'une commission formée par la section de géométrie et présidée par le secrétaire perpétuel, l'illustre mathématicien M. Bertrand, est au-dessus de tous les soupçons.

Il n'y a eu ni fraude, ni surprise : c'est une chose bien entendue. En faisant aujourd'hui, publiquement cette délaration, l'Académie n'étonnera personne.

Mais il reste quelques points douteux dans la question pendante : le sujet proposé par l'Académie était-il absolument nouveau ? N'avait-il pas été traité, il y a douze ans, par un vieux et fort habile géomètre, M. Smith, savant anglais, le demi-lauréat du jour ?

Le travail de M. Smith ne fut-il pas publié, et n'a-t-il pu être utilement consulté par son concurrent, l'étudiant allemand ?

L'Académie ne pourra répondre que d'une manière évasive à ces trois propositions que le rapporteur, M. Camille Jordan, a trop prudemment laissées de côté dans sa réponse à la presse.

Sans doute l'Académie avouera que le sujet avait déjà été traité par M. Smith qui avait publié un beau mémoire que tout savant pouvait consulter. Mais ajoutera-t-elle, ce premier travail n'était, pour ainsi dire, qu'une esquisse de la question qui, proposée par nous pour notre grand prix, a donné naissance à de beaux développements, et a fourni à la science deux travaux fort beaux tous les deux et se distinguant l'un de l'autre par les méthodes de recherches : on a ainsi servi les intérêts de la science. C'est là un fait incontestable ; mais il ne reste pas moins debout l'objection suivante : le premier travail de M. Smith contenait en germes toute la solution du problème ; cet embryon, fruit des longues recherches et du génie inventif d'un vieux et fort habile savant, du demi-lauréat M. Smith, a pu et même dû servir de point de départ aux investigations du jeune étudiant allemand. Cette objection ne pourra jamais être détruite.

Comment se fait-il que celui ou ceux des membres de la commission, que la section de géométrie a chargés de proposer le sujet, aient fixé leur choix sur une question traitée, publiée, connue ? De deux choses l'une : ou bien l'Académie savait que le sujet était loin d'être nouveau ; et alors son choix paraîtra toujours singulier ; ou bien elle n'avait pas de données suffisantes sur l'état de la question, et alors on pourra lui reprocher de ne pas être au courant des beaux travaux publiés à l'étranger. On ne peut sortir de ce dilemne.

On a aussi reproché à l'Académie d'avoir couronné un mort : M. Smith, qui a quitté notre vallée de larmes depuis un an et demi, dit-on. Ce reproche est sans valeur : l'Académie lit, classe, couronne les Mémoires qui, on le sait, ne portent pas le nom de leurs auteurs, mais contiennent une devise qui est reproduite sur une enveloppe cachetée, et c'est celle-ci qui renferme le nom du candidat. C'est seulement lorsque le classement est arrêté qu'on ouvre les plis cachetés annexés aux mémoires couronnés ; et c'est alors que l'on connaît le nom des lauréats.

Telle est l'histoire vraie du grand prix de mathématiques qui donne lieu, en ce moment, à une petite polémique entre les presses française et étrangère, et notre Académie des sciences. Nous désirons mais nous ne croyons pas que les déclarations officielles qui seront faites aujourd'hui à l'Académie, réussissent à détruire les objections que nous avons nettement formulées et à dissiper tous les doutes qui préoccupent les esprits.

Em. Nol.

'How is it that the one or those of the members of the commission, the geometry section, charged with proposing the subject, fixed their choice on a question treated, published and known?'

LE FIGARO, PARIS April 16th, 1883.

'The moral of the affair evidently is that greater care must henceforth be taken in the selection of subjects for prizes.'

THE TIMES, LONDON April 16th, 1883.

Figure 31: Le Figaro, Paris – April 16th, 1883

[11]

10 This letter, from Hermite to Kronecker, is in private hands. A copy of the correspondence is kept in the Archives of the French Academy of Sciences. I thank Professor Catherine Goldstein, CNRS, Institut de mathématiques de Jussieu-Paris Rive Gauche, for sharing with me her transcription of this letter.

11 Figure 31:
Le Figaro, Paris – April 16th, 1883.
https://gallica.bnf.fr
The Times, London – April 16th, 1883.
https://www.britishnewspaperarchive.co.uk

As a postscript to this story, on January 5th 1900, David Hilbert was seeking some topic ideas for his address to the International Congress of Mathematicians later that year. Hermann Minkowski responded that the only thing that came to mind was Smith's Presidential address to the LMS (Section 2.9).[12] We do not know whether this had any influence on David Hilbert, but it certainly indicates that Hermann Minkowski, whose criticism of other mathematicians, including his supervisor Ferdinand von Lindemann, was often scathing, respected Smith.[13]

Many research articles have been published on Minkowski's early work on quadratic forms, including on his prize memoir of 1882.[14] Comparatively few articles have been published on Henry Smith's contribution to the arithmetical theory of quadratic forms, least of all on his prize memoir. Building on the material discussed in Chapters 3, 4 and 5, and as a starting point for future work, the following is an introduction to Smith's prize memoir: *Mémoire sur la Représentation des Nombres par des Sommes de Cinq Carrés* (1882) in which he presents his notation for the classification of forms of n indeterminates.

6.2 MÉMOIRE SUR LA REPRÉSENTATION DES NOMBRES PAR DES SOMMES DE CINQ CARRÉS

Smith's prize memoir of 1882 was initially published by the *Académie des Sciences* in 1887 (Sciences, 1887). It was reproduced in the second volume of Smith's collected works, edited by James W.L. Glaisher, in 1894 (Smith, 1894b, pp. 623–680). Smith introduces his memoir with the following overview.

> Overall, I would have liked to reproduce the general results contained in the notice of 1867, in subjecting them to careful examination and adding rigorous demonstrations to them. But the time for such extensive work being limited, I have restricted myself, as much as possible, within the limits of the question proposed by the Academy. However, from the beginning, I have dealt with quadratic forms of n indeterminates, because the properties of quaternary forms, on which the solution of the problem actually depends, are easier to grasp when they are stated in a perfectly general manner (*ibid*, p. 623).

He states that the methods employed in his memoir are the ones that he developed in his earlier publications from 1861, 1867 and 1868. (Smith, 1861, Smith, 1867, Smith, 1868). Smith's prize memoir of 1882 has the following fundamental concepts and methods.

12 See (Rüdenberg and Zassenhaus, 1973, p. 120).
13 See (Wilson, 2021, p. 119).
14 See (Schwermer, 2007, pp. 483–504).

1. The classification of quadratic forms in n indeterminates (Smith, 1868).

2. The *Smith normal form* of a matrix with integer entries (Smith, 1861).

3. The method for the summation of the series

$$\sum \left(\frac{n}{m}\right) \frac{1}{m^2}$$

This method is attributed to Lejeune Dirichlet (Dirichlet, 1839, 1840b,a). Smith used this method to demonstrate Eisenstein's formula for the weight of a genus of ternary quadratic forms (Smith, 1867).

In 1867 Smith provided a complete classification of ternary quadratic forms (Section 5.2). For a *primitive* ternary quadratic form f he defined the discriminant, contravariant and primitive contravariant of f as $\Omega^2\Delta$, ΩF and F respectively. The numbers Ω and Δ are the *arithmetical invariants* of f. Two primitive forms of the same invariants $[\Omega, \Delta]$ are said to belong to the same order when they and their primitive contravariants are alike properly or alike improperly primitive. Finally, using his identities, he successfully subdivided an order of ternary quadratic forms into genera.

In 1882 Smith, in the first article of his prize memoir, provides the definitions for a complete classification of quadratic forms in n indeterminants. He had already presented these details in as early as 1864 in his memoir *On the orders and genera of quadratic forms containing more than three indeterminates* (Smith, 1864b, pp. 199–203). He defined a quadratic form f_1 in n indeterminants and, what he termed, a concomitant of the i^{th} species of f_1.

DEFINITION (Smith, 1864b, p. 199).
Let

$$f_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^{(1)} x_i x_j$$

denote a quadratic form containing n indeterminates $x_1, x_2, \ldots\ldots\ldots, x_n$ with integer coefficients $A_{i,j}$.

⊠

DEFINITION (Concomitant) (*ibid*, p. 199).
Let $A^{(1)}$ be the symmetrical $n \times n$ matrix of this form.
Let $A^{(i)}$ be the i^{th} derived matrix of $A^{(1)}$, a symmetrical $I \times I$ matrix where $I = C(n, i)$. The elements of $A^{(i)}$ are the *i-rowed minors* of $A^{(1)}$. The quadratic form

$$f_i = \sum_{i=1}^{I} \sum_{j=1}^{I} A_{ij}^{(i)} X_i X_j$$

will be a concomitant of the i^{th} species of f_1.

⊠

I have provided the following overview of Smith's definition of a concomitant of the (n-1)th species of f_1 (Figure 32).

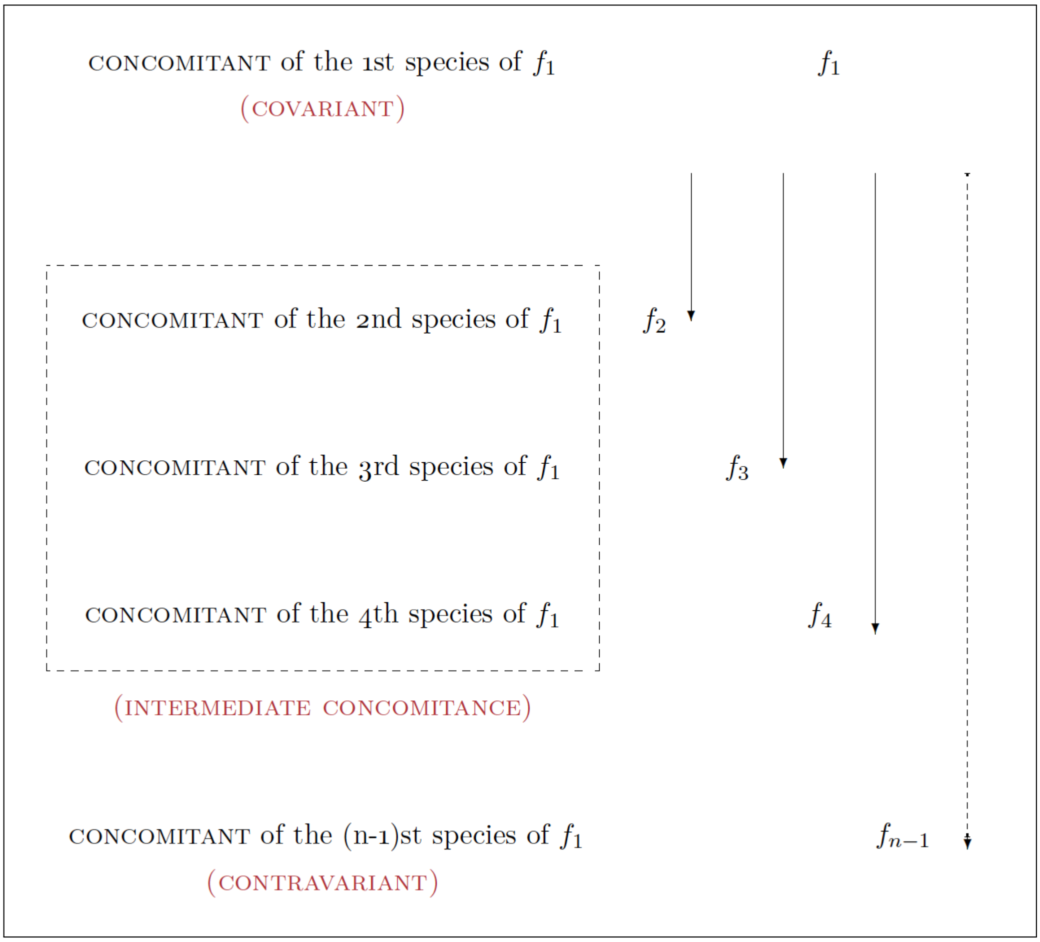

Figure 32: Concomitant of the (n-1)th species of f_1

Smith provides the reader with the following example followed by his definition of the discriminant and the *arithmetical invariants* of f_1.

EXAMPLE (*ibid*, p. 200)
Let

$$\begin{aligned} f_1 &= a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 \\ &+ 2b_1x_1x_2 + 2b_2x_1x_3 + 2b_3x_1x_4 + 2b_4x_2x_3 + 2b_5x_2x_4 + 2b_6x_3x_4 \end{aligned}$$

be a quadratic form f_1 containing four indeterminates. The form f_2 may be defined by the equation

$$\begin{aligned} f_2 &= (b_1^2 - a_1a_2)X_1^2 + (b_2^2 - a_1a_3)X_2^2 + (b_3^2 - a_1a_4)X_3^2 \\ &+ (b_4^2 - a_2a_3)X_4^2 + (b_5^2 - a_2a_4)X_5^2 + (b_6^2 - a_3a_4)X_6^2 \\ &+ 2(b_1b_2 - a_1b_4)X_1X_2 + 2(b_1b_3 - a_1b_5)X_1X_3 \\ &- 2(b_1b_4 - a_2b_2)X_1X_4 - 2(b_1b_5 - a_2b_3)X_1X_5 \\ &- 2(b_2b_5 - b_3b_4)X_1X_6 + 2(b_2b_3 - a_1b_6)X_2X_3 \\ &+ 2(b_2b_4 - a_3b_1)X_2X_4 - 2(b_1b_6 - b_3b_4)X_2X_5 \\ &- 2(b_2b_6 - a_3b_3)X_2X_6 - 2(b_1b_6 - b_2b_5)X_3X_4 \\ &+ 2(b_3b_5 - a_4b_1)X_3X_5 + 2(b_3b_6 - a_4b_2)X_3X_6 \\ &+ 2(b_4b_5 - a_2b_6)X_4X_5 - 2(b_4b_6 - a_2b_5)X_4X_6 \\ &+ 2(b_5b_6 - a_4b_4)X_5X_6 \end{aligned}$$

is the concomitant of the 2nd species of f_1.

⊠

DEFINITION (*ibid*, p. 199)
Let

$$f_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}^{(1)} x_i x_j$$

denote a quadratic form containing n indeterminates $x_1, x_2, ..., x_n$ with integer coefficients $A_{i,j}$. Let $A^{(1)}$ be the symmetrical $n \times n$ matrix of this form. We assume f_1 to be a positive definite form. The *discriminant* of f_1, i.e. the determinant of the matrix $(A_{i,j})_{n\times n}$ is represented by $\bigtriangledown_n$. The greatest common divisor of the minors of the orders $n-1, n-2, ..., 2, 1$ in the same matrix are denoted by $\bigtriangledown_{n-1}, \bigtriangledown_{n-2}, ..., \bigtriangledown_2, \bigtriangledown_1$. These numbers are arithmetical *invariants* of the form f_1, that is they remain unchanged when f_1 is transformed by any linear transformation of determinant unity where the coefficients are integral numbers. We assume f_1 to be a *primitive* form, hence $\bigtriangledown_1 = 1$. Instead of the numbers $\bigtriangledown_s$, it will be an advantage to instead consider the sequence

$$\frac{\bigtriangledown_2}{\bigtriangledown_1} \div \frac{\bigtriangledown_1}{\bigtriangledown_2} \quad , \quad \frac{\bigtriangledown_3}{\bigtriangledown_2} \div \frac{\bigtriangledown_2}{\bigtriangledown_1} \quad , \quad .... \quad , \quad \frac{\bigtriangledown_n}{\bigtriangledown_{n-1}} \div \frac{\bigtriangledown_{n-1}}{\bigtriangledown_{n-2}}$$

which are always integral and represented by $I_1, I_2, ..., I_{n-1}$. These numbers are the first, second,.........,(n-1)th *invariants* of the form f_1.

⊠

To provide clarity I have chosen this example to determine the discriminant and the invariants I_1, I_2, I_3 for a primitive quadratic form f_1 containing four indeterminates. f_1 is termed *primitive* when, in this case, the eight integers $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ admit of no common divisor other than unity.

EXAMPLE Let

$$f_1 = 9x_1^2 + 7x_2^2 + 3x_3^2 + 13x_4^2 + 12x_1x_2 + 2x_1x_3 + 20x_1x_4 + 10x_2x_4$$

be a quadratic form containing four indeterminates, the form f_2 defined by the equation

$$\begin{aligned} f_2 &= 27X_1^2 + 26X_2^2 + 17X_3^2 + 21X_4^2 + 66X_5^2 + 39X_6^2 \\ &- 12X_1X_2 - 30X_1X_3 - 14X_1X_4 - 80X_1X_5 \\ &+ 10X_1X_6 - 20X_2X_3 + 36X_2X_4 - 60X_2X_6 \\ &- 10X_3X_4 + 56X_3X_5 + 26X_3X_6 + 30X_4X_6 \end{aligned}$$

is the concomitant of the 2nd species of f_1. The form f_3 defined by the equation

$$\begin{aligned} f_3 &= 198X_1^2 + 38X_2^2 + 26X_3^2 + 74X_4^2 \\ &- 84X_1X_2 - 66X_1X_3 - 120X_1X_4 \\ &+ 28X_2X_3 + 50X_2X_4 + 40X_3X_4 \end{aligned}$$

is the concomitant of the 3rd species of f_1. In this example

$$A^{(1)} = \left\| \begin{array}{cccc} 9 & 6 & 1 & 10 \\ 6 & 7 & 0 & 5 \\ 1 & 0 & 3 & 0 \\ 10 & 5 & 0 & 13 \end{array} \right\|$$

f_1 has discriminant 12. f_2 has discriminant 12^3. f_3 has discriminant 12^3. f_1 has discriminant 12 with invariants $\bigtriangledown_4, \bigtriangledown_3, \bigtriangledown_2, \bigtriangledown_1 = 12, 2, 1, 1$ and $I_1, I_2, I_3 = 1, 2, 3$.

⊠

Smith proceeds to define the primitive concomitant θ_i followed by the definition of order of forms in n indeterminates.

DEFINITION (Smith, 1868, p. 198)
Let

$$\theta_i = \frac{1}{\nabla_i} f_i$$

The forms $\theta_1, \theta_2, ..., \theta_{n-1}$ are the *primitive concomitants*, and the last is the *primitive contravariant*, of f_1 of θ_1. Each on of them is either properly primitive or improperly primitive. Two primitive forms of the same invariants $I_1, I_2, ..., I_{n-1}$ are said to belong to the same order when they and their primitive concomitants are alike properly or alike improperly primitive.

⊠

Finally, Smith's identities in n indeterminates

$$f_1(x_1, x_2, ..., x_r) f_1(y_1, y_2, ..., y_r) - \frac{1}{4}\left(x_1 \frac{\partial f_1}{\partial y_1} + x_2 \frac{\partial f_1}{\partial y_2} + ... + x_r \frac{\partial f_1}{\partial y_r}\right)^2 = f_2(X_1, X_2, ...)$$

$$f_2(x_1, x_2, ..., x_r) f_2(y_1, y_2, ..., y_r) - \frac{1}{4}\left(x_1 \frac{\partial f_2}{\partial y_1} + x_2 \frac{\partial f_2}{\partial y_2} + ... + x_r \frac{\partial f_2}{\partial y_r}\right)^2 = f_1(X_1, X_2, ...)$$

lead to the subdivision of the orders into genera.

Note that $X_1, X_2, ...$ are the determinants of the matrix

$$\left\| \begin{matrix} x_1 & x_2 & ...... & x_r \\ y_1 & y_2 & ...... & y_r \end{matrix} \right\|$$

taken in a suitable order.

6.3 CONCLUSION

The circumstances surrounding the *Grand Prix des Sciences Mathématiques* of 1882 represents an interesting episode in the history of mathematics. Many accounts of the sequence of events have been written and it is still a story that is spoken about today. Despite the unfavourable press reports, Hermann Minkowski became a number theorist and a mathematician of great distinction and was never guilty of any wrongdoing. History has been kinder to Minkowski than Smith as his name is associated with many contributions to mathematics, while Smith's name is still unfamiliar to even to most professional mathematicians. This was attributed to their different traditions. Minkowski worked in a large and talented school of mathematicians whose research in the theory of numbers had developed over a century from Gauss. Smith, on the other hand, was isolated in a country, only beginning to gain its footing in the subject. The awarding of the prize to Smith brought him recognition for his mathematical achievement, and surprise to many of his close friends at Oxford (Section 2.6).

The controversy may be attributed to national barriers working in both directions. Smith's memoirs on the theory of numbers, although published by the *Royal Society*, were of interest to very few mathematicians in Britain at that time. They would have been of great interest to many Continental mathematicians, had they been aware of them. The first announcement of the *French Académie des Sciences* prize problem in 1881 was overlooked in Britain because it was published in France. The controversy was also an indication of the standing of the theory of numbers in Victorian Britain. Smith was arguably the strongest mathematician in Oxford at that time but, despite the attempts by the BAAS to promote the subject during the 1860's, he was alone with his subject at Oxford. Consequently, it would seem, no other Oxford mathematician could have brought the early announcement of the prize to his attention.[15] The only exception to this would have been James W.L. Glaisher of Cambridge, who was the first person Smith turned to seeking advice. Advances in the theory of numbers in Britain was well behind that on the Continent, as indicated in Charles Hermite's initial response to Smith. Smith knew, perhaps better than anyone else in Britain, the importance of endeavours to form connections with European mathematicians and learn of their mathematical advances.

> So it was the Smith, who during his lifetime had been famous for avoiding any kind of malicious gossip, and for his ability to reconcile opposing factions, was, after his death, at the centre of an international controversy which would surly have been abhorrent to him (Hannabuss, 1983, pp. 901–903).

15 For an account of Henry Smith's mathematical colleagues at Oxford, William Dorkin (1814–1869) and Bartholomew Price (1818–1898) see (Wilson, 2021, pp. 99–101).

7 CONCLUSION

This thesis begins in Bantry, County Cork, Ireland about 1800, and ends at the *Académie des Sciences* in Paris, France in 1883. I was initially interested in Henry Smith's memoirs on the theory of numbers but, as this research developed, his life in mathematics became as important as the mathematics he wrote. James W.L. Glaisher believed that Smith's devotion to the subject he loved would not have endured without the balance of the life he lived. The details of Smith's life and mathematics in this thesis would support that assertion.

The details of Henry Smith's Irish heritage are interesting. Despite the death of his father, while Henry was a child, his early childhood was a happy one. His early achievements as a student at Oxford may be attributed, not just to his academic talent, but to the dedication of his mother who sought, through education, to make a better life for her children. Their close family upbringing ensured Smith would have the companionship and support of his sister Eleanor throughout his life. His early academic appointments at Balliol College, at just 23 years of age, were important as they secured a path for him through his academic life. His election to the Savilian Professorship, at just 34 years of age, marked the beginning of a decade of important publications on the theory of numbers. All of these factors ensured that Smith's life at Oxford, surrounded by friends and colleagues, was a content and fulfilled one.

Smith's decision to attend Oxford University in 1845, as it would transpire, was an excellent choice. Later the academic freedom he enjoyed at Oxford meant that he could devote himself to mathematics and absorb Continental developments. However, his caring personality and sense of public duty meant that his time for mathematics was sacrificed to various offices, royal commissions and scientific societies. Many authors have suggested that this was a factor in Henry Smith being less well known than his contemporaries. William Spottiswoode believed that the time Henry Smith devoted to scientific societies did not adversely affect the quality of his mathematical output. He admitted that Henry Smith did not publish the number of mathematical memoirs of which he was capable, but those he did publish were of high value. Perhaps the truth is to be found in the personal letters of his sister Eleanor who revealed that, towards the end of his life, the workload he imposed on himself became harmful to his health.

The theory of numbers was outside the scope of British mathematics during the 19th century. The decision of the BAAS in 1859 to commission Henry Smith to write his *Reports on the Theory of Numbers* was a timely one. His final report in 1865 was followed by a sequence of memoirs on the arithmetical theory of quadratic forms. In this thesis I have tried to illustrate the beauty of his accomplishment under the theme of the classification of ternary forms. His mathematical influences were very important. By extending, and publishing, these results in more that three indeterminates lead Henry Smith to the circumstances of the *Grand Prix des Sciences Mathématiques* of 1882. The award of the *French Académie des Sciences* brought Henry Smith international recognition for his work, but it came with a controversy not of his making. The *Prix Décernés* and short report (3 pages) of the adjudicating Commission states that the analysis of the prize memoirs of Henry Smith and Hermann Minkowski were carried out simultaneously. It highlights some of the similarities and differences that existed between these memoirs. This report presents us with the basis of a very interesting consideration, and further work. Can some of the differences between these two memoirs, that are referred to in this report, be extracted in some detail? The initial article of Henry Smith's prize memoir presents the rich notation, and definition of order and genera, for quadratic forms in n indeterminates. These details and presentation show a natural extension of Henry Smith's earlier classification of ternary quadratic forms. The 'powerful analytical methods' of his prize memoir, as referred to by James Glaisher, may be found by considering the *Smith normal form* of a matrix with integer entries, and the method for the summation of the series

$$\sum \left(\frac{n}{m}\right) \frac{1}{m^2}.$$

A

REDUCTION AND EQUIVALENCE OF BINARY QUADRATIC FORMS

Henry Smith's *Report on the Theory of Numbers* Part III (1861) contains a brief systematic résumé of the general theory of reduction and equivalence of binary quadratic forms as they appear in the *Disquisitiones Arithmeticae* (Science, 1861, pp. 292–324, Smith, 1894a, pp. 163–207).[1] The following are short extracts from this report which pertain to chapter 3. Distinguishing between forms of a negative and positive determinant, two basic problems are based on the definition of a *reduced form*. Firstly, to construct a system of reduced forms for a given determinant D and, secondly, to decide if two given forms of the same determinant are equivalent. The transformations which send one of two equivalent forms to the other are also important. The theory of the representation of numbers by quadratic forms reduces to these important problems. Illustrative examples throughout this appendix are the author's own.

A.1 ELEMENTARY DEFINITIONS

ARTICLE 85 (*ibid*, p. 170)
A binary quadratic form f may be represented as[2]

$$f = ax^2 + 2bxy + cy^2$$

where f is termed *primitive* (i.e. that the three integers a, b, c admit of no common divisor other than unity), and that its *discriminant* is different from zero. This *discriminant*, or the determinant of the matrix

$$-\begin{vmatrix} a & b \\ b & c \end{vmatrix}$$

is represented by D. A primitive form f is *properly primitive* when at least one of the coefficients a, c is odd; it is *improperly primitive* when those coefficients are both even. In an *improperly primitive* form b is odd or the form would not be primitive. The binary quadratic form f_1 becomes a new binary quadratic

1 For an account of binary quadratic forms in section V of the *Disquisitiones Arithmeticae* see (Goldstein, Schappacher, and Schwermer, 2007, pp. 8–13).
2 Symbolised by the formula $(a, b, c)(x, y)^2$ or, when it is not necessary to specify the indeterminates, by the simpler formula (a, b, c).

form f_2 when new variables are introduced. If x, y are the variables for the form f_1 and letting

$$x = \alpha X + \beta Y$$
$$y = \alpha' X + \beta' Y$$

where $\alpha, \beta, \alpha', \beta'$ are four particular integers and X, Y are the new variables. Forms f_1 and f_2 are said to be *equivalent* when one may be transformed into the other by a linear transformation of determinant unity i.e.

$$\begin{vmatrix} \alpha & \beta \\ \alpha' & \beta' \end{vmatrix} = \pm 1$$

Forms are *properly equivalent* if the determinant of this transformation is $+1$. Forms are *improperly equivalent* if the determinant is -1. We only consider *proper equivalence* of *positive definite forms.* A *positive definite form* is one in which the numbers represented by the form are positive, i.e., the first coefficient a of the form is positive. All equivalent forms are said to constitute a *class.* Equivalent forms represent the same integers and have the same discriminant. However, it is not true that forms of the same discriminant are necessarily equivalent. A *reduced form* is a form representing a class of equivalent forms. To decide whether two given forms of the same discriminant are equivalent, and hence members of the same class, we compare their *reduced forms.* All classes with the same discriminant D and the same greatest common divisor constitute an *order.*

A.2 FORMS OF NEGATIVE DETERMINANT

ARTICLE 92 (*ibid*, p. 182)
A form (a, b, c) of negative determinant D which satisfy the following conditions is called a *reduced form*

1. $|2b| \leq |a|$,
2. $|2b| \leq |c|$,
3. $|a| \leq |c|$.

The third condition (combined with the first implies the second) is an artificial restriction intended to enable a precise enunciation of the theorem that every class contains one, and only one reduced form. To obtain the *reduced form* equivalent to a given form (a, b, c) we form a series of *contiguous forms*, beginning with the given form and ending with the reduced form.

Two form (a, b, c) and (a', b', c') of the same negative determinant D are *contiguous forms* when

$$c = a' \quad , \quad b + b' \equiv 0 (\text{mod } a')$$

Two contiguous forms are always equivalent; for if $b + b' = \mu a'$, the former passes to the latter by the transformation

$$\begin{pmatrix} 0 & -1 \\ 1 & \mu \end{pmatrix}$$

Furthermore, the transformation T_n which passes (a_0, b_0, a_1) into the equivalent reduced form (a_n, b_n, a_{n+1}) is

$$\begin{pmatrix} 0 & -1 \\ 1 & \mu_1 \end{pmatrix} \times \begin{pmatrix} 0 & -1 \\ 1 & \mu_2 \end{pmatrix} \times \ldots\ldots\ldots\ldots \times \begin{pmatrix} 0 & -1 \\ 1 & \mu_n \end{pmatrix}$$

where $\mu_i = (b_{i-1} + b_i)/a_i$.

ARTICLE 95 (*ibid*, p. 191)
The important theorem, that for every positive or negative determinant the number of classes is finite, is a consequence of the theory of reduction. To establish its truth, it is sufficient to employ the reduction of Lagrange (*ibid*, Article 92, pp. 182–185), which is applicable to forms of a positive determinant no less than to forms of a negative determinant. For every class of forms of determinant D there exists at least one form where the coefficients satisfy the inequalities $|2b| \leq |a|$ and $|2b| \leq |c|$. If D is negative these inequalities give $ac \leq -\frac{4}{3}D$ and $b \leq \sqrt{-\frac{1}{3}D}$. If D is positive these inequalities give $ac \leq D$ and $b \leq \sqrt{\frac{1}{5}D}$. The number of forms whose coefficients satisfy these inequalities is evidently limited. Hence the number of non-equivalent classes is finite.[3]

⊠

3 The first condition for a reduced form of negative determinant D implies that $4b^2 \leq a^2$ and the third condition implies $a^2 \leq ac$. Now $4b^2 \leq ac$ or $3b^2 \leq ac - b^2$. Hence

$$|b| \leq \sqrt{-\frac{1}{3}D}$$

From this it also follows that $3ac = -3D + 3b^2 \leq -3D - D \leq -4D$ and since $a^2 \leq ac$,

$$a \leq \sqrt{-\frac{4}{3}D}.$$

Two basic problems may be based on the definition of a reduced form and on the resulting inequalities. Firstly, to construct a system of representative quadratic forms of determinant D by writing down all the forms whose coefficients satisfy these inequalities. Secondly, to decide if two quadratic forms are equivalent they must have the same determinant and belong to the same *class*. Hence we compare their *reduced forms*. The following examples are basic illustrations of these problems.

Example

Let $D = -24$. Firstly

$$b \leq \sqrt{-\frac{1}{3}D} \leq \sqrt{\frac{24}{3}} \leq \sqrt{8}$$

Hence b has the following values $0, \pm 1, \pm 2$. Since D is negative, $|a||c| = b^2 - D$. The numbers $24, 25, 28$ are decomposed into two factors in all possible ways.

$$\begin{array}{lllllllll} 24 & = & 1 \times 24 & = & 2 \times 12 & = & 3 \times 8 & = & 4 \times 6 \\ 25 & = & 1 \times 25 & = & 5 \times 5 & & & & \\ 28 & = & 1 \times 28 & = & 2 \times 14 & = & 4 \times 7 & & \end{array}$$

Since the first factor is always a and the second c this gives the following forms.

$(1, 0, 24), (2, 0, 12), (3, 0, 8), (4, 0, 6), (1, \pm 1, 24), (2, \pm 1, 12), (3, \pm 1, 8), (4, \pm 1, 6),$
$(1, \pm 2, 24), (2, \pm 2, 12), (3, \pm 2, 8), (4, \pm 2, 6), (1, 0, 25), (5, 0, 5), (1, \pm 1, 25), (5, \pm 1, 5),$
$(1, \pm 2, 25), (5, \pm 2, 5), (1, 0, 28), (2, 0, 14), (4, 0, 7), (1, \pm 1, 28), (2, \pm 1, 14), (4, \pm 1, 7),$
$(1, \pm 2, 28), (2, \pm 2, 14), (4, \pm 2, 7).$

Eliminate from this list the forms where $|2b| \not\leq a$, where $D \neq -24$, and forms which are not primitive. The remaining forms are the positive, reduced, primitive forms of determinant $D = -24$. They are

$$(1, 0, 24), (3, 0, 8), (5, 1, 5), (4, 2, 7).$$

⊠

EXAMPLE

Are the forms $(35, 26, 20)$ and $(29, 47, 77)$ equivalent?

Both have determinant $D = -24$.

To find the reduced equivalent of each form.

Firstly

$$\boxed{\phi_0 = (35, 26, 20)} \longrightarrow \phi_1 = (20, 14, 11) \qquad \mu = 2$$

$$\phi_1 = (20, 14, 11) \longrightarrow \phi_2 = (11, 8, 8) \qquad \mu = 2$$

$$\phi_2 = (11, 8, 8) \longrightarrow \phi_3 = (8, 0, 3) \qquad \mu = 1$$

$$\phi_3 = (8, 0, 3) \longrightarrow \boxed{\phi_4 = (3, 0, 8)} \qquad \mu = 0$$

Secondly

$$\boxed{\phi_0 = (29, 47, 77)} \longrightarrow \phi_1 = (77, 30, 12) \qquad \mu = 1$$

$$\phi_1 = (77, 30, 12) \longrightarrow \phi_2 = (12, 6, 5) \qquad \mu = 3$$

$$\phi_2 = (12, 6, 5) \longrightarrow \phi_3 = (5, -1, 5) \qquad \mu = 1$$

$$\phi_3 = (5, -1, 5) \longrightarrow \boxed{\phi_4 = (5, 1, 5)} \qquad \mu = 0$$

These forms are not equivalent as they belong to different classes, i.e.

$$(35, 26, 20) \longrightarrow (3, 0, 8) \quad \text{under} \quad \begin{pmatrix} -1 & 2 \\ 1 & -3 \end{pmatrix}$$

$$(29, 47, 77) \longrightarrow (5, 1, 5) \quad \text{under} \quad \begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix}$$

⊠

The problem of the representation of numbers by quadratic forms depends first, on the solution of a quadratic congruence, and, secondly, on the problem of the equivalence of forms.

Article 86 (*ibid*, p. 172)
To obtain all the *primitive representations* of a given number M by a given form (a, b, c) we investigate all the values of the expression $\sqrt{D} (\text{mod } M)$. If $\Omega_1, \Omega_2, \ldots\ldots\ldots$ be those values, we next compare each of the forms

$$\left(M, \Omega, \frac{\Omega^2 - D}{M}\right) \quad (*_A)$$

with (a, b, c). If none of them are equivalent to (a, b, c), M does not admit of a *primitive representations* by (a, b, c). But if one or more of them, say $(*_A) \sim (a, b, c)$ we let

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be the formula exhibiting all the transformations of (a, b, c) into

$$\left(M, \Omega_1, \frac{\Omega_1^2 - D}{M}\right)$$

then all the *primitive representations* of a given number M by (a, b, c), which appertain to the value $\Omega_1 (\text{mod } M)$ are contained in the formula $(a, b, c)(\alpha, \gamma)^2 = M$.

Example
To obtain all the *primitive representations* of $M = 1001$ by the form (a, b, c) of determinant $D = -794$, we investigate all the values of $\Omega^2 \equiv -794 (\text{mod } 1001)$. Solving yields $\Omega = 47, 96, 135, 278, 723, 866, 905, 954 (\text{mod } 1001)$. For each congruence class we compare (a, b, c) with each of the forms

$$\left(M, \Omega, \frac{\Omega^2 - D}{M}\right)$$

Say, for example, $\Omega \equiv 905 (\text{mod } 1001)$.

$$\boxed{\phi_0 = (1001, 905, 819)} \longrightarrow \phi_1 = (819, -86, 10) \qquad \mu = 1$$

$$\phi_1 = (819, -86, 10) \longrightarrow \boxed{\phi_2 = (10, -4, 81)} \qquad \mu = -9$$

Now $(10, -4, 81)$ is the reduced form with

$$(1001, 905, 819) \longrightarrow (10, -4, 81) \text{ under } \begin{pmatrix} -1 & 9 \\ 1 & -10 \end{pmatrix}$$

Furthermore, from the inverse transformation, $(10, -4, 81)(-10, -1)^2 = 1001$.

⊠

A.3 FORMS OF POSITIVE AND NOT SQUARE DETERMINANT

Henry Smith's *Report on the Theory of Numbers* Part III (1861) also provides a résumé of the writing of Gauss and Dirichlet on the problem of equivalence for forms of a positive and not square determinant (article 93). He outlines that Gauss devoted articles 183–196 of the *Disquisitiones Arithmeticae* to the problem and how, in a memoir on the subject from 1854, Dirichlet remarked that the demonstrations relating to it may be greatly abbreviated by employing certain known results of the theory of continued fractions.[4] In his report Henry Smith clarifies that the 'method does not differ materially from that proposed by Lejeune Dirichlet; nor indeed is it, in principle, very distinct from that of Gauss, the connection of which with the theory of continued fractions he has suppressed' (Smith, 1894a, p. 185).

ARTICLE 93 (*ibid*, p. 188)
A form (a, b, c) with positive determinant D is termed a *reduced form* when

$$\left|\frac{-b-\sqrt{D}}{c}\right| > 1 \quad \text{and} \quad \left|\frac{-b+\sqrt{D}}{c}\right| < 1$$

Consequently a *reduced form* will satisfy the following conditions:

1. $\mid a \mid < 2\sqrt{D},\ \mid b \mid < 2\sqrt{D},\ \mid c \mid < 2\sqrt{D},$
2. $\sqrt{D} - b < \mid a \mid < \sqrt{D} + b,$
3. $\sqrt{D} - b < \mid c \mid < \sqrt{D} + b.$

To obtain the *reduced form* equivalent to a given form (a, b, c) we again form a series of *contiguous forms*, beginning with the given form and ending with the reduced form. The problem is much harder to solve when the determinant is positive as the reduced forms are not, in general, all non-equivalent.

⊠

Again, two basic problems may be based on this definition of a reduced form and on the resulting inequalities. Firstly, to construct a system of representative quadratic forms of determinant D by writing down all the forms whose coefficients satisfy these inequalities. Secondly, the more difficult problem of deciding if two quadratic forms are equivalent. For positive determinants, there are always several distinct but equivalent *reduced forms*.

4 *Vereinfachung der Theorie der binären quadratischen Formen von positiver Determinante*, Memoirs of the Academy of Berlin, 1854.

Example

Let $D = 13$.
Since $ac = b^2 - D$ and $b^2 < D$, c and a must have opposite sign.
For this example b will have the following values $1, 2, 3$.
The numbers $-12, -9, -4$ are decomposed into two factors in all possible ways.

$$\begin{aligned} -12 &= \pm 1 \times \mp 12 = \pm 2 \times \mp 6 = \pm 3 \times \mp 4 \\ -9 &= \pm 1 \times \mp 9 = \pm 3 \times \mp 3 \\ -4 &= \pm 1 \times \mp 4 = \pm 2 \times \mp 2 \end{aligned}$$

Since the first factor is always a and the second c this gives the following forms.

$(\pm 1, 1, \mp 12), (\pm 1, 2, \mp 12), (\pm 1, 3, \mp 12), (\pm 2, 1, \mp 6), (\pm 2, 2, \mp 6), (\pm 2, 3, \mp 6),$
$(\pm 3, 1, \mp 4), (\pm 3, 2, \mp 4), (\pm 3, 3, \mp 4), (\pm 1, 1, \mp 9), (\pm 1, 2, \mp 9), (\pm 1, 3, \mp 9),$
$(\pm 3, 1, \mp 3), (\pm 3, 2, \mp 3), (\pm 3, 3, \mp 3), (\pm 1, 1, \mp 4), (\pm 1, 2, \mp 4), (\pm 1, 3, \mp 4),$
$(\pm 2, 1, \mp 2), (\pm 2, 2, \mp 2), (\pm 2, 3, \mp 2), (\pm 4, 1, \mp 3), (\pm 4, 3, \mp 1).$

Eliminate from this list the forms where $|\, a \,| \not< 2\sqrt{D},\ |\, b \,| \not< 2\sqrt{D},\ |\, c \,| \not< 2\sqrt{D}$, where $D \neq 13$, and the forms which are not primitive. The remaining forms are the positive, reduced, primitive forms of determinant $D = 13$. They are

$$(3, 1, -4), (-4, 1, 3), (4, 1, -3), (-3, 1, 4), (-3, 2, 3), (3, 2, -3), (-1, 3, 4),$$
$$(1, 3, -4), (4, 3, -1), (-4, 3, 1), (2, 3, -2), (-2, 3, 2).$$

⊠

To decide if two quadratic forms with positive determinant are equivalent they must have the same determinant and belong to the same *class*. Hence we compare their *reduced forms*. The characteristic feature of the following method is the introduction of *irrational* numbers.

Article 93 (*ibid*, p. 185)

Let (a, b, c) denote a *primitive* binary quadratic form whose determinant $D = b^2 - ac$ is positive. The corresponding quadratic equation $a + 2b\omega + c\omega^2 = 0$, denoted as $[a, b, c]$, has first and second roots defined respectively as

$$\frac{-b - \sqrt{D}}{a} \quad \text{and} \quad \frac{-b + \sqrt{D}}{a}$$

In what follows we shall say that roots ω, ω' of the respective forms (a, b, c) and (a', b', c') have the same denomination if they are both first roots or both second roots, and opposite denomination if one is a first root and the other is a second root. If the two forms (a, b, c) and (a', b', c') have the same determinant, and if they have roots ω, ω' of the same denomination connected by the equation

$$\omega = \frac{\gamma + \delta\omega'}{\alpha + \beta\omega'}$$

in which the four numbers $\alpha, \beta, \gamma, \delta$ satisfy $\alpha\delta - \beta\gamma = +1$, then the two forms are *properly equivalent*. Furthermore (a, b, c) goes to (a', b', c') under the transformation

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

The the two forms are *improperly equivalent* when roots ω, ω' are of different denomination with $\alpha\delta - \beta\gamma = -1$.

Article 93 (*ibid*, p. 186)
If ω, Ω are two irrational quantities related by the equation

$$\omega = \frac{\gamma + \delta\Omega}{\alpha + \beta\Omega}$$

in which the four numbers $\alpha, \beta, \gamma, \delta$ satisfy $\alpha\delta - \beta\gamma = \pm 1$, the developments of ω and Ω in a continued fraction will ultimately coincide, and the same quotient will occupy an even or an uneven place in both developments alike, if $\alpha\delta - \beta\gamma = +1$, but an even place in the one, and an uneven place in the other, if $\alpha\delta - \beta\gamma = -1$.

Example
For $\phi_0 = (3, 1, -4)$, with $D = 13$, we obtain the following period of ten forms by reduction.

$\phi_0 = (3, 1, -4)$	$\longrightarrow$	$\phi_1 = (-4, 3, 1)$	$\mu = -1$
$\phi_1 = (-4, 3, 1)$	$\longrightarrow$	$\phi_2 = (1, 3, -4)$	$\mu = +6$
$\phi_2 = (1, 3, -4)$	$\longrightarrow$	$\phi_3 = (-4, 1, 3)$	$\mu = -1$
$\phi_3 = (-4, 1, 3)$	$\longrightarrow$	$\phi_4 = (3, 2, -3)$	$\mu = +1$
$\phi_4 = (3, 2, -3)$	$\longrightarrow$	$\phi_5 = (-3, 1, 4)$	$\mu = -1$
$\phi_5 = (-3, 1, 4)$	$\longrightarrow$	$\phi_6 = (4, 3, -1)$	$\mu = +1$
$\phi_1 = (4, 3, -1)$	$\longrightarrow$	$\phi_7 = (-1, 3, 4)$	$\mu = -6$
$\phi_7 = (-1, 3, 4)$	$\longrightarrow$	$\phi_8 = (4, 1, -3)$	$\mu = +1$
$\phi_8 = (4, 1, -3)$	$\longrightarrow$	$\phi_9 = (-3, 2, 3)$	$\mu = -1$
$\phi_9 = (-3, 2, 3)$	$\longrightarrow$	$\phi_{10} = (3, 1, -4)$	$\mu = +1$

⊠

Article 93 (*ibid*, p. 187)
A quadratic form $(\alpha_0, \beta_0, \alpha_1)$ of positive determinant is said to be *reduced* when the roots of $[\alpha_0, \beta_0, \alpha_1]$ are of opposite sign and the absolute value of the first root being greater than unity, the other less in absolute magnitude than unity. A series of reduced forms equivalent to any proposed form (a_0, b_0, a_1) can always be found. For, if the first root of $[a_0, b_0, a_1]$ be developed in a continued fraction, and if its period of equations (beginning with an equation occupying an uneven place in the series of transformed equations) be represented as before by

$$[\alpha_0, \beta_0, \alpha_1], [\alpha_1, \beta_1, \alpha_2], [\alpha_2, \beta_2, \alpha_3], ..., [\alpha_{2k-1}, \beta_{2k-1}, \alpha_0]$$

the forms

$$(\alpha_0, \beta_0, \alpha_1), (\alpha_1, -\beta_1, \alpha_2), (\alpha_2, -\beta_2, \alpha_3), ..., (\alpha_{2k-1}, -\beta_{2k-1}, \alpha_0)$$

will be all reduced and all equivalent to (a_0, b_0, a_1). These forms, so deduced from the development of the first root of the equation $[a_0, b_0, a_1]$, we shall term the period of forms equivalent to (a_0, b_0, a_1), or, more briefly, the period of (a_0, b_0, a_1) . It will be seen that each form of the period is contiguous to that which precedes it, and that the first is contiguous to the last.

Example
Let $\phi_0 = (3, 1, -4)$ with determinant $D = 13$.
The corresponding quadratic equation $3 + 2\omega - 4\omega^2 = 0$ has first root

$$\frac{1+\sqrt{13}}{4}$$

Developing as a continued fraction gives

$$\frac{1+\sqrt{13}}{4} = [1; 6, 1, 1, 1, 1, ...]$$

The corresponding period of transformed quadratic equations are

$$[3, 1, -4], [-4, 3, 1], [1, 3, -4], [-4, 1, 3], [3, 2, -3], [-3, 1, 4], \\ [4, 3, -1], [-1, 3, 4], [4, 1, -3], [-3, 2, 3].$$

Note that every equation of the period has one of its roots positive and greater than unity, the other negative and less in absolute magnitude than unity i.e.

$$\left|\frac{-b_0 - \sqrt{D}}{a_1}\right| > 1 \quad \text{and} \quad \left|\frac{-b_0 + \sqrt{D}}{a_1}\right| < 1$$

that is $\sqrt{D} - b_0 < | \, a_1 \, | < \sqrt{D} + b_0$, the condition satisfied by the coefficients of a reduced form of positive determinant.

⊠

Two equivalent reduced forms with positive determinant belong to the same period. Two reduced forms cannot be equivalent when they belong to different periods. With this notion in place we now have a method of deciding whether two given forms with the same positive determinant are equivalent. Are the forms $(713, 60, 5)$ and $(62, 95, 145)$, which both have determinant $D = 35$, equivalent? Firstly, find a reduced equivalent of each form. Secondly, generate the reduced forms of determinant $D = 35$. Next subdivide the reduced forms into periods. The original forms are equivalent if and only if the reduced forms belong to the same period. If the forms are equivalent we can obtain a transformation which sends one form to the other.

EXAMPLE

Are the forms $(713, 60, 5)$ and $(62, 95, 145)$ equivalent?
Both have determinant $D = 35$.
Firstly, find the reduced equivalent of each form.
The form (a, b, c) is reduced when

$$0 < \sqrt{D} - b < |\, c \,| < \sqrt{D} + b$$

Firstly

$$\boxed{(713, 60, 5)} \longrightarrow \boxed{(5, 5, -2)} \qquad \mu = 13$$

Now $(5, 5, -2)$ is the reduced form with

$$(713, 60, 5) \longrightarrow (5, 5, -2) \quad \text{under} \quad \begin{pmatrix} 0 & -1 \\ 1 & 13 \end{pmatrix}.$$

Secondly $(62, 95, 145)$

$$\boxed{(62, 95, 145)} \longrightarrow (145, -95, 62) \qquad \mu = +0$$
$$(145, -95, 62) \longrightarrow (62, -29, 13) \qquad \mu = -2$$
$$(62, -29, 13) \longrightarrow (13, 3, -2) \qquad \mu = -2$$
$$(13, 3, -2) \longrightarrow \boxed{(-2, 5, 5)} \qquad \mu = -4$$

Now $(-2, 5, 5)$ is the reduced form with

$$(62, 95, 145) \longrightarrow (-2, 5, 5) \quad \text{under} \quad \begin{pmatrix} -3 & 10 \\ 2 & -7 \end{pmatrix}.$$

There are 8 reduced forms of determinant $D = 35$.

$$(\pm 1, 5, \mp 10) \qquad (\pm 2, 5, \mp 5)$$
$$(\pm 10, 5, \mp 1) \qquad (\pm 5, 5, \mp 2)$$

Now for the subdivision of the reduced forms into periods.
For $\phi_0 = (1, 5, -10)$ we obtain the following period of two forms:

$$\boxed{\phi_0 = (1, 5, -10)} \longrightarrow \phi_1 = (-10, 5, 1) \qquad \mu = -1$$
$$\phi_1 = (-10, 5, 1) \longrightarrow \boxed{\phi_2 = (1, 5, -10)} \qquad \mu = +10$$

For $\psi_0 = (-1, 5, 10)$ we obtain the following period of two forms:

$$\boxed{\psi_0 = (-1, 5, 10)} \longrightarrow \psi_1 = (10, 5, -1) \qquad \mu = +1$$
$$\psi_1 = (10, 5, -1) \longrightarrow \boxed{\psi_2 = (-1, 5, 10)} \qquad \mu = -10$$

For $\chi_0 = (2, 5, -5)$ we obtain the following period of two forms:

$$\boxed{\chi_0 = (2, 5, -5)} \longrightarrow \chi_1 = (-5, 5, 2) \qquad \mu = -2$$
$$\chi_1 = (-5, 5, 2) \longrightarrow \boxed{\chi_2 = (2, 5, -5)} \qquad \mu = +5$$

For $\theta_0 = (5, 5, -2)$ we obtain the following period of two forms:

$$\boxed{\chi_0 = (5, 5, -2)} \longrightarrow \chi_1 = (-2, 5, 5) \qquad \mu = -5$$
$$\chi_1 = (-2, 5, 5) \longrightarrow \boxed{\chi_2 = (5, 5, -2)} \qquad \mu = +2$$

There are four periods of determinant $D = 35$ each with two forms. The two reduced forms $(5, 5, -2)$ and $(-2, 5, 5)$ belong to the same two form period. Consequently, the two given forms $(713, 60, 5)$ and $(62, 95, 145)$ are equivalent. Furthermore

$$(713, 60, 5) \longrightarrow (62, 95, 145) \text{ under } \begin{pmatrix} -3 & -5 \\ 21 & 68 \end{pmatrix}.$$

$\boxtimes$

Example

To obtain all the *primitive representations* of $M = 39$ by the form (a, b, c) of determinant $D = 79$, we investigate all the values of $\Omega^2 \equiv 79 \pmod{39} \equiv 1 \pmod{39}$. Solving yields $\Omega = 1, 14, 25, 38 \pmod{39}$. For each congruence class we compare (a, b, c) with each of the forms

$$\left(M, \Omega, \frac{\Omega^2 - D}{M}\right)$$

Say, for example, $\Omega \equiv 25 \pmod{39}$.

$$
\begin{array}{lcll}
\boxed{(35,25,14)} & \longrightarrow & (14,3,-5) & \mu = +2 \\
(14,3,-5) & \longrightarrow & (-5,2,15) & \mu = -1 \\
(-5,2,15) & \longrightarrow & (15,13,6) & \mu = +1 \\
(15,13,6) & \longrightarrow & (6,5,-9) & \mu = +3 \\
(6,5,-9) & \longrightarrow & (-9,4,7) & \mu = -1 \\
(-9,4,7) & \longrightarrow & \boxed{(7,3,-10)} & \mu = +1
\end{array}
$$

Now $(7, 3, -10)$ is the reduced form with

$$(35, 25, 14) \longrightarrow (7, 3, -10) \quad \text{under} \quad \begin{pmatrix} -7 & 6 \\ 8 & -7 \end{pmatrix}.$$

Furthermore, from the inverse transformation, $(7, 3, -10)(-7, -8)^2 = 39$.

⊠

Article 100 (*ibid*, p. 212)

Let (a, b, c) be a primitive form of the positive determinant D.
Let $(a, b, c)(x_0, y_0)^2 = M$ a positive number represented by (a, b, c).
Let $m = \gcd(a, 2b, c)$.
$[T, U]$ the least positive solution of $T^2 - DU^2 = m^2$, so that if

$$x_n = \frac{1}{m}\Big[T_n x_0 - U_n(bx_0 + cy_0)\Big] \quad , \quad y_n = \frac{1}{m}\Big[T_n y_0 + U_n(ax_0 + by_0)\Big]$$

the two formulae $[x_n, y_n]$ and $[-x_n, -y_n]$ will together express every representation of M, which belongs to the same set as $[x_0, y_0]$.
Similarly, let $[x'_n, y'_n]$ and $[-x'_n, -y'_n]$ denote a complete set of representations of the positive number M' by (a, b, c).

B

QUADRATIC RESIDUES

B.1 QUADRATIC RESIDUES AND NONRESIDUES

Definition: (Quadratic residues and nonresidues). Let $m \in \mathbb{N}$ and $a \in \mathbb{Z}$ be such that $\gcd(a, m) = 1$. Then a is called a quadratic residue modulo m if the congruence

$$x^2 \equiv a (\text{mod } m) \quad (*_A)$$

has a solution and a is called a quadratic nonresidue modulo m if $(*_A)$ has no solution.

Remark:

i Note that, by definition, integers a that do not satisfy the condition $\gcd(a, m) = 1$ are not classified as quadratic residues or nonresidues. In particular, 0 is considered neither a quadratic residue nor a quadratic nonresidue (even though, for $a = 0$, $(*_A)$ has a solution, namely $x = 0$.

ii While the definition of quadratic residues and nonresidues allow the modulus m to be an arbitrary positive integer, in the following we will focus exclusively on the case when m is an odd prime p.

Proposition: (Number of solutions to quadratic congruences). Let p be an odd prime, and let $a \in \mathbb{Z}$ with $\gcd(a, m) = 1$.

i If a is a quadratic nonresidue modulo p, the congruence $(*_A)$ has no solution.

ii If a is a quadratic residue modulo p, the congruence $(*_A)$ has exactly two incongruent solutions x modulo p. More precisely, if x_0 is one solution, then a second, incongruent, solution is given by $p - x_0$.

Proposition: (Number of quadratic residues and nonresidues). Let p be an odd prime. Then among the integers $1, 2, \ldots\ldots, p-1$, exactly half (i.e., $(p-1)/2$) are quadratic residues modulo p, and exactly half are quadratic nonresidues modulo p.

B.2 THE LEGENDRE SYMBOL

DEFINITION: (The Legendre Symbol). Let p be an odd prime, and let $a \in \mathbb{Z}$ with $\gcd(a, p) = 1$ (or, equivalently, $p \nmid a$). The Legendre symbol of a modulo p, denoted by $\left(\frac{a}{p}\right)$, is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p. \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

REMARK: Note that the modulus in this definition, and all the results that follow, is restricted to odd primes (i.e., a prime other that 2). One can extend the definition, and most of the results, to composite moduli, but things get a lot more complicated then.

PROPOSITION: (Properties of the Legendre Symbol) Let p be an odd prime, and let $a, b \in \mathbb{Z}$ with $\gcd(a, p) = 1$ and $\gcd(b, p) = 1$.

i If $a \equiv b (\text{mod } p)$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$.

ii $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

iii $\left(\frac{a^2}{p}\right) = 1$

iv $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = \begin{cases} 1 & \text{if } p \equiv 1 (\text{mod } 4). \\ -1 & \text{if } p \equiv 3 (\text{mod } 4). \end{cases}$

v $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = \begin{cases} 1 & \text{if } p \equiv 1, 7 (\text{mod } 8). \\ -1 & \text{if } p \equiv 3, 5 (\text{mod } 8). \end{cases}$

PROPOSITION: (Euler's Criterion). Let p be an odd prime, and let $a \in \mathbb{Z}$ with $\gcd(a, p) = 1$. Then a is a quadratic residue modulo p if $a^{(p-1)/2} \equiv 1 (\text{mod } p)$, and a quadratic nonresidue if $a^{(p-1)/2} \equiv -1 (\text{mod } p)$; equivalently

$$\left(\frac{a}{p}\right) = (-1)^{(p-1)/2} (\text{mod } p)$$

B.3 THE LAW OF QUADRATIC RECIPROCITY

THEOREM: (Quadratic reciprocity law – Gauss 1795). Let p and q be distinct odd primes. Then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

Equivalently

$$\left(\frac{p}{q}\right) = \begin{cases} -\left(\frac{q}{p}\right) & \text{if } p \equiv 3 (\text{mod } 4) \text{ and } q \equiv 3 (\text{mod } 4). \\ \left(\frac{q}{p}\right) & \text{otherwise.} \end{cases}$$

REMARK:

i The first form of the reciprocity law is the cleaner and more elegant form, and the one in which the law is usually stated. However, for applications, the second form is more useful. In this form the law says that numerator and denominator in a Legendre symbol (assuming both are distinct odd primes) can be interchanged in all cases except when both numerator and denominator are congruent to 3 modulo 4, in which case the sign of the Legendre symbol flips after interchanging numerator and denominator. Put differently, this form states that p is a quadratic residue modulo q if and only if q is a quadratic residue modulo p, except in the case when both p and q are congruent to 3 modulo 4; in the latter case p is a quadratic residue modulo q if and only if q is a quadratic nonresidue modulo p.

ii Note that the reciprocity law requires numerator and denominator to be distinct odd primes. In particular, it does not apply directly to cases where the numerator is composite, negative, or an even number. However, these cases can be reduced to the prime case using the multiplicativity of the Legendre symbol along with the special values at -1 and 2.

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = \begin{cases} 1 & \text{if } p \equiv 1 (\text{mod } 4). \\ -1 & \text{if } p \equiv 3 (\text{mod } 4). \end{cases}$$

and

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = \begin{cases} 1 & \text{if } p \equiv 1, 7 (\text{mod } 8). \\ -1 & \text{if } p \equiv 3, 5 (\text{mod } 8). \end{cases}$$

In fact, these last two relations are called the First Supplementary Law and the Second Supplementary Law, as they 'supplement' the quadratic reciprocity law.

C

PROPERLY PRIMITIVE, REDUCED, POSITIVE TERNARY QUADRATIC FORMS

In 1851 Eisenstein published tables of primitive reduced positive ternary quadratic forms. He calculated these tables by simplifying Seeber's inequalities for a reduced ternary quadratic form by replacing them with linear inequalities (Eisenstein, 1851). The first table [*Section of*] of properly primitive reduced positive ternary quadratic forms of discriminant -1 to -100 and -358 (Table 20). The second table [*Section of*] of improperly primitive reduced positive ternary quadratic forms of discriminant -2 to -100. (Table 21). Eisenstein also included the number δ of transformations of the form onto itself.

Definition (Eisenstein, 1847a, p. 120) A transformation which maps a form f onto itself is called an *automorphic transformation*. Let δ denote the number of *positive automorphics* of the form f. The *weight of a form* m is the reciprocal of the number of its positive automorphics, i.e.

$$m = \frac{1}{\delta}$$

Table 20: Properly primitive, reduced, positive ternary quadratic forms $f = ax^2 + a'y^2 + a''z^2 + 2byz + 2b'xz + 2b''xy$ of discriminant $-D$ and the number δ of their automorphs. Separate list for $ax^2 + a'y^2 + a''z^2$ (Eisenstein, 1851, pp. 169–185)

D	a	a'	a''	b	b'	b''	δ
3	1	2	2	−1	0	0	12
5	1	2	3	−1	0	0	4
7	1	2	4	−1	0	0	4
	2	2	3	1	1	1	6
8	1	3	3	−1	0	0	4
	2	2	3	−1	−1	0	8
9	1	2	5	−1	0	0	4
	2	2	3	0	0	−1	12
10	2	2	3	0	−1	0	4
11	1	2	6	−1	0	0	4
	1	3	4	−1	0	0	2
12	1	4	4	−2	0	0	12
	2	3	3	1	1	1	4
13	1	2	7	−1	0	0	4
	2	2	5	1	1	1	6
	2	3	3	−1	0	−1	2
14	1	3	5	−1	0	0	2
15	1	2	8	−1	0	0	4
	1	4	4	−1	0	0	4
	2	2	5	0	0	−1	12
	2	3	3	0	0	−1	4
16	1	4	5	−2	0	0	4
	2	2	5	−1	−1	0	8
	2	3	3	−1	0	0	4
	3	3	3	−1	−1	−1	24
17	1	2	9	−1	0	0	4
	1	3	6	−1	0	0	2
	2	3	4	1	1	1	2
18	2	2	5	0	−1	0	4
	2	3	4	−1	0	−1	2
19	1	2	10	−1	0	0	4
	1	4	5	−1	0	0	2
	2	2	7	1	1	1	6
	2	3	4	−1	−1	0	2

D	a	a'	a''
1	1	1	1
2	1	1	2
3	1	1	3
4	1	1	4
	1	2	2
5	1	1	5
6	1	1	6
	1	2	3
7	1	1	7
8	1	1	8
	1	2	4
9	1	1	9
	1	3	3
10	1	1	10
	1	2	5
11	1	1	11
12	1	1	12
	1	2	6
	1	3	4
	2	2	3
13	1	1	13
14	1	1	14
	1	2	7
15	1	1	15
	1	3	5
16	1	1	16
	1	2	8
	1	4	4
17	1	1	17
18	1	1	18
	1	2	9
	1	3	6
	2	3	3
19	1	1	19

Continued Overleaf........

D	a	a'	a''	b	b'	b''	δ
20	1	3	7	−1	0	0	2
	1	4	6	−2	0	0	4
	2	3	4	0	0	−1	4
	3	3	3	1	1	1	6
21	1	2	11	−1	0	0	4
	1	5	5	−2	0	0	4
	2	2	7	0	0	−1	12
	2	3	4	0	−1	0	4
	3	3	3	0	−1	−1	2
22	2	3	4	−1	0	0	2
	2	3	5	1	1	1	2
23	1	2	12	−1	0	0	4
	1	3	8	−1	0	0	2
	1	4	6	−1	0	0	2
	2	3	5	−1	0	−1	2
24	1	4	7	−2	0	0	4
	1	5	5	−1	0	0	4
	2	2	7	−1	−1	0	8
	3	3	3	0	0	−1	4
	3	3	4	−1	−1	−1	4
25	1	2	13	−1	0	0	4
	2	2	9	1	1	1	6
	2	3	5	−1	−1	0	2
	2	3	5	0	0	−1	4
26	1	3	9	−1	0	0	2
	1	5	6	−2	0	0	2
	2	2	7	0	−1	0	4
27	1	2	14	−1	0	0	4
	1	4	7	−1	0	0	2
	1	6	6	−3	0	0	12
	2	2	9	0	0	−1	12
	2	3	5	0	−1	0	4
	2	3	6	1	1	1	2
	2	4	5	2	1	1	2
28	1	4	8	−2	0	0	4
	2	3	5	−1	0	0	2
	2	3	6	−1	0	−1	2
	2	4	5	−2	−1	0	4
	3	3	4	1	1	1	2

D	a	a'	a''
20	1	1	20
	1	2	10
	1	4	5
	2	2	5
21	1	1	21
	1	3	7
22	1	1	22
	1	2	11
23	1	1	23
24	1	1	24
	1	2	12
	1	3	8
	1	4	6
	2	3	4
25	1	1	25
	1	5	5
26	1	1	26
	1	2	13
27	1	1	27
	1	3	9
28	1	1	28
	1	2	14
	1	4	7
	2	2	7

Table 21: Improperly primitive, reduced, positive ternary quadratic forms $f = ax^2 + a'y^2 + a''z^2 + 2byz + 2b'xz + 2b''xy$ of discriminant $-D$ and the number δ of their automorphs. (Eisenstein, 1851, pp. 186–189)

D	a	a'	a''	b	b'	b''	δ
4	2	2	2	1	1	1	24
6	2	2	2	0	0	−1	12
10	2	2	4	1	1	1	6
12	2	2	4	−1	−1	0	8
	2	2	4	0	0	−1	12
14	2	2	4	0	−1	0	4
16	2	2	6	1	1	1	6
18	2	2	6	0	0	−1	12
20	2	2	6	−1	−1	0	8
	2	4	4	2	1	1	4
22	2	2	6	0	−1	0	4
	2	2	8	0	0	−1	6
24	2	2	8	0	0	−1	12
	2	4	4	1	1	1	4
26	2	4	4	−1	0	−1	2
28	2	2	8	−1	−1	0	8
	2	2	10	1	1	1	6
	2	2	4	0	0	−1	4
30	2	2	8	0	−1	0	4
	2	2	10	0	0	−1	12
	2	4	4	−1	0	0	4
34	2	2	12	1	1	1	6
	2	4	6	2	1	1	2
36	2	2	10	−1	−1	0	8
	2	2	12	0	0	−1	12
	2	4	6	−2	−1	0	4
	4	4	4	1	2	2	8
38	2	2	10	0	−1	0	4
	2	4	6	1	1	1	2
40	2	2	14	1	1	1	6
	2	4	6	−1	0	−1	2
42	2	2	14	0	0	−1	12
	2	4	6	−1	−1	0	2
	2	4	6	0	0	−1	4

D

HENRY SMITH'S IDENTITIES

Two primitive ternary quadratic forms of the same invariants $[\Omega, \Delta]$ are said to belong to the same order when they and their primitive contravariants are alike properly or alike improperly primitive. Henry Smith uses the following identities to achieve a subdivision of the orders into genera. The following is an extract from Paul Bachmann's *Die Arithmetik der Quadratischen Formen* (Volume 4, Part 1, 1898) (Bachmann, 1898, pp. 7–10).

$$f(x_1, y_1, z_1).f(x_2, y_2, z_2) - \frac{1}{4}\left(x_1 \frac{\partial f}{\partial x_2} + y_1 \frac{\partial f}{\partial y_2} + z_1 \frac{\partial f}{\partial z_2}\right)^2$$
$$= \Omega F(y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2)$$

$$F(x_1, y_1, z_1).F(x_2, y_2, z_2) - \frac{1}{4}\left(x_1 \frac{\partial F}{\partial x_2} + y_1 \frac{\partial F}{\partial y_2} + z_1 \frac{\partial F}{\partial z_2}\right)^2$$
$$= \Delta f(y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2)$$

These important identities may be established as follows. Let

$$f(x_1, y_1, z_1) = a x_1^2 + a' y_1^2 + a'' z_1^2 + 2b y_1 z_1 + 2b' x_1 z_1 + 2b'' x_1 y_1$$
$$f(x_2, y_2, z_2) = a x_2^2 + a' y_2^2 + a'' z_2^2 + 2b y_2 z_2 + 2b' x_2 z_2 + 2b'' x_2 y_2$$

$$f^0(x) = \frac{1}{2}\frac{\partial f}{\partial x_1} = a x_1 + b'' y_1 + b' z_1 \quad , \quad f^0(y) = \frac{1}{2}\frac{\partial f}{\partial x_2} = a x_2 + b'' y_2 + b' z_2$$

$$f^1(x) = \frac{1}{2}\frac{\partial f}{\partial y_1} = b'' x_1 + a' y_1 + b z_1 \quad , \quad f^1(y) = \frac{1}{2}\frac{\partial f}{\partial y_2} = b'' x_2 + a' y_2 + b z_2$$

$$f^2(x) = \frac{1}{2}\frac{\partial f}{\partial z_1} = b' x_1 + b y_1 + a'' z_1 \quad , \quad f^2(y) = \frac{1}{2}\frac{\partial f}{\partial z_2} = b' x_2 + b y_2 + a'' z_2$$

$$f(x_1, y_1, z_1) = f^0(x).x_1 + f^1(x).y_1 + f^2(x).z_1$$
$$f(x_2, y_2, z_2) = f^0(y).x_2 + f^1(y).y_2 + f^2(y).z_3$$

From the theory of determinants, we have

$$\begin{vmatrix} ax_1 + a'y_1 + a''z_1 & ax_2 + a'y_2 + a''z_2 \\ bx_1 + b'y_1 + b''z_1 & bx_2 + b'y_2 + b''z_2 \end{vmatrix} =$$

$$\begin{vmatrix} a' & a'' \\ b' & b'' \end{vmatrix} \cdot \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} + \begin{vmatrix} a'' & a \\ b'' & b \end{vmatrix} \cdot \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} + \begin{vmatrix} a & a' \\ b & b' \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

Applying this theorem to

$$\begin{vmatrix} f(x_1, y_1, z_1) & \\ & f(x_2, y_2, z_2) \end{vmatrix} = \begin{vmatrix} f^0(x)x_1 + f^1(x)y_1 + f^2(x)z_1 & f^0(x)x_1 + f^1(x)y_1 + f^2(x)z_1 \\ f^0(y)x_2 + f^1(y)y_2 + f^2(y)z_2 & f^0(y)x_2 + f^1(y)y_2 + f^2(y)z_2 \end{vmatrix}$$

$$= \begin{vmatrix} f^1(x) & f^2(x) \\ f^1(y) & f^2(y) \end{vmatrix} \cdot \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \qquad (*_A)$$

$$+ \begin{vmatrix} f^2(x) & f^0(x) \\ f^2(y) & f^0(y) \end{vmatrix} \cdot \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} \qquad (*_B)$$

$$+ \begin{vmatrix} f^0(x) & f^1(x) \\ f^0(y) & f^1(y) \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \qquad (*_C)$$

Now

$$(*_A) \quad \begin{vmatrix} f^1(x) & f^2(x) \\ f^1(y) & f^2(y) \end{vmatrix} = \begin{vmatrix} b''x_1 + a'y_1 + bz_1 & b'x_2 + by_2 + a''z_2 \\ b''x_1 + a'y_1 + bz_1 & b'x_2 + by_2 + a''z_2 \end{vmatrix}$$
$$= (b''x_1 + a'y_1 + bz_1)(b'x_2 + by_2 + a''z_2) - (b''x_1 + a'y_1 + bz_1)(b'x_2 + by_2 + a''z_2)$$

$$(*_B) \quad \begin{vmatrix} f^2(x) & f^0(x) \\ f^2(y) & f^0(y) \end{vmatrix} = \begin{vmatrix} b'x_1 + by_1 + a''z_1 & ax_1 + b''y_1 + b'z_1 \\ b'x_2 + by_2 + a''z_2 & ax_2 + b''y_2 + b'z_2 \end{vmatrix}$$
$$= (b'x_1 + by_1 + a''z_1)(ax_2 + b''y_2 + b'z_2) - (b'x_2 + by_2 + a''z_2)(ax_1 + b''y_1 + b'z_1)$$

$$(*_C) \quad \begin{vmatrix} f^0(x) & f^1(x) \\ f^0(y) & f^1(y) \end{vmatrix} = \begin{vmatrix} ax_1 + b''y_1 + b'z_1 & b''x_1 + a'y_1 + bz_1 \\ ax_2 + b''y_2 + b'z_2 & b''x_2 + a'y_2 + bz_2 \end{vmatrix}$$
$$= (ax_1 + b''y_1 + b'z_1)(b''x_2 + a'y_2 + bz_2) - (ax_2 + b''y_2 + b'z_2)(b''x_1 + a'y_1 + bz_1)$$

The following identity is important

$$f^0(y)x_1 + f^1(y)x_2 + f^2(y)x_3 = f^0(x)y_1 + f^1(x)y_1 + f^2(x)y_3$$

Hence

$$f(x_1, y_1, z_1).f(x_2, y_2, z_2) - \frac{1}{4}\left(x_1\frac{\partial f}{\partial x_2} + y_1\frac{\partial f}{\partial y_2} + z_1\frac{\partial f}{\partial z_2}\right)^2$$

$$= \Omega F(y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, x_1y_2 - y_1x_2)$$

$$F(x_1, y_1, z_1).F(x_2, y_2, z_2) - \frac{1}{4}\left(x_1\frac{\partial F}{\partial x_2} + y_1\frac{\partial F}{\partial y_2} + z_1\frac{\partial F}{\partial z_2}\right)^2$$

$$= \Delta f(y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, x_1y_2 - y_1x_2)$$

E

SUPPLEMENTARY THEOREM AND DEMONSTRATION

Theorem: (Smith, 1868, p. 259).
There exists pairs of forms φ and Φ, equivalent to f and F, and satisfying the congruences

$$\begin{aligned}\varphi &\equiv \alpha x^2+\beta\Omega y^2+\gamma\Omega\Delta z^2 (\text{mod } \triangledown)\\ \Phi &\equiv \beta\gamma\Omega\Delta x^2+\alpha\gamma\Delta y^2+\alpha\beta z^2 (\text{mod } \triangledown)\\ \alpha\beta\gamma &\equiv 1(\text{mod } \triangledown)\end{aligned}$$

for any proposed modulus $\triangledown$; but this modulus must be odd, if either f or F is improperly primitive.

Demonstration: (Smith, 1868, p. 259).
Let $\nabla' = \nabla\Omega^2\Delta$ where ∇ is an arbitrary constant. Assume A'' is prime to ∇', i.e., $\gcd(A'', \nabla') = 1$. Assume $A'' \equiv \Omega(\text{mod } 4)$ if $\Omega\Delta$ is odd. Let

$$\gamma \equiv \frac{1}{A''}(\text{mod } \nabla')$$

From the definition of f and its contravariant, we have

$$\begin{aligned}aB'+b''B+b'A'' &= 0\\ b''B'+a'B+bA'' &= 0\\ b'B'+bB+a''A'' &= \Omega^2\Delta\end{aligned}$$

The system of linear congruences

$$\begin{aligned}ax+b''y+b' &\equiv 0(\text{mod } \nabla')\\ b''x+a'y+b &\equiv 0(\text{mod } \nabla')\\ b'x+by+a'' &\equiv \gamma\Omega\Delta(\text{mod } \nabla')\end{aligned}$$

are satisfied when x and y are determined by

$$\begin{aligned}A''x &\equiv \frac{B'}{\Omega}(\text{mod}\nabla')\\ A''y &\equiv \frac{B}{\Omega}(\text{mod}\nabla')\end{aligned}$$

The system is resoluble admitting Ω incongruous solutions.
Let $x \equiv \lambda(\text{mod } \nabla')$ and $y \equiv \mu(\text{mod } \nabla')$ be any one of these solutions.
We transform f into an equivalent form f_1 by the substitution $x = x + \lambda z$ and $y = y + \mu z$.
The coefficients a_1, b_1'', a_1' are the same as a, b'', a', the coefficients a_1'', b_1, b_1' are

respectively congruous for the modulus ∇' to $\gamma\Omega\Delta, 0, 0$.
So f_1 satisfies the congruence

$$f_1 \equiv ax^2 + 2b''xy + a'y^2 + \gamma\Omega\Delta z^2 (\text{mod } \nabla')$$

The binary form $ax^2 + 2b''xy + a'y^2$ is *primitive*. Now for this primitive binary form we assume a is prime to ∇', i.e., $\gcd(a, \nabla') = 1$. Let

$$\beta \equiv \frac{A''}{a} (\text{mod } \nabla')$$

The system of linear congruences

$$\begin{aligned} ax + b'' &\equiv 0 (\text{mod } \nabla') \\ b''x + a' &\equiv \beta\Omega (\text{mod } \nabla') \end{aligned}$$

is resoluble and admits one solutions.
Let $x \equiv \lambda (\text{mod } \nabla')$ be that solution.
We transform f_1 into an equivalent form φ by the substitution $x = x + \lambda y$.
The coefficient a is the same as a_1, the coefficients a', b'' are respectively congruous for the modulus ∇' to $\beta\Omega, 0$.
So φ satisfies the congruence $\varphi \equiv ax^2 + \beta\Omega y^2 + \gamma\Omega\Delta z^2 (\text{mod } \nabla')$.

⊠

Example: Let

$$\begin{aligned} f &= 3x^2 + 4y^2 + 5z^2 + 2yz + 2xz + 2xy \\ F &= 19x^2 + 14y^2 + 5z^2 - 4yz - 6xz - 8xy \end{aligned}$$

f is a form of the invariants $[\Omega, \Delta] = [1, 50]$.
F is a form of the invariants $[\Delta, \Omega] = [50, 1]$.
Let $\nabla' = \nabla\Omega^2\Delta$ where ∇ is an arbitrary constant. Let $\nabla = 1$. Let

$$\gamma \equiv \frac{1}{A''} (\text{mod } \nabla')$$

Now $11\gamma \equiv 1 (\text{mod } 50)$ hence $\gamma = 41$. The system of linear congruences

$$\begin{aligned} 3x + y + 1 &\equiv 0 (\text{mod } 50) \\ x + 4y + 1 &\equiv 0 (\text{mod } 50) \\ x + y + 5 &\equiv 0 (\text{mod } 50) \end{aligned}$$

are satisfied when $x \equiv 27 (\text{mod } 50)$ and $y \equiv 18 (\text{mod } 50)$. We transform f into an equivalent form f_1 by the substitution $x = x + 27z$, $y = y + 18z$.

$$\begin{aligned} f \equiv\ & 3(x + 27z)^2 + 4(y + 18z)^2 + 5z^2 + 2(y + 18z)z \\ & +2(x + 27z)z + 2(x + 27z)(y + 18z) (\text{mod } 50) \end{aligned}$$

Hence $f_1 \equiv 3x^2 + 2xy + 4y^2 (\text{mod } 50)$. Let

$$\beta \equiv \frac{A''}{a} (\text{mod } \nabla')$$

Now $3\beta \equiv 11 \pmod{50}$ hence $\beta = 37$. The system of linear congruences

$$\begin{aligned} 3x + 1 &\equiv 0 \pmod{50} \\ x + 4 &\equiv 37 \pmod{50} \end{aligned}$$

are satisfied when $x \equiv 33 \pmod{50}$. We transform f_1 into an equivalent form φ by the substitution $x = x + 33y$.

$$f_1 \equiv 3(x + 33y)^2 + 2(x + 33y)y + 4y^2$$

Hence $\varphi \equiv 3x^2 + 37y^2 \pmod{50}$.

⊠

RESTRICTIONS – THE SIMULTANEOUS CHARACTER

B. (Smith, 1868, p. 265)
Let $\Omega \equiv 2 (\text{mod } 4)$ and $\Delta \equiv 1 (\text{mod } 2)$.
The simultaneous character is $(-1)^{\frac{1}{8}(f^2-1)}\Psi$. $*$

If	$* = (-1)^{\frac{1}{8}(\Delta^2-1)}$	$* = -(-1)^{\frac{1}{8}(\Delta^2-1)}$
$M \equiv \Omega' (\text{mod } 4)$	$m \equiv 5\Delta, 7\Delta (\text{mod } 8)$	$m \equiv \Delta, 3\Delta (\text{mod } 8)$
$M \equiv 3\Omega' (\text{mod } 4)$	$m \equiv \Delta, 7\Delta (\text{mod } 8)$	$m \equiv 3\Delta, 5\Delta (\text{mod } 8)$

Except when Ω and Δ are both odd it will be found that, in the case of any two properly primitive forms f and F, every representation of an odd number by either of the two is simultaneous with the representation of odd numbers by the other. The restrictions imposed on the numbers m and M by the simultaneous character are as follows. If $(-1)^{\frac{1}{8}(f^2-1)}\Psi = (-1)^{\frac{1}{8}(\Delta^2-1)}$ then f cannot represent numbers congruous to $3\Delta (\text{mod } 8)$. If $(-1)^{\frac{1}{8}(f^2-1)}\Psi = -(-1)^{\frac{1}{8}(\Delta^2-1)}$ then f cannot represent numbers congruous to $7\Delta (\text{mod } 8)$. This is because it cannot represent them simultaneously with the representation of odd numbers.

C. (Smith, 1868, p. 265)
Let $\Omega \equiv 1 (\text{mod } 2)$ and $\Delta \equiv 2 (\text{mod } 4)$.
The simultaneous character is $(-1)^{\frac{1}{8}(F^2-1)}\Psi$. $*$

If	$* = (-1)^{\frac{1}{8}(\Omega^2-1)}$	$* = -(-1)^{\frac{1}{8}(\Omega^2-1)}$
$m \equiv \Delta' (\text{mod } 4)$	$M \equiv 5\Omega, 7\Omega (\text{mod } 8)$	$M \equiv \Omega, 3\Omega (\text{mod } 8)$
$m \equiv 3\Delta' (\text{mod } 4)$	$M \equiv \Omega, 7\Delta (\text{mod } 8)$	$M \equiv 3\Omega, 5\Omega (\text{mod } 8)$

Since the contravariant of f and F, and the invariants Ω and Δ are everywhere simultaneously interchangeable, the restrictions imposed on the numbers m and M by the simultaneous character are the reciprocal of the previous table. If $(-1)^{\frac{1}{8}(F^2-1)}\Psi = (-1)^{\frac{1}{8}(\Omega^2-1)}$ then F cannot represent numbers congruous to $3\Omega (\text{mod } 8)$. If $(-1)^{\frac{1}{8}(F^2-1)}\Psi = -(-1)^{\frac{1}{8}(\Omega^2-1)}$ then F cannot represent numbers congruous to $7\Omega (\text{mod } 8)$. This is because it cannot represent them simultaneously with the representation of odd numbers.

D. (Smith, 1868, p. 265)
Let $\Omega \equiv 2 \pmod 4$ and $\Delta \equiv 2 \pmod 4$.
The simultaneous character is $(-1)^{\frac{1}{8}(F^2-1)}\Psi$. $*$

If	$* = (-1)^{\frac{1}{8}(\Delta'^2-1)+\frac{1}{8}(\Omega'^2-1)}$	$* = -(-1)^{\frac{1}{8}(\Delta'^2-1)+\frac{1}{8}(\Omega'^2-1)}$
$m \equiv \Delta' \pmod 8$	$M \equiv 5\Omega', 7\Omega' \pmod 8$	$M \equiv \Omega', 3\Omega' \pmod 8$
$m \equiv 3\Delta' \pmod 8$	$M \equiv 3\Omega', 5\Omega' \pmod 8$	$M \equiv \Omega', 7\Omega' \pmod 8$
$m \equiv 5\Delta' \pmod 8$	$M \equiv \Omega', 3\Omega' \pmod 8$	$M \equiv 5\Omega', 7\Omega' \pmod 8$
$m \equiv 7\Delta' \pmod 8$	$M \equiv \Omega', 7\Omega' \pmod 8$	$M \equiv 3\Omega', 5\Omega' \pmod 8$

The restrictions imposed on the numbers m and M by the simultaneous character are displayed above. Subject to these restriction f may represent odd numbers congruous to $\Delta', 3\Delta', 5\Delta', 7\Delta' \pmod 8$ and F may represent odd numbers congruous to $\Omega', 3\Omega', 5\Omega', 7\Omega' \pmod 8$.

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G

TABLE OF COMPLETE GENERIC CHARACTERS

(Smith, 1868, p. 267)
Case [P]: $(-1)^{\frac{1}{2}(f-1)}$ and $(-1)^{\frac{1}{2}(F-1)}$ are both characters.
Ψ is not an independent character but is retained in the table only because it serves to express the condition of possibility.

Case [Q]: If $(-1)^{\frac{1}{2}(f-1)}$ and Ψ, but not $(-1)^{\frac{1}{2}(F-1)}$, are inscribed as characters, Ψ represents the character $(-1)^{\frac{1}{2}(F-1)}$, or is not a character at all, according as $(-1)^{\frac{1}{2}(f-1)} = (-1)^{\frac{1}{2}(\Delta_1-1)}$ or $= -(-1)^{\frac{1}{2}(\Delta_1-1)}$. This corresponds to the cases in table I when $(-1)^{\frac{1}{2}(F-1)}$ is, or is not a character, according as $(-1)^{\frac{1}{2}(f-1)} = (-1)^{\frac{1}{2}(\Delta_1-1)}$ or $= -(-1)^{\frac{1}{2}(\Delta_1-1)}$.

Similarly, if $(-1)^{\frac{1}{2}(F-1)}$ and Ψ, but not $(-1)^{\frac{1}{2}(f-1)}$, are inscribed as characters, Ψ represents the character $(-1)^{\frac{1}{2}(f-1)}$, or is not a character at all (simply $+1$), according as $(-1)^{\frac{1}{2}(F-1)} = (-1)^{\frac{1}{2}(\Omega_1-1)}$ or $= -(-1)^{\frac{1}{2}(\Omega_1-1)}$. This corresponds to the cases in table I when $(-1)^{\frac{1}{2}(f-1)}$ is, or is not a character, according as $(-1)^{\frac{1}{2}(F-1)} = (-1)^{\frac{1}{2}(\Omega_1-1)}$ or $= -(-1)^{\frac{1}{2}(\Omega_1-1)}$. Also in this case the symbol Ψ takes the place of $*$ and $\dagger$ in table I. It also serves to express the condition of possibility.

Case [R]: If $(-1)^{\frac{1}{2}(f-1)}$ and $(-1)^{\frac{1}{8}(F^2-1)}\Psi$, but not $(-1)^{\frac{1}{2}(F-1)}$, are inscribed as characters, $(-1)^{\frac{1}{8}(F^2-1)}\Psi$ represents the character $(-1)^{\frac{1}{2}(F-1)+\frac{1}{8}(F^2-1)}$, or $(-1)^{\frac{1}{8}(F^2-1)}$, according as $(-1)^{\frac{1}{2}(f-1)} = (-1)^{\frac{1}{2}(\Delta_1-1)}$ or $= -(-1)^{\frac{1}{2}(\Delta_1-1)}$. This corresponds to the cases in table I. The symbol Ψ also serves to express the condition of possibility.

Case [S]: Neither $(-1)^{\frac{1}{2}(f-1)}$ nor $(-1)^{\frac{1}{2}(F-1)}$ is a character.

In the case of [Q] and [R] the units Ψ, $(-1)^{\frac{1}{8}(f^2-1)}\Psi$, $(-1)^{\frac{1}{8}(F^2-1)}\Psi$, which properly represent simultaneous characters of the forms f and F, are employed to represent supplementary characters. This use of these symbols is admissible, because, when $\Omega\Delta$ is even (as in the case of [Q] and [R]), every representation of an odd number by f and F is simultaneous with the representation of odd numbers by F and f.

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H

CONDITION OF POSSIBILITY FOR TERNARY QUADRATIC FORMS

(Smith, 1868, p. 266)
Let f and F be properly primitive forms.
f is a form of the invariants $[\Omega, \Delta]$.
F is a form of the invariants $[\Delta, \Omega]$.
Let $\alpha = +1$ or -1 according as Ω is of the form $\Omega_1\Omega_2^2$ or $2\Omega_1\Omega_2^2$.
Let $\beta = +1$ or -1 according as Δ is of the form $\Delta_1\Delta_2^2$ or $2\Delta_1\Delta_2^2$.

The condition of possibility is

$$\Psi \times \alpha^{\frac{1}{8}(f^2-1)} \times \beta^{\frac{1}{8}(F^2-1)} \times \left(\frac{f}{\Omega_1}\right) \times \left(\frac{F}{\Delta_1}\right) = (-1)^{\frac{1}{2}(\Omega_1+1)\frac{1}{2}(\Delta_1+1)}$$

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Demonstration The complete generic character of a form f is determined by the characters a and A''. Let f and F both be properly primitive. Suppose a and A'' are positive (odd) integers prime to one each other and to $\Omega\Delta$. The adjoint of $f = \Omega F$, i.e.

$$(a'a''-b^2)x^2+(a''a-b'^2)y^2+(aa'-b''^2)z^2+\dots\dots\dots = \Omega\left[Ax^2+A'y^2+A''z^2+\dots\dots\dots\right]$$

Now $aa'-b''^2 = \Omega A''$. Hence $b''^2 \equiv -\Omega A'' (\text{mod } a)$ and

$$\begin{aligned}\left(\frac{-\Omega A''}{a}\right) &= 1\\ \left(\frac{-\Omega}{a}\right)\left(\frac{A''}{a}\right) &= 1 \quad (*_A)\end{aligned}$$

The adjoint of $F = \Delta f$, i.e.

$$(A'A''-B^2)x^2+(A''A-B'^2)y^2+(AA'-B''^2)z^2+\dots\dots\dots = \Delta\left[ax^2+a'y^2+a''z^2+\dots\dots\dots\right]$$

Now $A'A''-B^2 = \Delta a$. Hence $B''^2 \equiv -\Delta a (\text{mod } A'')$ and

$$\begin{aligned}\left(\frac{-\Delta a}{A''}\right) &= 1\\ \left(\frac{-\Delta}{A''}\right)\left(\frac{a}{A''}\right) &= 1 \quad (*_B)\end{aligned}$$

Multiplying $(*_A)$ and $(*_B)$ together and observing that, by the law of quadratic reciprocity

$$\begin{aligned}\left(\frac{A''}{a}\right)\left(\frac{a}{A''}\right) &= (-1)^{\frac{1}{2}(a-1)\frac{1}{2}(A''-1)} \\ \left(\frac{-\Omega}{a}\right)\left(\frac{-\Delta}{A''}\right) &= (-1)^{\frac{1}{2}(a-1)\frac{1}{2}(A''-1)} \quad *\end{aligned}$$

Let α and β be as defined above. Now

$$\begin{aligned}\left(\frac{-\Omega}{a}\right) &= (-1)^{\frac{1}{2}(a-1)}\alpha^{\frac{1}{8}(a^2-1)}\left(\frac{\Omega_1}{a}\right) \\ \left(\frac{\Omega_1}{a}\right) &= (-1)^{\frac{1}{2}(a-1)\frac{1}{2}(\Omega_1-1)}\left(\frac{a}{\Omega_1}\right) \\ \left(\frac{-\Omega}{a}\right) &= (-1)^{\frac{1}{2}(a-1)\frac{1}{2}(\Omega_1+1)}\alpha^{\frac{1}{8}(a^2-1)}\left(\frac{a}{\Omega_1}\right) \quad (*_A)\end{aligned}$$

Also

$$\begin{aligned}\left(\frac{-\Delta}{A''}\right) &= (-1)^{\frac{1}{2}(A''-1)}\beta^{\frac{1}{8}(A''^2-1)}\left(\frac{\Delta_1}{A''}\right) \\ \left(\frac{\Delta_1}{A''}\right) &= (-1)^{\frac{1}{2}(A''-1)\frac{1}{2}(\Delta_1-1)}\left(\frac{A''}{\Delta_1}\right) \\ \left(\frac{-\Delta}{A''}\right) &= (-1)^{\frac{1}{2}(A''-1)\frac{1}{2}(\Delta_1+1)}\beta^{\frac{1}{8}(A''^2-1)}\left(\frac{A''}{\Delta_1}\right) \quad (*_B)\end{aligned}$$

Inserting $(*_A)$ and $(*_B)$ into $*$ we get

$$\begin{aligned}(-1)^{\frac{1}{2}(a-1)\frac{1}{2}(A''-1)}(-1)^{\frac{1}{2}(a-1)\frac{1}{2}(\Omega_1+1)}(-1)^{\frac{1}{2}(A''-1)\frac{1}{2}(\Delta_1+1)}\alpha^{\frac{1}{8}(a^2-1)}\beta^{\frac{1}{8}(A''^2-1)}\left(\frac{A''}{\Delta_1}\right)\left(\frac{a}{\Omega_1}\right) &= 1 \\ (-1)^{\frac{1}{2}(\Omega_1+1)\frac{1}{2}(\Delta_1+1)}(-1)^{\frac{1}{2}(\Omega_1+A'')\frac{1}{2}(\Delta_1+a)}\alpha^{\frac{1}{8}(a^2-1)}\beta^{\frac{1}{8}(A''^2-1)}\left(\frac{A''}{\Delta_1}\right)\left(\frac{a}{\Omega_1}\right) &= 1\end{aligned}$$

For odd integers $(-1)^{\frac{1}{2}(\Omega_1+A'')\frac{1}{2}(\Delta_1+a)} = (-1)^{\frac{1}{2}(\Omega_1 A''+1)\frac{1}{2}(\Delta_1 a+1)} = \Psi$.
Writing f and F for a and A'' respectively, the condition of possibility as

$$\Psi \times \alpha^{\frac{1}{8}(f^2-1)} \times \beta^{\frac{1}{8}(F^2-1)} \times \left(\frac{f}{\Omega_1}\right) \times \left(\frac{F}{\Delta_1}\right) = (-1)^{\frac{1}{2}(\Omega_1+1)\frac{1}{2}(\Delta_1+1)}$$

Every generic character which satisfies this condition is a character of an actual existing genus.

I

GRAND PRIX DES SCIENCES MATHÉMATIQUES – LETTERS

Henry Smith wrote to Charles Hermite drawing attention to an oversight by the French *Académie des Sciences*. The following is Charles Hermite's reply to Henry Smith (Smith, 1894a, p. lxvi).

Paris, 26 Février, 1882.

Mon Cher Monsieur

Aucun des membres de la commission qui a proposé pour sujet du prix des sciences mathématiques en 1882 la démonstration des théorèmes d'Eisenstein sur la décomposition des nombres en cinq carrés n'avait connaissance de vos travaux contenant depuis bien des années cette démonstration et dont j'ai pour la première fois connaissance par votre lettre. L'embarras n'est point pour vous, mais pour le rapporteur des mémoires envoyés au concours, et si j'étais ce rapporteur je n'hésiterais pas un moment à faire d'abord l'aveu complet de l'ignorance où il s'est trouvé de vos publications, et ensuite à proclamer hautement que vous aviez donné la solution de la question proposée. Une circonstance pourrait ôter tout embarras et rendre sa tâche facile autant qu'agréable. S'il avait en effet à rendre compte d'un mémoire adressé par vous-même dans lequel vous rappelleriez vos anciennes recherches en les complétant, vous voyez que justice vous serait rendue en même temps que les intentions de l'Académie seraient remplies puisqu'on lui annoncerait la solution complète de la question proposée. Jusqu'ici je n'ai pas eu connaissance qu'aucune pièce ait été envoyée, ce qui s'explique par la direction du courant mathématique qui ne se porte plus maintenant vers l'arithmétique. Vous êtes seul en Angleterre à marcher dans la voie ouverte par Eisenstein. M. Kronecker est seul en Allemagne; et chez nous M. Poincaré, qui a jeté en avant quelques idées heureuses sur ce qu'il appelle les invariants arithmétiques, semble maintenant ne plus songer qu'aux fonctions Fuchsiennes et aux équations différentielles. Vous jugerez s'il vous convient de répondre à l'appel de l'Académie à ceux qui aiment l'Arithmétique; en tout cas soyez assuré que la commission aura par moi connaissance de vos travanx si elle a se prononcer et à faire un rapport à l'Académie sur des mémoires soumis à son examen . . . Je vous renouvelle, mon cher Monsieur, l'expression de ma plus haute estime et de mes sentiments bien sincèrement dévoués.

Charles Hermite.

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Charles Hermite writes to Rudolf Lipschitz (1832–1903) in praise of Hermann Minkowski's 1882 memoir to the *French Académie des Sciences* (Section of letter) (Goldstein, 2019, p. 17).

Paris, 12 Mai, 1883.

Monsieur,

Le mémoire couronné de Mr Minkowski étant écrit en allemand, a été lu et étudié par Mr Camille Jordan qui m'en a rendu compte. Ce n'est point à mon jugement une oeuvre aussi considérable que les mémoires de Rosenhaim et de Mr Kummer, mais je ne doute point que le jeune géomètre n'ait devant lui un grand avenir, et qu'il ne justifie pleinement votre confiance, si vous réalisez votre intention de vous l'attacher comme professeur extraordinaire. Son travail nous a paru plus complet et meilleur à certains égards que celui de Mr Smith; il révèle une science algébrique profonde, et un talent d'invention qui promet de belles et importantes découvertes dans l'avenir. Je pense donc que vous servez la cause de la science en lui facilitant son entrée dans la carrière universitaire, et qu'il est digne de votre appui, dès à présent, et que plus tard il le sera encore davantage. J'ai bien de la peine à vous suivre, dans la recherche extrêmement difficile ou vous êtes engagé; jamais je n'ai eu à lutter contre des obstacles de cette nature, et je ne pourrais certainement point faire les efforts que vous ont demandé les tentatives que vous me communiquez. Ce sont des questions beaucoup plus simples qui m'occupent en ce moment, et au point qu'il m'en coute beaucoup de m'en détacher pour préparer mes leçons. L'arithmétique est une sirène; en l'écoutant je m'abandonne, je me laisse aller à la dérive et je cours sur les écueils..........

Charles Hermite.

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