BLOCKS OF DEFECT ZERO AND PRODUCTS OF ELEMENTS OF ORDER p

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ABSTRACT. Suppose that G is a finite group and that F is a field of characteristic $p > 0$ which is a splitting field for all subgroups of G. Let e_0 be the sum of the block idempotents of defect zero in FG, and let Ω be the set of solutions to $g^p = 1$ in G. We show that $e_0 = (\Omega^+)^2$, when p is odd, and $e_0 = (\Omega^+)^3$, when $p = 2$. In the latter case $(\Omega^+)^2 = R^+$, where R is the set of real elements of 2-defect zero. So $e_0 = \Omega^+ R^+ = (R^+)^2$. We also show that $e_0 = \Omega^+\Omega_4^+ = (\Omega_4^+)^2$, when $p = 2$, where Ω_4 is the set of solutions to $g^4 = 1$. These results give us various criteria for the existence of p-blocks of defect zero.

1. INTRODUCTION

Let G be a finite group, and let F be a field of characteristic $p > 0$, which is a splitting field for all subgroups of G. Identify $g \in G$ with its image in the group algebra FG. If $X \subseteq G$, then $X^+ := \sum_{x \in X} x$ is the sum of the elements of X in FG, and $\Omega(X) := \{x \in X \mid x^p = 1_G\}$. For convenience, we will use Ω in place of $\Omega(G)$. Let e_0 be the sum of the block idempotents in FG of defect zero, and let G_p ($G_{p'}$) denote the set of p-elements (p-regular elements) of G. In [T71], Y. Tsushima proves that

$$e_0 = (G_p^+)^2. \tag{1.1}$$

The motivation for this paper comes from this theorem and a result of R. Knörr [Kn89, 2.9]. It is a theorem of R. Brauer that an irreducible character χ of G lies in a p-block of defect zero if and only if χ vanishes on the non-trivial elements of G_p. Knörr's result shows that one need only consider whether χ vanishes on the non-trivial elements of Ω. We reprove a version of this result in Corollary 2.3 below. Lemma 2.1 is crucial to our proof, and indeed to the rest of the paper. The idea behind this lemma came from a proof of Knörr's result due to G. R. Robinson [R89]. An immediate consequence is Proposition 2.4, which shows that $e_0 = (\Omega^+)^3$.

At this point the theory diverges, depending on whether p is odd or even. Using a fomula of B. Külshammer, we prove in Theorem 3.7 that $e_0 = (\Omega^+)^2$, when $p \neq 2$. When $p = 2$, we use an old idea of Brauer and Fowler to show that $(\Omega^+)^2 = R^+$, where R is the set of real elements of G of 2-defect zero. A proof is given in Proposition 4.1. This result, together with Example 7.4, shows that Theorem 3.7 is false when $p = 2$.

Section 5 gives more general consequences of Lemma 2.1, using a chain of ideals defined by Külshammer in [K91]. One such consequence is Corollary 5.9, where it

Date: July 29, 1998.
1991 *Mathematics Subject Classification.* 20C20.
The author was supported by a Forbairt Postdoctoral Fellowship while writing this paper.

is shown that $e_0 = \Omega^+\Omega_4^+ = (\Omega_4^+)^2$, when $p = 2$, where Ω_4 is the set of solutions to $x^4 = 1_G$ in G.

In Proposition 6.3, we show that the number of p-blocks of defect zero is the rank of a certain square matrix, refining a result of Robinson [R83]. Section 7 summarizes our results and presents some simple examples.

We now give some notation which will be used throughout the paper.

Let Cl be the collection of conjugacy classes of G, and let Cl_0 be the subcollection of classes of p-defect zero. Choose a fixed element $g_C \in C$, for each $C \in \mathrm{Cl}$. The centre of the ring FG will be denoted by Z. Let

$$Z_0 = \sum_{C \in \mathrm{Cl}_0} FC^+$$

be the ideal of Z spanned by the class sums of p-defect zero. Since Z is a commutative ring, the nilpotent elements form an ideal which coincides with the Jacobson Radical $\mathrm{J}(Z)$.

Let Bl denote the set of block (or central primitive) idempotents in FG, and let Bl_0 denote the subset of block idempotents of defect zero. If $e \in \mathrm{Bl}$, then Ze is a local ring with Jacobson Radical $\mathrm{J}(Ze)$ of codimension 1. So the map $\lambda_e \colon z \to ze \mod \mathrm{J}(Ze)$, for $z \in Z$, is a linear character of Z. Set

$$E := \sum_{e \in \mathrm{Bl}} Fe, \qquad E_0 := \sum_{e \in \mathrm{Bl}_0} Fe.$$

Then E is the maximal semi-simple subalgebra of Z, and $E_0 = E \cap Z_0$ is the ideal of E spanned by the idempotents of defect zero. Also $Z = E \bigoplus \mathrm{J}(Z)$ as F-algebras. We note that

$$Z_0\,\mathrm{J}(Z) = 0_{FG}, \tag{1.2}$$

since it follows from [F82, VI.4.6] that $Z_0 \subseteq \mathrm{Soc}(FG) := \{x \in FG \mid x\,\mathrm{J}(FG) = 0_{FG}\}$, and it is certainly true that $\mathrm{J}(Z) \subseteq \mathrm{J}(FG)$. Furthermore,

$$E_0 = Z_0^2, \tag{1.3}$$

by [IW73, Lemma 2].

If $X \subseteq G$, set

$$\mathrm{C}(X) := \{g \in G \mid \; x^g = x \text{ for all } x \in X\}.$$

We will use $\mathrm{C}(g)$ in place of $\mathrm{C}(\{g\})$, whenever g is an element of G.

If n is an integer then $\overline{n}$ will denote its residue modulo p.

2. Properties of Ω^+

We first prove a basic result about Ω.

Lemma 2.1. $(\Omega^+)^2 \in Z_0$.

Proof. We may write

$$(\Omega^+)^2 = \sum_{g \in G} \overline{|\Phi(g)|}\; g, \tag{2.2}$$

where $\Phi(g) = \{(a, b) \in \Omega \times \Omega \mid ab = g\}$, for each $g \in G$. Clearly $|\Phi(g)|$ is a class function of G.

Fix an element g of G, and a Sylow p-subgroup P of $\mathrm{C}(g)$. The group P acts by conjugation on the set $\Phi(g)$. So

$$\overline{|\Phi(g)|} = \overline{|\Phi_P(g)|},$$

where $\Phi_P(g) = \{(a,b) \in \Omega(\mathrm{C}(P)) \times \Omega(\mathrm{C}(P)) \mid ab = g\}$.

Now $\Omega(Z(P))$ acts freely on $\Phi_P(g)$ via $(a,b)z := (az, z^{-1}b)$, for $a,b \in \Phi_P(g)$ and $z \in \Omega(Z(P))$. So $|\Omega(Z(P))|$ divides $|\Phi_P(g)|$. Thus $\overline{|\Phi_P(g)|} = 0_F$, unless $P =< 1_G >$. We conclude that

$$(\Omega^+)^2 = \sum_{\substack{g \in G \\ p \not| |\mathrm{C}(g)|}} \overline{|\Phi(g)|}\; g = \sum_{C \in \mathrm{Cl}_0} \overline{|\Phi(g_C)|}\; C^+.$$

This proves the lemma. □

The next result was first noted by Knörr in [Kn89, 2.9].

Corollary 2.3. *Let $e \in \mathrm{Bl}$. Then*

$$\lambda_e(\Omega^+) = \begin{cases} 1_F, & \textit{if } e \textit{ has defect zero;} \\ 0_F, & \textit{if } e \textit{ has positive defect.} \end{cases}$$

Proof. Suppose that e has defect zero. Then λ_e vanishes on all p-singular class sums, by [NT89, 4.7.4]. In particular, $\lambda_e(\Omega^+) = \lambda_e(1_G) = 1_F$.

Suppose that e has positive defect. Then λ_e vanishes on Z_0, by [NT89, 3.6.27]. Hence

$$(\lambda_e(\Omega^+))^2 = \lambda_e((\Omega^+)^2) = 0_F.$$

We conclude that $\lambda_e(\Omega^+) = 0_F$. □

Note that e_0 is supported on the classes of defect zero i.e. $e_0 \in Z_0$. The following result will be improved in Theorem 3.7, for $p \neq 2$.

Proposition 2.4. $e_0 = (\Omega^+)^n$, *for $n \geq 3$.*

Proof. Suppose that $e \in \mathrm{Bl}$. If $e \notin \mathrm{Bl}_0$, then $\Omega^+ e \in \ker(\lambda_e)e \subseteq \mathrm{J}(Z)$ by Corollary 2.3, and $(\Omega^+)^2 \in Z_0$ by Lemma 2.1. So $0_{FG} = (\Omega^+)^3 e = (\Omega^+)^4 e = \ldots$, using (1.2). If $e \in \mathrm{Bl}_0$, then $\Omega^+ e = e$ by Corollary 2.3. So $e = (\Omega^+)^2 e = (\Omega^+)^3 e = \ldots$. Hence $(\Omega^+)^n = \sum_{e \in \mathrm{Bl}} (\Omega^+)^n e = \sum_{e \in \mathrm{Bl}_0} e = e_0$, for $n \geq 3$. □

3. The p-Power Map

Suppose that $\mathcal{O}$ is a commutative associative ring with identity $1_\mathcal{O}$. We can define a form $(\ ,\ )$ on $\mathcal{O}G$ by setting

$$(x,y) := \begin{cases} 1_\mathcal{O}, & \text{if } xy = 1_G, \\ 0_\mathcal{O}, & \text{if } xy \neq 1_G, \end{cases} \quad \text{for all } x,y \in G,$$

and extending $\mathcal{O}$-bilinearly to $\mathcal{O}G$. It is readily established that $(\ ,\ )$ is an associative non-degenerate symmetric bilinear form on $\mathcal{O}G$. i.e. for all $a,b,c \in \mathcal{O}G$ we have

$$\begin{aligned} &(ab,c) &&= (a,bc), \\ &(a,x) &&= 0_\mathcal{O}, \ \text{for all } x \in \mathcal{O}G \quad \Longrightarrow \quad a = 0_{\mathcal{O}G}, \\ &(a,b) &&= (b,a), \\ &(\lambda a, b) &&= \lambda(a,b), \ \text{for all } \lambda \in \mathcal{O}, \end{aligned} \tag{3.1}$$

We note that

$$a = \sum_{g \in G} (a, g^{-1})g, \quad \text{for all } a \in \mathcal{O}G,$$

and in particular

$$A^+B^+ = \sum_{C \in \mathrm{Cl}} (A^+B^+, g_C^{-1})\ C^+, \quad \text{for all } A, B \in \mathrm{Cl}.$$

The $\mathcal{O}$-elements (A^+B^+, g_C^{-1}) are called the *class multiplication constants* of $Z(\mathcal{O}G)$. It is clear that the class multiplication constants for FG are obtained by reducing the corresponding constants for $\mathbb{Z}G$ modulo p.

Let

$$K = KFG := F\{ab - ba \mid a, b \in FG\}$$

be the *commutator subspace* of FG. It is straightforward to show that

$$K = Z^\perp := \{k \in FG \mid (z, k) = 0_F, \text{ for all } z \in Z\} \tag{3.2}$$

is the *dual space* of Z with respect to the bilinear form (,). In fact, this holds for any finite dimensional algebra which possesses a non-degenerate associative bilinear form. The following well-known result is due to Brauer.

Lemma 3.3. *If $a, b \in FG$, then*

$$\begin{array}{ccc} (a+b)^p & \equiv & a^p + b^p \mod K; \\ a \in K & \Longrightarrow & a^p \in K. \end{array}$$

Proof. See [F82, I.16.3(ii)]. □

For $g \in G_{p'}$ and $z \in Z$, we define $g^{p^{-1}}$ to be the unique p-regular element whose p^{th} power is g. In [K91, (49),(55)], Külshammer gives an expression for z^p when $z \in Z$. We need, and prove, only the following special case of this result.

Lemma 3.4. *Suppose that $z \in Z$ and $g \in G_{p'}$. Then $(z^p, g) = (\Omega^+ z, g^{p^{-1}})^p$.*

Proof. We have

$$\begin{aligned} (z^p, g) &= (1_G, z^p g), && \text{by (3.1)} \\ &= \big(1_G, (zg^{p^{-1}})^p\big), && \text{as } z \text{ and } g^{p^{-1}} \text{ commute} \\ &= \big(1_G, \textstyle\sum_{x \in G} (x^{-1}, zg^{p^{-1}})^p x^p\big), && \text{by Lemma 3.3 and (3.2)} \\ &= \big(\textstyle\sum_{x \in \Omega} (x^{-1}, zg^{p^{-1}})\big)^p, && \text{as } F \text{ has characteristic } p \\ &= (\Omega^+, zg^{p^{-1}})^p, && \text{by (3.1)} \\ &= (\Omega^+ z, g^{p^{-1}})^p, && \text{by (3.1).} \end{aligned}$$

□

Suppose that $C \in \mathrm{Cl}$ and that $e \in \mathrm{Bl}$. It is a result of K. Iizuka [I61], that $eC^+ \in FS(C)$, where $S(C)$ is the p-section of G containing C. It follows that

$$(e, g) = \left\{ \begin{array}{ll} 0_F, & \text{for all } g \in G \backslash G_{p'}, \\ (e\Omega^+, g), & \text{for all } g \in G_{p'}. \end{array} \right. \tag{3.5}$$

The next result seems to have been originally proved by M. Osima [O55].

Corollary 3.6. *Suppose that e is an idempotent in Z. Then $(e, g) = (e, g^p)^{p^{-1}}$, for all $g \in G_{p'}$.*

Proof. Let g be a p-regular element of G. Then

$$\begin{aligned}(e,g) &= (e\Omega^+,g), && \text{by (3.5)}\\ &= (e^p,g^p)^{p^{-1}}, && \text{by Lemma 3.4}\\ &= (e,g^p)^{p^{-1}}, && \text{as } e \text{ is an idempotent.}\end{aligned}$$

□

We can apply the previous two results most effectively when p is an odd prime.

Theorem 3.7. *$e_0 = (\Omega^+)^2$, if $p \neq 2$.*

Proof. Suppose that $g \in G_{p'}$. Then

$$\begin{aligned}(e_0,g) &= (e_0,g^p)^{p^{-1}}, && \text{by Corollary 3.6}\\ &= ((\Omega^+)^p,g^p)^{p^{-1}}, && \text{using the fact that } p>2 \text{ and Proposition 2.4}\\ &= ((\Omega^+)^2,g), && \text{by Lemma 3.4.}\end{aligned}$$

Suppose that $g \in G\backslash G_{p'}$. Then

$$(e_0,g) = 0_F = ((\Omega^+)^2,g), \quad \text{by (3.5) and Lemma 2.1.}$$

□

4. Real conjugacy classes and 2-blocks of defect zero

In this section we let $p = 2$. Recall that R is the set of real elements of 2-defect zero in G.

Suppose that $H \leq G$. Set $\mathrm{I}(H) := \Omega(H)\backslash\{1_G\}$. So $\mathrm{I}(H)$ is the set of involutions in H. We note that $|\mathrm{I}(H)|$ is odd, by Sylow's theorem, provided $2 \mid |H|$. Let $\mathrm{C}(g)^* := \{x \in G \mid g^x \in \{g, g^{-1}\}\}$ be the *extended centralizer* of g in G.

Proposition 4.1. $(\Omega^+)^2 = R^+$. *Hence* $e_0 = \Omega^+R^+ = (R^+)^2$.

Proof. If $|G|$ is odd, the result is trivial, since $\Omega = \{1_G\} = R$. So we will assume that $2 \mid |G|$.

If $a, b \in \Omega$, then ab is real, since $a^{-1}(ab)a = ba = (ab)^{-1}$. Thus, by Lemma 2.1, $(\Omega^+)^2$ is a linear combination of real elements of 2-defect zero.

Suppose that g is a real element of 2-defect zero. Using the notation of Lemma 2.1, the map $(a,b) \to a$, for $(a,b) \in \Phi(g)$, yields a bijection between $\Phi(g)$ and $\mathrm{I}(\mathrm{C}(g)^*)$. But $|\mathrm{I}(\mathrm{C}(g)^*)|$ is odd, since $2 \mid |\mathrm{C}(g)^*|$. This completes the proof. □

Note 4.2. Using this proposition, it is straightforward to show that every 2-block of defect zero has a real defect class. This generalizes [G88, 1.2].

Note 4.3. It is easy to generate examples where the number of 2-blocks of defect zero exceeds the number of real classes of 2-defect zero. See Example 7.5. However, the number of real 2-blocks of defect zero does not exceed the number of real classes of 2-defect zero, by [G88, 3.1].

We can obtain the equality $e_0 = (R^+)^2$ directly using an idea in [KM97]. If $C \in \mathrm{Cl}$, set $C^\circ := \{g^{-1} \mid g \in C\}$. If $\chi \in \mathrm{Irr}(G)$, let $e_\chi := |G|^{-1}\chi(1_G)\sum_{g\in G}\chi(g^{-1})g$ be the corresponding primitive idempotent of $Z(\mathbb{C}G)$. Kellersch and Meyberg study the *Casimir Element* $\mathcal{C}$, given by the equation

$$\sum_{\chi\in\mathrm{Irr}(G)} \left(\frac{|G|}{\chi(1_G)}\right)^2 e_\chi \;=\; \mathcal{C} \;=\; \sum_{C\in\mathrm{Cl}} \frac{|G|}{|C|}\, C^+C^{\circ+}, \quad \text{in } Z(\mathbb{C}G). \tag{4.4}$$

Let ζ be a $|G|^{th}$-root of unity in $\mathbb{C}$, and let $\mathcal{O}$ be the localization of the algebraic integers of $\mathbb{Q}(\zeta)$ at some prime ideal containing 2. Then $\mathcal{O}$ is a local principal ideal domain with a unique maximal ideal $J(\mathcal{O})$. The residue field $\mathcal{O}/J(\mathcal{O})$ is a splitting field for all subgroups of G, of characteristic 2. If $\chi \in \mathrm{Irr}(G)$, then $|G|\chi(1_G)^{-1}e_\chi \in \mathcal{O}$ and $|G|\chi(1_G)^{-1} \in J(\mathcal{O})$ unless χ has 2-defect zero. If $C \in \mathrm{Cl}$, then $|G|/|C| \in J(\mathcal{O})$ unless $C \in \mathrm{Cl}_0$. Reducing (4.4) modulo $J(\mathcal{O})$, and abusing notation slightly, we obtain

$$\sum_{C \in \mathrm{Cl}_0} C^+ C^{\circ +} \;=\; \overline{\mathcal{C}} \;=\; e_0.$$

If $C \in \mathrm{Cl}_0$, then $C^+C^{\circ+} + C^{\circ+}C^+ = 0$. So the contribution of C to $\overline{\mathcal{C}}$ is zero unless C is a real class. If C is real, then its contribution is $C^+C^{\circ+} = (C^+)^2$. We conclude that

$$e_0 = \overline{\mathcal{C}} = \sum_{C \in \mathrm{Cl} \cap R} (C^+)^2 = (R^+)^2.$$

5. Külshammer's Ideals

In this section we prove results about certain ideals of Z studied by Külshammer in [K91].

Let n be an integer ≥ 1. For $X \subseteq G$, set $\Omega_{p^n}(G, X) := \{g \in G \mid g^{p^n} \in X\}$. For convenience we will use Ω_{p^n} in place of $\Omega_{p^n}(G, \{1_G\})$. Let p^N be the exponent of a Sylow p-subgroup of G. Note that $\Omega_{p^n} = G_p$, whenever $n \geq N$.

Set $T_nFG = T_n := \{x \in FG \mid x^{p^n} \in K\}$. Then T_n is a Z-submodule of FG which contains $K = Z^\perp$. Hence its dual, $T_n^\perp$, is an ideal of Z. By [K91, (36),(38)] we have

$$T_n^\perp = F\{\Omega_{p^n}(G, C)^+ \mid C \in \mathrm{Cl}\}.$$

Moreover,

$$Z \supseteq T_1^\perp \supseteq T_2^\perp \supseteq \cdots \supseteq T_N^\perp = \mathrm{Soc}(FG) \cap Z,$$

by [K91, (36),(37)].

The proofs of Lemma 2.1, Corollary 2.3 and Proposition 2.4 can be adapted, without difficulty, to show the following three results.

Lemma 5.1. $(T_n^\perp)^2 \subseteq Z_0$.

Corollary 5.2. *Suppose that $C \in \mathrm{Cl}$ and $e \in \mathrm{Bl}$. Then*

$$\lambda_e(\Omega_{p^n}(G, C)^+) = \begin{cases} 0_F, & \text{if } e \text{ has positive defect;} \\ \lambda_e((\Omega_{p^n}(G, C) \cap G_{p'})^+), & \text{if } e \text{ has defect zero.} \end{cases}$$

In particular, $\Omega_{p^n}(G, C)^+ \in \mathrm{J}(Z)$, if C is a p-singular conjugacy class. Hence

$$\Omega_{p^n}^+ \equiv \Omega^+ \mod \mathrm{J}(T_1^\perp). \tag{5.3}$$

Proposition 5.4. $\mathrm{J}(T_n^\perp)^3 = 0$. *In particular, $e_0 = (\Omega_{p^n}^+)^m$, if $m \geq 3$.*

For $g \in G_{p'}$, let $g^{p^{-n}}$ denote the unique p-regular element of G whose p^{nth} power is g. We can adapt the proof of Lemma 3.4 to show:

Lemma 5.5. *Suppose that $z \in Z$ and $g \in G_{p'}$. Then $(z^{p^n}, g) = (\Omega_{p^n}^+ z, g^{p^{-n}})^{p^n}$.*

The next result gives a description of $\mathrm{J}(T_n^\perp)$.

Proposition 5.6. *Suppose that* $m \geq 1$. *Then*

$$\{z \in T_n^{\perp} \mid \Omega_{p^m}^{+} z = 0\} = \{z \in T_n^{\perp} \mid z^{p^m} = 0\},$$

Hence

$$\mathrm{J}(T_n^{\perp}) = \{z \in T_n^{\perp} \mid \Omega_{p^m}^{+} z = 0\} \quad \begin{cases} \text{when } p = 2 \text{ and } m \geq 2; \\ \text{when } p \neq 2 \text{ and } m \geq 1. \end{cases}$$

Proof. Let $z \in T_n^{\perp}$. Lemma 5.1 implies that both $\Omega_{p^m}^{+} z$ and z^{p^m} lie in $FG_{p'}$. Also

$$(\Omega_{p^m}^{+} z, g) = (z^{p^m}, g^{p^m})^{p^{-m}}, \quad \text{for all } g \in G_{p'},$$

by Lemma 5.5. But $g \to g^{p^m}$ is a bijective map on $G_{p'}$. Thus $\Omega_{p^m}^{+} z = 0$ if and only if $z^{p^m} = 0$.

The last statement follows from Proposition 5.4. □

This allows us to prove:

Theorem 5.7.

$$e_0 = \Omega_{p^m}^{+} \Omega_{p^n}^{+}, \quad \begin{cases} \text{when } p = 2 \text{ and } m \geq 2; \\ \text{when } p \neq 2 \text{ and } m \geq 1. \end{cases}$$

Proof. It follows from Proposition 2.4 and (5.3) that $\Omega_{p^n}^{+} \equiv e_0 \bmod \mathrm{J}(T_1^{\perp})$. Suppose that $p = 2$ and $m \geq 2$, or that $p \neq 2$ and $m \geq 1$. Then $\Omega_{p^m}^{+}\Omega_{p^n}^{+} = \Omega_{p^m}^{+} e_0$, using Proposition 5.6. But Proposition 5.4 implies that $\Omega_{p^m}^{+} e_0 = e_0$. This completes the proof. □

The following corollary is Tsushima's result (1.1).

Corollary 5.8. $e_0 = (G_p^{+})^2$.

Proof. This follows from Theorem 5.7, once we note that $G_p = \Omega_{p^{N+2}}$. □

Corollary 5.9. *Suppose that* $p = 2$. *Then* $e_0 = \Omega^{+}\Omega_4^{+} = (\Omega_4^{+})^2$.

Proof. This is just Theorem 5.7 with $p = 2$, $m = 2$ and $n = 1$ or 2. □

Let $\sqrt[p^n]{1_G}$ denote the set of elements of G of order p^n.

Corollary 5.10. *Suppose that* $n \geq 2$. *Then*

$$\left. \begin{array}{r} \Omega^{+} \sqrt[p^m]{1_G}^{+} \\ \sqrt[p^n]{1_G}^{+} \sqrt[p^m]{1_G}^{+} \end{array} \right\} = 0_{FG}, \quad \begin{cases} \text{if } p = 2 \text{ and } m \geq 3; \\ \text{if } p \neq 2 \text{ and } m \geq 2. \end{cases}$$

Proof. Suppose that $p = 2$ and $m \geq 3$, or $p \neq 2$ and $m \geq 2$. Then

$$\begin{aligned} \Omega^{+} \sqrt[p^m]{1_G}^{+} &= \Omega^{+}(\Omega_{p^m}^{+} - \Omega_{p^{m-1}}^{+}) \\ &= e_0 - e_0, \quad \text{by Theorem 5.7} \\ &= 0_{FG}. \end{aligned}$$

Also,

$$\begin{aligned} \sqrt[p^n]{1_G}^{+} \sqrt[p^m]{1_G}^{+} &= (\Omega_{p^n}^{+} - \Omega_{p^{n-1}}^{+})(\Omega_{p^m}^{+} - \Omega_{p^{m-1}}^{+}) \\ &= e_0 - e_0 - e_0 + e_0, \quad \text{by Theorem 5.7} \\ &= 0_{FG}. \end{aligned}$$

□

Finally, we note the following:

Proposition 5.11. *Suppose that $p = 2$. Then $e_0 = R^+$ if and only if $(\sqrt[4]{1_G}^+)^2 = 0_{FG}$.*

Proof. We have $(\sqrt[4]{1_G}^+)^2 = (\Omega_4^+ - \Omega^+)^2 = (\Omega_4^+)^2 - (\Omega^+)^2 = e_0 - R^+$, using Corollary 5.9 and Proposition 4.1. The result follows. □

6. The number of p-blocks of defect zero

In this section we prove a result which is really a corollary to Proposition 2.4.

Let $C_1, \ldots, C_r$ be a full list of the conjugacy classes of G of p-defect zero. So $C_1^+, \ldots, C_r^+$ is an F-basis for Z_0. Let $\mathbb{Z}_{(p)}$ denote the localization of the ring of integers $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$. We will use $\overline{x}$ to denote the image of $x \in \mathbb{Z}_{(p)}$ modulo the ideal $p\mathbb{Z}_{(p)}$.

Suppose that S is a fixed Sylow p-subgroup of G. For $1 \leq i, j \leq r$, define

$$\begin{aligned} \Omega_{i,j} &:= \{(u, v) \in C_i \times C_j \mid uv^{-1} \in \Omega\} \\ \Omega_{i,j}^S &:= \{(u, v) \in C_i \times C_j \mid uv^{-1} \in \Omega(S)\}. \end{aligned}$$

Both $|\Omega_{i,j}|/|C_j|$ and $|\Omega_{i,j}^S|/|C_j|$ lie in $\mathbb{Z}_{(p)}$, because S acts fixed point free on $\Omega_{i,j}$ and $\Omega_{i,j}^S$. Let A be the $r \times r$ matrix whose i, j^{th}-entry is

$$a_{i,j}^S = a_{i,j} := \overline{|\Omega_{i,j}^S|/|C_j|}.$$

Lemma 6.1. *Suppose that $1 \leq i, j \leq r$.*

$$|\Omega_{i,j}^S|/|C_j| \equiv |\Omega_{i,j}|/|C_j|, \quad \operatorname{mod} p\mathbb{Z}_{(p)}. \tag{6.2}$$

Hence $a_{i,j}$ is the coefficient of g_{C_j} in $C_i^+\Omega^+$.

Proof. Let Syl denote the set of Sylow p-subgroups of G and let Syl_s denote the set of Sylow p-subgroups which contain a fixed $s \in G_p$. Then $|\mathrm{Syl}_s| \equiv 1 \mod p$, by a well-known generalization of Sylow's Theorem. Hence

$$|\Omega_{i,j}|/|C_j| = \sum_{S \in \mathrm{Syl}} |\Omega_{i,j}^S|/|C_j|, \quad \operatorname{mod} p\mathbb{Z}_{(p)}.$$

But $|\Omega_{i,j}^S|/|C_j| = |\Omega_{i,j}^T|/|C_j|$, for all $S, T \in \mathrm{Syl}$. Sylow's Theorem now gives (6.2).

Also

$$\begin{aligned} \overline{|\Omega_{i,j}|/|C_j|} &= \frac{(C_i^+ C_j^{\circ +}, \Omega^+)}{|C_j|} \\ &= \frac{(\Omega^+ C_i^+, C_j^{\circ +})}{|C_j|}. \end{aligned}$$

The last statement of the lemma follows from (6.2). □

The main result of this section is the following refinement of a result in [R83].

Proposition 6.3. *The number of p-blocks of G of defect zero is the p-rank of A.*

Proof. Suppose that $z \in Z_0$ and that $e \in \mathrm{Bl}$. Then $\lambda_e(z) = 0_F$, unless e has defect zero. So $Z_0 = E_0 \bigoplus \mathrm{J}(Z_0)$ as F-algebras. Also e_0 acts as the identity on E_0, and $e_0 \,\mathrm{J}(Z_0) = 0_{FG}$ by (1.2). Thus

$$e_0 Z_0 = e_0 E_0 = E_0. \tag{6.4}$$

Now $\Omega^+ = e_0 + j$, for some $j \in \mathrm{J}(Z)$, by Proposition 2.4. So $\Omega^+ C_i^+ = e_0 C_i^+ + jC_i^+ = e_0 C_i^+$, for $i = 1, \ldots, r$, using (1.2). Hence

$$e_0 Z_0 = \Omega^+ Z_0. \tag{6.5}$$

The proposition follows from (6.4), (6.5) and Lemma 6.1. ☐

7. Existence of p-Blocks of Defect Zero and Examples

For $g \in G$ set

$$\Phi(g) := \begin{cases} |\{(u,v,w) \in G \times G \times G \mid u^2 = v^2 = w^2 = 1_G, uvw = g\}|, & \text{if } p = 2; \\ |\{(u,v) \in G \times G \mid u^p = v^p = 1_G, uv = g\}|, & \text{if } p \neq 2. \end{cases}$$

We collect the results of Sections 2, 3, 4 and 5 in the following theorem. This refines Corollary 1 of [T71].

Theorem 7.1. *G has a p-block of defect zero if and only if G has an element g for which $\Phi(g) \not\equiv 0 \mod p$. Any such g is of p-defect zero. Moreover,*

(1) for $n \geq 1$, $m \geq 2$ and $p = 2$ we have

$$\begin{aligned} \Phi(g) &= |\{(u,v) \in G \times G \mid \ u^2 = 1_G,\ v \text{ real of 2-defect zero},\ uv = g\}| \\ &= |\{(u,v) \in G \times G \mid \ u, v \text{ real of 2-defect zero},\ uv = g\}| \\ &= |\{(u,v) \in G \times G \mid \ u^{2^n} = v^{2^m} = 1_G,\ uv = g\}|; \end{aligned}$$

(2) for $n \geq 1$, $m \geq 1$ and $p \neq 2$ we have

$$\Phi(g) = |\{(u,v) \in G \times G \mid \ u^{p^n} = v^{p^m} = 1_G,\ uv = g\}|.$$

Proof. The group G has a p-block of defect zero if and only if $e_0 \neq 0_{FG}$. But

$$e_0 = \sum_{g \in G} \overline{\Phi(g)} g,$$

by Proposition 2.4 and Theorem 3.7. This proves the first statement.

The equalities in (1) follow from Proposition 4.1 and Theorem 5.7.

The equality in (2) follows from Theorem 5.7. ☐

Corollary 7.2. *Let L be a simple group of Lie type and odd characteristic p, and let $\hat{L}$ be the associated universal covering group. Then*

$$\Phi(g) \equiv \begin{cases} \pm |L|_{p'}^{-1} \mod p, & \text{if } g \in L \text{ has } p\text{-defect zero;} \\ \pm 1 \mod p, & \text{if } p \not| |\hat{L}|/|L| \text{ and } g \in \hat{L} \text{ has } p\text{-defect zero;} \\ 0 \mod p, & \text{if } g \in L \text{ or } g \in \hat{L}, \text{ and } g \text{ has positive } p\text{-defect.} \end{cases}$$

In particular, every element of p-defect zero is a product of two elements of order p.

Proof. By Theorem 8.6.1 and the discussion on pp. 197–199 of [Ca72], the p'-part of $|\hat{L}|$ is congruent to $\pm 1 \mod p$. By Theorem 8.2 of [S63], the Steinberg Character χ is the only character of $\hat{L}$ that lies in a p-block of defect zero, and Theorem 8.4 of that paper shows that $\chi(1) = |\hat{L}|_p$, while $\chi(g) = \pm 1$, for all g of p-defect zero.

Hence

$$\begin{aligned} e_0 = e_\chi &= \chi(1)|\hat{L}|^{-1} \sum_{C \in \mathrm{Cl}_0} \chi(g_C^{-1}) C^+ \\ &= |\hat{L}|_{p'}^{-1} \sum_{C \in \mathrm{Cl}_0} \pm C^+ = \sum_{C \in \mathrm{Cl}_0} \pm C^+. \end{aligned}$$

We can use a similar argument for L. The result follows from Theorem 3.7. □

The last conclusion of this corollary is false when $p = 2$. For example, the group $A_3(2) \cong A_8$ has non-real elements of 2-defect zero. None of these elements is expressible as a product of two involutions.

We use the notation of [Co85] for the following examples.

Example 7.3. *The Mathieu group M_{12} has no real classes of 2-defect zero, and two non-real classes, $11A, 11B$, of 2-defect zero. In particular, by Proposition 4.1 it cannot have any 2-blocks of defect zero.*

Many of the other sporadic simple groups which lack a 2-block of defect zero *do* have real classes of 2-defect zero, as the following example illustrates.

Example 7.4. *The Mathieu group M_{22} has one real class, $5A$, and four non-real classes, $7A, 7B, 11A, 11B$, of 2-defect zero. It has no 2-blocks of defect zero. In particular, Theorem 3.7 is false when $p = 2$.*

The last example shows that the number of 2-blocks of defect zero may exceed the number of real classes of 2-defect zero.

Example 7.5. *The group $U_4(3)$ has two real classes $3D, 5A$, and six non-real classes $7A, 7B, 9A, 9B, 9C, 9D$, of 2-defect zero. It has one real irreducible character, χ_{20}, of 2-defect zero, and two non-real irreducible characters, χ_{17} and χ_{18}, of 2-defect zero.*

8. Acknowledgement

I thank Rod Gow for bringing [R89] to my attention, for pointing out Corollary 7.2 and for his encouragement and hospitality while I was writing this paper. I also thank the Referee for useful comments about the proofs of Lemma 2.1 and Propositions 2.4 and 4.1.

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