SYLOW INTERSECTIONS, DOUBLE COSETS, AND 2-BLOCKS

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1. Notation and statement of results

Throughout G will be a finite group and F will be a finite field of characteristic $p > 0$, although we are mainly interested in the case $p = 2$. For convenience we assume that F is a splitting field for all subgroups of G. We let $\mathbb{Z}_{(p)}$ denote the localization of the integers $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$. If $x \in \mathbb{Z}_{(p)}$, then x^* will denote its image modulo the unique maximal ideal of $\mathbb{Z}_{(p)}$. We regard x^* as lying in the prime field $\mathrm{GF}(p)$ of F.

The elements of G may be identified with the members of a distinguished basis of the group algebra FG. Thus each $x \in FG$ is of the form $x = \sum_{g\in G} \beta(x,g)g$, where $\beta(x,g) \in F$, for $g \in G$. We define the element x^o of FG as $x^o := \sum_{g\in G} \beta(x,g^{-1})g$. The map $x \to x^o$ is an anti-isomorphism of FG, and its restriction to the centre Z of FG is an involutory isomorphism. We use $\mathcal{K}^+$ to denote the sum of the elements in a G-conjugacy class $\mathcal{K}$ in FG.

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The set of all such class sums forms an F-basis for Z. If $x \in Z$ and $g \in \mathcal{K}$, we will use $\beta(x, \mathcal{K}^+)$ in place of $\beta(x, g)$.

By a p-block B of G we mean a direct F-algebra summands of FG. Associated with B there is a primitive idempotent $e \in Z$, and an F-epimorphism $\omega : Z \to F$. We indicate these associations by $B \leftrightarrow e \leftrightarrow \omega$, and call e the *block idempotent*, and ω the *central character*, of B. Set $B^o := \{x \in FG \mid x^o \in B\}$ and $\omega^o(x) := \omega(x^o)$. Then $B^o \leftrightarrow e^o \leftrightarrow \omega^o$ is a p-block of G. We say that B is a *real block* of G if $B = B^o$.

R. Brauer showed how to associate with B a G-conjugation family of p-groups, which he called the *defect groups* of B. Let D be a defect group of B. Then D is not arbitrarily embedded in G. For instance Brauer proved that D is the largest normal p-subgroup of its normalizer $\mathbf{N}(D)$. J. A. Green [5] showed that there exists $g \in G$ and a Sylow p-subgroup S of G, such that $S \cap S^g = D$, and M. F. O'Reilly [9] showed that g could be chosen to be p-regular with defect group D. Here a defect group of g means a Sylow p-subgroup of the centralizer $\mathbf{C}(g)$ of g in G.

Let e_D denote the sum of the block idempotents associated with the p-blocks of G which have defect group D, and let $\mathcal{S}yl$ denote the collection of Sylow p-subgroups of G. We prove the following partial converse to these results:

Theorem 2.9. *Let $p = 2$ and let $g \in G$ be 2-regular with defect group D. Then $\beta(e_D, g) = |\{P \in \mathcal{S}yl \mid P \cap P^g = D, PgP = Pg^{-1}P\}|\, 1_F$.*

Now G has a 2-block with defect group D if and only if $\beta(e_D, g) \neq 0$ for some 2-regular element g with defect group D. So 2.9 furnishes a necessary and sufficient condition for G to have a 2-block with defect group D.

If $g \in G$, set $\mathbf{C}^*(g) := \{x \in G \,|\, g^x \in \{g, g^{-1}\}\}$. We call the Sylow 2-subgroups of $\mathbf{C}^*(g)$ the *extended defect groups* of g. Let $\mathcal{K}$ be the conjugacy class of G. The extended defect groups of the elements of $\mathcal{K}$ form a single G-orbit. which we call the *extended defect groups* of $\mathcal{K}$. If E is an extended defect group of g, then $D := \mathbf{C}_E(g)$ is a group of g which is contained in E, and $|E : D| \leq 2$. We call (D, E) a *defect pair* for g. The defect pairs of the elements of $\mathcal{K}$ form a single G-orbit. We call g a *real element* if it is G-conjugate to g^{-1}. Theorem 2.9 can be refined for real elements as follows:

Theorem 3.1. *Let $p = 2$ and let $g \in G$ be real and 2-regular with defect pair (D, E). Then $\beta(e_D, g) = |\{P \in \mathcal{S}yl \,|\, P \cap P^g = D, E \leq P\}|\, 1_F$.*

We use this theorem to give an alternative proof of Theorem 4.8 of [4] and also to provide a self-contained treatment of some results of M. Herzog.

Let $\mathcal{K}$ be a conjugacy class of G. We call $\mathcal{K}$ a *real class* if it coincides with its inverse class $\mathcal{K}^o := \{g \in G \,|\, g^{-1} \in \mathcal{K}\}$. We call a real class $\mathcal{K}$ *properly real* if $g^2 \neq 1$ for $g \in \mathcal{K}$.

Suppose that $\mathcal{K}$ and $\mathcal{L}$ are conjugacy classes of G. We write

$$\mathcal{K} \leq \mathcal{L},$$

if each defect group of $\mathcal{K}$ is contained in some defect group of $\mathcal{L}$. Suppose in addition that $\mathcal{K}$ is properly real and that $\mathcal{L}$ is real. We write

$$\mathcal{K} \preceq \mathcal{L}, \tag{1.1}$$

if for each defect pair (D, E) of $\mathcal{K}$, there exists $l \in \mathcal{L}$ such that $D \leq \mathbf{C}(l)$ and $E \leq \mathbf{C}^*(l)$, but $E \not\leq \mathbf{C}(l)$ if $l^2 \neq 1$.

Let $B \leftrightarrow e \leftrightarrow \omega$ be a 2-block of G. It is well-known that there exists a 2-regular class $\mathcal{L}$ of G such that $\beta(e, \mathcal{L}^+) \neq 0$ and $\omega(\mathcal{L}^+) \neq 0$. Any such $\mathcal{L}$ is called a *defect class* for B. The min-max theorem [7, 15.31] states that

$$(1.2)\qquad \begin{aligned} \omega(\mathcal{K}^+) \neq 0 \implies \mathcal{L} \leq \mathcal{K}, \\ \beta(e, \mathcal{K}^+) \neq 0 \implies \mathcal{K} \leq \mathcal{L}, \end{aligned} \qquad \text{for each class } \mathcal{K} \text{ of } G.$$

Suppose that B is a real 2-block of G. R. Gow showed in [3] that B has a defect class $\mathcal{K}$ which is real. Let (D, E) be a defect pair for $\mathcal{K}$. Gow proved that the extended defect groups of the real defect classes of B are G-conjugate to E. For this reason he referred to the G-conjugates of E as the extended defect groups of B. We call (D, E) a *defect pair* for B. Theorem 2.1 of [3] can be extended in the following way:

Theorem 4.3 (Min-Max for Real 2-Blocks). *Let $B \leftrightarrow e \leftrightarrow \omega$ be a real non-principal 2-block of G and let $\mathcal{L}$ be a real defect class of B. Then*

$$\begin{aligned} \omega(\mathcal{K}^+) \neq 0 \implies \mathcal{L} \preceq \mathcal{K}, \\ \beta(e, \mathcal{K}^+) \neq 0 \implies \mathcal{K} \preceq \mathcal{L}, \end{aligned} \qquad \textit{for each real class } \mathcal{K} \textit{ of } G.$$

Throughout the paper S will be a fixed Sylow 2-subgroup of G. Let $D \leq E$ be subgroups of S with $|E : D| = 2$. Let $S\backslash G/S$ denote a set of representatives for the double cosets of S in G. If $x, y \in G$ lie in the same (S, S)-double coset, then the groups $S \cap S^x$ and $S \cap S^y$ are S-conjugate. We say that SgS is a *self-dual double coset* if $SgS = Sg^{-1}S$. Lemma 5.1 furnishes a 2-subgroup $(S \cap S^g)^*$ of G which contains $S \cap S^g$ as a subgroup of index 2, whenever SgS is self-dual and distinct from S. Moreover, if $x, y \in SgS$, then $(S \cap S^x)^*$ and $(S \cap S^y)^*$ are conjugate in S.

If H and K are subgroups of G, we write $H =_G K$ if some G-conjugate of H equals K, and we write $H \leq_G K$ if some G-conjugate of H is contained in K. We let $\sum_x^D$ denote a sum which ranges over those elements x of $S\backslash G/S$

for which $S \cap S^x =_G D$, and let $\sum_{x \equiv x^{-1}}^{D}$ denote the restriction of this sum to the self-dual double cosets.

Suppose that $\{\mathcal{K}_1, \ldots, \mathcal{K}_v\}$ is a complete list of the real 2-regular classes of G which have defect pair (D, E). We call a self-dual double coset SgS a (D, E)-*double coset* if there exists $x \in Sg$ which simultaneously satisfies:

(1) $x \in \mathcal{K}_1 \bigcup \cdots \bigcup \mathcal{K}_v$;

(2) $(S \cap S^g, (S \cap S^g)^*)$ is a defect pair for x.

Let $x_1, \ldots, x_w$ be a (possibly empty) set of representatives for the (D, E)-double cosets of S. Suppose that $w \neq 0$. We define an $v \times w$ integer matrix N by setting the i, j-th entry of N to be the number N_{ij} of y_i in $\mathcal{K}_i \cap Sx_j$ such that $(S \cap S^{x_j}, (S \cap S^{x_j})^*)$ is a defect pair for y_i. When $p = 2$, Theorem A of [11] can be refined as follows:

Theorem 5.2. *The number of real* 2*-blocks of* G *which have defect pair* (D, E) *is zero, if* $w = 0$*, and is the* 2*-rank of the matrix* $N \cdot N^T$*, if* $w \neq 0$.

2. Sylow intersections and 2-blocks

Our starting point is Proposition 3.1 of [4]:

Lemma 2.1. *Let* $B \leftrightarrow e \leftrightarrow \omega$ *be a* p*-block of* G *which has defect group* D *and let* $\mathcal{K}$ *be a* p*-regular class of* G *which has defect group* D. *Then*

$$\beta(e, \mathcal{K}^+) = \left(\frac{\dim(B)}{|G|\,|\mathcal{K}|}\right)^* \omega(\mathcal{K}^{o+}).$$

Let JZ denote the Jacobson radical of Z and let EZ denote the F-span of the idempotents in Z. Then JZ is an ideal of Z, EZ is a direct sum of copies of F (as an F-algebra) and $Z = JZ \bigoplus EZ$ as F-algebras. The *Robinson map* is the natural F-algebra projection $\epsilon : Z \to EZ$ with respect to this decomposition. Let $z \in Z$ and let n be a positive integer such that $g^{p^n} = 1_G$, for each p-element g of G, and $\lambda^{p^n} = \lambda$, for each $\lambda \in F$. Then $\epsilon(z) = z^{p^n}$. We also have

$$\epsilon(z) = \sum_B \omega(z)e, \tag{2.2}$$

where $B \leftrightarrow e \leftrightarrow \omega$ ranges over the p-blocks of G. See [12] for further details.

For the rest of the paper we take $p = 2$ and $\mathrm{Char}(F) = 2$. Recall that e_D denotes the sum of the block idempotents in Z which have defect group D. We combine Lemma 2.1 and (2.2) as follows:

Corollary 2.3. *Let $\mathcal{K}$ be a 2-regular class of G which has defect group D. Then $\beta(e_D, \mathcal{K}^+) = \beta(\epsilon(\mathcal{K}^{o+}), \mathcal{K}^+)$.*

Proof. It follows from (1.2) and (2.2) that

$$\beta(\epsilon(\mathcal{K}^{o+}), \mathcal{K}^+) = \sum_B \omega(\mathcal{K}^{o+})\beta(e, \mathcal{K}^+),$$

where $B \leftrightarrow e \leftrightarrow \omega$ ranges over the 2-blocks of G which have defect group D. Also $(\dim(B)/|G||\mathcal{K}|)^* = 1_F$, for each such B. Thus

$$\begin{aligned}
\beta(\epsilon(\mathcal{K}^{o+}), \mathcal{K}^+) &= \sum \beta(e, \mathcal{K}^+)^2, && \text{using Lemma 2.1} \\
&= (\sum \beta(e, \mathcal{K}^+))^2, && \text{as } F \text{ has characteristic 2} \\
&= \beta(e_D, \mathcal{K}^+)^2 \\
&= \beta(e_D, \mathcal{K}^+), && \text{as } \beta(e_D, \mathcal{K}^+) \in \mathrm{GF}(2).
\end{aligned}$$

□

If $\mathcal{K}$ and $\mathcal{L}$ are 2-regular classes which have defect group D then

$$\beta(\epsilon(\mathcal{K}^+), \mathcal{L}^+) = \sum_x^D |\mathcal{K} \cap Sx|\,|\mathcal{L} \cap Sx|\,1_F, \tag{2.4}$$

using 1.3.3 and 1.3.4 of [12]. This allows us to prove:

Proposition 2.5. *Let $\mathcal{K}$ be a 2-regular class with defect group D. Then* $\beta(e_D, \mathcal{K}^+) = \sum_{x \equiv x^{-1}}^{D} |\mathcal{K} \cap Sx|\, 1_F$.

Proof. By Corollary 2.3 and (2.4) we have

$$\beta(e_D, \mathcal{K}^+) \;=\; \sum_{x}^{D} |\mathcal{K}^o \cap Sx|\, |\mathcal{K} \cap Sx|\, 1_F. \tag{2.6}$$

Let $x \in G$. The map $sx \leftrightarrow (sx)^{-s}$, for $s \in S$, establishes a bijection between the sets $\mathcal{K} \cap Sx$ and $\mathcal{K}^o \cap Sx^{-1}$. So

$$|\mathcal{K} \cap Sx| = |\mathcal{K}^o \cap Sx^{-1}|. \tag{2.7}$$

Suppose $SxS \neq Sx^{-1}S$. Then the contribution of these cosets to (2.6) is

$$|\mathcal{K}^o \cap Sx|\, |\mathcal{K} \cap Sx| 1_F + |\mathcal{K}^o \cap Sx^{-1}|\, |\mathcal{K} \cap Sx^{-1}| 1_F \;=\; 2 \cdot |\mathcal{K}^o \cap Sx|\, |\mathcal{K} \cap Sx| 1_F \;=\; 0_F.$$

It follows that

$$\beta(e_D, \mathcal{K}^+) \;=\; \sum_{x \equiv x^{-1}}^{D} |\mathcal{K}^o \cap Sx|\, |\mathcal{K} \cap Sx|\, 1_F. \tag{2.8}$$

Suppose that $SxS = Sx^{-1}S$. Then

$$\begin{aligned} |\mathcal{K}^o \cap Sx| &= |\mathcal{K}^o \cap Sx^{-1}|, && \text{as } Sx^{-1} \text{ and } Sx \text{ are } S\text{-conjugate} \\ &= |\mathcal{K} \cap Sx|, && \text{by (2.7).} \end{aligned}$$

We conclude from (2.8) and the fact that the prime field of F is GF(2) that

$$\beta(e_D, \mathcal{K}^+) \;=\; \sum_{x \equiv x^{-1}}^{D} |\mathcal{K} \cap Sx|^2\, 1_F \;=\; \sum_{x \equiv x^{-1}}^{D} |\mathcal{K} \cap Sx|\, 1_F.$$

□

Proof of Theorem 2.9. Recall that g is a 2-regular element of G with defect group D. Let $\mathcal{K}$ be the class of G which contains g. We shall compute $|\{(k,P)\in\mathcal{K}\times\mathcal{S}yl \mid PkP = Pk^{-1}P, P\cap P^k =_G D\}|$ in two different ways. On the one hand it equals $|\mathcal{K}|\,|\mu(g)|$, where

$$\mu(g) := \{P\in\mathcal{S}yl \mid PgP = Pg^{-1}P, P\cap P^g =_G D\}.$$

On the other hand it equals $|\mathcal{S}yl|\sum^{D}_{x\equiv x^{-1}}|\mathcal{K}\cap SxS|$. The double coset SxS is a union of $|S:S\cap S^x|$ right cosets of S, and each of these is S-conjugate to Sx. It follows that $|\mathcal{K}\cap SxS| = |S:S\cap S^x||\mathcal{K}\cap Sx|$. Also $|S:S\cap S^x| = |S:D|$, whenever $S\cap S^x$ is G-conjugate to D. But $|\mathcal{S}yl|$ is odd, by Sylow's Theorem. Thus

$$\begin{aligned}
|\mu(g)|\,1_F &= \frac{|S:D|}{|\mathcal{K}|}\sum^{D}_{x\equiv x^{-1}}|\mathcal{K}\cap Sx|\,1_F\\
&= \sum^{D}_{x\equiv x^{-1}}|\mathcal{K}\cap Sx|\,1_F, \quad \text{as } \mathcal{K} \text{ has defect group } D\\
&= \beta(e_D,g), \quad \text{by Proposition 2.5.}
\end{aligned}$$

We claim that D acts by conjugation on $\mu(g)$. For, suppose that $P\in\mu(g)$ and $d\in D$. Then $dg = gd$. So $P^d\cap P^{dg} = (P\cap P^g)^d =_G D$, and $P^dgP^d = (PgP)^d = (Pg^{-1}P)^d = P^dg^{-1}P^d$. Thus $P^d\in\mu(g)$, which proves our claim.

Each D-orbit in $\mu(g)$ has 2-power order, and P is stabilized by D if and only if $D\le P$. But $D\le P$ implies that $D\le P\cap P^g$. Since $P\cap P^g =_G D$, it follows that P is stabilized by D if and only if $P\cap P^g = D$. We conclude that

$$|\mu(g)| \equiv |\{P\in\mathcal{S}yl \mid P\cap P^g = D, PgP = Pg^{-1}P\}| \pmod 2,$$

from which the theorem follows.

$\square$

Theorem 2.9 has no obvious analogue for odd primes. For instance, if P is a Sylow 3-subgroup of $\mathrm{PSL}_3(2)$ and g is an element of order 4, the set $\{P \in \mathcal{S}yl \mid P \cap P^g = \{1\}, PgP = Pg^{-1}P\}$ has cardinality 4. However, g has 3-defect zero and appears with zero multiplicity in the sum of the 3-block idempotents of defect zero.

We indicate how our methods may be used to sharpen Corollary 2 of [11]:

Theorem 2.10. *Let g be a 2-regular element of G which has defect group D. Suppose that $P \cap P^g = D$, for each Sylow 2-subgroup P of G which contains D. Then g lies in a defect class of some real 2-block of G. In particular, G has a real 2-block with defect group D.*

Proof. Let r_D denote the sum of the real 2-block idempotents of G which have defect group D, and let $\mathcal{K}$ be the conjugacy class of G which contains g. We can show that

$$\beta(r_D, \mathcal{K}^+) = \beta(\epsilon(\mathcal{K}^+), \mathcal{K}^+),$$

by modifying the proof of Corollary 2.3. We can then adapt the proofs of Proposition 2.5 and Theorem 2.9 to show that

$$\beta(r_D, g) = |\{P \in \mathcal{S}yl \mid P \cap P^g = D\}|\, 1_F. \tag{2.11}$$

The number of Sylow 2-subgroups of G which contain D is odd, by a well known generalization of Sylow's Theorem. It then follows from our hypothesis, and (2.11), that $\beta(r_D, g) = 1_F$. So G has a real 2-block $B \leftrightarrow e \leftrightarrow \omega$ which has defect group D, and $\beta(e, \mathcal{K}^+) = \beta(e, g) \neq 0_F$. Also $\omega(\mathcal{K}^+) = \omega(\mathcal{K}^{o+}) \neq 0_F$, by Lemma 2.1. This completes the proof.

□

3. Real 2-regular classes and 2-blocks

In this section we prove Theorem 3.1 and give a number of applications.

Proof of Theorem 3.1. Recall that g is a 2-regular element of G with defect pair (D, E). Note that if $E \leq P$, then $PgP = Pg^{-1}P$.

We claim that E acts on the set $\phi(g) := \{P \in \mathcal{S}yl \mid P \cap P^g = D, PgP = Pg^{-1}P\}$ by conjugation. For, suppose that $P \in \phi(g)$. Then D normalizes P. If $e \in E \backslash D$ then $g^e = g^{-1}$. So $P^e g P^e = (Pg^{-1}P)^e = (PgP)^e = P^e g^{-1} P^e$. Moreover $eg = g^{-1}e$ normalizes D. Thus $P^e \cap P^{eg} = (P^g \cap P)^{g^{-1}e} = D^{g^{-1}e} = D$. This shows that $P^e \in \phi(g)$, which proves our claim.

Each E-orbit on $\phi(g)$ has cardinality 1 or 2. Since P is a Sylow 2-subgroup of G, it is stablized by E if and only if $E \leq P$. We conclude that

$$|\phi(g)| \equiv |\{P \in \mathcal{S}yl \mid P \cap P^g = D, E \leq P\}| \,(\mathrm{mod} 2).$$

The result now follows from Theorem 2.9.

□

In our first application of Theorem 3.1, we give another proof of [4, 4.8].

Theorem 3.2. *Let D be a 2-subgroup of G. Suppose that no subgroup of $\mathbf{N}(D)/D$ is isomorphic to a dihedral group of order 8. Then $\beta(e_D, g) = 1_F$, for each real 2-regular element g of G which has defect group D. In particular, the following are equivalent:*

(a). *G has a real 2-regular element with defect group D;*

(b). *G has a 2-block with defect group D;*

(c). *G has a real 2-block with defect group D.*

Proof. The implications (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a) follow as in [4, 4.8].

Suppose that D is a Sylow 2-subgroup of G. Then the principal 2-block $B_0 \leftrightarrow e_0 \leftrightarrow \omega_0$ is the only real block with defect group D, and the identity class is the only real 2-regular class with defect group D. Also $\beta(e, 1^+) = \beta(e^o, 1^+)$, for each non-real 2-block idempotent. It follows that $\beta(e_D, 1^+) = \beta(e_0, 1^+) = \omega_0(1^+) = 1_F$, using Lemma 2.1 (this also follows from a theorem of R. Brauer).

Suppose that D is not a Sylow 2-subgroup of G. Let g be a real 2-regular element with defect pair (D, E). The first statement and the implication (a) $\Longrightarrow$ (c) will follow from Theorem 3.1, if we can show that $P \cap P^g = D$, whenever P is a Sylow 2-subgroup of G which contains E.

Assume for the sake of contradiction that there exists $P \in \mathcal{S}yl$ with $E \leq P$ and $P \cap P^g > D$. Let $x \in E \backslash D$, and set $y := x^{-1}g = g^{-1}x^{-1}$. Then $y \in \mathbf{N}(D)$, since $x \in E \leq \mathbf{N}(D)$ and $g \in \mathbf{C}(D)$. Also $y \in \mathbf{N}(P \cap P^g)$, since $(P \cap P^g)^y = P^{x^{-1}g} \cap P^{x^{-1}} = P^g \cap P$.

The 2-group $\langle y \rangle$ acts on the nontrivial 2-group $\mathbf{N}_{P \cap P^g}(D)/D$. So we can choose $n \in \mathbf{N}_{P\cap P^g}(D) \backslash D$ such that $n^2 \in D$ and $[n, y] \in D$. Our hypothesis on $\mathbf{N}(D)/D$ forces $[n, x] \in D$. Therefore $[n, g] = [n, xy] = [n, x][n, y]^{x^{-1}}$ lies in D. It follows that g centralizes $\langle D, n \rangle$, since $\langle g \rangle$ is a 2'-group which acts trivially on every factor of $1 \leq D < \langle D, n \rangle$. This contradicts the fact that D is a defect group of g. The theorem follows.

□

In our next application we give self contained proofs of a number of results on extremal 2-blocks which are due to M. Herzog [6]. We call G a *CI-group* if every intersection of distinct Sylow 2-subgroups of G is centralized by some Sylow 2-subgroup of G. It is straightforward to show that every subgroup and factor group of a CI-group is a CI-group. Let S and T be Sylow 2-subgroups of G. We say that $S \cap T$ is a *maximal Sylow intersection* in G if $S \neq T$ and whenever $S \cap T \leq P \cap Q$, where $P \neq Q$ are Sylow 2-subgroups of G, then $S \cap T = P \cap Q$.

Lemma 3.3. *Let G be a CI-group. Suppose that $S \neq T$ are Sylow 2-subgroups of G. Then $S \cap T$ is centralized by every 2-group which contains it.*

Proof. Let R be a 2-subgroup of G which contains $S \cap T$. Since $S \neq T$, we may assume that $R \neq T$. Then $R \cap T \geq S \cap T$. It is no loss to assume that $R = S$ and moreover that $S \cap T$ is a maximal Sylow intersection in G.

Now $\mathbf{Z}(S) \leq \mathbf{C}(S \cap T)$. So we can find a Sylow 2-subgroup X of G which centralizes $S \cap T$ and contains $\mathbf{Z}(S)$. Then $S \cap T \leq X$, since X normalizes $S \cap T$. It follows that $S \cap T \leq S \cap X$. If $S = X$ we are done. So assume that $S \neq X$. Then $S \cap T = S \cap X$, as $S \cap T$ is a maximal Sylow intersection. In particular $\mathbf{Z}(S) \leq S \cap T$. But $S \cap T \leq \mathbf{Z}(X)$ and $|\mathbf{Z}(S)| = |\mathbf{Z}(X)|$. So $\mathbf{Z}(S) = S \cap T = \mathbf{Z}(X)$.

□

Here is our main result:

Theorem 3.4. *Let G be a CI-group. Then $\beta(e_D, g) = 1_F$, for each real 2-regular element $g \in G$ which has defect group D. In particular, the statements (a), (b) and (c) of Theorem 3.2 are equivalent.*

Proof. The implications (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a) follow as in Theorem 4.8 of [4].

Let g be a real 2-regular element of G which has defect pair (D, E). Choose $s \in E \backslash D$ and set $t := sg$. Then s and t are 2-elements which invert g and $s^2 = t^2$ lies in D. Let S be a Sylow 2-subgroup of G which contains E. Then $t \in \mathrm{N}(S \cap S^g)$ since $(S \cap S^g)^t = S^{st} \cap S^{sgt} = S^g \cap S^{t^2} = S^g \cap S$. So $\langle S \cap S^g, t \rangle$ is a 2-group which contains $S \cap S^g$. We deduce from Lemma 3.3 that t centralizes

$S \cap S^g$. Also $s \in S$ also centralizes $S \cap S^g$, again using Lemma 3.3. So $S \cap S^g$ is a 2-subgroup of $\mathbf{C}(g)$. It follows that $S \cap S^g = D$, as $D \leq S \cap S^g$ and D is a Sylow 2-subgroup of $\mathbf{C}(g)$. The first statement and the implication (a) $\Longrightarrow$ (c) now follow as in Theorem 3.2.

□

We can now prove:

Proposition 3.5. *Let G be a CI-group and let D be a maximal Sylow intersection in G. Then G has a real 2-block with defect group D.*

Proof. Note that D is the largest normal 2-subgroup of $\mathbf{N}(D)$, and also that it is not a Sylow 2-subgroup of $\mathbf{N}(D)$.

We claim that $\mathbf{N}(D)$ has a nonidentity real 2-regular element. For suppose otherwise. Then $\mathbf{N}(D)/D$ has no nonidentity real 2-regular elements. It follows from the Baer-Suzuki theorem that $\mathbf{N}(D)/D$ has a nontrivial normal 2-subgroup, which contradicts the first paragraph.

Theorem 3.4 now shows that $\mathbf{N}(D)$ has a real 2-block b with non-maximal defect. But b has a defect group which contains D, by a theorem of R. Brauer. It follows that D is a defect group of b. The proposition now follows from Brauer's first main theorem.

□

Theorems 1 and 2 of [6] are consequences of the following corollaries:

Corollary 3.6. *Let G be a finite group. Then G has a normal Sylow 2-subgroup if and only if G is a CI-group with no real non-principal 2-blocks.*

Proof. The 'only if' part is straightforward.

Suppose that G is a CI-group which has no real non-principal 2-blocks. Proposition 3.5 implies that G has no maximal Sylow intersections. So G has a normal Sylow 2-subgroup.

□

We call G a *TI-group* if every pair of distinct Sylow 2-subgroups of G intersect in the identity.

Corollary 3.7. *Let G be a finite group. Then G is a TI-group if and only if G is a CI-group and all real non-principal 2-blocks of G have defect 0.*

Proof. The 'only if' part is straightforward.

Suppose that G is a CI-group and all real non-principal 2-blocks of G have defect 0. We may assume that G does not have a normal Sylow 2-subgroup. Let D be a maximal Sylow intersection in G. Then G has a real 2-block with defect group D, by Proposition 3.5. It follows from the hypothesis that $D = \{1\}$. So G is a TI-group.

□

4. Extended defect groups for real 2-Blocks

In this section we introduce the notion of defect pairs for real 2-blocks. We defined the relation $\preceq$ in (1.1). Now $\preceq$ is almost a partial order, in the sense that if $\mathcal{K}$ and $\mathcal{L}$ are properly real classes and if $\mathcal{M}$ is a real class, then

$$\mathcal{K} \preceq \mathcal{L} \quad \text{and} \quad \mathcal{L} \preceq \mathcal{M} \quad \Longrightarrow \quad \mathcal{K} \preceq \mathcal{M}.$$

Also

$$(4.1) \quad \mathcal{K} \preceq \mathcal{L} \quad \text{and} \quad \mathcal{L} \preceq \mathcal{K} \quad \Longrightarrow \quad \mathcal{K} \text{ and } \mathcal{L} \text{ have the same defect pairs.}$$

Set $[\mathcal{K}] := \mathcal{K} \cup \mathcal{K}^o$, for each class $\mathcal{K}$ of G, and let

$$Z^* := \sum F[\mathcal{K}]^+,$$

where $\mathcal{K}$ ranges over the classes of G. Then Z^* is a subalgebra of Z, as it coincides with the set of fixed points of the involutary automorphism $x \to x^o$ of Z. Each real 2-block idempotent of FG lies in Z^*. By inspecting the proof of Theorem 2.1 of [3], we see that the following is true:

Proposition 4.2. *Suppose that $\mathcal{L}$ is a real class of G and that $\mathcal{K}^+$ is a properly real class which lies in the ideal of Z^* generated by $\mathcal{L}^+$. Then $\mathcal{K} \preceq \mathcal{L}$.*

R. Gow showed in [3, 1.2] that if $B \leftrightarrow e \leftrightarrow \omega$ is a real 2-block of G, then there exists a real 2-regular class $\mathcal{K}$ of G such that $\beta(e, \mathcal{K}^+) \neq 0$ and $\omega(\mathcal{K}^+) \neq 0$. He called any such class a real defect class for B. We will call the defect pairs of the real defect classes of B the *defect pairs* of B.

Proof of Theorem 4.3. Suppose that $\omega(\mathcal{K}^+) \neq 0$. Then $e = \omega(\mathcal{K}^+)^{-1}\epsilon(\mathcal{K}^+)e$. Also $\epsilon(\mathcal{K}^+) = (\mathcal{K}^+)^{2^n}$, for some integer $n > 0$, as in Section 2. So e lies in the ideal of Z^* which is generated by $\mathcal{K}^+$. But $\beta(e, \mathcal{L}^+) \neq 0$. So $\mathcal{L} \preceq \mathcal{K}$, by Proposition 4.2.

Suppose that $\beta(e, \mathcal{K}^+) \neq 0$. Then, using the fact that $\omega(\mathcal{L}^+) \neq 0$, the argument of the previous paragraph shows that $\mathcal{K} \preceq \mathcal{L}$.

□

Let P, Q, R and S be subgroups of G. We say that the pairs (P, Q) and (R, S) are *conjugate in* G if there exists $g \in G$ such that $R = P^g$ and $S = Q^g$. Our corollary is an immediate consequence of (4.1) and Theorem 4.3:

Corollary 4.4. *The defect pairs of a real 2-block are conjugate in G.*

5. The number of real 2-blocks with a given defect pair

We begin this section with a result which associates a certain S-orbit of 2-groups to each self-dual (S,S)-double coset.

Lemma 5.1. *Suppose that $x \in G\backslash S$ and that $SxS = Sx^{-1}S$. Set $(S \cap S^g)^* := (S \cap S^g) \cup (Sg \cap g^{-1}S)$, for each $g \in SxS$. Then $(S \cap S^g)^*$ is a 2-subgroup of G which contains $S \cap S^g$ as a subgroup of index 2. Moroever the $(S \cap S^g)^*$ forms a single S-conjugation orbit, and $Sg \cap g^{-1}S$ coincides with the set $\{y \in Sg \mid y^2 \in S \cap S^g\}$.*

Proof. First we show that $Sg \cap g^{-1}S$ is nonempty. We may write $g^{-1} = sgt$, for certain $s, t \in S$. Then $sg = g^{-1}t$ is an element of $Sg \cap g^{-1}S$.

We claim that $Sg \cap g^{-1}S$ is a right $(S \cap S^g)$-coset. Let $a = b^g \in S \cap S^g$, where $a, b \in S$, and let $cg = g^{-1}d \in Sg \cap g^{-1}S$, where $c, d \in S$. Then $(cg)a$ also lies in $Sg \cap g^{-1}S$ since $g^{-1}(da) = (cg)a = (cb)g$. Also $(cg)(sg)^{-1}$ lies in $S \cap S^g$ since $cs^{-1} = (cg)(g^{-1}s^{-1}) = (g^{-1}d)(t^{-1}g) = (dt^{-1})^g$. This proves our claim.

Now $sg \in \mathbf{N}(S \cap S^g)$, since $(S \cap S^g)^{sg} = S^{sg} \cap S^t = S^g \cap S$. It follows from this and the previous paragraphs that $(S \cap S^g)^*$ is a subgroup of G, which contains $S \cap S^g$ as a subgroup of index 2.

Write $g = uxv$, where $u, v \in S$. Then it is clear that $(S \cap S^g)^* = (S \cap S^x)^{*v}$. So the 2-groups $(S \cap S^g)^*$ form a single S-orbit of subgroups of G.

Finally, suppose that $y = zg$, for $z \in S$. If $y^2 \in S \cap S^g$, then $y^2 = u^g$, for some $u \in S$. Thus $y = (g^{-1}ug)(g^{-1}z) = g^{-1}uz$ lies in $Sg \cap g^{-1}S$. Conversely, suppose that $y \in Sg \cap g^{-1}S$. Then $y = g^{-1}v$, for some $v \in S$. Hence $zv = y^2 = (vz)^g$ lies in $S \cap S^g$. This proves the last statement of the lemma.

□

The subgroups $(S \cap S^g)^*$ have appeared in the literature on self-inverse double cosets. See for example 12.13.(ii) of [2].

Let $\mathcal{K}$ and $\mathcal{L}$ be 2-regular conjugacy classes of G, which have defect groups D and Q respectively. Let x be any element of G. The 2-group $S \cap S^x$ acts by conjugation on $\mathcal{K} \cap Sx$ and on $\mathcal{L} \cap Sx$. So $|\mathcal{K} \cap Sx| \equiv |\mathbf{C}(\mathcal{K}, x)| \,(\mathrm{mod} 2)$, where $\mathbf{C}(\mathcal{K}, x) := \mathcal{K} \cap Sx \cap \mathbf{C}(S \cap S^x)$. Let $\mathbf{O}(\mathcal{L}, x)$ denote the set of those orbits of $S \cap S^x$ on $\mathcal{L} \cap Sx$ which have a representative l such that $S \cap S^x$ contains a Sylow 2-subgroup of $\mathbf{C}(l)$. Now $|\mathcal{L} \cap Sx| = \sum |S \cap S^x : \mathbf{C}_{S \cap S^x}(l)|$. So $(|Q|\,|\mathcal{L} \cap Sx|/|S \cap S^x|)$ is an integer which has the same parity as $|\mathbf{O}(\mathcal{L}, x)|$.

We will use $\sum_x^{Q<D}$ denote a sum which ranges over those double cosets SxS for which $Q \leq_G S \cap S^x \leq_G D$, and $\sum_{x \equiv x^{-1}}^{Q<D}$ to denote the restriction of this sum to the self-dual double cosets. These sums are empty unless $Q \leq_G D$. It follows from [12, 1.3.3 and 1.3.4], and the fact that $\mathrm{Char}(F) = 2$, that

$$\beta(\epsilon(\mathcal{K}^+), \mathcal{L}^+) = \sum_x^{Q<D} |\mathbf{C}(\mathcal{K}, x)|\,|\mathbf{O}(\mathcal{L}, x)|\,1_F. \tag{5.2}$$

Let $\mathbf{C}^*(\mathcal{K}, x)$ denote the set of elements of $\mathbf{C}(\mathcal{K}, x)$ which are inverted by some element of $(S \cap S^x)^*$, and let $\mathbf{O}^*(\mathcal{L}, x)$ denote the set of orbits in $\mathbf{O}(\mathcal{L}, x)$ whose elements are inverted by some element of $(S \cap S^x)^*$.

Proposition 5.3. *Suppose that $\mathcal{K}$ and $\mathcal{L}$ are real 2-regular classes of G, with defect groups D and Q respectively. Then*

$$\beta(\epsilon(\mathcal{K}^+), \mathcal{L}^+) = \sum_{x \equiv x^{-1}}^{Q<D} |\mathbf{C}^*(\mathcal{K}, x)|\,|\mathbf{O}^*(\mathcal{L}, x)|\,1_F.$$

Proof. By pairing each double coset in (5.2) with its dual, as in the proof of Proposition 2.5, we see that

$$\beta(\epsilon(\mathcal{K}^+), \mathcal{L}^+) = \sum_{x \equiv x^{-1}}^{Q<D} |\mathbf{C}(\mathcal{K}, x)|\, |\mathbf{O}(\mathcal{L}, x)|\, 1_F.$$

Suppose that $SxS = Sx^{-1}S$, where $x \in G$. Let $sx \in Sx$ and $tx = x^{-1}u \in Sx \cap x^{-1}S$, where $s, t, u \in S$. Then $(sx)^{-tx} = (u^{-1}x)(x^{-1}s^{-1})(tx) = u^{-1}s^{-1}tx$ also lies in Sx. Set

$$y \cdot z := \begin{cases} y^z, & \text{if } z \in S \cap S^x; \\ (y^{-1})^z, & \text{if } z \in (S \cap S^x)^* \backslash (S \cap S^x). \end{cases}$$

for each $y \in Sx$ and $z \in (S \cap S^x)^*$. It is straightforward to show that this defines an action of the 2-group $(S \cap S^x)^*$ on Sx.

Now $(S \cap S^x)^*$ stabilizes $\mathbf{C}(\mathcal{K}, x)$, and also each $S \cap S^x$-orbit in $\mathbf{O}(\mathcal{L}, x)$. Since $(S \cap S^x)^*$ is a 2-group, this implies that

$$|\mathbf{C}(\mathcal{K}, x)| \equiv |\mathbf{C}^*(\mathcal{K}, x)| \,(\mathrm{mod} 2) \quad \text{and} \quad |\mathbf{O}(\mathcal{L}, x)| \equiv |\mathbf{O}^*(\mathcal{L}, x)| \,(\mathrm{mod} 2).$$

The proposition follows from this.

□

Proof of Theorem 5.2. Recall the notation established in Section 1.

Let $B_1 \leftrightarrow e_1 \leftrightarrow \omega_1, \ldots, B_u \leftrightarrow e_u \leftrightarrow \omega_u$, be a complete list of the (real) 2-blocks of G which have defect pair (D, E). Suppose that $1 \leq i, j \leq u$. Then

$$\delta_{ij} = \omega_j(e_i) = \sum \beta(e_i, \mathcal{K}^+)\omega_j(\mathcal{K}^+), \tag{5.5}$$

where $\mathcal{K}$ runs through the conjugacy classes of G. Suppose that $\mathcal{K} \neq \mathcal{K}^o$. Then the contribution of $\mathcal{K}$ and $\mathcal{K}^o$ to (5.5) is

$$\beta(e_j, \mathcal{K}^+)\omega_i(\mathcal{K}^+) + \beta(e_j, \mathcal{K}^{o+})\omega_i(\mathcal{K}^{o+}) = 2 \cdot \beta(e_j, \mathcal{K}^{o+})\omega_i(\mathcal{K}^{o+}) = 0.$$

Also any real class which occurs with non-zero multiplicity in (5.5) is 2-regular and is not the trivial class. So any such class is properly real. It follows from Theorem 4.3 that

$$\delta_{ij} = \sum_{k=1}^{v} \beta(e_i, \mathcal{K}_k^+)\omega_j(\mathcal{K}_k^+). \tag{5.6}$$

Form the $u \times v$-matrices A and B by setting the i,j-th entry of A as $A_{ij} = \beta(e_i, \mathcal{K}_j^+)$ and the i,j-th entry of B as $B_{ij} = \omega_i(\mathcal{K}_j^+)$. Then AB^T is the $u \times u$ identity matrix, by (5.6). It follows that the $v \times v$-matrix $B^T A$ has rank u.

Suppose that $B \leftrightarrow e \leftrightarrow \omega$ is a non-real 2-block of G and that $1 \leq i, j \leq v$. Then, since $\mathcal{K}_i = \mathcal{K}_i^o$ and $\mathcal{K}_j = \mathcal{K}_j^o$, the contribution of e and e^o to $\beta(\epsilon(\mathcal{K}_i^+), \mathcal{K}_j^+)$ is

$$\omega(\mathcal{K}_i^+)\beta(e, \mathcal{K}_j^+) + \omega^o(\mathcal{K}_i^+)\beta(e^o, \mathcal{K}_j^+) = 2 \cdot \omega(\mathcal{K}_i^+)\beta(e, \mathcal{K}_j^+) = 0.$$

Thus

$$\beta(\epsilon(\mathcal{K}_i^+), \mathcal{K}_j^+) = \beta\left(\sum \omega(\mathcal{K}_i^+)e, \mathcal{K}_j^+\right) = \sum \omega(\mathcal{K}_i^+)\beta(e, \mathcal{K}_j^+),$$

where $B \leftrightarrow e \leftrightarrow \omega$ runs through the real 2-blocks of G. So by Theorem 4.3 we have

$$\beta(\epsilon(\mathcal{K}_i^+), \mathcal{K}_j^+) = \sum_{k=1}^{u} \omega_k(\mathcal{K}_i^+)\beta(e_k, \mathcal{K}_j^+).$$

The sum on the right hand side is the i,j-th entry of the matrix $B^T A$. We conclude that the $v \times v$ matrix M with i,j-th entry $M_{ij} = \beta(\epsilon(\mathcal{K}_i^+), \mathcal{K}_j^+)$ has rank u.

It now follows from Proposition 5.3 that $u = 0$ if $w = 0$, and

$$M_{i,j} = \sum_{k=0}^{w} \mathbf{C}^*(\mathcal{K}_i, x_k)\mathbf{C}^*(\mathcal{K}_j, x_k),$$

if $w > 0$. But $\mathbf{C}^*(\mathcal{K}_i, x_k) = N_{ik}$ and $\mathbf{C}^*(\mathcal{K}_j, x_k) = N_{jk}$. We conclude that $M = N \cdot N^T$, which completes the proof.

□

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