THE IMPORTANCE OF BEING BEAUTIFUL IN MATHEMATICS

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1. Introduction

In this article I will discuss beauty in mathematics and I will present a case for why I consider beauty to be arguably the most important feature of mathematics. However, I will first make some general comments about mathematics that are relevant to my discussion.

Mathematics essentially comprises an abundance of ideas. Number, triangle and limit are just some examples of the myriad ideas in mathematics. I find from experience in teaching mathematics and promoting mathematics among the general public that it's a big surprise for many people when they hear that number is an idea that cannot be sensed with our five physical senses. Numbers are indispensable in today's society and appear practically everywhere from football scores to phone numbers to the time of day. One of my favourite football scores, which I refer to in some talks, is the 'celebrated' result:

Louth 1-9 v 1-7 Cork in 1957.

I will return to this football score later.

The reason number appears practically everywhere is because a number is actually an idea and not something physical. Many people think that they can physically see the number two when it's written on the blackboard but this is not so. The number two cannot be physically sensed because it's an idea.

Mathematical ideas like number can only be really 'seen' with the 'eyes of the mind' because that is how one 'sees' ideas. Think of a sheet of music which is important and useful but it is nowhere near as interesting, beautiful or powerful as the music it represents. One can appreciate music without reading the sheet of music. Similarly, mathematical notation and symbols on a blackboard are just like the sheet of music; they are important and useful but they are nowhere near as interesting, beautiful or powerful as the actual mathematics (ideas) they represent. The number 2 on the blackboard is purely a symbol to represent the idea we call two. Many people claim they do not see mathematics in the physical world and this is because they are looking with the wrong eyes. These people are not looking with the eyes of their mind. For example if you look at a car with your physical eyes you do not really see mathematics, but if you look with the eyes of your mind you may see an abundance of mathematical ideas that are crucial for the design and operation of the car.

So what is this idea we call two? If one looks at the history of number one sees that the powerful idea of number did not come about overnight. As with most potent mathematical ideas, its creation involved much imagination and creativity and it took a long time for the idea to evolve into something close to its current state around 2500 BC. Here is one way to think of what the number two is:

> *Think of all pairs of objects that exist; they all have something in common and this common thing is the idea we call two.*

One can think of any positive whole number in a similar way. Note that this idea of two is different from two sheep, two cars etc. The seemingly simple statement that

$$20+31=51$$

is actually an abstract statement, since it deals with ideas rather than concrete objects, and solves infinitely many problems (since you can pick any object you want to count) in one go. This illustrates the incredible practical power of abstraction and many people do not realise that they use abstraction all the time, e.g. when adding. Note that it's not physically possible to solve infinitely many different problems and yet, Hey Presto! it can be done in the abstract in one go. It borders on magic that it can be done.

Abstraction essentially means that we work with ideas and also try to deal with many seemingly different problems/situations in one go, in the abstract, by discarding superfluous information and retaining the important common features, which will be ideas. Many people tend to think of abstraction as the antithesis of practicality but as the above example of addition shows, abstraction can be the most powerful way to solve practical problems because it essentially means you try to solve many seemingly different problems in one go, in the abstract, as opposed to solving all the different problems separately. The latter approach of solving the different problems separately is what people did as relatively recently as less than five thousand years ago by using different physical tokens for counting different objects. For example, they used circular tokens for counting sheep and cylindrical tokens for counting jars of oil etc. Nowadays, of course, thanks to abstraction, we just do it in one go as 20+31=51 and it doesn't matter whether we are counting sheep or jars of oil. Clearly, there are much more advanced examples of abstraction but the 20+31=51 example captures the essential feature of abstraction. See [1] for more on abstraction.

These surprises (that number is an idea and addition is an example of abstraction) can actually be very positive experiences for some people and these surprises don't confuse them; in fact it can change their perception of mathematics for the better and make them more comfortable with other more complicated ideas because they are now already comfortable with one abstract mathematical idea, i.e. number. These surprises also enhance the understanding, awareness and appreciation of mathematics for many people. Some people also find it fascinating to know that the idea of number was not always known to humans and was actually created by somebody around 2500 BC. As I said above, before 2500 BC the idea of number had not been created and people used different physical tokens to count different objects.

Now, back to that pleasing football score:

Louth 1-9 v 1-7 Cork in 1957

Sometimes I use this result, and other examples, to illustrate how number is an idea and why it is so prevalent in today's society. I comment on how the same symbol 9 is used in two different places to indicate two different things. One refers to 9 very satisfying points scored by Louth, while the other refers to 9 hundreds of years. The reason for this is that 9 is just a symbol to represent an idea and that idea can slot into infinitely many different situations. This is one reason why mathematical ideas and abstraction are so powerful and ubiquitous in society today.

Some other important features of the above scoreline are that it was the last time that Louth won the All Ireland senior football title, it shows the smallest county defeating the largest county and I could go on!

I will now move on to the main topic of this article.

2. Beauty in mathematics

The beauty in mathematics typically lies in the beauty of ideas because, as already discussed, mathematics consists of an abundance of ideas. Our notion of beauty usually relates to our five senses, like a beautiful vision or a beautiful sound etc. The notion of beauty in relation to our five senses clearly plays a very important and fundamental role in our society. However, I believe that ideas (which may be unrelated to our five senses) may also have beauty and this is where you will typically find the beauty in mathematics. Thus, in order to experience beauty in mathematics, you typically need to look, not with your physical eyes, but with the 'eyes of your mind' because that is how you 'see' ideas.

From my experience in the teaching of mathematics and the promotion of mathematics among the general public, I have found that the concept of beauty in mathematics shocks many people. However, after a quick example (like the big sum for a little boy below) or two and a little chat the very same people have changed their perception of mathematics for the better and agree that beauty is a feature of mathematics. One of the reasons why many people are shocked when I mention beauty in mathematics is because they expect the usual notion of beauty in relation to our five senses but as I said above the beauty in mathematics typically cannot be sensed with our five senses.

Around 2,500 years ago the Classical Greeks reckoned there were three ingredients in beauty and these were:

lucidity, simplicity and restraint.

Note that "simplicity" above typically means "simplicity in hindsight", because it may not be easy to come up with the idea initially. On the contrary, it may require much creativity and imagination to come up with the idea initially. These three ingredients above might not necessarily give a complete recipe for beauty for everybody, or maybe a recipe for beauty doesn't even exist. However, it can be interesting to have these ingredients in the back of your mind when you encounter beauty in mathematics. Also, for the Classical Greeks, the three ingredients applied to beauty, not just in mathematics, but in many of their interests like literature, art, sculpture, music, architecture etc.

3. Some examples of beauty in mathematics

Example 1. Big sum for a little boy

Here is a simple example of what I consider to be beauty in mathematics. A German boy, Karl Friedrich Gauss (1777-1855), was in his first arithmetic class in the late 18th century and the teacher had to leave for about 15 minutes. The teacher asked the pupils to add up all the numbers from 1 to 100 assuming that would keep them busy while he was gone. Gauss put up his hand before the teacher left the room. Gauss had the answer and his solution exhibits both beauty and practical power. Gauss observed that:

$$1+100=101,$$
$$2+99=101,$$
$$3+98=101,$$
$$\ldots$$
$$\ldots$$
$$\ldots$$
$$50+51=101$$

and so the sum of all the numbers from 1 to 100 is 50 times 101 which is 5050. Notice how Gauss' solution exploits the symmetry in the problem and flows very smoothly. Compare it to the direct brute force approach of 1+2+3+4.... which is very cumbersome and would take a long time. Both approaches will give the same answer but Gauss' solution is elegant and the other is tedious. Gauss' approach is also much more powerful than the 1+2+3... approach because his idea can be generalised to solve more complicated problems, but you cannot really do much more with the 1+2+3... approach. This power of the beauty in mathematics happens frequently. For those people who are shocked by the notion of beauty in mathematics, this example from Gauss usually changes their perception of mathematics very quickly for the better and they then agree that beauty can be a feature of mathematics.

Example 2. The Seven bridges of Königsberg

This is the famous Seven bridges of Königsberg puzzle. Königsberg, which is now called Kalin-ingrad in Russia, was a city in East Prussia during the eighteenth century. The city was on the banks of the River Pregel and the four parts of the city, denoted by A, B, C, and D, were linked by seven bridges. See Figure 1

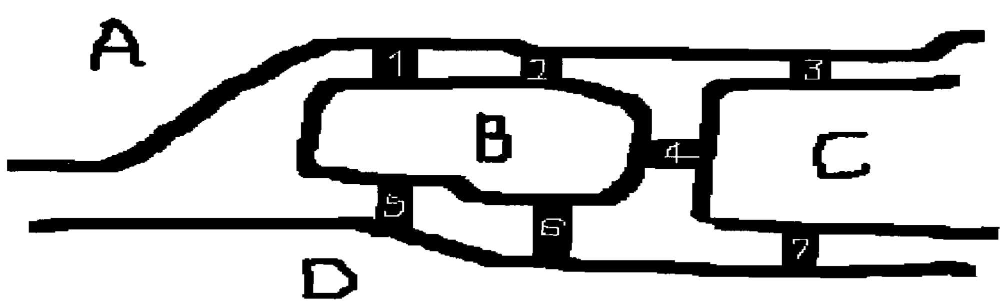

Figure 1

On Sundays people liked to walk around the city and the following question arose:

> Is it possible for one to return to their starting point, anywhere in the city, by cross-ing each bridge exactly once?

It's a bit like the Dublin puzzle which asks:
"Can you walk from one side of Dublin to the other without passing a pub?□
I suppose you could call it 'The infinite pubs of Dublin' puzzle! Anyway, nobody could solve the Königsberg puzzle until the famous Swiss mathematician, Euler (1707-1783), heard about the puzzle and solved it in 1736.

Euler proved that it's impossible to return to your starting point by crossing each bridge exactly once. So, how did Euler's proof go? Well, suppose for convenience that your starting point is in A. The same argument will work for B, C and D. Now, Euler observed that a necessary condition for being able to return to your starting point after crossing each bridge exactly once is that there must be an even number of bridges linked to A. The reason for this is that you must leave A on some bridge and then come back to A on a different bridge, then leave and come back again etc. If you think about it you will see that if there was an odd number of bridges linked to A, then you would have no last bridge to come back on. Now, one can see that there is actually an odd number of bridges linked to A and so you cannot return to your starting point after crossing each bridge exactly once. I think Euler's proof is ingenious and is the epitome of elegance. The Classical Greeks would consider it beautiful too because it certainly has those three ingredients of lucidity, simplicity (in hindsight) and restraint.

Euler's solution is a famous example of elegance in the history of mathematics. Furthermore, his solution of a seemingly trivial puzzle led to a whole new area in mathematics called network theory (or graph theory) which is now indispensable for understanding and designing telecommunication networks, computer circuits, complicated timetables (like our university timetable here in Maynooth) and much more. This is a great example of how an elegant solution of a seemingly innocent puzzle can lead to a major breakthrough in mathematics which in turn can produce very powerful solutions to all sorts of important problems in engineering, science and many other areas.

Notice that nowadays one could just throw this puzzle at a computer and the computer would just check all the millions of possible routes and conclude that it's impossible to return to your starting point by crossing each bridge exactly once. However, I don't see any elegance in a computer churning out the word 'Impossible'. The computer approach provides no insight into why it's impossible and furthermore doesn't give you any new ideas that could be applied elsewhere. However, Euler's approach provides insight into why it's impossible and his idea, as I said above, led to a whole new area in mathematics that is now indispensable for solving many important problems in engineering, science and many other areas. So, maybe it's just as well there were no computers in Euler's time! Also, it's interesting to note that there are no longer seven bridges in Königsberg because the city was bombed heavily during the second world war. Only three of the original bridges are left and two of the others have been rebuilt. Apparently, it is now possible to return to your starting point by crossing each bridge exactly once, unlike back in Euler's time in 1736!

Euler was the most prolific mathematician ever, in terms of number of publications, until the Hungarian mathematician, Erdös (1913-1996), passed him out recently. Erdös was so prolific that apparently, on a long train journey once, he ended up chatting with the train conductor, who was not a mathematician, and between the two of them they solved a previously unsolved problem and published it later! Euler was not only prolific in mathematics; he also had thirteen children. Actually, he once said that some of his greatest mathematical ideas came to him while he had a sleeping baby on his lap. Note that one could base an outdoor mathematical activity on the

Königsberg puzzle by finding a place near your school, like an appropriate variety of paths in a park, and ask a similar question as in the Königsberg puzzle above.

Example 3. The walk along a mountain path

This strange looking puzzle seems to have nothing to do with mathematics and yet it is one of my favourite examples of beauty in mathematical thinking. The puzzle goes as follows:

> *Deirdre starts walking along a mountain path from her house to Ciara's house at 9 a.m. on Saturday morning. Deirdre stays overnight at Ciara's house and starts walking back along the same mountain path at 9 a.m. on Sunday morning. Is there a point on the mountain path where Deirdre passes at the same time on both days?*

Notice that there are no assumptions made about Deirdre's speed on either day. She may walk faster on one day than the other; we don't know and it doesn't matter. The solution to this puzzle involves thinking outside the box in a big way. Here is the solution: Imagine the Saturday walk and the Sunday walk starting simultaneously and you will see that the two walks must intersect at some point X. This point X is a point on the path where Deirdre passes at the same time on both days. That's it! Now, that solution has beauty.

Example 4. An extraordinary equation

The following equation is widely regarded to be the most beautiful equation in mathematics:

$$e^{i\pi} + 1 = 0 \qquad (*)$$

Why? Well, essentially because it embraces the five most important numbers in mathematics and it does so in quite a lucid and relatively simple way. The five numbers all have very different origins and yet it's quite extraordinary that one relatively 'simple' relationship embraces them all. Each of the five numbers

$$0,\ 1,\ \pi,\ e,\ i$$

has a fascinating history. The most interesting book title related to these numbers is undoubtedly '*Zero: The biography of a dangerous idea*' by Charles Seife. Read it and you will see why zero was and still is a dangerous idea. Second in the league of interesting book titles for these numbers is '*An imaginary tale: the story of* $\sqrt{-1}$ ' by Paul Nahin.

Equation (*) above can be proved by setting $\theta = \pi$ in the equation: $e^{i\theta} = \cos\theta + i\sin\theta$.

Equation (*) is not only very aesthetically pleasing, but it also is very useful in a practical way. For example, it plays a fundamental role in helping us understand how things change periodically in time. In fact, the electricity supply industry, which utilises alternating current to provide electricity, uses equation (*) and its consequences every time it designs and operates a power station. So, quite literally, in this case mathematical beauty definitely has practical power!

Example 5. Magic

This example provides a taste of the magic in mathematics. I will present this example in the form of a trick below. Tricks can often be a good way to stimulate students. They can also provide an intriguing setting for the discussion of mathematics. The trick below has many important applications to science. For example, the trick relates to why students can listen to their favourite music on a CD and why a CD supposedly has no flaws/scratches etc. like the old LPs. The trick also relates to why we can view images from Mars!

Here is the trick:

Create an audience of students. Ask a volunteer to set up a square with five rows and five columns of cards (or anything that has a front and a back side that are different), with a random number of cards face up and face down. Ask the volunteer to turn one of the cards over while you are not looking. The trick is that you will be able to say which card was turned over. However, just before the student turns the card over, you suggest adding in one card to each of the five original rows and one card to each of the five original columns in order to make your problem more difficult. This action is crucial to the trick but you don't let the audience know this. You carefully, yet seemingly carelessly, append a new card to each of the five rows and a new card to each of the five columns such that the number of cards face up in each of the first five new rows is even and the number of cards face up in each of the first five new columns is even.

You then look away and let the volunteer turn one card over. You look back at the cards (and wave your magic wand!) and simply silently count the number of cards face up in each of the first five new rows and each of the first five new columns and note where you get an odd answer. This will tell you where the overturned card lies. It will seem like magic.

The above trick can also be performed by using zeros and ones on the blackboard instead of cards face up and face down.

I have performed this trick many times in my public promotion of mathematics in schools and the general public and the trick definitely makes a big impression on people. I feel the idea behind this trick has a certain beauty to it. Where there is beauty in mathematics, practical power will often follow, and so it's not surprising that this idea also has important practical applications. How does the above trick relate to applications in science? Well, in the trick you are using a basic version of a technique that is fundamental in the powerful practical area of 'error correction in codes'. This is the technique where information is appended to the code (message) by the transmitter, in order that the receiver of the message will be able to detect a possible error, due to physical interference etc, and hopefully correct the error. The analogue of the error in the above trick is the overturned card and you were able to detect where the error lies essentially by appending extra information before the card was turned over.

Error correction in codes is crucial in the performance of compact discs. Take a CD from your music collection. The sound is digitally stored on the CD. This digital information can be thought of as a code (message) consisting of zeros and ones, just like the face-up cards and face-down cards in the trick above. Extra information is also appended to the CD as in the trick above to give the total code on the CD. A laser beam in your CD player transmits this total code to a decoder. The decoder receives the total code and attempts to detect any errors which may have been caused by dirt or a scratch etc. This detection process is an advanced version of the method used in the trick above. When an error is detected it can then be corrected so that the sound emanating from your CD player is correct. This is a far cry from the needle on the turntable!

Error correction in codes is also fundamental in analysing information transmitted from spacecraft. For example, when a spacecraft takes photos of Mars, the information is digitally stored like in the CD above. This information (and the extra appended information like above) is transmitted to earth. Any errors caused along the way, like radio interference etc, can be detected and corrected like above. We can then see the correct images of Mars.

Example 6. It's a knock out

Recall that I said I could go on, in relation to the Louth v Cork scoreline. Well, this example is related to that scoreline. How many games took place in the 1957 All-Ireland senior football championship before the Louth captain, Dermot O'Brien, lifted the Sam Maguire to the cheers of all the jubilant Louth fans at Croke Park? It's not obvious, is it?

Here is a similar, yet seemingly more difficult problem: Pick any knock-out tournament you want; it could be football, tennis etc. Suppose there are 127 teams involved and that each game produces one winner who proceeds to the next round and one loser that cannot return to the tournament later on. So, there are no draws, replays or GAA-backdoor-like features. How many games must be played before the champion lifts the trophy? Generalise this to the case where you replace 127 by any positive number n.

This looks like quite a complicated problem because you don't know if some teams have byes into later rounds and you don't have any information on the structure of the tournament other than what is mentioned above which doesn't seem like enough information. Nevertheless, there will be a beautiful two-line solution to this problem. This is a good example of how, by looking at the problem in a completely different way, the solution just simply pops out. Another feature of this problem is the following metaphor which I sometimes mention in my promotion/teaching of mathematics:

You might feel like you are banging your head against a brick wall and there is no way through to the other side. However, maybe there is an unlocked door somewhere in the brick wall and you just need to gently push it open and there you are, on the other side.

Instead of concentrating on the start of the tournament and looking forward in time, like most people do, we will go to the end of the tournament to produce the elegant two-line solution:

> The 'champion lifting the trophy' is equivalent to 'exactly 126 losers' which is equivalent to 'exactly 126 games played'! Consequently, the answer is 126.

Similarly, the general solution to the n-team problem is n-1.

It's interesting to note that, in the Junior Certificate Syllabus, one of the general objectives in mathematics education is that the students should appreciate mathematics as a result of being able to acknowledge the beauty. There are many other examples of beauty in mathematics. Some require more advanced material and some don't. Here are just two more examples, of many: Euclid's elegant proof that there are infinitely many primes and the aesthetically pleasing proof by Hippasus that $\sqrt{2}$ is irrational.

4. Why beauty is arguably the most important feature of mathematics

From my experience teaching a course on the history of mathematics I feel that beauty in mathematics is arguably the most important feature of mathematics. I will present a case for this opinion shortly. Five other important features of mathematics are:

a) Deductive reasoning. See Reason (iii) below for more on this.

b) Abstraction. See section 1 above for more on this.

c) The practical power of mathematics, i.e. the powerful applications of mathematics to science, engineering, navigation, meteorology, finance and many other areas.

d) Research. Historically, research in mathematics has been very vibrant with mathematicians trying to solve many unsolved problems and also developing new theories. The motivation for mathematical research can come from a problem in the physical world or just from pure human imagination. One can play 'Who wants to be a Millionaire?' in mathematical research! How? Well, go to www.claymath.org and check out the Clay Mathematics Institute's Millennium Problems. There is a million dollars prize money for solving any of the seven Millennium Problems. Let me know if you solve any and I would be happy to be your agent! One of the unsolved Millennium problems relates to the Navier-Stokes equation, which is partially named after an Irish mathematician. George Stokes (1819-1903) was born in Skreen, Co. Sligo. The Navier-Stokes equation is important in many practical problems including the stability of ships and can also be used to model ocean currents. Coincidentally Stokes did a lot of work on fluid dynamics, related to waves and ocean currents, and now Skreen is close to some of the best waves for surfing in Europe (e.g. Easkey).

e) Freedom. The notion of freedom in mathematics shocks many people. However, as Cantor (1845-1918) once said,

"The essence of mathematics lies in its freedom".

The reason freedom is an important feature of mathematics is because one is free to conceive of any ideas one wants in mathematics. Whether or not these ideas will lead to anything interesting or useful is another matter. Historically, the major breakthroughs in mathematics have typically happened because the great mathematicians were free to conceive of any ideas they wanted even if they broke with conventions and seemed bizarre to other mathematicians and the general public. Three examples, of many, are the discovery that $\sqrt{2}$ was irrational by Hippasus in Ancient Greece, the discovery of Non-Euclidean Geometry in the 19th century which liberated geometry and the creation of Quaternions by Hamilton on the banks of the Royal Canal in Dublin in 1843 which liberated algebra from arithmetic. See section 6 for more on Quaternions.

Mathematics is so much more than mere numbers, techniques and formulas. Techniques on their own are usually devoid of stimulation and beauty. The art of doing mathematics may involve any of the following: creativity, imagination, inspiration, ingenuity, surprise, mystery, beauty, intuition, insight, subtlety, fun, a wild thought, wonder, symmetry, harmony, aesthetic pleasure, originality, a great sense of achievement, a profound idea, a simple and yet powerful idea, deep concentration and hard work.

As I will outline below, features (a), (b), (c), and (d) above are all intimately related to beauty in mathematics.

I will now present a case for why I believe that beauty is, arguably, the most important feature of mathematics. I will give four reasons.

Reason (i) *The quest for beauty has often been the motivation for why the great mathematicians do research in mathematics.*

Intellectual curiosity, the quest for beauty and the need to understand and solve important practical problems (in science and many other areas) are some of the motivating elements for doing mathematics. From my experience in teaching a course on the history of mathematics, I feel that the search for beauty has often been the motivation for why the great mathematicians do research in mathematics. I will let some of these mathematicians speak for themselves:

Ireland's greatest mathematician, William Rowan Hamilton (1805-1865), was also a poet and regarded *"Mathematics as an aesthetic creation, akin to poetry, with its own mysteries and moments of profound revelation".*

He also wrote: *"For mathematics, as well as poetry, has its own enthusiasm and holds its own communion, with the sublimity and beauty of the universe".*

Hardy (1877-1947), once wrote:

> *"The mathematician's patterns, like the painter's or poet's, must be beautiful, the ideas, like the colours or the words must fit together in a harmonious way. Beauty is the first test; there is no permanent place in the world for ugly mathematics. It may be hard to define mathematical beauty, but that is just as true of beauty of any kind - we may not quite know what we mean by a beautiful poem, but that does not prevent us from recognising one when we read it".*

The great French mathematician, Poincare (1854-1912), said:

> *"The mathematician does not do mathematics because it's useful, he studies it because he delights in it and he delights in it because it's beautiful".*

Somebody once wrote: *"Many mathematicians do research out of a desire for mathematical elegance and the thrill of exploring the unknown".*

Archimedes (287-212 BC) is widely regarded as one of the three greatest mathematicians of all time. The historian, Plutarch, once wrote about Archimedes:

> *"He, i.e. Archimedes, regarded the business of engineering, and indeed of every art which ministers to the material needs of life, as an ignoble and sordid activity, and he concentrated his ambition exclusively upon those speculations whose beauty and subtlety are untainted by the claims of necessity. These studies, he believed, are incomparably superior to any others, since here the grandeur and beauty of the subject matter vie for our admiration with the cogency and precision of the methods of proof".*

This is quite a remarkable statement when one considers that Archimedes' mathematics was, and still is, exceptionally powerful when applied to the areas of engineering, science and many other areas.

Archimedes and Hamilton are great examples of people who pursued mathematics for its aesthetic qualities and yet their mathematics has turned out to be incredibly powerful when applied to science, engineering and many other important practical areas. There are many other examples of such people. The moral of the story here is that the practical power of mathematics can be an offspring of the search for beauty. This leads us on to reason (ii) below.

Reason (ii) *The practical power of mathematics is often an offspring of the search for beauty in mathematics.*

See examples 1, 2, 4 and 5 in section 3 where one can see the practical power of the beauty in mathematics. The quest for beauty in mathematics is what has motivated many of the great mathematicians and yet their mathematics has turned out to be incredibly powerful in science and many other areas. Very often this search for beauty in mathematics has led to new ideas and discoveries of new theories that have fundamentally changed our understanding of the physical world and are now indispensable in the physical world. It's clear from the history of mathematics that the practical power of mathematics is often an offspring of the quest for beauty in mathematics. For example, in the sixteenth century the Polish mathematician, Copernicus, was convinced that the universe was a systematic harmonious structure framed on the basis of mathematical principles, designed by God. This pursuit for an aesthetic harmonious mathematical structure led Copernicus to his famous heliocentric theory which stated that the earth and the planets revolved around the sun as opposed to the earlier belief that the earth was the centre of the universe with the sun revolving around the earth. Copernicus had no experimental evidence for his theory. The motivation for his theory was purely aesthetic because the mathematics describing the sun-centred universe was more aesthetically pleasing than the mathematics describing the earth-centred universe. Galileo and Kepler would later pursue Copernicus' ideas and provide experimental evidence that the earth revolved around the sun. This shocked the world and revolutionised science and society.

As we know, Hamilton's motivation for doing research in mathematics was the search for beauty and yet his mathematics has turned out to be incredibly powerful when applied to science and many other areas. For example, his fundamental theory of dynamics was indispensable for the creation of Quantum Mechanics which is how we now understand the physical world at the microscopic level. Also, his famous Hamiltonian function is fundamental to many aspects of physics. Here is what Hamilton wrote about his new 'General method of dynamics' in 1834:

> *"The difficulty is therefore at least transferred from the integration of many equations of one class to the integration of two of another; and even if it should be thought that no practical facility is gained, yet an intellectual pleasure may result from the reduction of the most complex and, probably, of all researches respecting the forces and motions of body, to the study of one characteristic function, the unfolding of one central relation..."*

It's clear that Hamilton didn't care if his new theory had practical applications. The important point for him is that it had 'intellectual pleasure', i.e. aesthetic pleasure. However, his new theory did turn out to have many powerful practical applications later on, e.g. in Quantum Mechanics as mentioned above. Again, here we have practical power being an offspring of the search for beauty in mathematics. In section 6 I also show some of the many powerful applications of Hamilton's Quaternions.

The Classical Greeks did mathematics for aesthetic pleasure, as we will see below in reason (iii). However, their mathematics has turned put to be exceptionally powerful in the practical world. Two examples, of many, are the ellipse and the parabola. They studied the abstract ellipse intensively for aesthetic pleasure and their results were exactly what Kepler needed two thousand years later to show that the orbits of the planets were ellipses with the sun at one of the foci. They also investigated the abstract parabola for aesthetic pleasure and their results later helped Galileo show that projectiles from the surface of the earth followed a parabolic trajectory. This solved a very important practical problem in the seventeenth century, around two thousand years after the Classical Greeks.

It's important to realise that in the applications of mathematics to the physical world, and elsewhere, mathematics does a lot more than solve problems: it can analyse, predict and prescribe: it can provide deeper insight and it can generate and explore new ideas. Mathematics has a rich history and it has played a very significant role in our civilisation.

Reason (iii) *Mathematics, as we know it today, was essentially born out of a pursuit for aesthetic pleasure and beauty by the Classical Greeks around 600 BC.*

The two main pillars of mathematics are Deductive Reasoning and Abstraction. Around 600 BC the Classical Greeks essentially created mathematics, as we know it today, based on these two pillars. Also, these two pillars, Deductive Reasoning and Abstraction, appealed to the Greeks for aesthetic reasons, as we will see below.

Deductive reasoning works as follows:
We start with premises (which are accepted facts) and then we make conclusions with certainty. It's this word certainty that makes deductive reasoning very special and distinguishes it from all other forms of reasoning. Deductive reasoning lies at the heart of a mathematical proof and means that a proof, once done correctly, is eternal. This made deductive reasoning very appealing to the Classical Greeks for aesthetic reasons because they were deep philosophers and found the quest for eternal truths to be aesthetically pleasing. Deductive reasoning is also one of the reasons why mathematics underpins so much of science. Notice that in some cases we don't care whether the premises mentioned above are true or not (e.g. in the 'proof by contradiction' approach we want to prove our premises are actually false). Note that 'potentially uncertain' things like intuition, conjectures, etc. also play an important role in the art of deductive reasoning because they

can give you powerful insight, targets for what to prove etc. Much creativity and imagination can also be involved in the art of deductive reasoning because there is no guaranteed approach that will always work. An example of an old proof that is eternal is Euclid's proof that there are infinitely many primes. Euclid's proof is as valid today as when it was first done around 2,300 years ago. Deductive reasoning has been called a celebration of the power of pure reason. See [1] for more on deductive reasoning.

See section 1 for a discussion about abstraction. The Classical Greeks were greatly attracted to abstract concepts and ideas which they considered to be eternal, perfect and aesthetically pleasing. Concrete physical things were, in their opinion, ephemeral and imperfect. They studied the abstract circle, i.e. the mathematical circle rather than a particular physical circle. However, many of their results about abstract concepts turned out to be fundamental in solving practical problems as we saw above in reason (ii). Note that the abstract circle has no thickness, colour or molecular structure whereas a physical circle does. Even though the abstract circle may be suggested by the physical circle, the Greeks emphasised that the abstract circle and the physical circle were two totally different creatures.

Deductive reasoning and abstraction are always around whenever mathematics appears. It's very revealing when you realise that mathematics, as we know it today, was born out of a pursuit of aesthetic pleasure and beauty by the Classical Greeks around 600 BC.

Reason (iv) *As Keats wrote: Beauty is truth, truth beauty.*

I feel that this reason is not necessarily as strong as the previous three reasons but I think it's worth mentioning. Deductive reasoning in mathematics is like finding truths, in a certain sense, that follow from accepted facts. In this way a big part of mathematics relates to searching for truths, in a certain sense, and therefore searching for beauty.

5. Some consequences of mathematical beauty elsewhere

Some beautiful visions and sounds can be a consequence of beauty in mathematics. For example, a physically beautiful piece of architecture may be based on the famous number called the Golden Ratio or a beautiful piece of Bach's music may be underpinned by the Fibonacci numbers. Also, certain aesthetically pleasing symmetries in mathematics may produce visually beautiful pieces of art. There are many other examples where beauty, related to our five physical senses, can be a consequence of beauty in mathematics.

6. The importance of being...

What the hell! Since this is a piece about beauty in mathematics, I might as well end with a bit of symmetry by circling back to the title of this article. Hamilton created a strange new number system called Quaternions on October 16, 1843 at Broombridge on the banks of the Royal Canal in Dublin. He has been called the Liberator of Algebra because his Quaternions did not satisfy the commutative rule of multiplication (ab=ba) in arithmetic and so they liberated algebra from the shackles of arithmetic. In Quaternions the order in which the numbers appear is important and this did not bother the creative Hamilton because this is usually what happens in the physical world. For example, consider an empty swimming pool and the two operations of diving in head-first and turning the water on. The order in which the operations take place is quite important! See [2] and [3] for more on Hamilton's life and works.

Quaternions now play a prominent role in many areas. One example, of many, is that they are heavily used in the computer animation industry and in special effects in movies. An example of this that always appeals to journalists, radio hosts and students of course, is that Lara Croft of Tomb Raider was created using Quaternions! Also, Quaternions were involved, through the Irish company Havok, in creating the renowned new special effects in the film, The Matrix Reloaded, and in the recent Bond film, Quantum of Solace. Havok won an Emmy award in the US in 2008 for pioneering new levels of realism and interactivity in movies and games. Havok were involved in creating the special effects for the movie, Poseidon, which was nominated for an OSCAR for its visual effects in 2007. Hamilton's WILDE thought on that famous day, October 16, 1843, with his creation of these strange four dimensional numbers, Quaternions, shocked the mathematical community and changed the whole landscape of mathematics forever because soon after the event many other mathematicians followed in Hamilton's footsteps and felt free to conceive of all sorts of seemingly strange number systems or algebraic structures (e.g. matrices) that did not satisfy the commutative rule (ab=ba) from arithmetic.

Quaternions also played a significant role in Maxwell's mathematical theory and prediction of electromagnetic waves in 1864. Thus, the inventions of radio, television, radar, X-rays and many other important products of our times are directly related to Hamilton.

I organise the annual Hamilton walk on October 16 where people retrace Hamilton's steps from Dunsink to Broombridge. It's an ideal event for Transition Year groups and many from the general public also participate. Typically, around 200 people come on the walk. Contact me if you are interested in bringing a school group. Many famous people including Fields Medallists and Nobel Prize winners have participated in the walk, which will be celebrating its 20th anniversary this year in 2009.

In 1855 Hamilton received a very unusual request. He refers to the event as follows:

> *"A very odd and original lady has also had a baby; such things, as you know will happen... Recently, when I met her for the first time in my life, she told me of this young Pagan as she called him. And she asked me to be a godfather, perhaps because she is an admirer of Wordsworth. However, I declined".*

The young pagan above was none other than Oscar Wilde. Lady Wilde soon became a close friend of Hamilton. Wordsworth is mentioned above because Hamilton and Wordsworth were good friends. In fact, Hamilton was the godfather of Wordsworth's son William. On a visit to the Dublin Writers' Museum in Parnell Street some years back, I noticed that Oscar Wilde was born on October 16. Who knows what Hamilton's reply to Lady Wilde might have been if he had known this?

References

[1] A Resource for Transition Year Mathematics Teachers, by F. Ó Cairbre, R. Watson and J. McKeon, published by the Department of Education and Science, 2006.

Note: Two free copies of this Resource Book were sent to every Second Level school by the Department of Education and Science in 2006.

[2] William Rowan Hamilton (1805-1865), Ireland's greatest mathematician, by F. Ó Cairbre, Ríocht na Midhe, 2000, XI, 124-150.

[3] William Rowan Hamilton and the Quaternion Walk, by Fiacre Ó Cairbre, Irish Mathematics Teachers' Association Newsletter 105, 2006, 3-13.