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Ollscoil na hÉireann Má Nuad

STABILITY RESULTS FOR
CONSTRAINED DYNAMICAL SYSTEMS

A dissertation submitted for the degree of
Doctor of Philosophy

By

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Dedicated to my parents and teachers

ABSTRACT

Differential-Algebraic Equations (DAE) provide an appropriate framework to model and analyse dynamic systems with constraints. This framework facilitates modelling of the system behaviour through natural physical variables of the system, while preserving the topological constraints of the system. The main purpose of this dissertation is to investigate stability properties of two important classes of DAEs. We consider some special cases of Linear Time Invariant (LTI) DAEs with control inputs and outputs, and also a special class of Linear switched DAEs. In the first part of the thesis, we consider LTI systems, where we focus on two properties: *passivity* and a generalization of passivity and small gain theorems called *mixed property*. These properties play an important role in the control design of large-scale interconnected systems. An important bottleneck for a design based on the aforementioned properties is their verification. Hence we intend to develop easily verifiable conditions to check passivity and mixedness of Single Input Single Output (SISO) and Multiple Input Multiple Output (MIMO) DAEs. For linear switched DAEs, we focus on the *Lyapunov stability* and this problem forms the basis for the second part of the thesis. In this part, we try to find conditions under which there exists a common Lyapunov function for all modes of the switched system, thus guaranteeing exponential stability of the switched system. These results are primarily developed for continuous-time systems. However, simulation and control design of a dynamic system requires a discrete-time representation of the system that we are interested in. Thus, it is critical to establish whether discrete-time systems, inherit fundamental properties of the continuous-time systems from which they are derived. Hence, the third part of our thesis is dedicated to the problems of preserving *passivity*, *mixedness* and *Lyapunov stability* under discretization. In this part, we examine several existing discretization methods and find conditions under which they preserve the stability properties discussed in the thesis.

PUBLICATIONS

Some ideas and figures have appeared in the following publications prepared during the course of this Doctorate:

PEER REVIEWED JOURNAL ARTICLES

- Shraavan Sajja, Robert Shorten and Ezra Zeheb, Comments and observations on the passivity of descriptor systems in state space, *Accepted by International Journal of Control*.
- Shraavan Sajja, Francesco Rossi, Patrizio Colaneri and Robert Shorten, Extensions of “Padé Discretization for Linear Systems With Polyhedral Lyapunov Functions” for generalised Jordan structures, *Accepted by IEEE Transactions on Automatic Control*.
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- Wynita Griggs, Shraavan Sajja, Robert Shorten and Brian Anderson, On Interconnections of Mixed Systems Using Classical Stability Theory, *Systems & Control Letters*, 61(5): 676–682, 2012.
- Robert Shorten, Martin Corless, Shraavan Sajja and Selim Solmaz, On Padé Approximations and the Preservation of Quadratic Stability for Switched Linear Systems, *Systems & Control Letters*, 60(9): 683–689, 2011.
- Ezra Zeheb, Robert Shorten and Shraavan Sajja, Strict positive realness of descriptor systems in state space, *International Journal of Control*, 83(9): 1799–1809, 2010. (**see errata online**)

PROCEEDINGS

- Wynita M. Griggs, S. Shraavan K. Sajja, Brian D. O. Anderson and Robert N. Shorten, Extending Small Gain and Passivity Theory For Large-Scale System Interconnections, in *Proc. IEEE American Control Conference*, pp. 6376-6381, 2012.
- Shraavan Sajja, Selim Solmaz, Robert Shorten, and Martin Corless, Preservation of Common Quadratic Lyapunov Functions and Padé Approximations, in *Proc. IEEE 49th Conference on Decision and Control*, pp. 7334-7338, 2010.

“LIVE AND LET LIVE”
— Mohandas Karamchand Gandhi
“HAPPINESS [is] ONLY REAL WHEN SHARED”
— Jon Krakauer, *Into the Wild*

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ACRONYMS

ODE	Ordinary Differential Equations
DAE	Differential-Algebraic Equations
LTI	Linear Time Invariant
LMI	Linear Matrix Inequalities
ISP	Input Strictly Passive
OSP	Output Strictly Passive
VSP	Very Strictly Passive
PR	Positive Real
SPR	Strictly Positive Real
SISO	Single Input Single Output
MIMO	Multiple Input Multiple Output
GHM	Generalized Hamiltonian Matrix
KYP	Kalman Yakubovich Popov
CQLF	Common Quadratic Lyapunov Function
DA	Digital to Analog
ZOH	Zero Order Hold
FOH	First Order Hold
RK	Runge-Kutta
GUES	Globally Uniformly Exponentially Stable
CLM	Common Lyapunov Matrix
CSM	Common Stein Matrix
QS	Quadratically Stable
OLHP	Open Left Half of the Complex Plane

DECLARATION

I herewith declare that I have produced this manuscript without the prohibited assistance of third parties and without making use of aids other than those specified.

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Sajja Surya Shravan Kumar

INTRODUCTION

Constraints are inevitable in real world systems. They can arise in the form of algebraic constraints dictated by the conservation laws for energy, mass or charge, or in the form of position constraints on a vehicle traversing a surface. In such cases, accurate mathematical models involve standard Ordinary Differential Equations (ODE) along with algebraic equations to describe these constraints. Such systems, consisting of both differential and algebraic equations are called DAEs. The main purpose of this dissertation is to investigate stability properties of a certain class of constrained dynamical systems, described by DAEs. In other words, the objective is to develop compact and easy to calculate conditions to evaluate the stability properties of some constrained dynamical systems. These conditions are primarily developed for continuous-time systems, however, as engineers, we rarely work with continuous-time systems exclusively. For simulation purposes, or for the purpose of control design, or in order to implement a controller, we must at some stage work with a discrete-time representation of the system that we are interested in. Thus, it is critical to establish whether discrete-time systems inherit fundamental properties of the continuous-time systems from which they are derived. Though discrete-time system design and analysis is a well-established field, the problem of preserving certain fundamental system properties under discretization is still an open question. The second objective of this thesis is to make a contribution in this direction by exploring discretization methods that preserve certain stability properties of continuous-time systems.

We broadly divide the thesis into three different parts: LTI Descriptor Systems, Linear Switched Descriptor Systems and Discretization Methods. In Sections 1.1, 1.2 and 1.3, we will first introduce the class of systems studied in each part of this work and then outline our contribution.

1.1 PART I: LTI DESCRIPTOR SYSTEMS

Computer-aided analysis of constrained dynamical systems like multibody mechanical systems or electrical networks is usually based on interconnection-oriented modelling. These automatic modelling approaches are employed to preserve the topological structure of the network [1]. Hence they lead to a modular approach towards modelling whereby the dynamic behaviour of each constituent subsystem is described by differential equations and the coupling of the subsystems by algebraic equations [2], further leading to DAE models. Traditional modelling methods eliminate such algebraic constraints to obtain models with a minimal set of unknowns, thereby losing useful network information. As a result there has

been significant interest in recent years towards a more flexible and general description of constrained dynamic systems given by DAEs of the form

$$F(t, x, \dot{x}) = 0, \quad \det \left[\frac{\partial F}{\partial \dot{x}} \right] \equiv 0 \quad (1.1)$$

where x is a common vector denoting the physical variables of an interconnected system and \dot{x} is its time derivative. To demonstrate these ideas, we consider the model of a simple RLC circuit generated using one such automatic modelling technique called Modified Nodal Analysis (for more details see [3]).

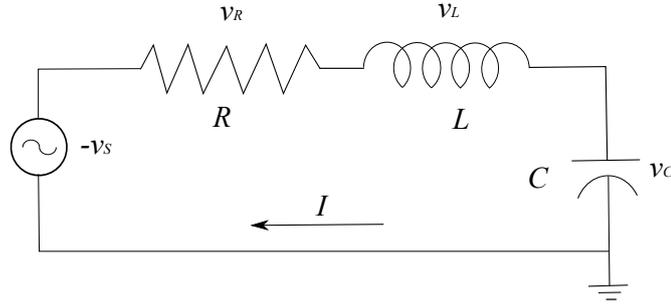


Figure 1: A simple RLC circuit.

Assuming ideal linear devices, the equations modelling the resistor, capacitor and inductor are

$$v_R = IR, \quad v_L = L\dot{I} \quad \text{and} \quad I = C\dot{v}_C,$$

respectively. Applying Kirchhoff's Voltage Law, we get the algebraic condition

$$-v_S + v_R + v_L + v_C = 0.$$

Thus the circuit model is

$$F(t, x, \dot{x}) = 0 : \begin{cases} \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{I} \\ \dot{v}_L \\ \dot{v}_C \\ \dot{v}_R \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -R & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} I \\ v_L \\ v_C \\ v_R \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} v_S, \end{cases}$$

and if we measure the voltage across the capacitor for output, we have

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I \\ v_L \\ v_C \\ v_R \end{bmatrix}.$$

Observe that the leading coefficient matrix $\left(\frac{\partial F}{\partial \dot{x}}\right)$ in the above state equation is singular. Such examples lead to an important subclass of (1.1), given by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t); \\ y(t) &= C^T x(t) + Du(t), \end{aligned} \quad (1.2)$$

where E and A are constant square matrices such that $\det[E] = 0$, $u(t)$, $y(t)$ are external input and output and B , C^T and D are constant matrices of appropriate dimension.

“Systems of the form (1.2) are called *LTI descriptor systems* because they arise from formulating the system equations in their natural physical variables” - [2, 4].

A popular subclass of *LTI* descriptor systems with an invertible matrix E ($\det[E] \neq 0$) are known as *regular* state space systems, and *LTI* descriptor systems are also known as generalized state space systems. Stability theory for *regular* systems is very rich, however, the results for *regular* systems cannot be easily generalized to descriptor systems owing to non-invertibility of the matrix E [5, 6, 7, 8, 9]. Passivity is one such property in the context of stability theory, and Part I of this dissertation aims at finding conditions under which *LTI* descriptor systems are passive. This is explained in more detail in the sequel.

1.1.1 Passivity of LTI descriptor systems

In this part of the thesis we consider the problem of finding the conditions under which *LTI* descriptor systems of the form (1.2) are passive. Passivity arises from the property of a physical system to dissipate energy.

A dynamical system with a state space model is passive if the energy absorbed by the network over any period of time $[0, t]$ is greater than or equal to the increase in the energy stored in the network over the same period; that is,

$$\int_0^t u^T(\tau)y(\tau)d\tau \geq V(x(t)) - V(x(0))$$

where $u^T(t)y(t)$ is the energy supply rate and $V(x(t))$ is the energy storage function in terms of the system state $x(t)$ -[10].

In the input-output framework for linear systems, passivity is equivalent to

$$\int_0^t u^T(\tau)y(\tau)d\tau \geq 0,$$

provided that the system is detectable [11]. Where detectability is a weaker version of observability, which allows unobservable modes as long as those are stable. When we deal

with **LTI** systems, the input-output relationships take the form of a rational transfer function (or a transfer function matrix) $H(s)$, $s \in \mathbb{C}$, and $Y(s) = H(s)U(s)$, where $U(s)$ and $Y(s)$ are the Laplace transforms of the time-functions $u(\cdot)$ and $y(\cdot)$. This allows the time domain conditions for passivity of **LTI** systems to be transformed into frequency domain conditions for positive realness of $H(s)$. The resulting conditions involve conditions on the poles of $H(s)$ and frequency dependent Linear Matrix Inequalities (**LMI**) of the form $H(j\omega) + H(j\omega)^* \geq 0 \forall \omega \in \mathbb{R}$, where $*$ is the complex conjugate transpose of a matrix. In this thesis we also consider stronger definitions of passivity like strict passivity and strict positive realness. More detailed mathematical formulations of all forms of passivity and positive realness are given in Chapter 2. The importance of such transfer functions can be attributed to some of the features below.

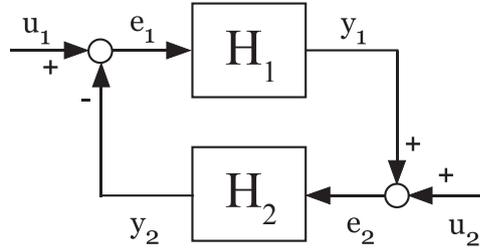


Figure 2: A negative feedback interconnection.

1. Suppose that two systems represented by transfer functions $H_1(s)$ and $H_2(s)$ are passive. Then the two systems, one obtained by the parallel interconnection and the other obtained by feedback interconnection, are both passive.
2. For a negative feedback interconnection of $H_1(s)$ and $H_2(s)$, where $H_1(s)$ is Positive Real (**PR**) and $H_2(s)$ is Strictly Positive Real (**SPR**) (or strictly passive), then the feedback interconnection is input-output stable.
3. If a **PR** transfer function $H_1(s)$ is connected via negative feedback with any nonlinear and/or time-varying controller device, then sufficient conditions for Lyapunov stability are provided by the Circle Criterion and Popov's Criterion.

Such attractive features combined with robustness against large variations of the system parameters, make passivity an important tool to analyse and synthesize stable control systems [12, 13, 14, 15].

The main bottleneck for passivity based control design and passivity enforcement is verification of passivity. The frequency domain conditions for verifying positive realness of a transfer function involves a numerical evaluation of $H(j\omega) + H(j\omega)^*$ for an infinite number of frequency points. This issue has led to a considerable amount of research for

regular system transfer functions of the form $H(s) = D + C^T(sI - A)^{-1}B$ with successful results. Owing to the non-invertibility of E in a descriptor system transfer function $H(s) = D + C^T(sE - A)^{-1}B$, the methods for *regular systems* cannot be easily generalized to descriptor systems. The main problem with testing positive realness of descriptor systems is that continuous-time descriptor system transfer functions might be improper in nature. This results in poles at infinity, which further leads to the problem of checking the signs of residues at infinity [16][12]. This problem involves decoupling the transfer function into proper and improper parts, using Weierstrass canonical form [17]. But direct transformation to the Weierstrass' canonical form can be numerically unstable and expensive [17], hence alternative methods have been explored.

SISO transfer functions: Algorithms to test (strict) positive realness of *regular* and descriptor versions of continuous-time transfer functions were proposed in [18]; these algorithms were based on generalized eigenvalue computation. They were further improved in [19], by removing the necessity of generalized eigenvalue computation. Computational methods to design (strictly) positive real transfer functions using optimization over linear matrix inequalities were proposed in [20]. Testing positive realness was further simplified in [21], by providing compact spectral conditions, however, the conditions derived in [21] are valid only for *regular* continuous-time transfer functions.

MIMO transfer functions: Methods for checking whether a **SISO** transfer function is **PR** (or **SPR**) cannot be easily generalized for **MIMO** transfer function matrices. Consequently, alternate methods to test the **PR** (or **SPR**) property of transfer function matrices were developed in [22], [23], [24], [25]. This work included checking the eigenvalues of a given Hamiltonian matrix on the negative real axis. But these methods fail when $D + D^T$ is singular and lead to generalized eigenvalue problems. This problem motivated new methods based on reciprocal transfer functions, proposed in [26], [27], [28]; to avoid solving for generalized eigenvalues, however, these methods are valid only for *regular* continuous-time transfer functions.

For **MIMO** descriptor systems, several passivity tests have been explored in [29], [30], [31], [32]. These approaches use Van Dooren's algorithm [33] or Orthogonal Reducing Equivalence Transformations (using singular value decompositions, QR factorizations and generalized Schur decompositions, see [30]) to decouple the descriptor system transfer functions and then check for their positive realness. Methods proposed in [31] use generalized controllable staircase forms to detect positive realness of descriptor systems. Another method using skew-Hamiltonian/Hamiltonian transformations was proposed in [34]. But the expensive nature of methods used for decoupling, like singular value decompositions, Schur and QR factorizations, keeps this area of research still relevant.

Recent progress in this field includes the works; [35], [17]. Passivity tests for descriptor systems based on the Generalized Hamiltonian Matrix (**GHM**) were proposed in [35]. In [17],

GHM methods were coupled with canonical projector techniques proposed in [36], [37] to formulate passivity tests for descriptor systems. Canonical projectors are used to decouple a descriptor system into its proper and improper constituents. These canonical projectors were constructed using fast sparse LU-based methods [17].

So far, the central approach in this area of research has been to decouple the descriptor system transfer function using efficient numerical methods and/or apply **GHM** methods to test passivity. In this thesis we avoid both the problems of generalized eigenvalue calculation for a **GHM** and decoupling the transfer function. Our methods are motivated in [26, 27, 28], which have successfully avoided generalized eigenvalue calculation for the **MIMO** case when $D + D^T$ is singular and also for the **SISO** case in [21].

1.1.2 Mixedness of LTI regular and descriptor systems

A situation that motivates the study of “mixed” systems [38, 39] is the one in which high frequency dynamics neglected for modelling purposes destroy the passivity properties of an otherwise passive system. These unmodelled dynamics will always be present in a real system. As such, the passivity theorem alone may not be adequate to show that the stability of the system interconnection is guaranteed [40]. The book [41], see also [42] and [43], described tools for establishing the stability of adaptive systems of the type examined in [40]; that is, where passivity-type properties hold only for low frequency signals.

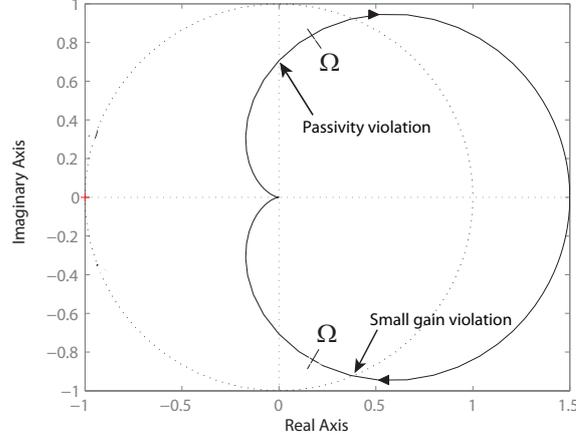
In particular, “mixed”, **LTI** systems, as defined in the frequency domain in [44], are systems that, in some frequency bands, have passivity-type properties, while in the remaining frequency bands, may lose these properties but instead have small gain-type properties; there exist no frequencies over which a “mixed” system has neither of the notions of these properties associated with it. In appropriate circumstances, multipliers (or weights) may be used to scale systems in feedback interconnections that do not exhibit mixtures of small gain and passivity properties into “mixed” systems frameworks [38].

The “mixed” property of an **LTI** system can be illustrated geometrically through Nyquist plots. The Nyquist plot of a passive system always lies in the right half plane, this fact is obvious from the condition

$$H(j\omega) + H(j\omega)^* \geq 0 \quad \forall \omega \in [-\infty, \infty]$$

for positive realness of a continuous-time **LTI** system. Similarly, when a continuous-time **LTI** has the small gain property, i.e.,

$$H(j\omega)^*H(j\omega) \leq 1 \quad \forall \omega \in [-\infty, \infty],$$

Figure 3: Nyquist diagram of $H(s)$.

then the Nyquist plot of $H(s)$ lies entirely inside the unit circle. Now let us consider an example of a “mixed” system with the transfer function

$$H(s) = \frac{3}{(s+1)(s+2)}$$

and Nyquist diagram as depicted in Figure 3. From the Nyquist diagram, it is obvious that there exists a frequency Ω such that,

- over the frequency band $[-\Omega, \Omega]$, $H(s)$ is passive and
- over the frequency bands $[-\infty, -\Omega]$ and $[\Omega, \infty]$, $H(s)$ has gain smaller than one.

In this thesis we explore the application of mixedness for large scale interconnected systems along the lines of passivity. In this pursuit we also correct an oversight in earlier proofs for stability of mixed systems from [38] and [39]. We also characterize “mixed” property for descriptor systems, while providing spectral conditions to test for mixedness.

1.2 PART II: LINEAR SWITCHED DESCRIPTOR SYSTEMS

In this part, we consider another important subclass of (1.1), given by

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x, \quad \sigma(t) \in \{1, \dots, N\}; \det[E_{\sigma(t)}] \equiv 0. \quad (1.3)$$

We assume throughout this paper that σ is a piecewise continuous switching signal with a finite number of discontinuities in any bounded time interval. To motivate this subclass, we

present an example from [45]. The switching circuit (a) in Figure 4 can be represented by the two modes (b) and (c). Modes (b) and (c) can be modelled using simple ODEs given by

$$\text{Mode (b): } \dot{I} = \frac{1}{L}v_S; \quad \text{Mode (c): } \begin{bmatrix} \dot{I} \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} I \\ v_C \end{bmatrix},$$

however, this ODE-based model has two main drawbacks:

1. there is no common state vector as the variable v_C is missing from Mode 1;
2. there is no description of the initial voltage of the capacitor after the switch.

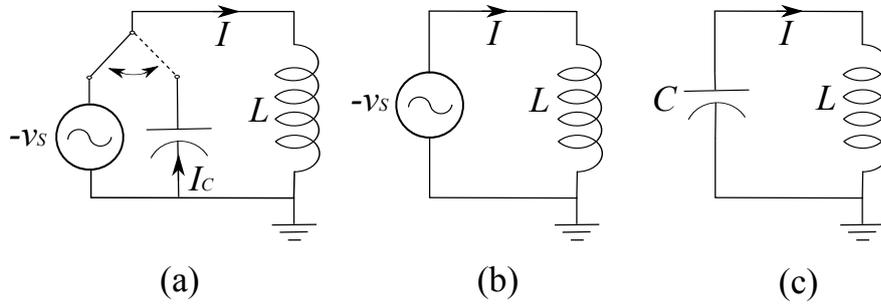


Figure 4: Circuit (a) described by the two modes (b) and (c)

These problems are the result of a modelling approach which eliminates algebraic constraints in order to obtain ODE formulations with a minimum number of variables describing the system, however, such issues can be easily avoided by using a switched descriptor framework as shown by the equations below.

$$\text{Mode (b): } \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{I} \\ \dot{v}_L \\ \dot{I}_C \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I \\ v_L \\ I_C \\ v_C \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} v_S,$$

$$\text{Mode (c): } \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{I} \\ \dot{v}_L \\ \dot{I}_C \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I \\ v_L \\ I_C \\ v_C \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} v_S.$$

Stability theory for switching descriptor systems is somewhat different to stability theory for *regular* dynamic systems; see [46, 47, 48, 49, 50, 51] for a discussion on the stability of switching descriptor systems using Lyapunov stability theory and distribution theory.

Roughly speaking, several mechanisms arise in switching descriptor systems in addition to the regular instability mechanisms found in regular switched systems. In particular, switching descriptor systems may give rise to discontinuities in the state leading to impulses. Thus,

1. the existence of a common Lyapunov function for all modes is not a sufficient condition for stability of a switching descriptor system as the Lyapunov function need not be continuous when transferring between two modes of the system;
2. impulsive behavior may arise when transferring between two modes of the system.

As impulses in engineering applications are usually disastrous, they need to be avoided while performing switching. Hence the switching function is chosen such that impulses are avoided, however, jumps in the states are common in practical scenarios, hence switching should ideally allow jumps while avoiding impulses.

1.2.1 Arbitrary switching

Exponential stability of a *regular* switched linear system ($E_i = I$ for all $i \in \{1, \dots, N\}$) under arbitrary switching is guaranteed by the existence of a common Lyapunov function. This statement is the result of a number of *converse theorems* which state that the existence of a common Lyapunov function is necessary and sufficient for the exponential stability of a *regular* switched linear system (see [52]). A similar result for switched descriptor systems of the form (1.3) has been proposed in [53]. This converse theorem is based on the distributional framework for switched descriptor systems developed in [51] and is valid only under the assumption that (1.3) has impulse-free solutions.

Our primary goal in this part is to obtain verifiable conditions on the matrices in $\mathcal{E} = \{E_1, \dots, E_N\}$ and $\mathcal{A} = \{A_1, \dots, A_N\}$ that guarantee the exponential stability of the switched system (1.3) for any switching signal. Considering the interesting properties of quadratic Lyapunov functions and their relationship with the Kalman Yakubovich Popov (KYP) Lemma, we primarily focus on the sufficient conditions for exponential stability obtained through showing the existence of an appropriate Common Quadratic Lyapunov Function (CQLF) for (1.3). In this thesis we explore the existence of a CQLF $V(x(t))$ for an arbitrary switching sequence $\{t_i\}$ such that

1. $\dot{V}(x(t)) < 0$ for $t \neq t_i$ and
2. $V(x(t_i^+)) \leq V(x(t_i))$ for $t = t_i$.

Similar sufficient conditions for asymptotic stability of linear switched descriptor systems were derived in [50], and they state that a common quadratic Lyapunov function guarantees stability under arbitrary switching when an additional condition involving certain spectral projections of (E_i, A_i) holds (for more details see Chapter 5), however, this condition is not

needed when the switching signal is chosen in such a way that no jumps occur.

Generalized quadratic Lyapunov functions: In this part we study the existence of generalized quadratic Lyapunov functions as a first step towards the study of exponential stability of switched linear descriptor systems. Contrary to *regular LTI* systems, there exist many different formulations of generalized quadratic Lyapunov functions proposed in [54, 55, 56, 57, 58, 59], however, we propose an equivalent, albeit modified, generalized quadratic Lyapunov function which provides a more convenient framework to study its existence. Thus enabling us to find useful criteria to determine whether a given collection of matrix pairs (E_i, A_i) has a generalized **CQLF**.

Special structures of matrices that guarantee existence of a CQLF: For *regular* switched systems with $E_i = I$ for all $i \in \{1, \dots, N\}$, there are some special cases where the structure of matrices $\{A_1, \dots, A_m\}$ guarantees the existence of a **CQLF**, e.g. if the A_i 's are normal ($A_i^T A_i = A_i A_i^T$) and Hurwitz, then $x^T x$ is a **CQLF**. Two important special cases considered by the switched system research community are given below.

1. Simultaneous triangularizability (commutativity): The special case where all the Hurwitz matrices A_i are upper triangular is illustrated through the following Theorem.

Theorem 1 [60],[61] *The set of systems $\{A_i\}$ has a CQLF if there exists a non-singular matrix $U \in \mathbb{C}^{n \times n}$ such that every $U^{-1}A_i U$ is upper (lower) triangular.*

When there is a matrix U such that $U^{-1}A_i U$ is in upper triangular form, the matrices A_1, \dots, A_m commute with each other, and thus it follows that the subsystems Σ_{A_i} have a **CQLF** [62], [63]. Analogous to *regular* switched systems, special structures of switched descriptor systems can be exploited to obtain interesting stability results. [46] extended the result on existence of a **CQLF** condition for *regular* commuting subsystems Σ_{A_i} from [62] to the switched descriptor system case. These results guarantee the existence of a **CQLF** of the form $V(x) = x^T E^T P x$ for switched descriptor system with two modes of operation ($\sigma(t) \in \{1, 2\}$). Results based on commutativity are further extended to discrete-time descriptor systems in [47],[48]. For further extensions based on the commutative property under a distributional framework, see [64].

2. Two systems with a rank-one difference: As a part of exploiting the special structure of matrices to obtain **CQLFs**, we recall this classic result regarding *regular* switched systems from [65, 66].

Theorem 2 [65, 66] *Let A_1 and A_2 be Hurwitz matrices in $\mathbb{R}^{n \times n}$, where the difference $A_1 - A_2$ has rank one. Then the switched system*

$$\dot{x}(t) = A(t)x(t); A(t) \in \mathcal{A} = \{A_1, A_2\} \quad (1.4)$$

has CQLF if and only if the matrix product $A_1 A_2$ has no negative real eigenvalues.

This condition can be recovered from Meyer’s version of the *KYP* Lemma; and it provides simple and easy to calculate necessary and sufficient conditions for the existence of a *CQLF*. Hence they promise simple and elegant results for switched descriptor systems upon extension. Thus, we explore the conditions under which analogous results can be obtained for switched descriptor systems.

1.2.2 *State dependent switching*

In certain situations, it is not necessary to guarantee stability for every possible switching signal; and a number of authors have considered questions related to the stability of switched linear systems under restricted switching regimes. For example, in certain scenarios, when the constituent subsystems are not stable, a suitable switching signal can be chosen to ensure exponential stability (see [46], [67]). Another important example of restricted switching is the state dependent switching, where the rule that determines when a switch in system dynamics may occur is determined by the value of the state-vector $x(t)$.

In this section of the thesis we consider state dependent switching. State dependent switching is necessary to avoid impulsive behaviour that may arise when the initial conditions of a system do not satisfy the system constraints. Impulsive behaviour of descriptor systems can be characterized by an important system property called index. Roughly speaking index of a descriptor system is the number of differentiations necessary to convert a *DAE* into an *ODE*. A more detailed discussion on the index of a descriptor system will follow in Chapter 2. Switched descriptor systems with different indices may have different constraints and thus different solution spaces; this may result in jumps and further impulsive behaviour at the switching instants. However, descriptor systems with index-one do not generate any impulses and hence, they are also known as impulse free descriptor systems.

In this thesis we deal with two special scenarios:

1. switching between index-zero (*regular* system) and index-one (impulse free) descriptor systems ;
2. switching between index-one (impulse free) and a particular class of index-two descriptor systems.

We found that when descriptor modes differ from each other by one index, they can be treated analogous to *regular* systems with rank-one difference, where one of the *regular* modes is marginally stable. This observation motivates us to use the following result in the context of switched descriptor systems.

Theorem 3 [68] *Suppose that A is Hurwitz and all the eigenvalues of $A - gh^T$ have negative real part, except one, which is zero. Suppose also that (A, g) is controllable and (A, h) is observable. Then, there exists a matrix $P = P^T > 0$ such that*

$$A^T P + PA < 0 \quad (1.5)$$

$$(A - gh^T)^T P + P(A - gh^T) \leq 0 \quad (1.6)$$

if and only if the matrix product $A(A - gh^T)$ has no real negative eigenvalues and exactly one zero eigenvalue.

1.3 PART III: DISCRETIZATION

Discretization methods play an important role in simulation and digital control of physical systems. Thus, it is of interest to explore how continuous-time systems can be transformed to a discrete form in a manner that preserves certain properties of the original system. Our main focus is to preserve the three properties described earlier: *passivity*, *mixedness* and *Lyapunov functions*. Initially we survey the discretization methods that preserve these properties for *regular* systems, and then we extend the successful methods to descriptor systems. Throughout the thesis, we only consider a constant sampling time interval h .

1.3.1 Passivity preserving discretization

Discretization methods for regular systems can be classified into two broad categories: (1) discretizing the continuous-time transfer functions and (2) discretizing the continuous-time state space model. We discuss both approaches in detail.

- **Discretizing Transfer Functions:** Discretization of continuous-time transfer functions can be further classified into two methods.

Approximating the discrete input signal through a Digital to Analog (DA) sampler

With Zero Order Hold (ZOH) and First Order Hold (FOH) elements on the input as DA samplers, the discrete-time transfer function can be obtained through the formulas

$$G_{ZOH}(z) = \frac{z-1}{z} \mathcal{Z} \left(\frac{1}{s} H(s) \right) \quad \text{and} \quad G_{FOH}(z) = \frac{(z-1)^2}{hz} \mathcal{Z} \left(\frac{1}{s^2} H(s) \right),$$

where \mathcal{Z} is the z transform and h is the sampling time. According to [69] positive realness of strictly proper transfer functions may not be preserved with ZOH, however, it was shown in [70] that positive realness can be preserved for continuous-time transfer of relative order zero via ZOH, provided that the direct input-output transmission gain is sufficiently large. These results were extended for the MIMO case in [71]. Passivity

preserving properties of FOH were explored in [71] while deducing that FOH elements preserve positive realness, and these results hold true even for strictly proper transfer functions. From the discussion so far, we can conclude that ZOH and FOH methods are only suitable for proper descriptor system transfer functions. This rules out improper transfer functions resulting from index two descriptor systems.

Numerical approximation of the integral using Euler's and Tustin's methods

Euler and Tustin methods are used to approximate the behaviour of a continuous-time transfer function by a discrete-time transfer function using integral numerical approximation. These methods require substitution of the integrator s^{-1} by approximations of the form: $\frac{h}{z-1}$, $\frac{z-1}{zh}$ and $\frac{2}{h} \frac{z-1}{z+1}$ corresponding to Euler's forward method, Euler's backward method and Tustin's method, respectively. J. Jiang [69] proved that Euler's forward method and Euler's backward method fail to preserve positive realness, however, Tustin's method preserves positive realness. We observe that Tustin's method may be a suitable method for discretizing PR and SPR descriptor system transfer functions. The effect of the Tustin transform on the Hamiltonian test matrices for descriptor systems will be explored in this part of the thesis.

- **Discretizing State Space Models:** Discretization of state space models with ZOH element on the input does not preserve passivity. Hence several different approaches have been proposed to implement passivity based digital controllers in [72, 73, 74]. Another new passivity preserving method for the state space models has been proposed in [75] and successfully used for passivity based network control in [76]. This method is based on the fact that the passivity property of a system depends on the input-output variables, hence non-passive models can be transformed into passive models by appropriate selection of input-output variables. Traditional discretization methods for the state space models do not change the output and consequentially degrade passivity of discrete-time systems. This problem has been solved by defining a modified output [75]. This new output is based on the average of the sampled output over the sampling period and produces passive discrete-time models whenever the continuous-time model is passive. This new output can be defined as

$$y^*(kh) = \int_{kh}^{kh+h} y(\tau) d\tau. \quad (1.7)$$

This method of preserving passivity has been studied only for *regular* systems, and there exists no equivalent method for preserving passivity of descriptor systems. This task will be carried out in Chapter 9 of this thesis.

1.3.2 Preserving mixedness under discretization

There exists no literature on discrete-time tests to check for “mixedness” and discretization methods which preserve “mixedness” for either *regular* systems or descriptor systems. In this part of the thesis, we focus on these two new problems.

1.3.3 Preserving Lyapunov Stability under Discretization

We begin our analysis with *regular* switched systems of the form

$$\Sigma_{sc} : \dot{x}(t) = A_c(t)x(t), \quad A_c(t) \in \mathcal{A}_c = \{A_{c1}, \dots, A_{cN}\}, \quad (1.8)$$

with its approximate discrete-time counterpart,

$$\Sigma_{sd} : x(k+1) = A_d(k)x(k), \quad A_d(k) \in \mathcal{A}_d = \{A_{d1}, \dots, A_{dN}\}. \quad (1.9)$$

Discretization of *regular* switched systems (or differential inclusions) has been studied by a number authors [77, 78, 79, 80, 81, 82, 83]. One of the main criteria for these discretization methods has been the convergence of the Hausdorff distance between reachability sets of continuous and discrete-time systems. These authors have primarily focused on obtaining classical convergence proofs for a class of Euler methods and multistep methods.

But the main goal of this part is to find a class of discretization methods $C : \Sigma_{sc} \rightarrow \Sigma_{sd}$ such that if V is a common Lyapunov function for Σ_{sc} , then the same function V is also a common Lyapunov function for Σ_{sd} . This is a very apt criterion for discretization of switched systems because the Converse Lyapunov Theorems state that existence of a common Lyapunov function is necessary and sufficient for the exponential stability of an arbitrarily switched *regular system* [52]. Some other interesting directions of research in this area have been on the topic of preserving positivity and co-positive Lyapunov functions of switched systems under discretization [84], and preserving positivity of descriptor systems [85]. Before proceeding further with our analysis of switched system discretization, we pose the same problem for a regular LTI subsystem.

Discretization of a regular LTI subsystem: Consider a continuous-time LTI system

$$\Sigma_c : \dot{x} = A_c x \quad (1.10)$$

and the corresponding discrete-time LTI system with sampling time $h > 0$,

$$\Sigma_d : x[(k+1)h] = C(A_c, h)x[kh]. \quad (1.11)$$

The closed form solution of (1.10) is given by $x(t) = e^{A_c(t-t_0)}x(t_0)$ for $t \geq t_0$ and this leads to $x[(k+1)h] = e^{A_c h}x[kh]$. Hence discretization of a continuous-time LTI system involves

the calculation of a matrix exponential, which may not yield satisfactory results [86]. One must therefore rely on numerical methods that are able to approximate the solution of a differential equation to any desired accuracy. In this thesis we mainly focus on single step Runge-Kutta (RK) methods [87]. One can study the behaviour of RK methods based on their stability functions (see the Dahlquist Criterion in [87]). The stability functions of Runge-Kutta methods are obtained from Padé approximation tables [88, 89]. These Padé approximations can also be seen as approximations for the matrix exponential $e^{A_c h}$. Initially we consider first order diagonal Padé approximations or equivalently RK methods with first-order diagonal Padé approximations as their stability functions, e.g. single stage Gauss methods. First-order diagonal Padé approximations are popularly known as bilinear transforms or Tustin transforms.

Bilinear Transform: Consider system (1.10) and its discrete-time equivalent (1.11). According to [90] if the discrete-time matrix A_d is obtained using the Tustin discretization method (bilinear transform) given by

$$A_d(h) = \left(I + A_c \frac{h}{2} \right) \left(I - A_c \frac{h}{2} \right)^{-1}, \quad (1.12)$$

then the following observations can be stated:

1. quadratic Lyapunov function class is preserved during discretization;
2. polyhedral Lyapunov function classes (vector norm based Lyapunov functions) are not preserved during discretization.

Observation 1. leads to a notable result concerning common quadratic Lyapunov functions, i.e. they are invariant under bilinear transform [90]. We show later that bilinear transform belongs to larger class of approximations called diagonal Padé approximations and the main disadvantage of bilinear transform is its low convergence rate $\mathcal{O}(h)$. This motivates the question whether other diagonal Padé approximations with increased numerical accuracy also preserve CQLFs? In this part of the thesis we address this question.

1.4 OUTLINE AND CONTRIBUTIONS

In this section we present the contributions made in this thesis and the corresponding publications for each part of the thesis.

PART I: LINEAR TIME INVARIANT DESCRIPTOR SYSTEMS

Passivity: In this part we develop easily verifiable, compact spectral conditions for checking the PR (or SPR) property of SISO and MIMO descriptor systems. To obtain our results, we use only elementary concepts from linear algebra and existing results on strict positive realness

for *regular systems*. This construction results in a test that involves only the evaluation of the eigenvalues of a matrix which is determined in an elementary manner from the matrices E, A, B, C, D ; while avoiding generalized eigenvalue calculation.

Mixed Property: In this part, we provide a proof of the stability result concerning simple feedback-loops consisting of two LTI “mixed” systems due to Griggs, Anderson and Lanzon [38, 39, 91]. We do so by applying classical Nyquist stability techniques (see Chapter 10). Our reasons for doing so are twofold. First, we correct an oversight in Theorems 1 and 6 of [38] and [39], respectively. In these, the system output signals were assumed to be bounded *a priori*. Secondly and importantly, these results pave the way to obtaining new sufficient conditions for the stability of large-scale interconnections of “mixed” systems. In this part we also propose a mathematical formulation for the “mixed” property of descriptor systems. A test, based on Hamiltonian matrices for determining whether *MIMO, regular LTI* systems have the property of “mixedness” was introduced in [44]. This test is further extended for the descriptor case. Some of this material has been published in the following articles.

1. Ezra Zeheb, Robert Shorten and Shravan Sajja, Strict positive realness of descriptor systems in state space, *International Journal of Control*, 83(9): 1799–1809, 2010. (**see errata online**)
2. Shravan Sajja, Robert Shorten and Ezra Zeheb, Comments and observations on the passivity of descriptor systems in state space, *Accepted by International Journal of Control*.
3. Wynita Griggs, Shravan Sajja, Robert Shorten and Brian Anderson, On Interconnections of Mixed Systems Using Classical Stability Theory, *Systems and Control Letters*, 61(5): 676–682, 2012.
4. Wynita M. Griggs, Shravan Sajja, Brian D. O. Anderson and Robert N. Shorten, Extending Small Gain and Passivity Theory For Large-Scale System Interconnections, in *Proc. IEEE American Control Conference*, pp. 6376-6381, 2012.

PART II: LINEAR SWITCHED DESCRIPTOR SYSTEMS

In this part we provide an alternate generalized Lyapunov equation for descriptor systems to suit our approach towards stability analysis of switched descriptor systems. Corresponding to the new generalized Lyapunov equation, we also propose alternate sufficient conditions for the stability of switched descriptor systems. We also derive a *KYP*-like Lemma for a special class of descriptor systems called index one systems. This *KYP*-like Lemma allows us to generate necessary and sufficient conditions for the existence of a *CQLF* for a switched descriptor system described by

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x, \quad \sigma(t) \in \{1, 2\}, \quad (1.13)$$

where each constituent system is index one, stable and the rank of $A_1^{-1}E_1 - A_2^{-1}E_2$ is one. Here, we show that if a simple eigenvalue condition holds, then the system is exponentially stable for arbitrary switching. We provide a state dependent switching rule associated with a simple spectral condition under which switching between index zero and index one or between index one and index two descriptor systems is exponentially stable. Our results for switched descriptor systems are based on the dimensionality reduction of descriptor systems using full rank decomposition. This result has been published in the following article.

5. Shrahan Sajja, Martin Corless, Ezra Zeheb and Robert Shorten, On dimensionality reduction and the stability of a class of switched descriptor system, *Accepted by Automatica*.

PART III: DISCRETIZATION

Passivity: In this part of the thesis we develop passivity preserving discretization methods for state space models of index one descriptor systems. We also show that Tustin’s method of discretizing transfer functions preserves positive realness of descriptor system transfer functions. We also present the output averaging of preserving passivity of a state space model in the context of descriptor systems.

Mixed Property: Here we develop discrete-time tests to check for “mixedness” of *regular* systems and descriptor systems. We also focus on discretization of “mixed” systems. As “mixedness” is a frequency domain property, we only focus on discretization of transfer functions. We find conditions under which a discretization method preserves “mixedness” and show that Tustin’s method is one of the suitable candidates for discretizing mixed systems.

Lyapunov Stability: In this part of the thesis, we prove that diagonal Padé approximations preserve CQLFs, irrespective of their order and sampling size h . We also show that the converse is not true. We further explore the conditions under which diagonal Padé approximations preserve polyhedral Lyapunov functions and show that there always exists a polyhedral Lyapunov function that can be preserved using diagonal Padé approximations. Finally, we derive generalized Padé approximations for descriptor systems and also show that numerical methods with diagonal Padé approximations as stability functions preserve generalized CQLFs. Our results on diagonal Padé approximations have been published in the following articles.

6. Shrahan Sajja, Francesco Rossi, Patrizio Colaneri and Robert Shorten, Extensions of “Padé Discretization for Linear Systems With Polyhedral Lyapunov Functions” for generalised Jordan structures, *Accepted by IEEE Transactions on Automatic Control*.

7. Robert Shorten, Martin Corless, Shravan Sajja and Selim Solmaz, On Padé Approximations and the Preservation of Quadratic Stability for Switched Linear Systems, *Systems and Control Letters*, 60(9): 683–689, 2011.
8. Shravan Sajja, Selim Solmaz, Robert Shorten, and Martin Corless, Preservation of Common Quadratic Lyapunov Functions and Padé Approximations, in *Proc. IEEE 49th Conference on Decision and Control*, pp. 7334-7338, 2010.

Part I

LINEAR TIME INVARIANT DESCRIPTOR SYSTEMS

In this part we give necessary and sufficient conditions for passivity of **LTI** descriptor systems in the form of simple spectral conditions. This work was carried out in collaboration with Prof. Martin Corless¹, Prof. Ezra Zeheb² and Prof. Robert Shorten³.

We also formulate the mixed property for **LTI** descriptor systems, and thereby correcting an error in the proof of a stability result concerning negative feedback-loops of mixed systems due to Griggs, Anderson and Lanzon [38, 39]. We do so using classical Nyquist arguments and then derive spectral conditions to test mixedness of descriptor systems. This work was carried out in collaboration with Dr. Wynita Griggs³.

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BACKGROUND AND PRELIMINARY RESULTS

In this chapter, we present some definitions and preliminary results for descriptor systems. We also introduce the mathematical formulation for passivity and mixedness of LTI systems.

2.1 PROPERTIES OF LTI DESCRIPTOR SYSTEMS

In this chapter, we consider descriptor systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t); x(t_0) = x_0 \quad (2.1)$$

$$y(t) = C^T x(t) + Du(t), \quad (2.2)$$

where $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{m \times m}$ and $\text{rank}(E) = r < n$. Now, we introduce some notation, definitions and preliminary results that shall be useful in analysing descriptor systems. We begin our analysis with an unforced LTI descriptor system ($u(t) \equiv 0$), described by

$$E\dot{x} = Ax, \quad (2.3)$$

where $E, A \in \mathbb{R}^{n \times n}$. In order for the origin to be attractive, we need A to be nonsingular [59]. If A was singular, there would be a non-zero vector v for which $Av = 0$; in that case, a solution starting at v would remain at v and the origin would not be attractive. *We assume throughout this thesis that A is nonsingular.* The following concepts are important in the study of descriptor systems.

Regular descriptor systems: The system is *regular* if $\det[sE - A]$ is not identically zero. *We further assume that all the descriptor systems considered in this thesis are regular.* Regular descriptor systems should not be confused with *regular systems* where $E = I$.

Eigenvalues of (E, A) : An *eigenvalue* of (E, A) is any complex number λ for which $\det[\lambda E - A] = 0$. When A is nonsingular, (E, A) has no eigenvalues at zero and for $\lambda \neq 0$

$$\det[\lambda E - A] = (-\lambda)^n \det[A] \det[\lambda^{-1}I - A^{-1}E]. \quad (2.4)$$

From this expression, it is clear that *the finite eigenvalues of (E, A) are simply the inverse of the nonzero eigenvalues of $A^{-1}E$.*

Consistency space: When E is nonsingular, the system is *regular* and has a unique continuous solution $x(\cdot)$ for any initial condition $x(t_0) = x_0$. When E is singular, this is no longer the case; there is not a solution for every initial condition. When A is invertible, system description (2.3) is equivalent to

$$x = A^{-1}E\dot{x}. \quad (2.5)$$

This means that $x(t)$ must always be in the subspace $\text{Im}(A^{-1}E)$; hence $\dot{x}(t)$ must be in $\text{Im}(A^{-1}E)$, which in turn implies that $x(t)$ must be in $\text{Im}((A^{-1}E)^2)$. By induction, we obtain that $x(t)$ is in $\text{Im}((A^{-1}E)^k)$ for all $k = 1, 2, \dots$. Since

$$\text{Im}((A^{-1}E)^{k+1}) \subset \text{Im}((A^{-1}E)^k)$$

and \mathbb{R}^n has finite dimension n , there exists $k^* \leq n$ such that

$$\text{Im}((A^{-1}E)^{k^*+1}) = \text{Im}((A^{-1}E)^{k^*}); \quad (2.6)$$

in that case, $\text{Im}((A^{-1}E)^k) = \text{Im}((A^{-1}E)^{k^*})$ for all $k \geq k^*$. Let

$$\mathcal{C} = \mathcal{C}(E, A) := \text{Im}((A^{-1}E)^{k^*}). \quad (2.7)$$

Since $\text{Im}((A^{-1}E)^{k^*+1}) = \text{Im}((A^{-1}E)^{k^*})$, we see that $A^{-1}E\mathcal{C} = \mathcal{C}$; this means that $A^{-1}E$ is a one-to-one mapping of \mathcal{C} onto itself; hence the kernel of E and \mathcal{C} intersect only at zero, i.e.,

$$\ker(E) \cap \mathcal{C}(E, A) = \{0\}, \quad (2.8)$$

If we let \tilde{A} be the inverse of the map $A^{-1}E$ restricted to \mathcal{C} , then (2.3), or equivalently (2.5), is equivalent to

$$\dot{x} = \tilde{A}x, \quad (2.9)$$

Thus the descriptor system is equivalent to the *regular system* (2.9), where $x(t)$ is in \mathcal{C} . We call $\mathcal{C}(E, A) = \mathcal{C}$ the *consistency space* for system (2.3) or (E, A) . Note that \tilde{A} is invertible on \mathcal{C} .

Hence, \mathcal{C} is the set of initial states x_0 for which the system (2.3) has a solution.

Index: The *index* of the system is the smallest integer k^* for which (2.6) holds. Clearly, a system is index-zero if and only if E is nonsingular. In this case, the system is equivalent to the regular system $\dot{x} = E^{-1}Ax$ and the consistency space is the whole state space. If E is singular, we make the following claim where the nullity of E is the dimension of the kernel of E and equals $n - r$ and $r = \text{rank}(E)$.

A system is index-one if and only if the number of zero eigenvalues of $A^{-1}E$ equals the nullity of E .

To see this, note that the number of zero of eigenvalues of $A^{-1}E$ is the algebraic multiplicity of zero as an eigenvalue of $A^{-1}E$, whereas the nullity of E (which equals the nullity of $A^{-1}E$) is the geometric multiplicity of zero as an eigenvalue of $A^{-1}E$. The geometric and algebraic and geometric multiplicities are equal if and only if $A^{-1}E$ and $(A^{-1}E)^2$ have the same nullity; this is equivalent to $\text{Im}((A^{-1}E)^2) = \text{Im}(A^{-1}E)$, that is, the system is index-one.

Theorem 4 (Weierstrass canonical form [92],[93]) *Let $\lambda E - A$ be a regular matrix pencil. Then there exist nonsingular matrices S and $T \in \mathbb{R}^{n \times n}$ such that*

$$SET = \begin{pmatrix} I_q & 0 \\ 0 & N \end{pmatrix} \quad SAT = \begin{pmatrix} J & 0 \\ 0 & I_{n-q} \end{pmatrix} \quad (2.10)$$

where I_q denotes the identity matrix of order q . The matrix J corresponds to the finite eigenvalues of $\lambda E - A$, whereas N is nilpotent and corresponds to the infinite eigenvalues. The matrices J and N can be assumed to in Jordan form.

When all eigenvalues of J have negative real parts, the pencil $\lambda E - A$ is said to be stable. The nilpotency index- μ of N , viz. $N^{\mu-1} \neq 0$ and $N^\mu = 0$, is called the index of the matrix pencil $\lambda E - A$. It should also be noted that both definitions of index are equivalent, i.e., $\mu = k^*$.

The concept of index defined above plays a major role in our analysis of descriptor systems, and it determines the switching strategy between the linear switched descriptor systems considered in Part II of this thesis. It also imposes a restriction on the smoothness of the control input for descriptor systems (see Theorem 6). There are several other definitions of index used in the literature, based on the nature of descriptor systems and their applications; for a detailed analysis and a comparison of various index concepts, see [94] [95].

Spectral Projections and Consistency Projectors:

Definition 1 [58] *The spectral projections onto the left and right of the deflating subspaces of $\lambda E - A$ corresponding to the finite eigenvalues are given by*

$$P_l = S^{-1} \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix} S \quad P_r = T \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix} T^{-1} \quad (2.11)$$

The projections onto the left and right of the deflating subspaces of $\lambda E - A$ corresponding to the eigenvalue at infinity are given by the complementary projectors $Q_l = I - P_l$ and $Q_r = I - P_r$.

Stability of descriptor systems:

Now we present different notions of stability for a general DAE of the form (1.1) [96], [97] under the assumption that \mathcal{C} is the consistency space of (1.1).

Definition 2 Stability in the sense of Lyapunov: $x^* = 0$ is a stable equilibrium point of (1.1) if, given any $\varepsilon > 0$, there is some $\delta > 0$ such that for any $x(t_0) \in \mathcal{C}$, $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \varepsilon$ for $t \geq t_0$ for all solutions $x(t)$ of the system.

In the above definition, x^* is *uniformly stable* if δ is not a function of t_0 .

Definition 3 Asymptotic stability: $x^* = 0$ is an asymptotically stable equilibrium point of (1.1) if

1. $x^* = 0$ is stable,
2. $x^* = 0$ locally attractive; i.e., there exists δ such that for any $x(t_0) \in \mathcal{C}$,

$$\|x(t_0)\| < \delta \Rightarrow \lim_{t \rightarrow \infty} x(t) = 0. \quad (2.12)$$

In order to quantify the rate of convergence, we define a different form of stability termed as exponential stability.

Definition 4 Exponential stability: $x^* = 0$ is exponentially stable equilibrium of (1.1) if there exist real constants $M \geq 1$, $\beta > 0$ such that for any $x(t_0) \in \mathcal{C}$,

$$\|x(t)\| \leq M e^{-\beta(t-t_0)} \|x(t_0)\| \quad (2.13)$$

for $t \geq t_0$ for all solutions $x(t)$ of (1.1).

We can say that x^* is *uniformly asymptotically stable* if $x^* = 0$ is asymptotically stable and δ is independent of t_0 . Further, it is required that the convergence in equation (2.12) is uniform. The notion of uniformity is important for time varying systems of the form (1.3), however for time invariant systems of the form (2.3), stability (asymptotic or exponential) implies uniform (asymptotic or exponential) stability. We say an equilibrium point x^* is *globally stable* if it is stable for all initial conditions $x(t_0) \in \mathcal{C}$. For the LTI descriptor systems under consideration in this part (2.3), the following Theorem provides the necessary and sufficient conditions for asymptotic stability.

Theorem 5 [59, 98] Let (E, A) be a regular matrix pair. The equilibrium point $x^* = 0$ of equation (2.3) is asymptotically stable if and only if all the finite eigenvalues of (E, A) lie in the open left half-plane.

State space formulation with $u \neq 0$: One is often interested in controlling descriptor systems through a control input u while measuring some internal state variables x to obtain a desired system behaviour, i.e.,

$$E\dot{x}(t) = Ax(t) + Bu(t); x(t_0) = x_0 \quad (2.14)$$

$$y(t) = C^T x(t) + Du(t), \quad (2.15)$$

where $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{m \times m}$ and $\text{rank}(E) = r < n$. In order to understand the solvability and consistency of the descriptor control system (2.14-2.15), we present the next Theorem based on the Weierstrass canonical form from Theorem 4.

Theorem 6 [99] *Let the pair (E, A) from (2.14) be regular and let S and T be non-singular matrices which transform (2.14) to Weierstrass canonical form, i.e.,*

$$SET = \begin{pmatrix} I_q & 0 \\ 0 & N \end{pmatrix} \quad SAT = \begin{pmatrix} J & 0 \\ 0 & I_{n-q} \end{pmatrix} \quad SB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad (2.16)$$

and set

$$T^{-1}x(t) = \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} \quad T^{-1}x_0 = \begin{pmatrix} \tilde{x}_{1,0} \\ \tilde{x}_{2,0} \end{pmatrix}. \quad (2.17)$$

Furthermore, let $\mu = \text{index}(E, A)$ and assume u is sufficiently smooth. Then we have the following:

1. the differential-algebraic equation (2.14) is solvable;
2. an initial condition (2.14) is consistent if and only if

$$\tilde{x}_{2,0} = - \sum_{i=0}^{\mu-1} N^i B_2 u^i(t_0); \quad (2.18)$$

in particular, the set of consistent initial values x_0 is nonempty.

3. every initial value problem with consistent initial condition is uniquely solvable.

It can be observed from (2.18) that for a classical smooth solution $x(t)$, it is necessary that u is μ times differentiable. Henceforth, in this thesis, we assume that descriptor control systems (descriptor systems with a non-zero input) of the form (2.14-2.15) are consistent and uniquely solvable.

Remark 1 *If we consider the case where $u \equiv 0$, then*

$$\tilde{x}_2 = - \sum_{i=0}^{\mu-1} N^i \tilde{x}_{2,0}. \quad (2.19)$$

Based on (2.19) we can observe that if $\tilde{x}_{2,0}$ represents a sudden jump (Heaviside step function, or the unit step function), then \tilde{x}_2 would consist of impulses (Dirac delta functions). However this is not an issue for index-one systems ($N = 0$), hence they are called **Impulse Free Systems**. For higher index systems, these impulses can be avoided through consistent initialization.

In order to study inconsistent initial values and impulsive solutions, we need a distributional framework. For a detailed study of distributional descriptor systems, see [100],[51]. Explicit representation of the solutions of (2.14) in terms of the original matrices E, A, B is possible using the concept of Drazin inverse for more details, see [99].

2.2 PASSIVITY OF LTI SYSTEMS

In this section, we provide some notation, mathematical definitions and preliminary results on passivity of LTI systems.

Notation: The notation $\Re[s]$ will be used to denote the real part of a complex number s . The conjugate of a complex number $s = a + jb$, where a and b are real, will be denoted by $s^* := a - jb$, where $j^2 = -1$. For a nonsingular matrix A , $A^{-*} := (A^{-1})^* = (A^*)^{-1}$, where A^* denotes the conjugate transpose of A . The largest and smallest singular values of a matrix A will be denoted by $\overline{\sigma}(A)$ and $\underline{\sigma}(A)$, respectively.

The notion of passivity for LTI systems is based on the input-output framework given by

$$Y(s) = H(s)U(s),$$

where $s \in \mathbb{C}$ is the Laplace variable, $U(s)$ and $Y(s)$ are the Laplace transforms of the time-functions $u(\cdot)$ and $y(\cdot)$. By slight abuse of notation, we use $H(s)$ for both scalar and matrix transfer functions. Hence we use the symbols $>$ (\geq) for positivity (non-negativity) for scalars and Hermitian positive definiteness (positive semi-definiteness) for matrices. In particular for transfer function matrices, $H^*(j\omega) := [H(j\omega)]^* = H^T(-j\omega)$. Also, \mathcal{RL}_∞ will be used to denote proper real rational transfer function matrices with no poles on the imaginary axis and \mathcal{RH}_∞ will be used to denote proper real rational transfer function matrices with no poles in the closed RHP.

Mathematical formulation of passivity:

Definition 5 [101] A system with input $u(\cdot)$ and output $y(\cdot)$, where $u(t), y(t) \in \mathbb{R}^m$, is passive if there is a constant $\beta \leq 0$ such that

$$\int_0^t y^T(\tau)u(\tau)d\tau \geq \beta \quad (2.20)$$

for all functions $u(\cdot)$ and all $t \geq 0$. If, in addition, there are constants $l \geq 0$ and $k \geq 0$ such that

$$\int_0^t y^T(\tau)u(\tau)d\tau \geq \beta + l \int_0^t u^T(\tau)u(\tau)d\tau + k \int_0^t y^T(\tau)y(\tau)d\tau \quad (2.21)$$

for all functions $u(\cdot)$ and all $t \geq 0$, then the system is *Output Strictly Passive (OSP)* if $k > 0$, *Input Strictly Passive (ISP)* if $l > 0$, and *Very Strictly Passive (VSP)* if $k > 0$ and $l > 0$.

The constant β is related to the initial conditions of a system and without loss of generality, we assume that the systems are initially relaxed, i.e., β is identically zero [101].

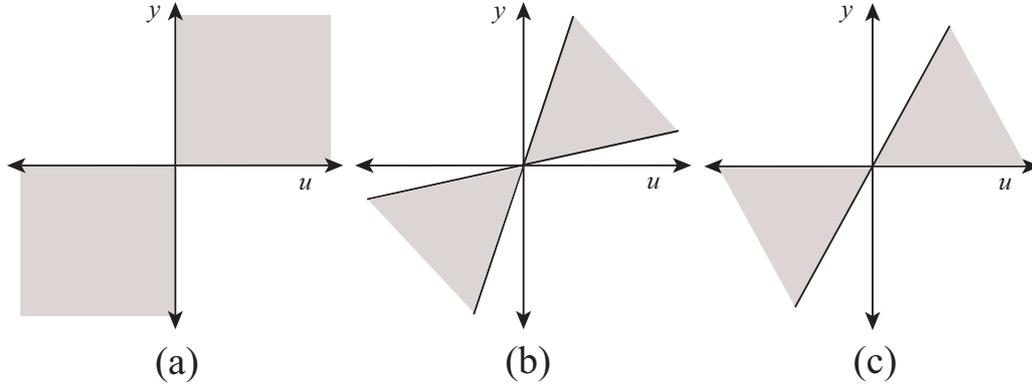


Figure 5: Graphs of systems Σ_1 , Σ_2 and Σ_3 lie in the shaded regions of Figures (a), (b) and (c) respectively.

From the above definition, a **SISO LTI** system Σ_1 , with input u_1 and output y_1 is passive if $u_1(t)y_1(t) \geq 0$ for every $t > 0$. Thus passivity can be interpreted as a restriction on the system input-output graph to lie in the shaded region illustrated in Figure 5-(a) [10]. Similarly for a **VSP** system Σ_2 with input u_2 and out y_2 , the input-output graph lies entirely in a region defined by $u_2(t)y_2(t) \geq \epsilon(u_2^2(t) + y_2^2(t))$ (assuming $k = l = \epsilon$) for every $t > 0$. This region is illustrated in Figure 5-(b). Other intermediate versions of passivity can also be interpreted in a similar way, for example, **OSP** systems (Σ_3) are restricted to region illustrated in Figure 5-(c).

Parseval's Theorem transforms time domain conditions (Definition 5) into frequency domain conditions under the assumption that $u(t)$ and $y(t)$ are Lebesgue integrable. Passivity of an **LTI** system is equivalent to positive realness of the corresponding transfer function $H(s)$ [102]. Now we define Positive Realness of a **LTI** transfer function matrices, these definitions can be easily extended to the scalar case.

Definition 6 [12] A transfer function matrix $H(s)$ is **PR** if and only if the following conditions hold.

1. $H(s)$ is analytic in $\{s \in \mathbb{C} : \Re(s) > 0\}$.
2. $H(j\omega) + H(j\omega)^* \geq 0 \quad \forall \quad \omega \in \mathbb{R} \cup \{\pm\infty\}$, with $j\omega$ not a pole of any element of $H(\cdot)$.
3. If $j\omega_0$ is a pole of any element of $H(\cdot)$, it is at most a simple pole, and the residue matrix, $K_0 = \lim_{s \rightarrow j\omega_0} (s - j\omega_0)H(s)$ in case ω_0 is finite, and $K_\infty = \lim_{\omega \rightarrow \infty} \frac{1}{j\omega}H(j\omega)$ in case ω_0 is infinite, is positive semi-definite Hermitian.

Frequency domain interpretations of other stricter forms of passivity are given below.

Definition 7 [101] Given a LTI system with transfer function matrix $H(s)$ analytic in the closed right half plane, the following assertions hold.

1. The system is *ISP* if and only if there exists a $l > 0$ such that $H(j\omega) + H(j\omega)^* \geq lI > 0$ for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$.
2. The system is *OSP* if and only if there exists a $k > 0$ such that $H(j\omega) + H(j\omega)^* \geq kH(j\omega)H(j\omega)^*$ for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$.
3. The system is *VSP* if and only if there exists $k, l > 0$ such that $H(j\omega) + H(j\omega)^* \geq kH(j\omega)H(j\omega)^* + lI$ for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$.

Negative feedback interconnection of a passive system and a strictly passive system is asymptotically stable. But according to the definitions given above, the concept of strict passivity is very restrictive, because $H(s)$ is strictly passive if and only if $H(j\omega) \geq \delta I > 0 \quad \forall \omega \in \mathbb{R}$. This is possible only for transfer functions with zero relative order. To relax this condition, the concept of *SPR* has been introduced such that the feedback combination of a passive system with an *SPR* one is also asymptotically stable [102]. There exists several notions of strict positive realness in the literature, however we primarily follow the definition given below.

Definition 8 [103] A transfer function $H(s)$ is *SPR* if there exists a scalar $\varepsilon > 0$ such that $H(s)$ is analytic in a region of the complex plane which includes those s for which $\Re(s) \geq -\varepsilon$ and

$$H(j\omega - \varepsilon) + H(j\omega - \varepsilon)^* \geq 0 \quad \forall \quad \omega \in \mathbb{R} \cup \{\pm\infty\}. \quad (2.22)$$

We say H is regular if $H(j\omega) + H(j\omega)^*$ is not identically zero for all $\omega \in \mathbb{R}$. For convenience, we will include regularity as a requirement for *SPR*.

This definition requires the existence of a positive ε such that $H(s - \varepsilon)$ is positive real. We call this *KYP-SPR*, as it is a notion of strict positive realness that is consistent with the *KYP* Lemma. Roughly speaking, the conditions for *KYP-SPR* and strict passivity coincide, except for an additional limit condition at infinity that must be satisfied in the *KYP-SPR* case (the so called side-condition). The ε -free definition of *SPR* given below is based on the result from [103].

Lemma 1 [103] *A transfer function $H(s)$ is SPR and regular if and only if the following conditions hold.*

1. *There exists $\beta > 0$ such that H is analytic in $\{s \in \mathbb{C} : \Re(s) > -\beta\}$.*
- 2.

$$H(j\omega) + H(j\omega)^* > 0 \quad \forall \quad \omega \in \mathbb{R} \quad (2.23)$$

- 3.

$$\lim_{|\omega| \rightarrow \infty} \omega^{2p} \det[H(j\omega) + H(j\omega)^*] \neq 0, \quad (2.24)$$

where p is the nullity of $H(j\infty) + H(j\infty)^*$. In either case, the above limit is positive.

2.2.1 Testing for passivity

Before proceeding to the methods to test passivity of descriptor systems, we present methods to test passivity of *regular systems*. Considering the compact, simple and algebraic nature of these methods, we will use them in the next chapter to develop similar passivity tests for descriptor systems. These methods can be classified into **SISO** and **MIMO** cases.

Passivity of Regular SISO Systems:

Consider a *regular SISO* system

$$\dot{x}(t) = Ax(t) + bu(t); \quad (2.25)$$

$$y(t) = c^T x(t) + du(t), \quad (2.26)$$

where $A \in \mathbb{R}^{n \times n}$, $b, c \in \mathbb{R}^n$, $d \in \mathbb{R}$, and assume that A is Hurwitz. The corresponding **SISO** transfer function is given by

$$H(s) = d + c^T (sI - A)^{-1} b. \quad (2.27)$$

Passivity tests for the **SISO** case are presented separately for $d > 0$ and $d = 0$.

Theorem 7 [21] *Consider the transfer function (2.27) with $d > 0$. $H(s)$ is SPR if and only if*

1. *A is Hurwitz and*
2. *the matrix $(A - \frac{1}{d}bc^T)A$ has no eigenvalues on the closed negative real axis $(-\infty, 0]$.*

$H(s)$ is PR if and only if

1. *the matrix $(A - \frac{1}{d}bc^T)A$ has no eigenvalue of odd (algebraic) multiplicity on the open negative real axis $(-\infty, 0)$ and*

2. all residues of $H(s)$ at poles on the imaginary axis are non-negative.

Theorem 8 [21] Consider the transfer function (2.27) with $d = 0$. $H(s)$ is *SPR* if and only if

1. A is Hurwitz and
2. • $c^T A b < 0$ and the matrix product $A(A - \frac{A b c^T A}{c^T A b})$ has no eigenvalue on the open negative real axis $(-\infty, 0)$ and exactly one zero eigenvalue;
(OR)
• $c^T A^{-1} b < 0$ and the matrix product $A^{-1}(A^{-1} - \frac{A^{-1} b c^T A^{-1}}{c^T A^{-1} b})$ has no eigenvalue on the open negative real axis $(-\infty, 0)$ and exactly one zero eigenvalue.

Let p be the smallest odd integer such that $c^T A^p b \neq 0$, then $H(s)$ is *PR* if and only if

1. $(-1)^{\frac{p+1}{2}} c^T A^p b > 0$;
2. the matrix $A \left(I - \frac{A^p b c^T}{c^T A^p b} \right) A$ has no eigenvalue of odd (algebraic) multiplicity on the open negative real axis $(-\infty, 0)$ and
3. all residues of $H(s)$ at poles on the imaginary axis are non-negative.

Passivity of Regular MIMO Systems:

Consider a regular MIMO system

$$\dot{x}(t) = Ax(t) + Bu(t); \quad (2.28)$$

$$y(t) = C^T x(t) + Du(t), \quad (2.29)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{m \times m}$, and assume that A is Hurwitz. The corresponding MIMO transfer function is given by

$$H(s) = D + C^T (sI - A)^{-1} B. \quad (2.30)$$

Again, we present these conditions separately for $D + D^T$ non-singular and $D + D^T$ singular.

Theorem 9 [22], [23] Let $A \in \mathbb{R}^{n \times n}$ be a real Hurwitz matrix and assume that $Q = D + D^T > 0$. Then the transfer function matrix

$$H(s) = D + C^T (sI - A)^{-1} B \quad (2.31)$$

is *SPR* if and only if the Hamiltonian matrix given by
$$\begin{bmatrix} -A + BQ^{-1}C^T & BQ^{-1}B^T \\ -CQ^{-1}C^T & A^T - CQ^{-1}B^T \end{bmatrix}$$
 has no eigenvalues on the imaginary axis.

Theorem 10 [26] *Let $A \in \mathbb{R}^{n \times n}$ be a real Hurwitz matrix with $D + D^T$ singular. Then the transfer function matrix*

$$H(s) = D + C^T (sI - A)^{-1} B \quad (2.32)$$

satisfies $H(j\omega) + H^(j\omega) > 0$ for all finite ω if and only if the $Q + Q^T > 0$, where $Q = D - C^T A^{-1} B$ and the Hamiltonian matrix given by*

$$N = \begin{bmatrix} -(A^{-1} + A^{-1} B Q^{-1} C^T A^{-1}) & A^{-1} B Q^{-1} B^T A^{-T} \\ -A^{-T} C Q^{-1} C^T A^{-1} & (A^{-1} + A^{-1} B Q^{-1} C^T A^{-1})^T \end{bmatrix}, \quad (2.33)$$

has no eigenvalues on the imaginary axis, except at the origin.

2.3 MIXED PROPERTY OF LTI SYSTEMS

LTI “mixed” systems, as defined in the frequency domain in [44], are systems that, in some frequency bands, have passivity-type properties, while in the remaining frequency bands, may lose these properties but instead have small gain-type properties; there exist no frequencies over which a “mixed” system has neither of the notions of these properties associated with it. In appropriate circumstances, multipliers (or weights) may be used to scale systems in feedback interconnections that do not exhibit mixtures of small gain and passivity properties into “mixed” systems frameworks [38]. Before defining the mixed property, we define an important property known as causality.

Definition 9 *A system is causal if and only if for all input pairs $u_1(t)$ and $u_2(t)$ such that*

$$u_1(t) = u_2(t), \quad t \leq t_0 \quad \forall t_0$$

the two corresponding outputs satisfy

$$y_1(t) = y_2(t), \quad t \leq t_0$$

If the above condition is not satisfied, then the system is able to anticipate the difference in the inputs before it occurs, which is not possible for physical systems. For LTI systems, simpler definitions of causality can be stated.

Definition 10 [104] *Let $h(t)$ be the impulse response for a system and $H(s) = \int_{-\infty}^{+\infty} h(t) e^{-st} dt$ be the corresponding bilateral laplace transform. Then the system is causal if and only if*

$$h(t) = 0 \quad t < 0, \quad (2.34)$$

or equivalently,

1. $H(s)$ is defined and analytic in a half-plane $\Re(s) > \sigma_0$ and

2. $H(s)$ grows not faster than a polynomial for $\Re(s) > \sigma_0$, for some σ_0 .

It is well known that if an LTI system is passive, then it is also causal [104], however, mixed property is based on violation of passivity. Hence, based on the above definition of causality and conditions imposed by stability, we restrict our definition of mixed property to \mathcal{RH}_∞ transfer function matrices.

Mathematical Formulation of “mixed” property:

Consider square transfer function matrix $M \in \mathcal{RH}_\infty$. Suppose that $a, b \in \mathbb{R}$.

Definition 11 [44] *A square transfer function matrix $M \in \mathcal{RH}_\infty$ is said to be input and output strictly passive over the frequency interval $[a, b]$, $(-\infty, a]$, $[b, \infty)$ or $(-\infty, \infty)$ if there exist $k, l > 0$ such that*

$$-kM^*(j\omega)M(j\omega) + M^*(j\omega) + M(j\omega) - lI \geq 0$$

for all $\omega \in [a, b]$, $(-\infty, a]$, $[b, \infty)$ or $(-\infty, \infty)$, respectively.

We will say that the system is input strictly passive over a frequency interval if Definition 11 is satisfied with $k = 0$; output strictly passive over a frequency interval if the definition is satisfied with $l = 0$; and passive over a frequency interval if it is satisfied with $k = l = 0$. Note that any $M(j\omega)$ satisfying Definition 11 over the frequency interval $(-\infty, a]$, $[b, \infty)$ or $(-\infty, \infty)$ must be such that $\lim_{\omega \rightarrow \pm\infty} \lambda_i[M^*(j\omega) + M(j\omega)] = c_{p_i} > 0$ for all i , where $\lambda_i \in \mathbb{R}$ denotes the i th eigenvalue of the Hermitian matrix $M^*(j\omega) + M(j\omega)$. Then $\lim_{\omega \rightarrow \pm\infty} \det[M^*(j\omega) + M(j\omega)] \neq 0$.

Definition 12 [44] *Define the system gain over the frequency interval $[a, b]$, $(-\infty, a]$, $[b, \infty)$ or $(-\infty, \infty)$ as*

$$\varepsilon := \inf\{\bar{\varepsilon} \in \mathbb{R}_+ : -M^*(j\omega)M(j\omega) + \bar{\varepsilon}^2 I \geq 0 \\ \text{for all } \omega \in [a, b], (-\infty, a], [b, \infty) \text{ or } (-\infty, \infty), \text{ respectively}\}.$$

The transfer function matrix $M \in \mathcal{RH}_\infty$ is said to have a gain of less than one over the frequency interval $[a, b]$, $(-\infty, a]$, $[b, \infty)$ or $(-\infty, \infty)$, respectively if $\varepsilon < 1$.

In the following, we present an important relationship, connecting passivity and small gain properties of a transfer function. This relationship will be used later to develop stability results for “mixed” regular and descriptor systems.

Lemma 2 (Scattering Property) *Suppose that $H \in \mathcal{RL}_\infty$ and that, at some $\omega \in \mathbb{R} \cup \{\pm\infty\}$, $H^*(j\omega) + H(j\omega) \geq 0$. Let $S(j\omega) := (H(j\omega) - I)(I + H(j\omega))^{-1}$. Then $-S^*(j\omega)S(j\omega) + I \geq 0$.*

For any system satisfying Definition 12 with gain of less than one over the frequency interval $(-\infty, a]$, $[b, \infty)$ or $(-\infty, \infty)$ it must be so that $\lim_{\omega \rightarrow \pm\infty} \lambda_i[-M^*(j\omega)M(j\omega) + I] = c_{s_i} > 0$ for all i , where $\lambda_i \in \mathbb{R}$ denotes the i th eigenvalue of the Hermitian matrix $-M^*(j\omega)M(j\omega) + I$. Then $\lim_{\omega \rightarrow \pm\infty} \det[-M^*(j\omega)M(j\omega) + I] \neq 0$. We now define a “mixed” system similarly to [44].

Definition 13 A square transfer function matrix $M \in \mathcal{RH}_\infty$ is said to be “mixed” if for each frequency $\omega \in \mathbb{R} \cup \{\pm\infty\}$: either

- (i) $-kM^*(j\omega)M(j\omega) + M^*(j\omega) + M(j\omega) - lI \geq 0$; and/or
- (ii) $-M^*(j\omega)M(j\omega) + \varepsilon^2 I \geq 0$.

The constants $k, l > 0$ and $\varepsilon < 1$ are independent of ω .

2.3.1 Testing for mixedness

Before analysing the methods to test “mixedness” of descriptor systems, we review the methods to test “mixedness” of regular systems.

“Mixedness” of Regular MIMO Systems:

A test for determining whether MIMO, LTI systems of the form (2.28-2.29) have the property of “mixedness” was introduced in [44]. This test was presented as a means of ascertaining “mixedness”; however, it is a procedure that can be adapted to test for other properties as well (for example, by setting the parameter (ε), denoting the gain of a system over a certain frequency interval, to a value of greater than one).

Lemma 3 Suppose that $k, l \in \mathbb{R}$ and define

$$M_1(j\omega) = -kM^*(j\omega)M(j\omega) + M^*(j\omega) + M(j\omega) - lI \quad (2.35)$$

Let $Y := I - kD$ and suppose that $X_1 := -kD^T D + D^T + D - lI$ is invertible. For some $\omega \in [a, b]$, the matrix $H_1(j\omega)$ has a zero eigenvalue if and only if the matrix H_1 has an eigenvalue on the imaginary axis between and including $-ja$ and $-jb$, where

$$N_1 := \begin{pmatrix} -A + BX_1^{-1}Y^T C^T & -BX_1^{-1}B^T \\ kCC^T + CYX_1^{-1}Y^T C^T & A^T - CYX_1^{-1}B^T \end{pmatrix}.$$

Lemma 4 Suppose that $\varepsilon \in \mathbb{R} \setminus \{0\}$ and define

$$M_2(j\omega) = -M^*(j\omega)M(j\omega) + \varepsilon^2 I. \quad (2.36)$$

Suppose that $X_2 := -D^T D + \varepsilon^2 I$ is invertible. For some $\omega \in [a, b]$, the matrix $M_2(j\omega)$ has a zero eigenvalue if and only if the matrix M_2 has an eigenvalue on the imaginary axis between and including $-ja$ and $-jb$, where

$$N_2 := \begin{pmatrix} -A - BX_2^{-1} D^T C^T & -BX_2^{-1} B^T \\ kCC^T + CDX_2^{-1} D^T C^T & A^T + CDX_2^{-1} B^T \end{pmatrix}.$$

We also note the following:

- there exist $k, l > 0$ such that $-kM(j\omega)^* M(j\omega) + M(j\omega)^* + M(j\omega) - lI \geq 0$ for all $\omega \in [a, b]$ if and only if $M(j\omega)^* + M(j\omega) > 0$ for all $\omega \in [a, b]$;
- under the assumption that $\det(Mj\infty)^* + Mj\infty) \neq 0$, there exist $k, l > 0$ such that $-kM(j\omega)^* M(j\omega) + M(j\omega)^* + M(j\omega) - lI \geq 0$ for all $\omega \in (-\infty, b], [a, \infty)$ or $(-\infty, \infty)$ if and only if $M(j\omega)^* + M(j\omega) > 0$ for all $\omega \in (-\infty, b], [a, \infty)$ or $(-\infty, \infty)$, respectively;
- there exists $\varepsilon < 1$ such that $-M(j\omega)^* M(j\omega) + \varepsilon^2 I \geq 0$ for all $\omega \in [a, b]$ if and only if $-M(j\omega)^* M(j\omega) + I > 0$ for all $\omega \in [a, b]$;
- under the assumption that $\det(-Mj\infty)^* Mj\infty + I) \neq 0$, there exists $\varepsilon < 1$ such that $-M(j\omega)^* M(j\omega) + \varepsilon^2 I \geq 0$ for all $\omega \in (-\infty, b], [a, \infty)$ or $(-\infty, \infty)$ if and only if $-M(j\omega)^* M(j\omega) + I > 0$ for all $\omega \in (-\infty, b], [a, \infty)$ or $(-\infty, \infty)$, respectively.

In other words, there are cases in which the free parameters k , l and ε can be eliminated from the test; that is, we can set $k = l = 0$ and $\varepsilon = 1$ when applying Lemmas 3 and 4.

Now let $k = l = 0$, then $M_1(j\omega) = M(j\omega) + M^*(j\omega)$. Similarly, let $\varepsilon = 1$, then $M_2(j\omega) = -M^*(j\omega)M(j\omega) + I$. Now consider Lemmas 3 and 4 and set

$$\Omega_p := \{\omega \in [-\infty, \infty] : N_1 \text{ has an eigenvalue on the imaginary axis } j\omega\}$$

$$\Omega_s := \{\omega \in [-\infty, \infty] : N_2 \text{ has an eigenvalue on the imaginary axis } j\omega\}.$$

Suppose that we divide the real axis $-\infty$ to ∞ into smaller intervals, where any elements of Ω_p and Ω_s are set as open interval endpoints, as follows:

$$\text{Division group 1} := (-\infty, \omega_{p_1}), (\omega_{p_1}, \omega_{p_2}), \dots, (\omega_{p_{n-1}}, \omega_{p_n}), (\omega_{p_n}, \infty)$$

$$\text{Division group 2} := (-\infty, \omega_{s_1}), (\omega_{s_1}, \omega_{s_2}), \dots, (\omega_{s_{m-1}}, \omega_{s_m}), (\omega_{s_m}, \infty)$$

where n = number of elements in Ω_p ; m = number of elements in Ω_s ; $\omega_{p_1}, \omega_{p_2}, \dots, \omega_{p_n}$ denote the elements of Ω_p listed in increasing order; and $\omega_{s_1}, \omega_{s_2}, \dots, \omega_{s_m}$ denote the elements of Ω_s listed in increasing order.

Let I_{N_1} denote the set of ω belonging to the intervals over which $M_1(j\omega) > 0$ and I_{N_2} denote the set of ω belonging to the intervals over which $M_2(j\omega) > 0$. Then we have the following result.

Theorem 11 *The following two statements are equivalent:*

- (i) *a continuous-time system with transfer function matrix $M(s) \in \mathcal{RH}_\infty$ is a “mixed” system;*
- (ii) $I_{N_1} \cup I_{N_2} = \{\omega \in \mathbb{R} : -\infty \leq \omega \leq \infty\}$.

In this chapter, we develop easily verifiable, compact spectral conditions for checking PR (or SPR) property of SISO and MIMO descriptor systems. To obtain our results, we use only elementary concepts from linear algebra and existing results on strict positive realness for regular systems. This construction results in a test that involves only the evaluation of the eigenvalues of a matrix that is determined in an elementary manner from the matrices E, A, B, C, D ; while avoiding generalized eigenvalue calculation.

3.1 INTRODUCTION

In this chapter, we consider passivity properties of descriptor systems given by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t); \\ y(t) &= C^T x(t) + Du(t), \end{aligned} \quad (3.1)$$

where $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$ and $\text{rank}(E) = p < n$. Passivity of the descriptor system defined above is equivalent positive realness of

$$H(s) = D + C^T (sE - A)^{-1} B. \quad (3.2)$$

The traditional approach towards checking passivity of descriptor systems requires transformation of the original system into a special form using involved numerical linear algebraic techniques. In this chapter, we use the ideas of full rank decomposition and reciprocal systems to obtain a convenient realization for our analysis. This realization yields an immediate connection to *regular systems* and reveals a direct link to the Kalman-Yacubovich-Popov Lemma. Most importantly, our approach results in construction of passivity tests that only involve the evaluation of simple eigenvalues of a matrix, determined in an elementary manner from E, A, B, C, D . Also, our passivity tests for descriptor systems for all finite frequencies do not need *a priori* knowledge of descriptor system index.

In this chapter, we consider passivity of index-one and index-two descriptor systems separately for both SISO and MIMO cases. Finally, we summarize all results and show that our passivity tests are independent of the index.

3.2 PRELIMINARY RESULTS

Here, we present some important properties of LTI descriptor systems regarding passivity and some preliminary results.

3.2.1 Passivity of descriptor systems: definitions

Special nature of descriptor system transfer functions can be observed using the Weierstrass canonical form (Theorem 4) to partition the matrices C^T and B as

$$\begin{bmatrix} C_1^T & C_2^T \end{bmatrix} = C^T T \text{ and } \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = SB. \quad (3.3)$$

Then we can express the descriptor system transfer function as

$$\begin{aligned} H(s) &= D + C^T (sE - A)^{-1} B \\ &= D + \begin{bmatrix} C_1^T & C_2^T \end{bmatrix} \begin{bmatrix} (sI_p - J_p)^{-1} & 0 \\ 0 & -(I_{n-p} - sN)^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ &= D - C_2^T B_2 + C_1^T (sI_p - J_p)^{-1} B_1 - \sum_{i=1}^{\infty} s^i C_2^T N^i B_2 \end{aligned}$$

In the above expression, $H_p(s) = D - C_2^T B_2 + C_1^T (sI_p - J_p)^{-1} B_1$ is the proper part and $H_\infty(s) = -\sum_{i=1}^{\infty} s^i C_2^T N^i B_2$ is the improper part. This decoupling helps us to conclude that descriptor systems having index greater than two cannot be PR (passive). This is clear from Definition 6, according to which: poles of a passive transfer function matrix at infinity, if they exist are simple and their residue is positive semi-definite. Hence, positive realness of an index-two transfer function matrix $H(s)$ is equivalent to positive realness of the proper part $H_p(s)$ and positive semi-definiteness of $-C_2^T N B_2$ [16]. This condition is obvious from the fact that for all finite ω ,

$$\begin{aligned} H(j\omega) + H(j\omega)^* &= H_p(j\omega) - (C_2^T N B_2)(j\omega) + H_p(j\omega)^* - (C_2^T N B_2)(-j\omega) \\ &= H_p(j\omega) + H_p(j\omega)^* \end{aligned} \quad (3.4)$$

For the index-one case, the transfer function matrices are proper, hence regular definitions of passivity apply.

From the definitions of passivity, the strict passivity and strict positive realness, and the above discussion on passivity of descriptor systems, it can be deduced that the condition common to all the definitions involves checking for positive definiteness (or semi-definiteness) of $H(j\omega) + H(j\omega)^*$. Other side conditions involve (easily verifiable) point conditions that arise due to the behaviour of $H(s)$ at infinity. Since the aforementioned condition is common

to all definitions, and since this must be checked for all ω , in this thesis, we primarily focus on the condition

$$H(j\omega) + H(j\omega)^* > 0 \quad \text{for all finite } \omega \quad (3.5)$$

for index-one and index-two descriptor systems in both **SISO** and **MIMO** cases. We will also discuss the behaviour of $H(s)$ at $j\omega = j\infty$ for each case of study.

The alternative definitions requiring the existence of a positive ε such that $H(s - \varepsilon)$ is positive real will be considered for a special case of strictly proper descriptor systems. We call this property **KYP – SPR**, as it is a notion of strict positive realness that is consistent with the **KYP** Lemma for *regular systems*.

3.2.2 Order reduction of LTI descriptor systems

A direct transformation to the Weierstrass canonical form can be numerically unstable and expensive [17], hence alternative methods of simplifying the analysis of descriptor systems will be analysed in this thesis. The following results are useful in reducing the stability problem of a descriptor system to that of a lower-order system. Before stating our results, we assume that all descriptor systems considered in this chapter satisfy the conditions that $\det[sE - A] \neq 0$ and that all finite poles of $\det[sE - A]$ are in the Open Left Half of the Complex Plane (**OLHP**), respectively.

Our results are based on the full rank decomposition of singular matrices explained below:

A pair of matrices (X, Y) is a decomposition of $E \in \mathbb{R}^{n \times n}$ if

$$E = XY^T. \quad (3.6)$$

If, in addition, X and Y both have full column rank, we say that (X, Y) is a **full rank decomposition** of E .

Note that if (X, Y) is a full rank decomposition of $E \in \mathbb{R}^{n \times n}$ and $\text{rank}(E) = r$, then, $X, Y \in \mathbb{R}^{n \times r}$ and $\text{rank}(X) = \text{rank}(Y) = r$.

Lemma 5 (Order reduction) Consider a descriptor system described by (2.3) and suppose that (X, Y) is a decomposition of E , then $x(\cdot)$ is a solution to system (2.3) if and only if $z(\cdot)$ is a solution to the descriptor system

$$\tilde{E}\dot{z} = z, \quad (3.7)$$

where

$$\tilde{E} = Y^T A^{-1} X. \quad (3.8)$$

and

$$z(t) = Y^T x(t) \quad (3.9)$$

such that $\mathcal{C}(\tilde{E}, I) = Y^T \mathcal{C}(E, A) = Y^T \mathcal{C}$. Hence, global uniformly exponential stability of the new system (6.10) and the original system (2.3) are equivalent.

PROOF: Consider any solution $x(\cdot)$ of the original system (2.3) and let

$$z(t) = Y^T x(t) \quad (3.10)$$

and $XY^T \dot{x} = Ax$. Since A is invertible, we can multiply both sides of the last equation by $Y^T A^{-1}$ to obtain $Y^T A^{-1} XY^T \dot{x} = Y^T x$ that is,

$$\tilde{E} \dot{z} = z, \quad (3.11)$$

where $\tilde{E} = Y^T A^{-1} X$. We now claim that there is a matrix T such that $x(t) = Tz(t)$ where $x(t)$ is in the consistency space \mathcal{C} of (E, A) . Since \mathcal{C} and the kernel of E intersect only at zero and the kernel of Y^T is contained in the kernel of E , it follows that \mathcal{C} and the kernel of Y^T intersect only at zero. This implies that the restriction of Y^T to \mathcal{C} yields a one-to-one map from \mathcal{C} onto the subspace $Y^T \mathcal{C}$. Thus, this map has an inverse map T from $Y^T \mathcal{C}$ to \mathcal{C} ; hence

$$x(t) = Tz(t). \quad (3.12)$$

We now show that the consistency space of (\tilde{E}, I) is $Y^T \mathcal{C}$. Considering k sufficiently large, we have

$$\begin{aligned} \mathcal{C} &= \text{Im}((A^{-1}E)^k), \\ \mathcal{C}(\tilde{E}, I) &= \text{Im}(\tilde{E}^k) = \text{Im}(\tilde{E}^{k+1}). \end{aligned}$$

Now note that

$$Y^T (A^{-1}E)^k = Y^T (A^{-1}XY^T)^k = (Y^T A^{-1}X)^k Y^T = \tilde{E}^k Y^T. \quad (3.13)$$

Since $\tilde{E} = Y^T A^{-1} X$, we must have

$$\text{Im}(\tilde{E}^k) = \text{Im}(\tilde{E}^{k+1}) = \text{Im}(\tilde{E}^k Y^T A^{-1} X) \subset \text{Im}(\tilde{E}^k Y^T) \subset \text{Im}(\tilde{E}^k).$$

This implies that $\text{Im}(\tilde{E}^k Y^T) = \text{Im}(\tilde{E}^k) = \mathcal{C}(\tilde{E}, I)$. It now follows from (6.18) that $Y^T \mathcal{C} = Y^T \text{Im}((A^{-1}E)^k) = \text{Im}(\tilde{E}^k Y^T)$. This yields the desired result that $\mathcal{C}(\tilde{E}, I) = Y^T \mathcal{C}$.

Consider now any continuous solution $z(\cdot)$ of the new descriptor system (2.3), and let $x(t) = Tz(t)$. We will show that $x(\cdot)$ is a solution of the original system. Since $z(t)$ is in

$Y^T \mathcal{C}$ and $x(t) = Tz(t)$, and since T is the inverse of Y^T restricted to \mathcal{C} , we see that $x(t) \in \mathcal{C}$ and $z(t) = Y^T x(t)$. Also

$$x = Tz = T\tilde{E}\dot{z} = TY^T A^{-1}XY^T \dot{x} = TY^T A^{-1}E\dot{x}.$$

Since x is in \mathcal{C} , we have $A^{-1}E\dot{x}$ in \mathcal{C} ; recalling that TY^T is the identity operator on \mathcal{C} , we obtain that $TY^T A^{-1}E\dot{x} = A^{-1}E\dot{x}$. Thus $E\dot{x} = Ax$. **Q.E.D.**

Lemma 6 (Stability of the reduced order system) *The descriptor pair (E, A) is stable if and only if the non-zero eigenvalues of \tilde{E} have negative real parts.*

PROOF: Since $\det(A) \neq 0$, the pair (E, A) has no zero eigenvalues.

$$\begin{aligned} \text{Also, } \det[\lambda E - A] &= (-1)^n \det[A] \det[I - \lambda A^{-1}XY^T] \\ &= (-1)^n \det[A] \det[I - \lambda Y^T A^{-1}X] \\ &= (-1)^n \det[A] \det[I - \lambda \tilde{E}] \\ &= (-1)^{n-r} \det(A) \det[\lambda \tilde{E} - I]. \end{aligned}$$

This gives the result that the eigenvalues of (E, A) and (\tilde{E}, I) are the same. Hence the eigenvalues of (E, A) are the inverses of the nonzero eigenvalues of \tilde{E} . Thus, (E, A) is stable if and only if the non-zero eigenvalues of \tilde{E} have negative real parts. **Q.E.D.**

And for $\lambda \neq 0$,

$$\begin{aligned} \det[\lambda I - A^{-1}E] &= \det[\lambda I - A^{-1}XY^T] \\ &= \lambda^n \det[I - \lambda^{-1}A^{-1}XY^T] \\ &= \lambda^n \det[I - \lambda^{-1}Y^T A^{-1}X] \\ &= \lambda^n \det[I - \lambda^{-1}\tilde{E}] \\ &= \lambda^{n-r} \det[\lambda I - \tilde{E}]. \end{aligned}$$

By continuity, the above equation also holds for $\lambda = 0$. Thus the eigenvalues of $A^{-1}E$ consist of $n - r$ zeros and the eigenvalues of \tilde{E} .

Lemma 7 (Index of the reduced order system) *The index of the equivalent reduced order system (\tilde{E}, I) is $k^* - 1$, where k^* is the index of (E, A) .*

PROOF: Recall that the index- k^* of (E, A) is the smallest integer k for which

$$\text{Im}((A^{-1}E)^{k+1}) = \text{Im}((A^{-1}E)^k). \quad (3.14)$$

Since E is singular, $k^* \geq 1$. Note that for any $k \geq 1$,

$$(A^{-1}E)^k = (A^{-1}XY^T)^k = A^{-1}X(Y^T A^{-1}X)^{k-1}Y^T = A^{-1}X\tilde{E}^{k-1}Y^T.$$

Since Y^T has full row rank, it now follows that

$$\text{Im}((A^{-1}E)^k) = A^{-1}X \text{Im}(\tilde{E}^{k-1}) \quad (3.15)$$

Since $A^{-1}X$ is full column rank, it follows that equation (3.14) is equivalent to

$$\text{Im}(\tilde{E}^k) = \text{Im}(\tilde{E}^{k-1});$$

hence the index of (\tilde{E}, I) is $k^* - 1$. **Q.E.D.**

The above lemmas are useful in reducing a descriptor system to an equivalent lower-order system with lower index. If $r < n$, then the new system (6.10) is of order r (number of state variables) and index- $(k^* - 1)$. One can iteratively apply Lemma 5 to achieve further order reduction, provided that there is a decomposition (\tilde{X}, \tilde{Y}) of \tilde{E} with $\tilde{X}, \tilde{Y} \in \mathbb{R}^{r \times \tilde{r}}$ with $\tilde{r} < r$. Since a square matrix always has a full rank decomposition, one can always iteratively reduce a single linear system (E, A) to a *regular system*. The above lemmas also yield the following Corollary.

Corollary 1 *Assume that (E, A) is a stable index-one descriptor system, and let (X, Y) be a full rank decomposition of E . Then, the matrix $Y^T A^{-1} X$ is Hurwitz stable (and consequently invertible).*

Comment 1 If E is singular, we make the following claim, where the nullity of E is the dimension of the kernel of E and equals $n - r$ with $r = \text{rank}(E)$.

A system is index-one if and only if the number of zero eigenvalues of $A^{-1}E$ equals the nullity of E .

To see this, note that the number of zero of eigenvalues of $A^{-1}E$ is the algebraic multiplicity of zero as an eigenvalue of $A^{-1}E$, whereas the nullity of E (which equals the nullity of $A^{-1}E$) is the geometric multiplicity of zero as an eigenvalue of $A^{-1}E$. The algebraic and geometric multiplicities are equal if and only if $A^{-1}E$ and $(A^{-1}E)^2$ have the same nullity; this is equivalent to $\text{Im}((A^{-1}E)^2) = \text{Im}(A^{-1}E)$, that is, the system is index one.

In this thesis, we also frequently use the *Matrix Inversion Lemma* as stated below.

Lemma 8 *For matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{n \times m}$ and $D \in \mathbb{R}^{m \times m}$ we have*

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}.$$

In the following, these results will be applied to index-one and index-two descriptor systems separately.

3.3 INDEX-ONE DESCRIPTOR SYSTEMS

The following result will allow us to deduce many properties of descriptor systems in an elementary manner. The evolution follows ideas introduced in [105].

Theorem 12 *Consider the following index-one descriptor transfer function:*

$$H(s) = D + C^T (sE - A)^{-1} B. \quad (3.16)$$

Let $E = XY^T$ be a full rank decomposition of the singular matrix E . Then $H(s)$ can be written as

$$H(s) = \tilde{D} + \tilde{C}^T (sI - \tilde{A})^{-1} \tilde{B} \quad (3.17)$$

with

$$\tilde{A} = (Y^T A^{-1} X)^{-1} \quad (3.18)$$

$$\tilde{B} = (Y^T A^{-1} X)^{-1} Y^T A^{-1} B \quad (3.19)$$

$$\tilde{C}^T = C^T A^{-1} X (Y^T A^{-1} X)^{-1} \quad (3.20)$$

$$\tilde{D} = D - C^T A^{-1} B + C^T A^{-1} X (Y^T A^{-1} X)^{-1} Y^T A^{-1} B. \quad (3.21)$$

PROOF : The proof follows by applying the matrix inversion lemma to (3.17) twice. We have.

$$\begin{aligned} H(s) &= D + C^T (XY^T s - A)^{-1} B \\ &= D - C^T \left(A^{-1} + A^{-1} X \left(\frac{1}{s} I - Y^T A^{-1} X \right)^{-1} Y^T A^{-1} \right) B \\ &= D - C^T A^{-1} B - C^T A^{-1} X \left(\frac{1}{s} I - Y^T A^{-1} X \right)^{-1} Y^T A^{-1} B \end{aligned}$$

Recall that the matrix $Y^T A^{-1} X$ is Hurwitz (and invertible) if the descriptor system is index-one. Now apply the matrix inversion lemma again. Thus $H(s) = D - C^T A^{-1} B + C^T A^{-1} X \left((Y^T A^{-1} X)^{-1} + (Y^T A^{-1} X)^{-1} (sI - Y^T A^{-1} X)^{-1} (Y^T A^{-1} X)^{-1} \right) Y^T A^{-1} B$. By gathering terms together we have that

$$H(s) = \tilde{D} + \tilde{C}^T (sI - \tilde{A})^{-1} \tilde{B}. \quad (3.22)$$

Q.E.D.

Comment 2 *Writing $H(s)$ as a regular transfer function allows spectral methods to be used to check for passivity. These methods involve building pairs of matrices from $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ for MIMO systems and $\tilde{A}, \tilde{b}, \tilde{c}, \tilde{d}$ for SISO systems, and checking eigenvalues of their products; see [106] [107] [108].*

Recall, we are interested in checking if

$$H(j\omega) + H(j\omega)^* > 0 \quad (3.23)$$

for all real and finite ω . In the case of **SISO** systems, this typically reduces to checking the eigenvalues of the matrix product

$$A^{-1}(A^{-1} + A^{-1}b(d - c^T A^{-1}b)^{-1}c^T A^{-1}) \quad (3.24)$$

[107], where $b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. In the case of a **MIMO** system, one obtains a similar condition on the Hamiltonian matrix [108]

$$\begin{bmatrix} -(A^{-1} + A^{-1}BQ^{-1}C^T A^{-1}) & A^{-1}BQ^{-1}B^T A^{-T} \\ -A^{-T}CQ^{-1}C^T A^{-1} & (A^{-1} + A^{-1}BQ^{-1}C^T A^{-1})^T \end{bmatrix}$$

with $Q = D - C^T A^{-1}B + D^T - B^T A^{-T}C$, $B, C \in \mathbb{R}^{n \times m}$ and $D \in \mathbb{R}^{m \times m}$. Applying these results to descriptor systems may appear to be problematic since the matrices appearing in $H(s)$ involve the full rank decomposition $E = XY^T$, which is not unique. We now show that the spectral conditions can be written in a manner that is independent of X and Y . We take care of these two cases in the following lemmas.

Lemma 9 *Define the matrices*

$$M^{-1} = A^{-1} + A^{-1}b(d - c^T A^{-1}b)^{-1}c^T A^{-1}, \quad (3.25)$$

$$\tilde{M}^{-1} = \tilde{A}^{-1} + \tilde{A}^{-1}\tilde{b}(\tilde{d} - \tilde{c}^T \tilde{A}^{-1}\tilde{b})^{-1}\tilde{c}^T \tilde{A}^{-1}. \quad (3.26)$$

Then all non-zero eigenvalues of the reduced order matrix product $\tilde{A}^{-1}\tilde{M}^{-1}$ coincide with the non-zero eigenvalues of $EA^{-1}EM^{-1}$.

PROOF : Observing that $\tilde{d} - \tilde{c}^T \tilde{A}^{-1}\tilde{b} = d - c^T A^{-1}b$, we have

$$\begin{aligned} \tilde{M}^{-1} &= \tilde{A}^{-1} + \tilde{A}^{-1}\tilde{b}(\tilde{d} - \tilde{c}^T \tilde{A}^{-1}\tilde{b})^{-1}\tilde{c}^T \tilde{A}^{-1} \\ &= (Y^T A^{-1}X) + Y^T A^{-1}b(d - c^T A^{-1}b)^{-1}c^T A^{-1}X \\ &= Y^T M^{-1}X. \end{aligned}$$

Hence, $\tilde{A}^{-1}\tilde{M}^{-1} = Y^T A^{-1}XY^T M^{-1}X$. Non-zero eigenvalues of any two matrix products RS^T and $S^T R$ coincide for any two matrices R and S of compatible dimensions [63]. Hence the non-zero eigenvalues of $Y^T A^{-1}XY^T M^{-1}X$ coincide with the non-zero eigenvalues of $XY^T A^{-1}XY^T M^{-1}$. Since $E = XY^T$, we have $XY^T A^{-1}XY^T M^{-1} = EA^{-1}EM^{-1}$. Also note that if $\tilde{A}^{-1}\tilde{M}^{-1}$ has η zero eigenvalues, then $EA^{-1}EM^{-1}$ has $n - r + \eta$ zero eigenvalues, where $r = \text{rank}(E)$. **Q.E.D.**

Lemma 10 Define the matrices

$$N = \begin{bmatrix} -(A^{-1} + A^{-1}BQ^{-1}C^T A^{-1}) & A^{-1}BQ^{-1}B^T A^{-T} \\ -A^{-T}CQ^{-1}C^T A^{-1} & (A^{-1} + A^{-1}BQ^{-1}C^T A^{-1})^T \end{bmatrix} \quad (3.27)$$

with $Q = D - C^T A^{-1} B + D^T - B^T A^{-T} C$,

$$\tilde{N} = \begin{bmatrix} -(\tilde{A}^{-1} + \tilde{A}^{-1}\tilde{B}\tilde{Q}^{-1}\tilde{C}^T \tilde{A}^{-1}) & \tilde{A}^{-1}\tilde{B}\tilde{Q}^{-1}\tilde{B}^T \tilde{A}^{-T} \\ -\tilde{A}^{-T}\tilde{C}\tilde{Q}^{-1}\tilde{C}^T \tilde{A}^{-1} & (\tilde{A}^{-1} + \tilde{A}^{-1}\tilde{B}\tilde{Q}^{-1}\tilde{C}^T \tilde{A}^{-1})^T \end{bmatrix} \quad (3.28)$$

with $\tilde{Q} = \tilde{D} - \tilde{C}^T \tilde{A}^{-1} \tilde{B} + \tilde{D}^T - \tilde{B}^T \tilde{A}^{-T} \tilde{C}$. Then all non-zero eigenvalues of the reduced-order Hamiltonian matrix \tilde{N} coincide with the non-zero eigenvalues of the matrix $\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} N$.

PROOF : Observing that $\tilde{Q} = Q$, we have

$$\begin{aligned} \tilde{N} &= \begin{bmatrix} -(\tilde{A}^{-1} + \tilde{A}^{-1}\tilde{B}\tilde{Q}^{-1}\tilde{C}^T \tilde{A}^{-1}) & \tilde{A}^{-1}\tilde{B}\tilde{Q}^{-1}\tilde{B}^T \tilde{A}^{-T} \\ -\tilde{A}^{-T}\tilde{C}\tilde{Q}^{-1}\tilde{C}^T \tilde{A}^{-1} & (\tilde{A}^{-1} + \tilde{A}^{-1}\tilde{B}\tilde{Q}^{-1}\tilde{C}^T \tilde{A}^{-1})^T \end{bmatrix} \\ &= \begin{bmatrix} -Y^T(A^{-1} + A^{-1}BQ^{-1}C^T A^{-1})X & Y^T(A^{-1}BQ^{-1}B^T A^{-T})Y \\ X^T(-A^{-T}CQ^{-1}C^T A^{-1})X & X^T(A^{-1} + A^{-1}BQ^{-1}C^T A^{-1})^T Y \end{bmatrix} \\ &= \begin{bmatrix} Y^T & 0 \\ 0 & X^T \end{bmatrix} \\ &\quad \times \begin{bmatrix} -(A^{-1} + A^{-1}BQ^{-1}C^T A^{-1}) & A^{-1}BQ^{-1}B^T A^{-T} \\ -A^{-T}CQ^{-1}C^T A^{-1} & (A^{-1} + A^{-1}BQ^{-1}C^T A^{-1})^T \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \\ &= \begin{bmatrix} Y^T & 0 \\ 0 & X^T \end{bmatrix} N \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \end{aligned}$$

Hence, the non-zero eigenvalues of \tilde{N} coincide with the non-zero eigenvalues of

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} Y^T & 0 \\ 0 & X^T \end{bmatrix} N = \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} N.$$

Also note that if \tilde{N} has η zero eigenvalues, then $\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} N$ has $2n - 2r + \eta$ zero eigenvalues, where $r = \text{rank}(E)$. **Q.E.D.**

Based on the observations of the previous section, we now give compact characterisations to check whether $H(j\omega) + H(j\omega)^* > 0$ for all finite ω . We begin with SISO systems and then proceed to study MIMO systems.

3.3.1 SISO index-one descriptor systems

Conditions for SISO systems can be obtained by recalling the following easily deduced fact [107].

Consider the SISO transfer function $H(j\omega) = d + c^T(sI - A)^{-1}b$ with A a Hurwitz matrix. Then $H(j\omega) + H(j\omega)^* > 0$ for all finite ω if and only if $d - c^T A^{-1}b > 0$ and the matrix product $A^{-1}M^{-1} = A^{-1}\left(A^{-1} + A^{-1}b(d - c^T A^{-1}b)^{-1}c^T A^{-1}\right)$ has no negative eigenvalues. This fact can be deduced from the expression

$$\begin{aligned} \Re[H(j\omega)] &= (d - c^T A^{-1}b) \cdot \det \left[\frac{1}{\omega^2} + A^{-2} \right]^{-1} \\ &\quad \times \det \left[\frac{1}{\omega^2} + A^{-1} \left(A^{-1} + A^{-1}b(d - c^T A^{-1}b)^{-1}c^T A^{-1} \right) \right]. \end{aligned}$$

This observation leads to the following result for scalar systems.

Theorem 13 Consider the scalar stable index-one descriptor transfer function with $d - c^T A^{-1}b > 0$,

$$H(s) = d + c^T(sE - A)^{-1}b.$$

$H(j\omega) + H(j\omega)^* > 0$ for all finite ω if and only if $EA^{-1}EM^{-1}$ has no negative eigenvalues.

PROOF : From Corollary 1 and Theorem 12, we can observe that $\tilde{A} = (Y^T A^{-1}X)^{-1}$ is Hurwitz. Also, observe that $\tilde{d} - \tilde{c}^T \tilde{A}^{-1} \tilde{b} = d - c^T A^{-1}b$. Now, we can apply Lemma 9 to obtain the desired result. **Q.E.D.**

SISO index-one transfer functions at infinity: The value of transfer function $H(s)$ at $s = j\infty$ is given by

$$\begin{aligned} H(j\infty) &= d - c^T A^{-1}b + c^T A^{-1}X(Y^T A^{-1}X)^{-1}Y^T A^{-1}b \\ &= (d - c^T A^{-1}b) \left[1 + (d - c^T A^{-1}b)^{-1}c^T A^{-1}X(Y^T A^{-1}X)^{-1}Y^T A^{-1}b \right] \\ &= (d - c^T A^{-1}b) \cdot \det \left[1 + Y^T A^{-1}b(d - c^T A^{-1}b)^{-1}c^T A^{-1}X(Y^T A^{-1}X)^{-1} \right] \\ &= (d - c^T A^{-1}b) \cdot \det \left[Y^T A^{-1}X + Y^T A^{-1}b(d - c^T A^{-1}b)^{-1}c^T A^{-1}X \right] \\ &\quad \times \det \left[(Y^T A^{-1}X)^{-1} \right]. \end{aligned}$$

Case 1: When $H(j\infty)$ is positive.

$H(j\infty)$ is positive if and only if $d - c^T A^{-1} b > 0$ and

$$\det [Y^T A^{-1} X + Y^T A^{-1} b (d - c^T A^{-1} b)^{-1} c^T A^{-1} X] \cdot \det [Y^T A^{-1} X] > 0.$$

This is possible if and only if

$$Y^T A^{-1} X + Y^T A^{-1} b (d - c^T A^{-1} b)^{-1} c^T A^{-1} X Y^T A^{-1} X$$

has no negative real eigenvalues and no zero eigenvalues. This is further equivalent to the condition that

$$X (Y^T A^{-1} + Y^T A^{-1} b (d - c^T A^{-1} b)^{-1} c^T A^{-1} X Y^T A^{-1}) = E M^{-1} E A^{-1}$$

has no negative real eigenvalues and at most $n - r$ zero eigenvalues. This leads to an improvement of Theorem 13, and we can state the following.

Theorem 14 Consider the scalar stable index-one descriptor transfer function with $d - c^T A^{-1} b > 0$,

$$H(s) = d + c^T (sE - A)^{-1} b. \quad (3.29)$$

Then $H(j\omega) + H(j\omega)^* > 0$ for all ω if and only if $EA^{-1}EM^{-1}$ has no negative eigenvalues and at most $n - r$ zero eigenvalues.

Case 2: When $H(j\infty)$ is zero.

When $H(j\infty) = 0$, the descriptor system transfer function is strictly proper, and hence we obtain conditions for strict positive realness of $H(s)$. For strict positive realness, we use the definition that there exists an $\varepsilon > 0$ such that $H(s - \varepsilon)$ is positive real. In this thesis, we term this definition **KYP-SPR**.

Recall again the conditions for *regular SISO* systems to be strictly positive real [107].

Consider the SISO transfer function $H(j\omega) = c^T (sI - A)^{-1} b$ with A a Hurwitz matrix. Then $H(s)$ is **KYP-SPR** if and only if $-c^T A^{-1} b > 0$ and the matrix product $A^{-1} (A^{-1} - A^{-1} b (c^T A^{-1} b)^{-1} c^T A^{-1})$ has no negative eigenvalues and at most one zero eigenvalue.

Theorem 15 Consider the scalar stable index-one descriptor transfer function with $d - c^T A^{-1} b > 0$,

$$H(s) = d + c^T (sE - A)^{-1} b. \quad (3.30)$$

Then $H(s)$ is **KYP-SPR** if and only if $EA^{-1}EM^{-1}$ has no negative eigenvalues and at most $n - r + 1$ zero eigenvalues.

PROOF: From Theorem 12,

$$\begin{aligned}
H(0) &= \tilde{d} - \tilde{c}^T \tilde{A}^{-1} \tilde{b} = d - c^T A^{-1} b > 0 \quad \text{and} \\
H(j\infty) &= \tilde{d} = 0 \\
&\Rightarrow d - c^T A^{-1} b + c^T A^{-1} X (Y^T A^{-1} X)^{-1} Y^T A^{-1} b = 0 \\
&\Rightarrow c^T A^{-1} X (Y^T A^{-1} X)^{-1} Y^T A^{-1} b = -(d - c^T A^{-1} b)
\end{aligned}$$

Also observe that $\tilde{c}^T \tilde{A}^{-1} \tilde{b} = c^T A^{-1} X (Y^T A^{-1} X)^{-1} Y^T A^{-1} b = -(d - c^T A^{-1} b)$, and we can apply Lemma 9 to obtain the desired result. **Q.E.D.**

3.3.2 MIMO index-one descriptor systems

MIMO systems can be dealt with by recalling the following observation from [106].

Consider the MIMO transfer function $H(s) = D + C^T (sI - A)^{-1} B$ with A a Hurwitz matrix, and $Q = D - C^T A^{-1} B$. Then $H(j\omega) + H(j\omega)^* > 0$ for all finite ω if and only if $Q + Q^T > 0$ and the matrix N in 2.33 has no eigenvalues on the imaginary axis except at the origin.

This observation leads to the following result for scalar systems.

Theorem 16 Consider the MIMO stable index-one descriptor transfer function

$$H(s) = D + C^T (sE - A)^{-1} B$$

with $Q = D - C^T A^{-1} B$ and $Q + Q^T > 0$. Then $H(j\omega) + H(j\omega)^* > 0$ for all finite ω if and only if $\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} N$ has no eigenvalues on the imaginary axis except at the origin.

PROOF : The proof follows directly from Lemma 10 and the fact that $Y^T A^{-1} X$ is a Hurwitz matrix. **Q.E.D.**

MIMO index-one transfer functions at infinity: From Theorem 12, the behaviour of $H(j\omega) + H(j\omega)^*$ at $j\omega = j\infty$ can be understood from

$$\begin{aligned}
\det[H(j\omega) + H(j\omega)^*] &= \det[\tilde{Q}] \cdot \det\left[\frac{1}{j\omega} I - \tilde{A}\right] \cdot \det\left[\frac{1}{j\omega} I + \tilde{A}^T\right] \\
&\quad \times \det\left[\frac{1}{j\omega} I + \tilde{N}\right], \tag{3.31}
\end{aligned}$$

where $\tilde{Q} = \tilde{D} - \tilde{C}^T \tilde{A}^{-1} \tilde{B} + \tilde{D}^T - \tilde{B}^T \tilde{A}^{-T} \tilde{C} > 0$.

Case 1: $H(j\infty) + H(j\infty)^*$ is positive definite.

Assume \tilde{N} has no imaginary eigenvalues, then by continuity $H(j\infty) + H(j\infty)^* > 0$ if and only if $\det[H(j\infty) + H(j\infty)^*] \neq 0$. Since $\tilde{A} = (Y^T A^{-1} X)^{-1}$ is Hurwitz,

$$\det[H(j\infty) + H(j\infty)^*] \neq 0 \Leftrightarrow \det[\tilde{N}] \neq 0.$$

Thus \tilde{N} has no zero eigenvalues, then $\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} N$ has at most $2n - 2r$ zero eigenvalues. This leads to an improvement of Theorem 16, and we can state that

Theorem 17 Consider the MIMO stable index-one descriptor transfer function

$$H(s) = D + C^T (sE - A)^{-1} B \quad (3.32)$$

with $Q = D - C^T A^{-1} B$ and $Q + Q^T > 0$. $H(j\omega) + H(j\omega)^* > 0$ for all ω if and only if $\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} N$ has no eigenvalues on the imaginary axis and at most $2n - 2r$ zero eigenvalues.

Case 2: $H(j\infty) + H(j\infty)^*$ is not positive definite.

When $H(j\infty) + H(j\infty)^* = \tilde{D} + \tilde{D}^T$ is not positive definite, we can obtain conditions for strict positive realness of $H(s)$. Strict positive realness of MIMO transfer functions is based on the additional side condition

$$\lim_{|\omega| \rightarrow \infty} \omega^{2\rho} \det[H(j\omega) + H(j\omega)^*] \neq 0. \quad (3.33)$$

where ρ is the nullity of $\tilde{D} + \tilde{D}^T$. In either case, the above limit is positive. Consider equation (3.31) and observe that $\tilde{Q} > 0$ and $\det \left[\frac{1}{j\omega} I - \tilde{A} \right] \cdot \det \left[\frac{1}{j\omega} I + \tilde{A}^T \right] \neq 0$ for very large ω . Hence the side condition is equivalent to

$$\lim_{|\omega| \rightarrow \infty} \omega^{2\rho} \det \left[\frac{1}{j\omega} I + \tilde{N} \right] = \lim_{|\omega| \rightarrow \infty} (j\omega)^\rho \det \left[\frac{1}{j\omega} I + \tilde{N} \right] \neq 0.$$

This condition can be modified as

$$\lim_{|\lambda| \rightarrow 0} \frac{1}{(j\lambda)^\rho} \det [j\lambda I + \tilde{N}] \neq 0. \quad (3.34)$$

Condition (3.34) is equivalent to the condition that \tilde{N} has no imaginary eigenvalues and ρ zero eigenvalues. Thus $H(s)$ is KYP-SPR if and only if $\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} N$ has no eigenvalues on the imaginary axis and at most $2n - 2r + \rho$ zero eigenvalues.

3.4 INDEX-TWO DESCRIPTOR SYSTEMS AND HIGHER

In this section, we consider index-two descriptor system transfer functions. Note that higher-index descriptor systems are not of interest to us since they cannot be strictly positive real [16]. If (E, A) is index-two descriptor system pair and (X, Y) is a full rank decomposition of the singular matrix E , then, from Lemma 7, $Y^T A^{-1} X$ is a singular matrix and $(Y^T A^{-1} X, I)$ is an index-one pair. Now we assume that (M, N) is a full rank decomposition of $Y^T A^{-1} X$, thus $(N^T M, I)$ is an index-0 descriptor system pair. Again, we focus on the condition that

$$H(j\omega) + H(j\omega)^* > 0$$

for all finite and real ω . The behaviour of index-two transfer functions at $\omega = \infty$ can be understood from the sign of the residues calculated for poles at infinity. Several methods have been proposed in the literature to calculate the residues and check for their signs [17], [16]. Hence we do not focus on these conditions.

Lemma 11 *Given an index-two descriptor system transfer function $H(s) = D + C^T (sE - A)^{-1} B$, it can be written as*

$$\begin{aligned} H(s) = D - C^T A^{-1} B & - s(C^T A^{-1} E A^{-1} B) \\ & - s(C^T A^{-1} X X_1) \left(\frac{1}{s} I - Y_1^T X_1 \right)^{-1} Y_1^T Y^T A^{-1} B, \end{aligned} \quad (3.35)$$

and the proper part of $H(s)$ can be expressed as

$$H_p(s) = D_p + C_p^T \left(\frac{1}{s} I - A_p \right)^{-1} B_p \quad (3.36)$$

such that $E = X Y^T$ and $Y^T A^{-1} X = X_1 Y_1^T$ are full rank decompositions of E and $Y^T A^{-1} X$, respectively, and

$$\begin{aligned} A_p &= Y_1^T X_1, & B_p &= (Y_1^T X_1)^{-1} Y_1^T Y^T A^{-1} B, \\ C_p^T &= -C^T A^{-1} X X_1, & D_p &= D - C^T A^{-1} B. \end{aligned}$$

PROOF : The proof follows by dual application of the matrix inversion lemma to the transfer function $H(s) = D + C^T (sE - A)^{-1} B$. From Theorem 12 we have

$$H(s) = D - C^T A^{-1} B - C^T A^{-1} X \left(\frac{1}{s} I - Y^T A^{-1} X \right)^{-1} Y^T A^{-1} B$$

Since the matrix $Y^T A^{-1} X$ is singular and $Y^T A^{-1} X = X_1 Y_1^T$, we obtain

$$H(s) = D - C^T A^{-1} B - s(C^T A^{-1} X) (I - s X_1 Y_1^T)^{-1} Y^T A^{-1} B.$$

Now, we apply the matrix inversion lemma again and use $E = XY^T$. Thus,

$$\begin{aligned} H(s) &= D - C^T A^{-1} B - s(C^T A^{-1} E A^{-1} B) \\ &\quad - s(C^T A^{-1} X X_1) \left(\frac{1}{s} I - Y_1^T X_1 \right)^{-1} Y_1^T Y^T A^{-1} B. \end{aligned}$$

From equation (3.35), we can observe that the transfer function of an index-two descriptor system has a simple pole at $s = \infty$. From equation (3.35), the residue of the pole at infinity can be easily calculated as

$$\begin{aligned} K_\infty &= \lim_{j\omega \rightarrow \infty} \frac{1}{j\omega} H(j\omega) \\ &= -C^T A^{-1} E A^{-1} B + C^T A^{-1} X X_1 (Y_1^T X_1)^{-1} Y_1^T Y^T A^{-1} B. \end{aligned}$$

Hence, the transfer function $H(s)$ of an index-two descriptor system can be decoupled into a proper transfer function $H_p(s)$ and the improper part sK_∞ given by

$$H(s) = H_p(s) + sK_\infty.$$

Then $H_p(s)$ can be calculated as shown below

$$\begin{aligned} H_p(s) &= H(s) - sK_\infty \\ &= D - C^T A^{-1} B - s(C^T A^{-1} X X_1) \left(\frac{1}{s} I - Y_1^T X_1 \right)^{-1} Y_1^T Y^T A^{-1} B \\ &\quad - s(C^T A^{-1} X X_1) (Y_1^T X_1)^{-1} Y_1^T Y^T A^{-1} B \\ &= D - C^T A^{-1} B \\ &\quad - s(C^T A^{-1} X X_1) \left(\left(\frac{1}{s} I - Y_1^T X_1 \right)^{-1} + (Y_1^T X_1)^{-1} \right) Y_1^T Y^T A^{-1} B \\ &= D - C^T A^{-1} B \\ &\quad - s(C^T A^{-1} X X_1) \left(\frac{1}{s} I - Y_1^T X_1 \right)^{-1} \left(\frac{1}{s} I \right) (Y_1^T X_1)^{-1} Y_1^T Y^T A^{-1} B \\ &= D - C^T A^{-1} B - (C^T A^{-1} X X_1) \left(\frac{1}{s} I - Y_1^T X_1 \right)^{-1} (Y_1^T X_1)^{-1} Y_1^T Y^T A^{-1} B. \end{aligned}$$

Thus the proper part of an index-two descriptor system can be expressed as

$$H_p(s) = D_p + C_p^T \left(\frac{1}{s} I - A_p \right)^{-1} B_p,$$

where

$$\begin{aligned} A_p &= Y_1^T X_1, & B_p &= (Y_1^T X_1)^{-1} Y_1^T Y^T A^{-1} B, \\ C_p^T &= -C^T A^{-1} X X_1, & D_p &= D - C^T A^{-1} B. \quad \mathbf{Q.E.D.} \end{aligned}$$

The following Lemmas illustrate spectral equivalences similar to earlier sections.

Lemma 12 Define the matrices

$$M = (A - bd^{-1}c^T), \quad (3.37)$$

$$M_p = (A_p - b_p d_p^{-1} c_p^T). \quad (3.38)$$

Then all non-zero eigenvalues of the reduced order matrix product $A_p M_p$ coincide with the non-zero eigenvalues of $EM^{-1}EA^{-1}$.

Proof : Consider

$$\begin{aligned} A_p M_p &= A_p (A_p - b_p d_p^{-1} c_p^T) \\ &= Y_1^T X_1 (Y_1^T X_1 + (Y_1^T X_1)^{-1} Y_1^T Y^T A^{-1} b (d - c^T A^{-1} b)^{-1} c^T A^{-1} X X_1) \\ &= ((Y_1^T X_1)^2 + Y_1^T Y^T A^{-1} b (d - c^T A^{-1} b)^{-1} c^T A^{-1} X X_1) \\ &= Y_1^T (X_1 Y_1^T + Y^T A^{-1} b (d - c^T A^{-1} b)^{-1} c^T A^{-1} X) X_1. \end{aligned}$$

Non-zero eigenvalues of any two matrix products RS^T and $S^T R$ coincide for any two matrices R and S of compatible dimensions [63]. Hence the non-zero eigenvalues of $A_p M_p$ coincide with the non-zero eigenvalues of

$$\begin{aligned} &(X_1 Y_1^T + Y^T A^{-1} b (d - c^T A^{-1} b)^{-1} c^T A^{-1} X) X_1 Y_1^T \\ &= (Y^T A^{-1} X + Y^T A^{-1} b (d - c^T A^{-1} b)^{-1} c^T A^{-1} X) Y^T A^{-1} X. \end{aligned}$$

Further eigenvalues of M_p coincide with the non-zero eigenvalues of

$$\begin{aligned} &XY^T (A^{-1} + A^{-1} b (d - c^T A^{-1} b)^{-1} c^T A^{-1}) XY^T A^{-1} \\ &= E (A^{-1} + A^{-1} b (d - c^T A^{-1} b)^{-1} c^T A^{-1}) EA^{-1} \\ &= E (A - bd^{-1}c^T)^{-1} EA^{-1}. \end{aligned}$$

It can be observed that if $A_p M_p$ has η number of zero eigenvalues, then $EM^{-1}EA^{-1}$ has $(n - r) + (r - q) + \eta = n - q + \eta$ number of zero eigenvalues, where n is dimension of matrix E , r is the rank of matrix E and q is the number of finite eigenvalues of (E, A) . **Q.E.D.**

Lemma 13 Define the matrices

$$\bar{N} = \begin{bmatrix} -(A^{-1} + A^{-1} B Q^{-1} C^T A^{-1}) & -A^{-1} B Q^{-1} B^T A^{-T} \\ A^{-T} C Q^{-1} C^T A^{-1} & (A^{-1} + A^{-1} B Q^{-1} C A^{-1})^T \end{bmatrix}$$

with $Q = D - C^T A^{-1} B + D^T - B^T A^{-T} C$,

$$N_p = \begin{bmatrix} -A_p + B_p Q_p^{-1} C_p^T & -B_p Q_p^{-1} B_p^T \\ C_p Q_p^{-1} C_p^T & (A_p - B_p Q_p^{-1} C_p^T)^T \end{bmatrix}$$

with $Q_p = D_p + D_p^T$. Then all non-zero eigenvalues of the reduced-order Hamiltonian matrix

N_p coincide with the non-zero eigenvalues of the matrix $\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \bar{N}$.

Proof : As $Q = Q_p$, we have

$$\begin{aligned}
N_p &= \begin{bmatrix} -A_p + B_p Q^{-1} C_p^T & -B_p Q^{-1} B_p^T \\ C_p Q^{-1} C_p^T & (A_p - B_p Q^{-1} C_p^T)^T \end{bmatrix} = \\
&\begin{bmatrix} -Y_1^T X_1 - (Y_1^T X_1)^{-1} Y_1^T Y^T A^{-1} B Q^{-1} C^T A^{-1} X X_1 & -(Y_1^T X_1)^{-1} Y_1^T Y^T A^{-1} B Q^{-1} B^T A^{-T} Y Y_1 (Y_1^T X_1)^{-T} \\ X_1^T X^T A^{-T} C Q^{-1} C^T A^{-1} X X_1 & (Y_1^T X_1)^T + X_1^T X^T A^{-T} C Q^{-1} B^T A^{-T} Y Y_1 (Y_1^T X_1)^{-T} \end{bmatrix} \\
&= \begin{bmatrix} (Y_1^T X_1)^{-1} Y_1^T & 0 \\ 0 & X_1^T \end{bmatrix} \times \\
&\begin{bmatrix} -Y^T A^{-1} X - Y^T A^{-1} B Q^{-1} C^T A^{-1} X & -Y^T A^{-1} B Q^{-1} B^T A^{-T} Y \\ X^T A^{-T} C Q^{-1} C^T A^{-1} X & (Y^T A^{-1} X + Y^T A^{-1} B Q^{-1} C^T A^{-1} X)^T \end{bmatrix} \\
&\times \begin{bmatrix} X_1 & 0 \\ 0 & Y_1 (Y_1^T X_1)^{-T} \end{bmatrix}.
\end{aligned}$$

The non-zero eigenvalues of N_p coincide with the non-zero eigenvalues of

$$\begin{aligned}
&\begin{bmatrix} -Y^T A^{-1} X - Y^T A^{-1} B Q^{-1} C^T A^{-1} X & -Y^T A^{-1} B Q^{-1} B^T A^{-T} Y \\ X^T A^{-T} C Q^{-1} C^T A^{-1} X & (Y^T A^{-1} X + Y^T A^{-1} B Q^{-1} C^T A^{-1} X)^T \end{bmatrix} \\
&\times \begin{bmatrix} X_1 (Y_1^T X_1)^{-1} Y_1^T & 0 \\ 0 & Y_1 (Y_1^T X_1)^{-T} X_1^T \end{bmatrix}
\end{aligned}$$

with an additional $2r - 2q$ zero eigenvalues.

We briefly digress here to show that given a square matrix $J \in \mathbb{R}^{r \times r}$ with $r - q$ zero eigenvalues and $R, S \in \mathbb{R}^{r \times q}$ such that $\text{rank}(R) = \text{rank}(S) = q$, the non zero eigenvalues of J and $JR(S^T R)^{-1} S^T$ coincide. To see this, consider

$$\begin{aligned}
\det[\lambda I - JR(S^T R)^{-1} S^T] &= \lambda^r \det\left[I - \frac{1}{\lambda} JR(S^T R)^{-1} S^T\right] \\
&= \lambda^r \det\left[I - \frac{1}{\lambda} (S^T R)^{-1} S^T JR\right] \\
&= \lambda^{r-q} \det[\lambda S^T R - S^T JR] \cdot \det[(S^T R)^{-1}].
\end{aligned}$$

Hence, non-zero eigenvalues of $JR(S^T R)^{-1} S^T$ are given by $\det[\lambda S^T R - S^T JR] = 0$. For $\det[\lambda S^T R - S^T JR] = 0$, there exist q non-zero scalars λ_i and non-zero vectors x_i for $i = 1, \dots, q$ such that

$$S^T J R x_i = \lambda_i S^T R x_i.$$

Let $y_i = R x_i$, then

$$S^T J y_i = \lambda_i S^T y_i \Rightarrow S^T (J y_i - \lambda_i y_i) = 0.$$

Thus $Jy_i - \lambda_i y_i$ is in the nullspace of S^T and $y_i = Rx_i \Rightarrow Jy_i - \lambda_i y_i$ is in the image space of R . The nullspace of S^T and the image of R coincide only at origin. Hence we have $Jy_i = \lambda_i y_i$ for $i = 1, \dots, q$.

From the above discussion we can conclude that the non-zero eigenvalues of N_p coincide with the non-zero eigenvalues of

$$\begin{aligned} & \begin{bmatrix} -Y^T A^{-1} X - Y^T A^{-1} B Q^{-1} C^T A^{-1} X & -Y^T A^{-1} B Q^{-1} B^T A^{-T} Y \\ X^T A^{-T} C Q^{-1} C^T A^{-1} X & (Y^T A^{-1} X + Y^T A^{-1} B Q^{-1} C^T A^{-1} X)^T \end{bmatrix} \\ &= \begin{bmatrix} Y^T & 0 \\ 0 & X^T \end{bmatrix} \begin{bmatrix} -(A^{-1} + A^{-1} B Q^{-1} C^T A^{-1}) & A^{-1} B Q^{-1} B^T A^{-T} \\ -A^{-T} C Q^{-1} C^T A^{-1} & (A^{-1} + A^{-1} B Q^{-1} C^T A^{-1})^T \end{bmatrix} \\ & \quad \times \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \\ &= \begin{bmatrix} Y^T & 0 \\ 0 & X^T \end{bmatrix} \bar{N} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}. \end{aligned}$$

Hence, the non-zero eigenvalues of N_p coincide with the non-zero eigenvalues of $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} Y^T & 0 \\ 0 & X^T \end{bmatrix} \bar{N} = \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \bar{N}$.

It can be observed that if N_p has 2η zero eigenvalues, then $\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \bar{N}$ has $(2n - 2r) + (2r - 2q) + 2\eta = 2n - 2q + 2\eta$ zero eigenvalues, where n is the dimension of the matrix E , r is the rank of the matrix E and q is the number of finite eigenvalues of (E, A) . **Q.E.D.**

3.4.1 SISO index-two descriptor systems

Conditions for SISO index-two descriptor systems can be obtained by recalling the fact that

$$H(j\omega) + H(j\omega)^* = H_p(j\omega) + H_p(j\omega)^*.$$

This observation leads to the following result.

Theorem 18 Consider the scalar stable index-two descriptor transfer function with $d - c^T A^{-1} b > 0$,

$$H(s) = d + c^T (sE - A)^{-1} b.$$

$H(j\omega) + H(j\omega)^* > 0$ for all finite ω if and only if $EA^{-1}EM^{-1}$ has no negative eigenvalues.

PROOF : The proof is obvious from Lemma 12 and the results for index-one proper transfer functions.

From spectral equivalence Lemma 12, we can also deduce that $EA^{-1}EM^{-1}$ can have at most $n - q$ zero eigenvalues, where q is the number of finite eigenvalues of (E, A) .

3.4.2 MIMO index-two descriptor systems

Passivity tests for MIMO systems can be deduced in a similar way using Lemma 13, however, we provide an alternate proof.

Theorem 19 Consider a stable index-two descriptor system $H(s) = D + C^T(sE - A)^{-1}B$. $H(j\omega) + H(j\omega)^* > 0$ for all finite ω if and only if:

- (i) $Q = D + D^T - C^T A^{-1} B - B^T A^{-T} C > 0$ and
- (ii) the matrix

$$\begin{bmatrix} -(A^{-1} + A^{-1} B Q^{-1} C^T A^{-1}) & -A^{-1} B Q^{-1} B^T A^{-T} \\ A^{-T} C Q^{-1} C^T A^{-1} & (A^{-1} + A^{-1} B Q^{-1} C A^{-1})^T \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \quad (3.39)$$

has no eigenvalues on the imaginary axis except at the origin.

PROOF : (i) Since $H(j\omega) + H(j\omega)^* > 0$, we have that $Q = H(0) + H(0)^* > 0$ is positive definite. Conversely, suppose now that Q is not positive definite. Then, $H(0) + H(0)^*$ cannot be positive definite.

- (ii) Suppose now that $Q > 0$. We have

$$\begin{aligned} H(j\omega) &= D - C^T A^{-1} B - C^T A^{-1} X \left(\frac{1}{j\omega} I - Y^T A^{-1} X \right)^{-1} Y^T A^{-1} B \\ &= \bar{D} + \bar{C}^T \left(\frac{1}{j\omega} I - \bar{A} \right)^{-1} \bar{B}, \end{aligned}$$

where $\bar{A} = Y^T A^{-1} X$, $\bar{B} = Y^T A^{-1} B$, $\bar{C}^T = -C^T A^{-1} X$, $\bar{D} = D - C^T A^{-1} B$, and this transfer function is defined everywhere except at $j\omega = j\infty$. This follows from the fact that the descriptor system is stable, and consequently $\bar{A} = Y^T A^{-1} X$ has eigenvalues in the open left half plane and at 0. Now we follow [106]. We wish to check that $H(j\omega) + H(j\omega)^* > 0$ for all finite $\omega \in \mathbb{R}$. Recall that (i) is assumed to hold. Then, the following transfer function is well defined for $\omega \neq 0$

$$H\left(\frac{1}{j\omega}\right) = \bar{D} + \bar{C}^T (j\omega I - \bar{A})^{-1} \bar{B},$$

and

$$H\left(\frac{1}{j\omega}\right) + H\left(\frac{1}{j\omega}\right)^* = \bar{D} + \bar{C}^T(j\omega I - \bar{A})^{-1}\bar{B} + \bar{D}^T - \bar{B}^T(j\omega I + \bar{A}^T)^{-1}\bar{C}.$$

Recall that (i) ensures that $Q = \bar{D} + \bar{D}^T > 0$. Thus $H(j\omega) + H(j\omega)^*$ is positive definite at $j\omega = 0$.

It also follows that the matrix Q is invertible. Thus, using a continuity argument, $H(j\omega) + H(j\omega)^*$ can only become negative definite or indefinite for finite ω if there exists a finite $\omega_0 \neq 0$ such that $\delta\left(\frac{1}{\omega_0}\right) = \det\left[H\left(\frac{1}{j\omega_0}\right) + H\left(\frac{1}{j\omega_0}\right)^*\right] = 0$. Thus we have $\delta\left(\frac{1}{\omega_0}\right)$

$$\begin{aligned} &= \det(Q) \cdot \det\left[I + [Q^{-1}\bar{C}^T, -Q^{-1}\bar{B}^T] \begin{bmatrix} (j\omega_0 I - \bar{A})^{-1} & 0 \\ 0 & (j\omega_0 I + \bar{A}^T)^{-1} \end{bmatrix} \begin{bmatrix} \bar{B} \\ \bar{C} \end{bmatrix}\right] \\ &= \det(Q) \cdot \det\left[I + \begin{bmatrix} (j\omega_0 I - \bar{A})^{-1} & 0 \\ 0 & (j\omega_0 I + \bar{A}^T)^{-1} \end{bmatrix} \begin{bmatrix} \bar{B} \\ \bar{C} \end{bmatrix} [Q^{-1}\bar{C}^T, -Q^{-1}\bar{B}^T]\right] \\ &= \det\left[j\omega_0 I + \begin{bmatrix} -\bar{A} & 0 \\ 0 & \bar{A}^T \end{bmatrix} + \begin{bmatrix} \bar{B} \\ \bar{C} \end{bmatrix} [Q^{-1}\bar{C}^T, -Q^{-1}\bar{B}^T]\right] \\ &\quad \times \det(Q) \cdot \det[(j\omega_0 I - \bar{A})^{-1}] \cdot \det[(j\omega_0 I + \bar{A}^T)^{-1}]. \end{aligned}$$

Since \bar{A} has no eigenvalues on the imaginary axis, except at the origin, and Q is non-singular, the non-existence of a finite $\omega_0 \neq 0$ such that $\delta\left(\frac{1}{\omega_0}\right) = 0$ is equivalent to the condition that the matrix

$$\bar{N} = \begin{bmatrix} -\bar{A} + \bar{B}Q^{-1}\bar{C}^T & -\bar{B}Q^{-1}\bar{B}^T \\ \bar{C}Q^{-1}\bar{C}^T & (\bar{A} - \bar{B}Q^{-1}\bar{C}^T)^T \end{bmatrix}$$

has no non-zero eigenvalues on the imaginary axis. Note:

$$\bar{N} = \begin{bmatrix} -Y^T A^{-1} X - Y^T A^{-1} B Q^{-1} C^T A^{-1} X & -Y^T A^{-1} B Q^{-1} B^T A^{-T} Y \\ X^T A^{-T} C Q^{-1} C^T A^{-1} X & (Y^T A^{-1} X + Y^T A^{-1} B Q^{-1} C^T A^{-1} X)^T \end{bmatrix}.$$

This matrix can be written as

$$\bar{N} = \begin{bmatrix} Y^T & 0 \\ 0 & X^T \end{bmatrix} \begin{bmatrix} -(A^{-1} + A^{-1} B Q^{-1} C^T A^{-1}) & -A^{-1} B Q^{-1} B^T A^{-T} \\ A^{-T} C Q^{-1} C^T A^{-1} & (A^{-1} + A^{-1} B Q^{-1} C^T A^{-1})^T \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}.$$

Finally, we use the fact that the non-zero eigenvalues of this matrix coincide with the matrix

$$\begin{bmatrix} -(A^{-1} + A^{-1} B Q^{-1} C^T A^{-1}) & -A^{-1} B Q^{-1} B^T A^{-T} \\ A^{-T} C Q^{-1} C^T A^{-1} & (A^{-1} + A^{-1} B Q^{-1} C^T A^{-1})^T \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} Y^T & 0 \\ 0 & X^T \end{bmatrix}.$$

Thus the condition becomes that the matrix

$$\begin{bmatrix} -(A^{-1} + A^{-1} B Q^{-1} C^T A^{-1}) & -A^{-1} B Q^{-1} B^T A^{-T} \\ A^{-T} C Q^{-1} C^T A^{-1} & (A^{-1} + A^{-1} B Q^{-1} C^T A^{-1})^T \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}$$

has no eigenvalues on the imaginary axis except at the origin. **Q.E.D.**

3.5 SUMMARY

Finally, we summarize all the results in this chapter through the following corollary. Observe that, testing for $H(j\omega) + H(j\omega)^* > 0$ for all finite ω does not require *a priori* knowledge of index of the descriptor system.

Corollary 2 Consider a stable *SISO* descriptor transfer function

$$H(s) = d + c^T (sE - A)^{-1} b.$$

Then $H(j\omega) + H(j\omega)^* > 0$ for all finite ω if and only if

- (i) $d - c^T A^{-1} b > 0$ and
- (ii) $EA^{-1}EM^{-1}$ has no negative eigenvalues.

For a stable *MIMO* descriptor system

$$H(s) = D + C^T (sE - A)^{-1} B.$$

Then $H(j\omega) + H(j\omega)^* > 0$ for all finite ω if and only if:

- (i) $Q = D + D^T - C^T A^{-1} B - B^T A^{-T} C > 0$ and
- (ii) $\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} N$ has no eigenvalues on the imaginary axis except at the origin.

3.6 EXAMPLES

To illustrate our results, we consider a simple *SISO* index-two descriptor system of the form (3.1), where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

and $d = -1$. Then $H(s) = d + c^T (sE - A)^{-1} b$ is an improper transfer function given by

$$H(s) = \frac{-s^3 - 4s^2 - 3s + 1}{s^2 + 3s + 2}.$$

It can be verified that $d - c^T A^{-1} B = 0.5 > 0$ and $\text{eig}(EA^{-1}EM^{-1}) = \{-1.3660, 0.3660, 00\}$.

The negative real eigenvalues of $EA^{-1}EM^{-1}$ give the zero crossing frequencies $\frac{1}{\omega_0} = 1.3660$ for $\Re \left[H \left(\frac{1}{j\omega} \right) \right]$. Thus, the zero crossing frequencies for $\Re [H(j\omega)]$ are $\omega_0 = \pm 0.8556$.

We can also test for passivity by calculating the eigenvalues of the matrix

$$\begin{bmatrix} -(A^{-1} + A^{-1}BQ^{-1}C^T A^{-1}) & -A^{-1}BQ^{-1}B^T A^{-T} \\ A^{-T}CQ^{-1}C^T A^{-1} & (A^{-1} + A^{-1}BQ^{-1}C^T A^{-1})^T \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}$$

given by $\{0, 0, 1.1687j, -1.1687j, 0.6050, 0, -0.6050, 0\}$. The imaginary eigenvalues of this matrix give the zero crossing frequencies $\frac{1}{\omega_0} = \pm 1.1687$ for $\det \left[H \left(\frac{1}{j\omega} \right) + H \left(\frac{1}{j\omega} \right)^* \right]$. Thus, the zero crossing frequencies for $\det [H(j\omega) + H(j\omega)^*]$ are $\omega_0 = \pm 0.8556$. These frequency values can be confirmed by plotting the real part of $H(j\omega)$ w.r.t. ω as shown in Figure 16.

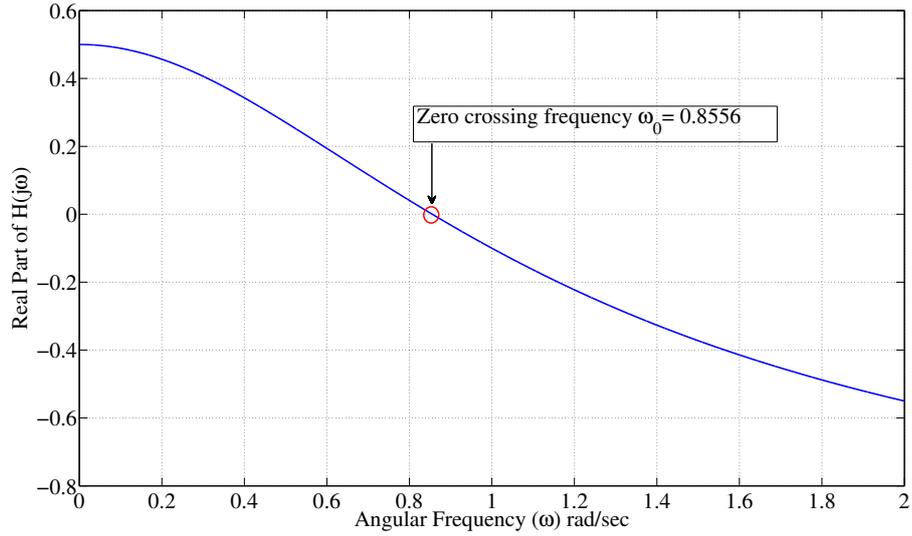


Figure 6: Plot showing the real part of $H(j\omega)$ w.r.t. ω

3.7 CONCLUSIONS

In this chapter, we considered passivity of **SISO** and **MIMO** descriptor systems and obtain easy to check spectral conditions to test for passivity. These conditions do not require the initial knowledge of the index of the descriptor system, and they also provide algebraic conditions to determine the frequency bands for which passivity might be lost.

MIXED PROPERTY

In this chapter, we derive stability results using classical Nyquist arguments for large-scale interconnections of “mixed” LTI systems. We compare our results with Moylan and Hill [15]. Our results indicate that, if one relaxes assumptions on the subsystems in an interconnection from assumptions of passivity or small gain to assumptions of “mixedness,” then the Moylan- and Hill-like conditions on the interconnection matrix become more stringent.

4.1 INTRODUCTION

In this chapter, we provide a proof of the stability result concerning simple feedback-loops consisting of two LTI “mixed” systems due to Griggs, Anderson and Lanzon [38, 39]. We do so by applying classical Nyquist stability techniques (see Section 4.3). Our reasons for doing so are twofold. First, we correct an error in Theorems 1 and 6 of [38] and [39], respectively. In these, the system output signals were assumed to be bounded *a priori*. Secondly and importantly, the result paves the way to obtaining new sufficient conditions for the stability of large-scale interconnections of “mixed” systems, which we derive in Section 9.4.2. Our large-scale interconnection results indicate that, as one relaxes the assumptions on the transfer function matrices of the systems, e.g. from assumptions of passivity to assumptions of “mixedness,” the Moylan- and Hill-like conditions on the interconnection matrix [15] become more severe.

Finally, we derive a necessary and sufficient condition (test) for determining whether or not a MIMO LTI descriptor system is “mixed”. The procedure involves the computation of two Hamiltonian matrices, one associated with any potentially passive aspects of the system and the other associated with the notion of small gain. The examination of the spectral characteristics of these Hamiltonian matrices, which are constructed from state-space matrices E, A, B, C, D , leads to the elimination of an element of frequency dependency from the test.

4.2 PRELIMINARY RESULTS

We will also require the following preliminary results.

Lemma 14 Suppose that $H_1 \in \mathcal{RL}_\infty$ and $H_2 \in \mathcal{RL}_\infty$. Suppose further that, at some $\omega \in \mathbb{R} \cup \{\pm\infty\}$, $H_1^*(j\omega) + H_1(j\omega) > 0$ and $H_2^*(j\omega) + H_2(j\omega) \geq 0$. Then

$$\det[I + H_1(j\omega)H_2(j\omega)] \neq 0.$$

PROOF: Since $H_1^*(j\omega) + H_1(j\omega) > 0$, $\Re[\lambda_i[H_1(j\omega)]] > 0 \forall i$ (where $\lambda_i[\cdot]$ denotes the i th eigenvalue) and so $H_1(j\omega)$ is nonsingular. Then

$$H_1^*(j\omega)(H_1^{-*}(j\omega) + H_1^{-1}(j\omega))H_1(j\omega) > 0$$

and observing the congruency, we can write

$$H_1^{-*}(j\omega) + H_1^{-1}(j\omega) > 0.$$

Then $H_1^{-*}(j\omega) + H_2^*(j\omega) + H_1^{-1}(j\omega) + H_2(j\omega) > 0$ and consequently $\Re[\lambda_i[H_1^{-1}(j\omega) + H_2(j\omega)]] > 0 \forall i$. Hence $\det[H_1^{-1}(j\omega) + H_2(j\omega)] \neq 0$ further implying $\det[I + H_1(j\omega)H_2(j\omega)] \neq 0$. **Q.E.D.**

Letting $H_1 = I$ and setting $H := H_2$ in the above Lemma statement, gives the following corollary.

Corollary 3 Suppose that $H \in \mathcal{RL}_\infty$ and that, at some $\omega \in \mathbb{R} \cup \{\pm\infty\}$, $H^*(j\omega) + H(j\omega) \geq 0$. Then $\det[I + H(j\omega)] \neq 0$.

Versions of the next corollary can be found in [109, Lemma 7 of Section VI.10] and [101, Theorem 2.3.4].

Corollary 4 Suppose that $H \in \mathcal{RL}_\infty$ and that, at some $\omega \in \mathbb{R} \cup \{\pm\infty\}$, $H^*(j\omega) + H(j\omega) \geq 0$. Let $S(j\omega) := (H(j\omega) - I)(I + H(j\omega))^{-1}$. Then $-S^*(j\omega)S(j\omega) + I \geq 0$.

PROOF: From Corollary 3, $\det[I + H(j\omega)] \neq 0$. Then

$$\begin{aligned} & 2(I + H(j\omega))^{-*}[H^*(j\omega) + H(j\omega)](I + H(j\omega))^{-1} \\ &= (I + H(j\omega))^{-*}[(I + H(j\omega))^*(I + H(j\omega)) - (H(j\omega) - I)^*(H(j\omega) - I)] \\ & \quad \times (I + H(j\omega))^{-1} \\ &= I - (I + H(j\omega))^{-*}(H(j\omega) - I)^*(H(j\omega) - I)(I + H(j\omega))^{-1} \\ &= I - S^*(j\omega)S(j\omega). \end{aligned}$$

Since $H^*(j\omega) + H(j\omega)$ and $I - S^*(j\omega)S(j\omega)$ are Hermitian-congruent, $-S^*(j\omega)S(j\omega) + I \geq 0$. **Q.E.D.**

An extension to Lemma 14 is given below.

Lemma 15 Suppose that $H_1 \in \mathcal{RH}_\infty$ and $H_2 \in \mathcal{RH}_\infty$. Suppose further that, at some $\omega \in \mathbb{R} \cup \{\pm\infty\}$, $H_1^*(j\omega) + H_1(j\omega) > H_1^*(j\omega)KH_1(j\omega)$ and $H_2^*(j\omega) + H_2(j\omega) \geq -K$, where $K \geq 0$ is a constant, symmetric matrix with entries in \mathbb{R} . Then $\det[I + \frac{1}{\kappa}H_1(j\omega)H_2(j\omega)] \neq 0$ for any $\kappa \geq 1$.

PROOF: Since $H_1^*(j\omega) + H_1(j\omega) > H_1^*(j\omega)KH_1(j\omega) \geq 0$, $H_1^*(j\omega) + H_1(j\omega) > 0$ and so $\Re[\lambda_i[H_1(j\omega)]] > 0 \forall i$. Then $H_1(j\omega)$ is nonsingular and hence

$$\begin{aligned} H_1^{-*}(j\omega) + H_1^{-1}(j\omega) &> K \\ \Rightarrow \kappa(H_1^{-*}(j\omega) + H_1^{-1}(j\omega)) &> K \quad \forall \kappa \geq 1 \end{aligned}$$

Adding $H_2^*(j\omega) + H_2(j\omega) \geq -K$ to the above inequality, we have

$$\kappa(H_1^{-*}(j\omega) + H_2^*(j\omega) + H_2(j\omega) + H_1^{-1}(j\omega)) > 0 \quad \forall \kappa \geq 1.$$

Hence $\Re[\lambda_i[\kappa H_1^{-1}(j\omega) + H_2(j\omega)]] > 0 \forall i$ and so $\det[\kappa H_1^{-1}(j\omega) + H_2(j\omega)] \neq 0$ further resulting in $\det[I + \frac{1}{\kappa}H_1(j\omega)H_2(j\omega)] \neq 0$.

Lastly, since our aim is to deduce the stability of interconnections of ‘‘mixed’’ systems using arguments based on classical Nyquist techniques, we state a MIMO version of the Nyquist stability theorem.

Theorem 20 [110, Theorem 5.8] [111, Remark 4 of Section 4.9.2] Consider the feedback interconnection of systems depicted in Figure 7. Suppose that $H_1 \in \mathcal{RH}_\infty$, $H_2 \in \mathcal{RH}_\infty$ and that the system interconnection is well-posed. Then the feedback-loop is stable if and only if the Nyquist plot of $\det[I + H_1(j\omega)H_2(j\omega)]$ for $-\infty \leq \omega \leq \infty$ does not make any encirclements of the origin.

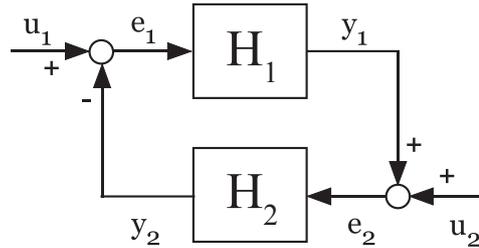


Figure 7: A negative feedback interconnection.

In the above theorem, well-posedness and stability are defined in the sense of [110, Sections 5.2 and 5.3]. Note, also, the following observations concerning the Nyquist plot of $\det[I + H_1(j\omega)H_2(j\omega)]$.

Observation 1 The Nyquist plot of $\det[I + H_1(j\omega)H_2(j\omega)]$ belongs to a family of Nyquist plots of $\det[I + \frac{1}{\kappa}H_1(j\omega)H_2(j\omega)]$, where $\kappa \in [1, \infty)$.

Observation 2 Each Nyquist plot of $\det[I + \frac{1}{\kappa}H_1(j\omega)H_2(j\omega)]$ is symmetrical about the real axis of the complex plane, where $\kappa \in [1, \infty)$.¹

Observation 3 As κ and ω vary continuously, the point in the complex plane on which the Nyquist plot of $\det[I + \frac{1}{\kappa}H_1(j\omega)H_2(j\omega)]$ lies varies continuously.

Observation 4 As $\kappa \rightarrow \infty$, $\det[I + \frac{1}{\kappa}H_1(j\omega)H_2(j\omega)] \rightarrow 1$.

Observation 5 Suppose that κ is very large such that $\det[I + \frac{1}{\kappa}H_1(j\omega)H_2(j\omega)]$ is almost equal to 1 for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$, further suppose that κ is continuously decreasing towards 1. Suppose that the Nyquist plot of $\det[I + H_1(j\omega)H_2(j\omega)]$ encircles the origin at least once. Then there must exist at least one κ_0 and one ω_0 for which $\det[I + \frac{1}{\kappa_0}H_1(j\omega_0)H_2(j\omega_0)] = 0$.

Thus, a sufficient condition for the Nyquist plot of $\det[I + H_1(j\omega)H_2(j\omega)]$ to make no encirclements of the origin is that, for all $\kappa \in [1, \infty)$ and all $\omega \in \mathbb{R} \cup \{\pm\infty\}$, $\det[I + \frac{1}{\kappa}H_1(j\omega)H_2(j\omega)] \neq 0$. Subsequently, we present scenarios in which this sufficient condition is satisfied and thus the stability of the feedback-loop is guaranteed.

4.3 SIMPLE FEEDBACK-LOOP

Now we provide the correct proof of the stability result on simple negative feedback interconnections of systems with “mixed” small gain and passivity properties due to Griggs, Anderson and Lanzon [38, 39]. We use the Nyquist discussion provided above.

Theorem 21 Suppose that $M_1 \in \mathcal{RH}_\infty$ and $M_2 \in \mathcal{RH}_\infty$ denote the transfer function matrices of “mixed” subsystems interconnected as depicted in Figure 8 and that this interconnection is well-posed. Suppose that there exist two distinct sets of frequency bands:

- (a) a set denoted by Ω_p that consists of frequency intervals over which both $M_1(j\omega)$ and $M_2(j\omega)$ have associated with them Property (i) as given in Definition 13; and
- (b) a set denoted by Ω_s that consists of frequency intervals over which both $M_1(j\omega)$ and $M_2(j\omega)$ have associated with them Property (ii) as given in Definition 13.

Furthermore, suppose that $\Omega_p \cup \Omega_s = \mathbb{R} \cup \{\pm\infty\}$. Then the negative feedback-loop is stable.

PROOF: Our aim is to show that, for all $\kappa \in [1, \infty)$ and all $\omega \in \mathbb{R} \cup \{\pm\infty\}$,

$$\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0.$$

From Section 4.2, this is a sufficient condition for stability. We do so by splitting our proof into two parts:

¹ $\det[I + \frac{1}{\kappa}H_1(-j\omega)H_2(-j\omega)] = \det[(I + \frac{1}{\kappa}H_2^*(j\omega)H_1^*(j\omega))^T] = \det[I + \frac{1}{\kappa}H_2^*(j\omega)H_1^*(j\omega)]$ (from [112, Equation 6.1.4]) = $\det[(I + \frac{1}{\kappa}H_1(j\omega)H_2(j\omega))^*] = \det[I + \frac{1}{\kappa}H_1(j\omega)H_2(j\omega)]$ (from [112, Exercise 6.1.6])

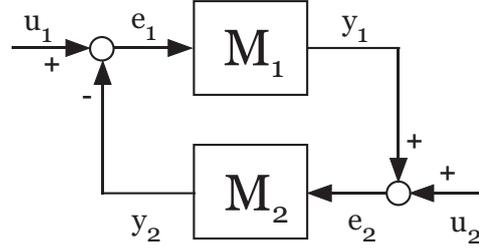


Figure 8: Negative feedback interconnection of “mixed” systems.

(i) first, we show that $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$ for all $\kappa \in [1, \infty)$ and all $\omega \in \Omega_s$; and

(ii) then, we show that $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$ for all $\kappa \in [1, \infty)$ and all $\omega \in \Omega_p$.

Part (i) for all $\omega \in \Omega_s$: From property (ii) of Definition 13, for $i = 1, 2$, there exists an $\varepsilon_i < 1$ such that $-M_i^*(j\omega)M_i(j\omega) + \varepsilon_i^2 I \geq 0$. This implies that, for $i = 1, 2$, $\bar{\sigma}(M_i(j\omega)) < 1$, which implies that $\bar{\sigma}(M_1(j\omega)M_2(j\omega)) < 1$ since $\bar{\sigma}(M_1(j\omega)M_2(j\omega)) \leq \bar{\sigma}(M_1(j\omega))\bar{\sigma}(M_2(j\omega))$. Now

$$0 < 1 - \bar{\sigma}(M_1(j\omega)M_2(j\omega)) \leq \underline{\sigma}(I + M_1(j\omega)M_2(j\omega))$$

from [110, Section 2.8], and so $\underline{\sigma}(I + M_1(j\omega)M_2(j\omega)) \neq 0$, which is equivalent to $\det[I + M_1(j\omega)M_2(j\omega)] \neq 0$. Furthermore, $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$ for any $\kappa > 1$. This is because $\bar{\sigma}(M_1(j\omega)M_2(j\omega)) < 1$ is equivalent to $\frac{1}{\kappa}\bar{\sigma}(M_1(j\omega)M_2(j\omega)) < \frac{1}{\kappa}$ (which is < 1) for any $\kappa > 1$, and so $\bar{\sigma}(\frac{1}{\kappa}M_1(j\omega)M_2(j\omega)) < 1$ for any $\kappa > 1$. Then

$$0 < 1 - \bar{\sigma}\left(\frac{1}{\kappa}M_1(j\omega)M_2(j\omega)\right) \leq \underline{\sigma}\left(I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)\right)$$

for any $\kappa > 1$, and from this the determinant inequality is immediate.

Part (ii) for all $\omega \in \Omega_p$: From property (i) of Definition 13, for $i = 1, 2$, there exist $k_i, l_i > 0$ such that $-k_i M_i^*(j\omega)M_i(j\omega) + M_i^*(j\omega) + M_i(j\omega) - l_i I \geq 0$. This implies that, for $i = 1, 2$, $M_i^*(j\omega) + M_i(j\omega) > 0$. Observe that $M_i^*(j\omega) + M_i(j\omega) > 0$ if and only if $\frac{1}{\sqrt{\kappa}}M_i^*(j\omega) + \frac{1}{\sqrt{\kappa}}M_i(j\omega) > 0$, where $\kappa > 0$. Then, from Lemma 14, $\det[I + \frac{1}{\kappa}M_1(j\omega)M_2(j\omega)] \neq 0$ for any $\kappa > 0$ and hence for any $\kappa \geq 1$. **Q.E.D.**

4.4 LARGE-SCALE INTERCONNECTIONS

Building on the techniques of the previous section, we derive sufficient conditions for the stability of large-scale interconnections of systems with mixtures of small gain and passivity properties. Consider a linear interconnection of N “mixed” systems with square transfer

function matrices denoted by $M_i \in \mathcal{RH}_\infty$, $i = 1, \dots, N$. The interconnection will be described by

$$e_i = u_i - \sum_{j=1}^N H_{ij} y_j,$$

where e_i is the input to subsystem i , $y_i = M_i e_i$ is the output of subsystem i , u_i is an external input and H_{ij} is a constant matrix. Writing

$$e := \begin{pmatrix} e_1 \\ \vdots \\ e_N \end{pmatrix}, \quad y := \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} \quad \text{and} \quad u := \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix},$$

the interconnection description may be written more compactly as

$$e = u - Hy, \tag{4.1}$$

where H is a matrix with block entries H_{ij} . Let $\tilde{M} := \text{diag}(M_1, \dots, M_N)$ such that $y = \tilde{M}e$. Eliminating y from (4.1), we have

$$e = (I + H\tilde{M})^{-1}u.$$

Then

$$y = \tilde{M}(I + H\tilde{M})^{-1}u. \tag{4.2}$$

This set-up is depicted in Figure 9. We will assume that the interconnection is well-posed and, similarly to Theorem 42, impose the following extra conditions on the systems in the interconnection. We require the existence of two distinct sets of frequency bands:

- (a) a set denoted by Ω_p that consists of frequency intervals over which every $M_i(j\omega)$ has property (i) as given in Definition 13 associated with it; and
- (b) a set denoted by Ω_s that consists of frequency intervals over which every $M_i(j\omega)$ has property (ii) as given in Definition 13 associated with it.

Again, we also require that $\Omega_p \cup \Omega_s = \mathbb{R} \cup \{\pm\infty\}$. In the following, $p_i, q_i \in \mathbb{R}$ for $i = 1, \dots, N$.

Theorem 22 *An interconnection of “mixed” subsystems, with input u and output y , as described above, is stable if there exist positive definite matrices $P := \text{diag}(p_1 I, \dots, p_N I)$ and $Q := \text{diag}(q_1 I, \dots, q_N I)$ such that $H^T Q + QH > 0$ and $-H^T P H + P > 0$.*

PROOF: Similarly to Section 4.3, our aim is to show that, for all $\kappa \in [1, \infty)$ and all $\omega \in \mathbb{R} \cup \{\pm\infty\}$, $\det[I + \frac{1}{\kappa} H\tilde{M}(j\omega)] \neq 0$. Again, we split our proof into two parts: (i) first, we

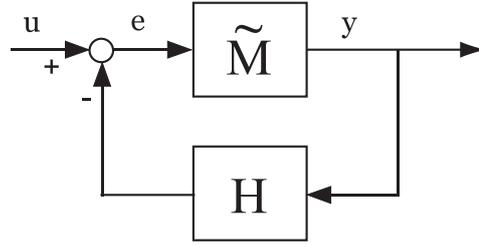


Figure 9: Large-scale interconnection of “mixed” systems.

show that $\det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)] \neq 0$ for all $\kappa \in [1, \infty)$ and all $\omega \in \Omega_s$; and (ii) then, we show that $\det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)] \neq 0$ for all $\kappa \in [1, \infty)$ and all $\omega \in \Omega_p$.

Part (i) for all $\omega \in \Omega_s$: Suppose that there exists a positive definite matrix P (as defined above) such that $-H^T P H + P > 0$. Let $\tilde{P} := P^{\frac{1}{2}}$ and note that $\tilde{P}^T = \tilde{P}$. Now $-H^T \tilde{P}^2 H + \tilde{P}^2 = \tilde{P}^T (-\tilde{P}^{-T} H^T \tilde{P}^T \tilde{P} H \tilde{P}^{-1} + I) \tilde{P}$ and so $-(\tilde{P} H \tilde{P}^{-1})^T \tilde{P} H \tilde{P}^{-1} + I > 0$ since $-H^T P H + P$ and $-(\tilde{P} H \tilde{P}^{-1})^T \tilde{P} H \tilde{P}^{-1} + I$ are Hermitian-congruent. Set $H_p := \tilde{P} H \tilde{P}^{-1}$. Then $-H_p^T H_p + I > 0$. Equivalently, $\bar{\sigma}(H_p) < 1$. From property (ii) of Definition 13, for $i = 1, \dots, N$, there exists an $\varepsilon_i < 1$ such that $-M_i^*(j\omega)M_i(j\omega) + \varepsilon_i^2 I \geq 0$. This implies that, for $i = 1, \dots, N$, $-M_i^*(j\omega)M_i(j\omega) + I > 0$. Since $\bar{\sigma}(M_i(j\omega)) < 1$, the same is true for $\tilde{M}(j\omega)$, i.e. $\bar{\sigma}(\tilde{M}(j\omega)) < 1$. Then $\bar{\sigma}(H_p \tilde{M}(j\omega)) < 1$. Now

$$0 < 1 - \bar{\sigma}(H_p \tilde{M}(j\omega)) \leq \underline{\sigma}(I + H_p \tilde{M}(j\omega))$$

from [110], and so $\underline{\sigma}(I + H_p \tilde{M}(j\omega)) \neq 0$, which is equivalent to $\det[I + H_p \tilde{M}(j\omega)] \neq 0$. Furthermore, $\det[I + \frac{1}{\kappa}H_p \tilde{M}(j\omega)] \neq 0$ for any $\kappa > 1$. This is because $\bar{\sigma}(H_p \tilde{M}(j\omega)) < 1$ is equivalent to $\frac{1}{\kappa} \bar{\sigma}(H_p \tilde{M}(j\omega)) < \frac{1}{\kappa}$ (which is < 1) for any $\kappa > 1$, and so $\bar{\sigma}(\frac{1}{\kappa}H_p \tilde{M}(j\omega)) < 1$ for any $\kappa > 1$. Then

$$0 < 1 - \bar{\sigma}\left(\frac{1}{\kappa}H_p \tilde{M}(j\omega)\right) \leq \underline{\sigma}\left(I + \frac{1}{\kappa}H_p \tilde{M}(j\omega)\right)$$

for any $\kappa > 1$. Finally, note that $\det[I + \frac{1}{\kappa}H_p \tilde{M}(j\omega)] = \det[\tilde{P}] \det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)] \det[\tilde{P}^{-1}]$ since \tilde{P}^{-1} and $\tilde{M}(j\omega)$ commute.

Part (ii) for all $\omega \in \Omega_p$: Suppose that there exists a positive definite matrix Q (as defined above) such that $H^T Q + Q H > 0$. Let $\tilde{Q} := Q^{\frac{1}{2}}$ and note that $\tilde{Q}^T = \tilde{Q}$. Now $H^T \tilde{Q}^2 + \tilde{Q}^2 H = \tilde{Q}^T (\tilde{Q}^{-T} H^T \tilde{Q}^T + \tilde{Q} H \tilde{Q}^{-1}) \tilde{Q}$ and so $\tilde{Q}^{-T} H^T \tilde{Q}^T + \tilde{Q} H \tilde{Q}^{-1} > 0$ since $H^T Q + Q H$ and $\tilde{Q}^{-T} H^T \tilde{Q}^T + \tilde{Q} H \tilde{Q}^{-1}$ are Hermitian-congruent. Set $H_Q := \tilde{Q} H \tilde{Q}^{-1}$. Then $H_Q^T + H_Q > 0$. From property (i) of Definition 13, for $i = 1, \dots, N$, there exist $k_i, l_i > 0$ such that

$$-k_i M_i^*(j\omega) M_i(j\omega) + M_i^*(j\omega) + M_i(j\omega) - l_i I \geq 0.$$

This implies that, for $i = 1, \dots, N$,

$$M_i^*(j\omega) + M_i(j\omega) > 0.$$

Hence, the same is true for $\tilde{M}(j\omega)$, i.e. $\tilde{M}^*(j\omega) + \tilde{M}(j\omega) > 0$. Observe that $\tilde{M}^*(j\omega) + \tilde{M}(j\omega) > 0$ if and only if $\frac{1}{\kappa}\tilde{M}^*(j\omega) + \frac{1}{\kappa}\tilde{M}(j\omega) > 0$, where $\kappa > 0$. Then, from Lemma 14, $\det[I + \frac{1}{\kappa}H_Q\tilde{M}(j\omega)] \neq 0$ for any $\kappa > 0$ and hence for any $\kappa \geq 1$. Finally, note that $\det[I + \frac{1}{\kappa}H_Q\tilde{M}(j\omega)] = \det[\tilde{Q}]\det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)]\det[\tilde{Q}^{-1}]$ since \tilde{Q}^{-1} and $\tilde{M}(j\omega)$ commute and so $\det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)] \neq 0$ for any $\kappa \geq 1$. **Q.E.D.**

Fixing $P = Q = I$ in the above theorem, gives the following result.

Corollary 5 *An interconnection of “mixed” subsystems, with input u and output y , as described above, is stable if $H^T + H > 0$ and $-H^T H + I > 0$.*

Our next version of the large-scale interconnected “mixed” systems stability result involves some relaxation of the assumptions on the interconnection structure described by the matrix H compared to the restrictions on H specified in Theorem 22. This relaxation is achieved by taking into consideration the values of k_i and ε_i associated with each of the “mixed” systems, where ε_i denotes the gain of the i th “mixed” system over frequencies in Ω_s , and k_i provides a measure of output strict passivity for the i th “mixed” system over frequencies in Ω_p . Suppose that $K := \text{diag}(k_1 I, \dots, k_N I)$ and $E := \text{diag}(\varepsilon_1 I, \dots, \varepsilon_N I)$, where $k_i > 0$ and $0 < \varepsilon_i < 1$ for $i = 1, \dots, N$.

Theorem 23 *An interconnection of “mixed” subsystems, with input u and output y , as described above, is stable if there exist positive definite matrices $P := \text{diag}(p_1 I, \dots, p_N I)$ and $Q := \text{diag}(q_1 I, \dots, q_N I)$ such that $H^T Q + QH + QK > 0$ and $-H^T P E^2 H + P > 0$.*

PROOF: The proof follows in a manner similar to that of the proof of Theorem 22. As before, we want to show that $\det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)] \neq 0$ for all $\kappa \in [1, \infty)$ and all $\omega \in \mathbb{R} \cup \{\pm\infty\}$.

Part (i) for all $\omega \in \Omega_s$: Suppose that there exists a positive definite matrix P (as defined above) such that $-H^T P E^2 H + P > 0$. Then, similarly to the proof of Theorem 22, we obtain $\bar{\sigma}(E H_P) < 1$, where $H_P := P^{\frac{1}{2}} H (P^{\frac{1}{2}})^{-1}$. From property (ii) of Definition 13, for $i = 1, \dots, N$, there exists an $\varepsilon_i < 1$ such that $-M_i^*(j\omega)M_i(j\omega) + \varepsilon_i^2 I \geq 0$. Equivalently, for $i = 1, \dots, N$, there exists an $\varepsilon_i < 1$ such that $-\frac{1}{\varepsilon_i^2}M_i^*(j\omega)M_i(j\omega) + I \geq 0$. Since $\bar{\sigma}(\frac{1}{\varepsilon_i}M_i(j\omega)) \leq 1$, $\bar{\sigma}(E^{-1}\tilde{M}(j\omega)) \leq 1$. Then $\bar{\sigma}(\tilde{M}(j\omega)H_P) < 1$ since E^{-1} and $\tilde{M}(j\omega)$ commute. Now

$$0 < 1 - \bar{\sigma}(\tilde{M}(j\omega)H_P) \leq \underline{\sigma}(I + \tilde{M}(j\omega)H_P)$$

from [110] and so $\underline{\sigma}(I + \tilde{M}(j\omega)H_P) \neq 0$ which is equivalent to $\det[I + \tilde{M}(j\omega)H_P] \neq 0$. Furthermore, $\det[I + \frac{1}{\kappa}\tilde{M}(j\omega)H_P] \neq 0$ for any $\kappa > 1$. This is because $\bar{\sigma}(\tilde{M}(j\omega)H_P) < 1$ is equivalent to $\frac{1}{\kappa}\bar{\sigma}(\tilde{M}(j\omega)H_P) < \frac{1}{\kappa}$ (which is < 1) for any $\kappa > 1$, and so $\bar{\sigma}(\frac{1}{\kappa}\tilde{M}(j\omega)H_P) < 1$ for any $\kappa > 1$. Then

$$0 < 1 - \bar{\sigma}\left(\frac{1}{\kappa}\tilde{M}(j\omega)H_P\right) \leq \underline{\sigma}\left(I + \frac{1}{\kappa}\tilde{M}(j\omega)H_P\right)$$

for any $\kappa > 1$. Finally, note that $\det[I + \frac{1}{\kappa}\tilde{M}(j\omega)H_P] = \det[I + \frac{1}{\kappa}H_P\tilde{M}(j\omega)]$ for any $\kappa \geq 1$ [112, Exercise 6.2.7] and that $\det[I + \frac{1}{\kappa}H_P\tilde{M}(j\omega)] = \det[P^{\frac{1}{2}}] \det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)] \det[(P^{\frac{1}{2}})^{-1}]$ since $(P^{\frac{1}{2}})^{-1}$ and $\tilde{M}(j\omega)$ commute.

Part (ii) for all $\omega \in \Omega_p$: Suppose that there exists a positive definite matrix Q (as defined above) such that $H^T Q + QH + QK > 0$. Similarly to the proof of Theorem 22, we obtain $H_Q^T + H_Q + K > 0$, where $H_Q := Q^{\frac{1}{2}}H(Q^{\frac{1}{2}})^{-1}$. From property (i) of Definition 13, for $i = 1, \dots, N$, there exist $k_i, l_i > 0$ such that $-k_i M_i^*(j\omega)M_i(j\omega) + M_i^*(j\omega) + M_i(j\omega) - l_i I \geq 0$. This implies that, for $i = 1, \dots, N$, $-k_i M_i^*(j\omega)M_i(j\omega) + M_i^*(j\omega) + M_i(j\omega) > 0$. Hence, $-\tilde{M}^*(j\omega)K\tilde{M}(j\omega) + \tilde{M}^*(j\omega) + \tilde{M}(j\omega) > 0$. Then, from Lemma 15, $\det[I + \frac{1}{\kappa}\tilde{M}(j\omega)H_Q] \neq 0$ for any $\kappa \geq 1$. Finally, note that $\det[I + \frac{1}{\kappa}\tilde{M}(j\omega)H_Q] = \det[I + \frac{1}{\kappa}H_Q\tilde{M}(j\omega)]$ [112, Exercise 6.2.7], and that $\det[I + \frac{1}{\kappa}H_Q\tilde{M}(j\omega)] = \det[Q^{\frac{1}{2}}] \det[I + \frac{1}{\kappa}H\tilde{M}(j\omega)] \det[(Q^{\frac{1}{2}})^{-1}]$ since $(Q^{\frac{1}{2}})^{-1}$ and $\tilde{M}(j\omega)$ commute. **Q.E.D.**

Set $P = Q = I$ in the above theorem, to obtain the following corollary.

Corollary 6 *An interconnection of “mixed” subsystems, with input u and output y , as described above, is stable if $H^T + H + K > 0$ and $-H^T E^2 H + I > 0$.*

4.5 EXAMPLES

We now compare our large-scale interconnected “mixed” systems stability results to the large-scale interconnected systems stability results of [15, Sections IV and V] (e.g. see [15, Theorems 4 and 5]) through the following example.

Consider an interconnected system from [15], depicted in Figure 10, with interconnection matrix

$$H = \begin{bmatrix} 1 & 0 & -\gamma \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Assume that H_1, H_2 and H_3 are passive and that $-8 < \gamma < 1$. According to [15], a sufficient condition for the stability of large-scale interconnections of passive systems is the existence of a positive definite diagonal matrix Q such that $H^T Q + QH > 0$. A necessary condition for this LMI to be feasible is that H has eigenvalues with positive real parts. Using the Robust Control Toolbox (MATLAB R2009a), we verify that, for any $-8 < \gamma < 1$, finding a solution to the LMI $H^T Q + QH > 0$ is indeed feasible.

Similarly, if H_1, H_2 and H_3 have gain less than one in [15], a sufficient condition for the stability of large-scale interconnections of systems with finite gain is the existence of a positive definite diagonal matrix P such that $-H^T P E^2 H + P > 0$.² A necessary condition

² Note that, in [15], the gains ε_i appearing in E are not necessarily less than one.

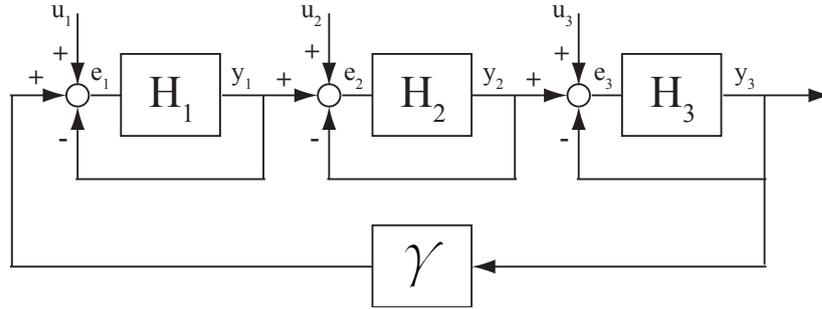


Figure 10: Interconnection 1.

for this LMI to be feasible is that EH has eigenvalues that lie inside the unit circle centred at the origin of the complex plane [113, Theorem 5.18].

Now, suppose that we relax the conditions on H_1 , H_2 and H_3 and assume that they are all “mixed” systems. For the same values of γ , we search for positive definite diagonal matrices P and Q that satisfy $H^T Q + QH > 0$ and $-H^T P H + P > 0$ simultaneously. We find that this LMI problem is not feasible for any $-8 < \gamma < 1$. Our results show that, as one relaxes the assumptions on the subsystems in the interconnection (from passivity or finite gain to “mixedness”), the [15]-like conditions on the interconnection matrix become more stringent, i.e. more restriction is imposed on the structure of the interconnection itself. For instance, in Theorems 22 and 23, the existence of solutions to a pair of LMIs as opposed to a single LMI is sufficient for stability.

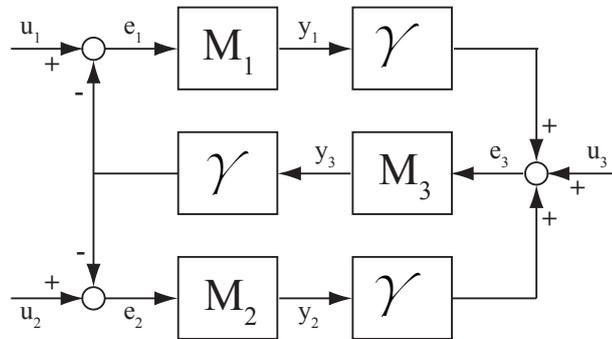


Figure 11: Interconnection 2.

We conclude this section with an example of a “mixed” systems interconnection for which stability is guaranteed. Consider the interconnection of the systems depicted in Figure 11,

and suppose that M_1, M_2 and M_3 are “mixed” with $k_1 = k_2 = k_3 = 0.01$. Let $\gamma = 0.5$. Then $K = 0.01I$ and

$$H = \begin{bmatrix} 0 & 0 & 0.5 \\ 0 & 0 & 0.5 \\ -0.5 & -0.5 & 0 \end{bmatrix}.$$

Since the eigenvalues of $H^T + H + K$ and $I - H^T H$ are positive, the interconnection is stable by Corollaries 5 and 6.

4.6 A TEST FOR “MIXEDNESS” OF DESCRIPTOR SYSTEMS

Given a system generalized state-space description of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= C^T x(t) + Du(t), \end{aligned}$$

where $E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{n \times m}, D \in \mathbb{R}^{m \times m}$ and $\text{rank}(E) = p < n$. We wish to determine whether or not the system is “mixed.” The aim is to construct a transfer function matrix $H(s)$ from the state-space data and determine whether or not there exist $k, l > 0$ and $\varepsilon < 1$ such that $H(s) = D + C^T (sE - A)^{-1} B$ satisfies (i) and/or (ii) from Definition 13 for each frequency $\omega \in \mathbb{R}$. However, we limit our methods to index-one descriptor systems with $H(j\infty) \neq 0$.

To construct a test independent of the frequency variable, we construct the matrices similar to N_1 and N_2 from Lemmas 3 and 4; using state space matrices E, A, B, C, D for $k, l > 0$ and $\varepsilon < 1$ and then calculate the eigenvalues of these matrices. Existences of purely imaginary eigenvalues indicate those frequencies at which the matrices $-kH(j\omega)^* H(j\omega) + H(j\omega)^* + H(j\omega) - lI$ and $H(j\omega)^* H(j\omega) + \varepsilon^2 I$ have zero eigenvalues. We employ the following Lemma to convert the descriptor system transfer function to a *regular* system transfer function.

Theorem 24 Consider the following index-one descriptor transfer function:

$$H(s) = D + C^T (sE - A)^{-1} B. \quad (4.3)$$

Let $E = XY^T$ be a full rank decomposition of the singular matrix E . Then $H(s)$ can be written

$$H(s) = \bar{D} + \bar{C}^T \left(\frac{1}{s} I - \bar{A} \right)^{-1} \bar{B} \quad (4.4)$$

with

$$\begin{aligned} \bar{A} &= Y^T A^{-1} X & \bar{C}^T &= -C^T A^{-1} X \\ \bar{B} &= Y^T A^{-1} B & \bar{D} &= D - C^T A^{-1} B. \end{aligned}$$

PROOF : The proof follows by applying the matrix inversion lemma to (4.4) We have.

$$\begin{aligned}
H(s) &= D + C^T (XY^T s - A)^{-1} B \\
&= D - C^T \left(A^{-1} + A^{-1} X \left(\frac{1}{s} I - Y^T A^{-1} X \right)^{-1} Y^T A^{-1} \right) B \\
&= D - C^T A^{-1} B - C^T A^{-1} X \left(\frac{1}{s} I - Y^T A^{-1} X \right)^{-1} Y^T A^{-1} B
\end{aligned}$$

Recall that the matrix $Y^T A^{-1} X$ is Hurwitz (and invertible) if the descriptor system is index-one. **Q.E.D.**

Using the Lemma stated above we re-write $H_1(j\omega) = -kH(j\omega)^* H(j\omega) + H(j\omega)^* + H(j\omega) - II$ as

$$\begin{aligned}
&-k \left(-\bar{B}^T \left(\frac{1}{j\omega} I + \bar{A}^T \right)^{-1} \bar{C} + \bar{D}^T \right) \left(\bar{C}^T \left(\frac{1}{j\omega} I - \bar{A} \right)^{-1} \bar{B} + \bar{D} \right) \\
&-\bar{B}^T \left(\frac{1}{j\omega} I + \bar{A}^T \right)^{-1} \bar{C} + \bar{D}^T + \bar{C}^T \left(\frac{1}{j\omega} I - \bar{A} \right)^{-1} \bar{B} + \bar{D} - II
\end{aligned}$$

These terms can be rearranged as

$$\begin{aligned}
H_1(j\omega) : &= \left((I - k\bar{D})^T \bar{C}^T \quad -\bar{B}^T \right) \left[\frac{1}{j\omega} I - \begin{pmatrix} \bar{A} & 0 \\ -k\bar{C}\bar{C}^T & -\bar{A}^T \end{pmatrix} \right]^{-1} \begin{pmatrix} \bar{B} \\ \bar{C}(I - k\bar{D}) \end{pmatrix} \\
&\quad -k\bar{D}^T \bar{D} + \bar{D}^T + \bar{D} - II. \tag{4.5}
\end{aligned}$$

Similarly, $H_2(j\omega) = -H(j\omega)^* H(j\omega) + \varepsilon^2 I$ can be arranged as

$$\begin{aligned}
H_2(j\omega) := &\left(-\bar{D}^T \bar{C}^T \quad -\bar{B}^T \right) \left[\frac{1}{j\omega} I - \begin{pmatrix} \bar{A} & 0 \\ -\bar{C}\bar{C}^T & -\bar{A}^T \end{pmatrix} \right]^{-1} \begin{pmatrix} \bar{B} \\ -\bar{C}\bar{D} \end{pmatrix} - \bar{D}^T \bar{D} + \varepsilon^2 I. \tag{4.6}
\end{aligned}$$

Now we formulate the test to check whether $H_1(j\omega)$ and $H_2(j\omega)$ have zero eigenvalues.

Lemma 16 Suppose $k, l \in \mathbb{R}$ and let $H(s) = D + C^T (sE - A)^{-1} B$ represent the transfer function matrix of a stable index-one descriptor system. Assume that $\bar{X}_1 = -k\bar{D}^T \bar{D} + \bar{D}^T + \bar{D} - II$ is invertible, and let $\bar{Y} = I - k\bar{D}$. Then $H_1(j\omega)$ has no zero eigenvalues over $\omega \in [a, b]$ if and only if the matrix N_{E1} given as

$$\begin{bmatrix} -(A^{-1} + A^{-1} B \bar{X}_1 \bar{Y}^T C^T A^{-1}) & A^{-1} B \bar{X}_1^{-1} B^T A^{-T} \\ (C^T A^{-1})^T (kI + \bar{Y} \bar{X}_1^{-1} \bar{Y}^T) (C^T A^{-1}) & (A^{-1} + A^{-1} B \bar{X}_1 \bar{Y}^T C^T A^{-1})^T \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \tag{4.7}$$

has no eigenvalues on the imaginary axis between and including $-ja$ and $-jb$, where $a, b \in \mathbb{R}$.

PROOF: Consider $\det[H_1(j\omega)]$ given as

$$\begin{aligned}
 & \det \left[\begin{pmatrix} \bar{Y}^T \bar{C}^T & -\bar{B}^T \end{pmatrix} \left[\frac{1}{j\omega} I - \begin{pmatrix} \bar{A} & 0 \\ -k\bar{C}\bar{C}^T & -\bar{A}^T \end{pmatrix} \right]^{-1} \begin{pmatrix} \bar{B} \\ \bar{C}\bar{Y} \end{pmatrix} + \bar{X}_1 \right] \\
 &= \det \left[I + \left[\frac{1}{j\omega} I - \begin{pmatrix} \bar{A} & 0 \\ -k\bar{C}\bar{C}^T & -\bar{A}^T \end{pmatrix} \right]^{-1} \begin{pmatrix} \bar{B} \\ \bar{C}\bar{Y} \end{pmatrix} \begin{pmatrix} \bar{X}_1^{-1} \bar{Y}^T \bar{C}^T & -\bar{X}_1^{-1} \bar{B}^T \end{pmatrix} \right] \\
 & \quad \times \det(\bar{X}_1) \\
 &= \det(\bar{X}_1) \cdot \det \left[\left(\frac{1}{j\omega} I - \bar{A} \right)^{-1} \right] \cdot \det \left[\left(\frac{1}{j\omega} I + \bar{A}^T \right)^{-1} \right] \cdot \det \left[\frac{1}{j\omega} I + \bar{N}_1 \right],
 \end{aligned}$$

where $\det \left[\frac{1}{j\omega} I + \bar{N}_1 \right]$ is given by (see Theorem 3)

$$\det \left[\frac{1}{j\omega} I + \begin{bmatrix} -\bar{A} + \bar{B}\bar{X}_1^{-1}\bar{Y}^T\bar{C}^T & -\bar{B}\bar{X}_1^{-1}\bar{B}^T \\ k\bar{C}\bar{C}^T + \bar{C}\bar{Y}\bar{X}_1^{-1}\bar{Y}^T\bar{C}^T & \bar{A}^T - \bar{C}\bar{Y}\bar{X}_1^{-1}\bar{B}^T \end{bmatrix} \right].$$

Since \bar{A} is Hurwitz, then $\det \left[\left(\frac{1}{j\omega} I - \bar{A} \right)^{-1} \right] \neq 0$ and $\det \left[\left(\frac{1}{j\omega} I + \bar{A}^T \right)^{-1} \right] \neq 0$ for all $\omega \in \mathbb{R}$, $H_1(j\omega)$ has a zero eigenvalue if and only if $\det(j\omega I + \bar{N}_1) = 0$, i.e. \bar{N}_1 has a purely imaginary eigenvalue. Of interest are only the frequencies $\omega \in [a, b]$, correspondingly, the eigenvalues of \bar{N}_1 that lie on the imaginary axis between and including $-ja$ and $-jb$. From spectral equivalence we can observe that non-zero eigenvalues of \bar{N}_1 coincide with non-zero eigenvalues of N_{E_1} given by

$$\begin{bmatrix} -A^{-1} - A^{-1}B\bar{X}_1\bar{Y}^T\bar{C}^T A^{-1} & A^{-1}B\bar{X}_1^{-1}B^T A^{-T} \\ (C^T A^{-1})^T (kI + \bar{Y}\bar{X}_1^{-1}\bar{Y}^T) (C^T A^{-1}) & (A^{-1} + A^{-1}B\bar{X}_1\bar{Y}^T\bar{C}^T A^{-1})^T \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}.$$

Q.E.D.

Lemma 17 Suppose $\varepsilon \in \mathbb{R}$ and let $H(s) = D + C^T(sE - A)^{-1}B$ represent the transfer function matrix of a stable index-one descriptor system. Assume that $\bar{X}_2 = -\bar{D}^T \bar{D} + \varepsilon^2 I$ is invertible. Then the matrix $H_2(j\omega)$ has no zero eigenvalues over $\omega \in [a, b]$ if and only if the matrix N_{E_2}

$$= \begin{bmatrix} -A^{-1} + A^{-1}B\bar{X}_2\bar{D}^T C^T A^{-1} & A^{-1}B\bar{X}_2^{-1}B^T A^{-T} \\ (C^T A^{-1})^T (kI + \bar{D}\bar{X}_2^{-1}\bar{D}^T) (C^T A^{-1}) & (A^{-1} - A^{-1}B\bar{X}_2\bar{D}^T C^T A^{-1})^T \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}$$

does not have any eigenvalues on the imaginary axis between and including $-ja$ and $-jb$.

PROOF: Consider $\det[H_2(j\omega)]$ given by

$$\begin{aligned}
& \det \left[\begin{pmatrix} -\bar{D}^T \bar{C}^T & -\bar{B}^T \end{pmatrix} \left[\frac{1}{j\omega} I - \begin{pmatrix} \bar{A} & 0 \\ -\bar{C} \bar{C}^T & -\bar{A}^T \end{pmatrix} \right]^{-1} \begin{pmatrix} \bar{B} \\ -\bar{C} \bar{D} \end{pmatrix} + \bar{X}_2 \right] \\
&= \det \left[I + \left[\frac{1}{j\omega} I - \begin{pmatrix} \bar{A} & 0 \\ -\bar{C} \bar{C}^T & -\bar{A}^T \end{pmatrix} \right]^{-1} \begin{pmatrix} \bar{B} \\ -\bar{C} \bar{D} \end{pmatrix} \begin{pmatrix} -\bar{X}_2^{-1} \bar{D}^T \bar{C}^T & -\bar{X}_2^{-1} \bar{B}^T \end{pmatrix} \right] \\
&\quad \times \det(\bar{X}_2) \\
&= \det(\bar{X}_2) \cdot \det \left[\left(\frac{1}{j\omega} I - \bar{A} \right)^{-1} \right] \cdot \det \left[\left(\frac{1}{j\omega} I + \bar{A}^T \right)^{-1} \right] \cdot \det \left[\frac{1}{j\omega} I + \bar{N}_2 \right],
\end{aligned}$$

where $\det \left[\frac{1}{j\omega} I + \bar{N}_2 \right]$ is given by (see Theorem 4)

$$\det \left[\frac{1}{j\omega} I + \begin{bmatrix} -\bar{A} - \bar{B} \bar{X}_2^{-1} \bar{D}^T \bar{C}^T & -\bar{B} \bar{X}_2^{-1} \bar{B}^T \\ \bar{C} \bar{C}^T + \bar{C} \bar{D} \bar{X}_2^{-1} \bar{D}^T \bar{C}^T & \bar{A}^T + \bar{C} \bar{D} \bar{X}_2^{-1} \bar{B}^T \end{bmatrix} \right].$$

Since \bar{A} is Hurwitz, then $\det \left[\left(\frac{1}{j\omega} I - \bar{A} \right)^{-1} \right] \neq 0$ and $\det \left[\left(\frac{1}{j\omega} I + \bar{A}^T \right)^{-1} \right] \neq 0$ for all $\omega \in \mathbb{R}$. Thus, $H_2(j\omega)$ has a zero eigenvalue if and only if $\det(j\omega I + \bar{N}_2) = 0$, ie: \bar{N}_2 has a purely imaginary eigenvalue. Of interest are only the frequencies $\omega \in [a, b]$, correspondingly, the eigenvalues of \bar{N}_2 that lie on the imaginary axis between and including $-ja$ and $-jb$. From spectral equivalence, we can observe that non zero eigenvalues of \bar{N}_2 coincide with non zero eigenvalues of N_{E2} given by

$$\begin{bmatrix} -A^{-1} + A^{-1} B \bar{X}_2^{-1} \bar{D}^T C^T A^{-1} & A^{-1} B \bar{X}_2^{-1} B^T A^{-T} \\ (C^T A^{-1})^T (kI + \bar{D} \bar{X}_2^{-1} \bar{D}^T) (C^T A^{-1}) & (A^{-1} - A^{-1} B \bar{X}_2^{-1} \bar{D}^T C^T A^{-1})^T \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}$$

Q.E.D.

Now, we use matrices N_{E1} , N_{E2} to construct a simple test to check ‘‘mixedness’’ of $H(s)$. Before proceeding, we note the following:

- there exist $k, l > 0$ such that $-kH(j\omega)^* H(j\omega) + H(j\omega)^* + H(j\omega) - lI \geq 0$ for all $\omega \in [a, b]$ if and only if $H(j\omega)^* + H(j\omega) > 0$ for all $\omega \in [a, b]$;
- under the assumption that $\det(H(j\infty)^* + H(j\infty)) \neq 0$, there exist $k, l > 0$ such that $-kH(j\omega)^* H(j\omega) + H(j\omega)^* + H(j\omega) - lI \geq 0$ for all $\omega \in (-\infty, b]$, $[a, \infty)$ or $(-\infty, \infty)$ if and only if $H(j\omega)^* + H(j\omega) > 0$ for all $\omega \in (-\infty, b]$, $[a, \infty)$ or $(-\infty, \infty)$, respectively;
- there exists $\varepsilon < 1$ such that $-H(j\omega)^* H(j\omega) + \varepsilon^2 I \geq 0$ for all $\omega \in [a, b]$ if and only if $-H(j\omega)^* H(j\omega) + I > 0$ for all $\omega \in [a, b]$;

- under the assumption that $\det(-H(j\infty)^*H(j\infty) + I) \neq 0$, there exists $\varepsilon < 1$ such that $-H(j\omega)^*H(j\omega) + \varepsilon^2 I \geq 0$ for all $\omega \in (-\infty, b]$, $[a, \infty)$ or $(-\infty, \infty)$ if and only if $-H(j\omega)^*H(j\omega) + I > 0$ for all $\omega \in (-\infty, b]$, $[a, \infty)$ or $(-\infty, \infty)$, respectively.

In other words, there are cases in which the free parameters k , l and ε can be eliminated from the test; that is, we can set $k = l = 0$ and $\varepsilon = 1$ when applying Lemmas 32 and 3. Under the assumptions that $\det(H(j\infty)^* + H(j\infty)) \neq 0$ and $\det(-H(j\infty)^*H(j\infty) + I) \neq 0$, Definition 13 becomes:

Definition 14 *An index-one descriptor system with transfer function matrix $H \in \mathcal{RH}_\infty^{m \times m}$ is said to be “mixed” if, at each frequency $\omega \in \mathbb{R}$, either*

- (i) $H(j\omega)^* + H(j\omega) > 0$; or
- (ii) $-H(j\omega)^*H(j\omega) + I > 0$;

or both (i) and (ii) hold.

Now let $k = l = 0$, then $H_1(j\omega) = H(j\omega) + H^*(j\omega)$. Similarly, let $\varepsilon = 1$, then $H_2(j\omega) = -H^*(j\omega)H(j\omega) + I$. Now consider Lemmas 16 and 17 and set

$$\begin{aligned}\Omega_p &:= \{\omega \in [-\infty, \infty] : N_{E1} \text{ has an eigenvalue on the imaginary axis at } j\omega\}, \\ \Omega_s &:= \{\omega \in [-\infty, \infty] : N_{E2} \text{ has an eigenvalue on the imaginary axis at } j\omega\}.\end{aligned}$$

Suppose that we divide the real axis $-\infty$ to ∞ into smaller intervals, where any elements of Ω_p and Ω_s are set as open interval endpoints, as follows

$$\begin{aligned}\text{Division group 1} &:= (-\infty, \omega_{p_1}), (\omega_{p_1}, \omega_{p_2}), \dots, (\omega_{p_{n-1}}, \omega_{p_n}), (\omega_{p_n}, \infty), \\ \text{Division group 2} &:= (-\infty, \omega_{s_1}), (\omega_{s_1}, \omega_{s_2}), \dots, (\omega_{s_{m-1}}, \omega_{s_m}), (\omega_{s_m}, \infty).\end{aligned}$$

where n is the number of elements in Ω_p ; m is the number of elements in Ω_s ; $\omega_{p_1}, \omega_{p_2}, \dots, \omega_{p_n}$ denote the elements of Ω_p listed in increasing order; and $\omega_{s_1}, \omega_{s_2}, \dots, \omega_{s_m}$ denote the elements of Ω_s listed in increasing order.

Let $I_{N_{E1}}$ denote the set of ω belonging to the intervals over which $H_1(j\omega) > 0$ and $I_{N_{E2}}$ denote the set of θ belonging to the intervals over which $H_2(j\omega) > 0$. Then we have the following result.

Theorem 25 *The following two statements are equivalent*

- (i) *an index-one continuous-time system with transfer function matrix $H(s) \in \mathcal{RH}_\infty$, is a “mixed” system;*
- (ii) $I_{N_{E1}} \cup I_{N_{E2}} = \{\theta \in \mathbb{R} : -\infty \leq \omega \leq \infty\}$.

4.7 CONCLUSIONS

The main contributions of this chapter include the derivation of sufficient conditions for the stability of (i) large-scale interconnections of “mixed” systems; and (ii) large-scale, time-varying interconnections of passive systems and systems with small gain. It was shown that relaxing the conditions on the subsystems in a large-scale interconnection (from conditions of passivity or small gain to ones of “mixedness”) results in a need to be extra attentive to the interconnection structure (i.e. more restriction is placed on it) should one wish to employ the conditions derived by Moylan and Hill to determine stability. We also derive a necessary and sufficient test for “mixedness” of descriptor systems.

Part II

SWITCHED DESCRIPTOR SYSTEMS

In this part, we consider the quadratic stability of switched descriptor systems. Initially we consider switching between a special class of descriptor systems having index-one and obtain spectral conditions sufficient to guarantee globally uniform exponential stability. Switching between descriptor systems having index-zero and index-one (or index-one and index-two) is also considered, and a state dependent switching rule is proposed to ensure globally uniform exponential stability. This work was carried out in collaboration with Prof. Martin Corless¹, Prof. Ezra Zeheb² and Prof. Robert Shorten³.

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BACKGROUND AND PRELIMINARY RESULTS

In this chapter, we present some important preliminary results on descriptor systems and regular switched systems. These results will be used in later chapters to derive sufficient conditions for global uniform exponential stability of switched descriptor systems.

5.1 INTRODUCTION

In the last decade, considerable research effort has been dedicated towards stability of switched descriptor systems. In this section, we introduce switched descriptor systems and also present some important developments in this field. In this thesis we are interested in analysing the stability of switching descriptor systems described by

$$\Sigma_s : E(t)\dot{x}(t) = A(t)x(t), \quad (5.1)$$

where $E(t) \in \mathcal{E} = \{A_1, \dots, A_N\}$ and $A(t) \in \mathcal{A} = \{A_1, \dots, A_N\}$. Here, we assume that the matrix pairs (E_i, A_i) always satisfy $\det(sE_i - A_i) \neq 0, \forall i \in \{1, \dots, N\}$. The logical rule that supervises switching between the constituent subsystems $E_i\dot{x}(t) = A_ix(t); i = \{1, 2, \dots, N\}$ generates switching signals, and is usually described by a piecewise constant mapping $\sigma : t \rightarrow \{1, 2, \dots, m\}$. The switching signal $\sigma(t)$ can be formally defined as below.

Definition 15 [66] *A switching signal $\sigma(t)$ is a piecewise constant function $\sigma : \mathbb{R}_+ \rightarrow \{1, 2, \dots, N\}$ with the following properties:*

1. *the points of discontinuity are the sequence of numbers $t_0, t_1, \dots, t_i, t_{i+1}, \dots$;*
2. *there exists a lower bound $\tau_{min} > 0$ for the interval between two consecutive discontinuities t_i, t_{i+1} , such that $t_{i+1} - t_k > \tau_{min}$, for all k ;*
3. *$\sigma(t)$ is continuous from the right, i.e. $\sigma(t) = i$ for $t_i \leq t < t_{i+1}$.*

Using the switching signal described above, we can define the linear switched descriptor systems as a linear time-varying system with piecewise constant linear dynamics given by a family of LTI systems of the form

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x, \quad \sigma(t) \in \{1, \dots, N\}. \quad (5.2)$$

Thus, if σ is continuous at t and $\sigma(t) = i$, the system must satisfy

$$E_i \dot{x}(t) = A_i x(t);$$

hence $x(t)$ must be in the consistency space of (E_i, A_i) .

To complete the description of a switching descriptor system, we must also specify how the system behaves at a point t_* of discontinuity of σ . If σ switches from i to j at t_* , then $x(t_*^-) := \lim_{t \rightarrow t_*, t < t_*} x(t)$ must be in $\mathcal{C}(E_i, A_i)$ and $x(t_*^+) := \lim_{t \rightarrow t_*, t > t_*} x(t)$ must be in $\mathcal{C}(E_j, A_j)$. If $x(t_*^-)$ is not in $\mathcal{C}(E_j, A_j)$, then one has to have a solution which is discontinuous at t_* , and to complete the description, one must specify how $x(t_*^+)$ is obtained from $x(t_*^-)$. Commonly, the switching condition on the state can be described by

$$x(t_*^+) = M_{ji} x(t_*^-) \quad (5.3)$$

when σ switches from i to j at t_* .

Also, switching may be restricted in the sense that one does not switch from i to j at any state $x(t_*^-)$ in $\mathcal{C}(E_i, A_i)$. In this case, the restriction may be described by

$$C_{ji} x(t_*^-) = 0; \quad (5.4)$$

An equivalent description of condition (5.4) can be obtained using the distributional framework [51]. In the distributional framework, the matrices M_{ji} are obtained using spectral projectors (see Definition 2.11).

Theorem 26 [51] *If the consistency projector corresponding to $\sigma(t) = i$ is defined as the spectral projection (see Definition 2.11) onto the right deflating subspace of the matrix pair (E_i, A_i) , then*

$$\Pi(E_i, A_i) = P_r = T \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix} T^{-1}. \quad (5.5)$$

Now consider the linear switched descriptor system (5.2), and assume that $\forall i, j \in \{1, \dots, N\}$: $E_i(I - \Pi_i)\Pi_j = 0$. Then every distributional solution of (5.2) is impulse free (see Remark 1) and is represented by a piecewise-smooth function $x: \mathbb{R} \rightarrow \mathbb{R}^n$. Furthermore, for all solutions $x: \mathbb{R} \rightarrow \mathbb{R}^n$,

$$\forall t \in \mathbb{R} : x(t) = \Pi_{\sigma(t)} x(t-). \quad (5.6)$$

5.2 GENERALIZED QUADRATIC LYAPUNOV FUNCTIONS

In the following, we introduce several different generalized quadratic Lyapunov functions used in the literature to study LTI descriptor systems. Studying the existence of such generalized quadratic Lyapunov functions is a good starting point in the study of switched linear

descriptor systems. We say that a scalar valued function V is a *generalized Lyapunov function* for the LTI system $E_i\dot{x} = A_ix$ if for all non-zero $x \in \mathcal{C} = \mathcal{C}(E_i, A_i)$, we have $V(x) > 0$ and $\dot{V}(x) < 0$; V is a *generalized quadratic Lyapunov function* if V is a Lyapunov function and V can be written as $V(x) = x^T P x$ for some symmetric matrix P ; in this case, we say that P is a *Lyapunov matrix* for the system. The existence of a Lyapunov matrix P guarantees that the system is Globally Uniformly Exponentially Stable (GUES), that is, there are constants $\alpha, \beta > 0$ such that every solution satisfies

$$\|x(t)\| \leq \beta e^{-\alpha(t-t_0)} \|x(t_0)\| \quad \text{for } t \geq t_0. \quad (5.7)$$

The next step towards exponential stability of linear switched descriptor systems would be to find useful criteria to determine whether a given collection of matrix pairs (E_i, A_i) have a common generalized quadratic Lyapunov function. Contrary to *regular LTI* systems, there exist many different formulations of generalized quadratic Lyapunov functions (see, e.g. [54], [55], [56], [57], [58]). We present two different formulations proposed by [59] and [54] for an *LTI* descriptor system given by

$$E\dot{x}(t) = Ax(t). \quad (5.8)$$

These generalized quadratic Lyapunov functions have been used extensively towards stability analysis of linear switched descriptor systems.

Theorem 27 [54] *A pair (E, A) is regular, stable and impulse-free (index-one) if and only if there exists $P \in \mathbb{R}^{n \times n}$ such that*

$$E^T P = P^T E \geq 0, \quad (5.9)$$

$$A^T P + PA < 0. \quad (5.10)$$

These matrix inequalities guarantee the existence of a quadratic Lyapunov function given by $V(x) = x^T E^T P x$.

Theorem 27 is valid only for index-one *LTI* descriptor systems, hence we present an alternative formulation from [59].

Definition 16 [59] *Consider the descriptor system (5.8) with regular matrix pair (E, A) and corresponding consistency space $\mathcal{C}(E, A) \subset \mathbb{R}^n$. Assume there exist a positive definite matrix $P = P^T \in \mathbb{C}^{n \times n}$ and a matrix $Q = Q^T \in \mathbb{C}^{n \times n}$ which is positive definite on $\mathcal{C}(E, A)$ such that the generalized Lyapunov equation*

$$A^T P E + E^T P A = -Q \quad (5.11)$$

is fulfilled. Then

$$V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} : x \mapsto (Ex)^T P Ex \quad (5.12)$$

is called a Lyapunov function for (5.8).

Note that this definition ensures that V is not increasing along solutions, i.e., for any solution $x: \mathbb{R} \rightarrow \mathbb{R}^n$ and all $t \in \mathbb{R}$,

$$\frac{d}{dt}V(x(t)) = -x^T(t)Qx(t) \leq 0$$

and equality only holds for $x(t) = 0$. Furthermore, the property $\ker(E) \cap \mathcal{C}(E,A) = \{0\}$ ensures that V is positive definite on $\mathcal{C}(E,A)$.

Theorem 28 [59] *A descriptor system (5.8) with regular matrix pair (E,A) is asymptotically stable if and only if there exists a Lyapunov function $V: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ for (5.8).*

In this thesis, we provide another alternative generalized quadratic Lyapunov function; which allows us to use the results from the well-understood theory of quadratic Lyapunov functions for regular LTI systems (see Chapter 7).

5.3 EXISTENCE OF A CQLF: TWO SYSTEMS WITH A RANK ONE DIFFERENCE

Analogous to *regular* switched systems, special structures of switched descriptor systems can be exploited to obtain conditions for the existence of a generalized CQLF. [46] extended the well-known result for regular commuting subsystems from [62] to the case of switched descriptor systems. In this thesis, we consider an alternate approach towards extending the results for *regular* linear switched systems to switched descriptor systems. As a part of presenting the necessary background material for our approach, we recall the classical KYP Lemma for regular SISO systems. In this thesis, we focus on obtaining a similar version of KYP Lemma for descriptor systems (see Chapter 7).

KYP Lemma relates the strict positive realness of a transfer function and the existence of quadratic Lyapunov functions [114]. Roughly speaking, Meyer's version of the KYP Lemma [115] can be stated as follows. Let $A \in \mathbb{R}^{n \times n}$ be a Hurwitz matrix. Let $b, c \in \mathbb{R}^n$ and d be a non-negative scalar. Let (A, b) , (A, c) be controllable/observable pairs, respectively. Then, there exists a positive definite matrix $P = P^T \in \mathbb{R}^{n \times n}$, $P > 0$, such that [116]

$$\begin{bmatrix} A & b \\ -c^T & -d \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & b \\ -c^T & -d \end{bmatrix} \leq 0, \quad (5.13)$$

$$A^T P + PA < 0, \quad (5.14)$$

if and only if $H(s) = d + c^T(sI - A)^{-1}b$ is KYP-SPR. There are many extensions of this lemma (for example relaxing the observability/controllability) assumption.

An important alternative statement of the KYP Lemma for SISO systems (b, c vectors) is that strict positive realness of $H(s)$ is equivalent to the existence of $P = P^T > 0$ satisfying either

1. $A^T P + PA < 0$ and $\left(A - \frac{bc^T}{d}\right)^T P + P \left(A - \frac{bc^T}{d}\right) < 0$, when d is strictly positive;
2. **(OR)** $(bc^T)^T P + P(bc^T) \leq 0$ when $d = 0$.

Next, we consider the two cases when $d \neq 0$ and $d = 0$ separately.

When $d \neq 0$, **KYP** strict positive realness is equivalent to the existence of a positive definite matrix P that simultaneously satisfies a pair of Lyapunov equations. When such a P exists, the function $V(x) = x^T P x$ is said to be a **CQLF** for the dynamic systems $\dot{x} = Ax$ and $\dot{x} = \left(A - \frac{bc^T}{d}\right)x$.

From Theorem 7, a regular **SISO** transfer is **SPR** for $d \neq 0$ if and only if the matrix $(A - \frac{1}{d}bc^T)A$ has no eigenvalues on the closed negative real axis $(-\infty, 0]$. These observations lead to the following result.

Theorem 29 [66] *Let A_1 and A_2 be Hurwitz matrices in $\mathbb{R}^{n \times n}$, where the difference $A_1 - A_2$ has rank one. Then the switched system*

$$\dot{x}(t) = A(t)x(t); A(t) \in \mathcal{A} = \{A_1, A_2\} \quad (5.15)$$

*has **CQLF** if and only if the matrix product $A_1 A_2$ has no negative real eigenvalues.*

This condition was originally derived as a frequency domain condition using the **SISO** Circle Criterion by [117], however it was later realised [65] that the condition has a natural and elegant formulation as illustrated in Theorem 29.

Now, we consider the case when $d = 0$. In this scenario, a regular **SISO** transfer is **SPR** if and only if $c^T A b < 0$, and the matrix product $A(A - \frac{A b c^T A}{c^T A b})$ has no eigenvalue on the open negative real axis $(-\infty, 0)$ and exactly one zero eigenvalue. Further, in [68] it was proved that the inequalities

$$\begin{aligned} A^T P + PA &< 0, \\ -\left(cb^T P + Pbc^T\right) &\leq 0. \end{aligned}$$

are equivalent to

$$\begin{aligned} A^T P + PA &< 0, \\ (A - gh^T)^T P + P(A - gh^T) &\leq 0, \end{aligned} \quad (5.16)$$

where $g = \frac{1}{c^T A b} A b$ and $h^T = c^T A$. These observations lead to the following result.

Theorem 30 [68] *Suppose that A is Hurwitz and all eigenvalues of $A - gh^T$ have negative real part, except one, which is zero. Suppose also that (A, g) is controllable and (A, h) is observable. Then, there exists a matrix $P = P^T > 0$ such that*

$$A^T P + PA < 0 \quad (5.17)$$

$$(A - gh^T)^T P + P(A - gh^T) \leq 0 \quad (5.18)$$

if and only if the matrix product $A(A - gh^T)$ has no real negative eigenvalues and exactly one zero eigenvalue.

Thus, Theorem 30 provides necessary and sufficient conditions for the existence of a CQLF for a pair of regular LTI systems $\dot{x}(t) = Ax(t)$ and $\dot{x}(t) = (A - gh^T)x(t)$, one of which is marginally stable.

The necessary and sufficient conditions stated above for regular switched systems are simple and easy to calculate. Hence they promise simple and elegant results for switched descriptor systems upon extension. Thus, we focus on developing similar necessary and sufficient spectral conditions for switched descriptor systems in Chapter 7.

SWITCHED DESCRIPTOR SYSTEMS: INDEX-ONE SYSTEMS AND ARBITRARY SWITCHING

In this chapter, we consider the quadratic stability of a class of switched descriptor systems. In some situations, using spectral characterisations of passivity, compact conditions for quadratic stability are obtained. Examples are given to illustrate our results.

6.1 INTRODUCTION

In this chapter, we consider switched descriptor systems having index-one. We primarily focus on exponential stability of a certain class of index-one switched descriptor systems. We first look at the conditions a quadratic form must satisfy to be a Lyapunov function for an LTI descriptor system. These quadratic forms are similar to the generalized Lyapunov equations proposed earlier in the literature (for more details see, background section 5.2). Based on this, we present conditions for the existence of a piecewise quadratic Lyapunov function for a switching descriptor system. The existence of such a function guarantees stability of the switching system. We further propose a result which permits one to determine the stability of a switching system by looking at the stability properties of a lower-order (state dimension) system. For index-one systems, the lower-order system is a regular switching system. We also derive a KYP-like Lemma for a special class of descriptor systems called index-one systems.

6.2 PRELIMINARY RESULTS

In this section, we present some general results on the stability of switched LTI descriptor systems. We first look at the conditions a quadratic form must satisfy to be a Lyapunov function for an LTI descriptor system. Based on this, we present conditions for the existence of a piecewise quadratic Lyapunov function for a switching descriptor system. The existence of such a function guarantees stability of the switching system.

6.2.1 Quadratic Lyapunov functions for LTI Descriptor Systems

Before we proceed to Lyapunov functions for switching descriptor systems, we present some basic Lyapunov theory in the context of LTI descriptor systems.

Suppose $V(x) = x^T P x$ is a quadratic Lyapunov function and P is a Lyapunov matrix for the descriptor system (2.3). Then, the requirement that $V(x) > 0$ all for non-zero x in \mathcal{C} is equivalent to P being positive-definite on \mathcal{C} , that is, $x^T P x > 0$ for all x in \mathcal{C} . For the requirement that $\dot{V}(x) < 0$ all for non-zero x in \mathcal{C} , we note that

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = \dot{x}^T (P A^{-1} E + E^T A^{-T} P) \dot{x} = -\dot{x}^T Q \dot{x},$$

where Q is given by

$$\boxed{P A^{-1} E + E^T A^{-T} P + Q = 0.} \quad (6.1)$$

Recall that the descriptor system (2.3), is equivalent to the *regular system* (2.9) where x is in \mathcal{C} and \tilde{A} is invertible on \mathcal{C} . Thus $\dot{V} = -x^T \tilde{A}^T Q \tilde{A} x$ and the requirement that $\dot{V}(x) < 0$ for all non-zero x in \mathcal{C} is equivalent to $\tilde{A}^T Q \tilde{A}$ being positive-definite on \mathcal{C} . Since \tilde{A} is a one-to-one mapping of \mathcal{C} onto \mathcal{C} , this requirement is equivalent to Q being positive-definite on \mathcal{C} . Thus, we have the following conclusion:

A symmetric matrix P is a Lyapunov matrix for the descriptor system $E\dot{x}(t) = Ax(t)$ if and only if the matrices P and Q (defined by (10.15)) are positive-definite on the consistency space of the system.

6.2.2 Quadratic stability of switching descriptor systems

Here, we consider a candidate Lyapunov function of the form $V(t, x) = x^T P_{\sigma(t)} x$, which is a time-invariant quadratic function for each constituent system, but which can switch as the system switches. Clearly, for each i , we need $V_i(x) = x^T P_i x$ to be a Lyapunov function for the system defined by (E_i, A_i) . Since we need V to decrease along trajectories, we also need $V(t_*^+, x(t_*^+)) \leq V(t_*^-, x(t_*^-))$ at points t_* of discontinuity of σ . Basically, the next lemma states that satisfaction of these conditions are sufficient to guarantee **GUES**.

Lemma 18 *Consider a switching descriptor system described by (5.2) and suppose that for each $i = 1, \dots, N$, there is a symmetric matrix P_i satisfying the following conditions*

- (a) *The matrix P_i is positive-definite on the consistency space of (E_i, A_i) .*
- (b) *The matrix $P_i A_i^{-1} E_i + E_i^T A_i^{-T} P_i$ is negative-definite on the consistency space of (E_i, A_i) .*
- (c) *If σ switches from i to j at t_* then,*

$$x(t_*^+)^T P_j x(t_*^+) \leq x(t_*^-)^T P_i x(t_*^-). \quad (6.2)$$

*Then the system is **GUES**.*

PROOF: Consider any solution $x(\cdot)$ of the system, and let $v(t) = x(t)^T P_{\sigma(t)} x(t)$. If t is a point of discontinuity of σ , then, by hypothesis (c),

$$v(t^+) \leq v(t^-). \quad (6.3)$$

If t is not a point of discontinuity of σ then,

$$E_i \dot{x}(t) = A_i x(t), \quad (6.4)$$

where $i = \sigma(t)$. It follows from the invertibility of A_i that $x(t) = A_i^{-1} E_i \dot{x}(t)$; hence

$$\dot{v} = \dot{x}^T P_i x + x^T P_i \dot{x} = \dot{x}^T (P_i A_i^{-1} E_i + E_i^T A_i^{-T} P_i) \dot{x} = -\dot{x}^T Q_i \dot{x}, \quad (6.5)$$

where

$$Q_i := -P_i A_i^{-1} E_i - E_i^T A_i^{-T} P_i. \quad (6.6)$$

Recall that system description (6.4) is equivalent to $\dot{x}(t) = \tilde{A}_i x(t)$, where $x(t)$ is in \mathcal{C}_i , the consistency space of (E_i, A_i) and \tilde{A}_i is some invertible map on \mathcal{C}_i . Hence

$$\dot{v} = -x^T \tilde{Q}_i x, \quad \text{where} \quad \tilde{Q}_i = \tilde{A}_i^T Q_i \tilde{A}_i.$$

Since \tilde{A}_i is invertible on \mathcal{C}_i and, by assumption, Q_i is positive-definite on \mathcal{C}_i , it follows that \tilde{Q}_i is positive-definite on \mathcal{C}_i . Recalling that P_i is positive-definite on \mathcal{C}_i , let

$$\alpha_i = \frac{1}{2} \min\{x^T \tilde{Q}_i x : x \in \mathcal{C}_i \text{ and } x^T P_i x = 1\}.$$

Then $\alpha_i > 0$ and $\dot{v} \leq -2\alpha_i v$. Now let

$$\alpha = \min\{\alpha_1, \dots, \alpha_n\}.$$

Then $\alpha > 0$ and

$$\dot{v}(t) \leq -2\alpha v(t) \quad (6.7)$$

when σ is continuous at t .

From this and the discontinuity condition (6.3), we can conclude that $v(t) \leq e^{-2\alpha(t-t_0)} v(t_0)$ for $t \geq t_0$. Since V is positive-definite on each consistency space, there are constants $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 \|x\|^2 \leq V(x) \leq \lambda_2 \|x\|^2$ whenever x is in any of the consistency spaces; hence every solution x satisfies

$$\|x(t)\| \leq \beta e^{-\alpha(t-t_0)} \|x(t_0)\| \quad (6.8)$$

for all $t \geq t_0$, where $\beta = \sqrt{\lambda_2/\lambda_1}$. This means that the system is **GUES. Q.E.D.**

Comment 3 If there exists a symmetric matrix P such that hypothesis (a)–(c) of the above lemma hold for system (5.2) with $P_i = P$ for $i = 1, \dots, N$, we say that this system is *quadratically stable* while P and $V(x) = x^T P x$ are *Lyapunov matrix* and *Lyapunov function*, respectively, for this system.

6.2.3 Order reduction for switching descriptor systems

The following result is useful in reducing the stability problem of a switching descriptor system to that of a lower-order system. Before stating this result, we recall that, if (X, Y) is a full rank decomposition of $E \in \mathbb{R}^{n \times n}$ and $\text{rank}(E) = r$, then, $X, Y \in \mathbb{R}^{n \times r}$ and $\text{rank}(X) = \text{rank}(Y) = r$.

Lemma 19 (Order reduction) *Consider a switching descriptor system described by (5.2) and switching conditions (5.3)-(5.4) when σ switches from i to j and suppose that (X_i, Y_i) is a decomposition of E_i with $Y_i \in \mathbb{R}^{n \times r}$ for $i = 1, \dots, N$. Then, there exist matrices T_1, \dots, T_N such that the following holds. A function $x(\cdot)$ is a solution to system (5.2)-(5.4) if and only if*

$$x(t) = T_{\sigma(t)} z(t) \quad (6.9)$$

for all t , where $z(\cdot)$ is a solution to the descriptor system

$$\tilde{E}_{\sigma(t)} \dot{z} = z \quad (6.10)$$

with switching conditions

$$z(t_*^+) = Y_j^T M_{ji} T_i z(t_*^-), \quad (6.11)$$

$$C_{ji} T_i z(t_*^-) = 0 \quad (6.12)$$

when σ switches from i to j where

$$\tilde{E}_i = Y_i^T A_i^{-1} X_i. \quad (6.13)$$

Moreover,

$$z(t) = Y_{\sigma(t)}^T x(t) \quad (6.14)$$

for all t , $\mathcal{C}(\tilde{E}_i, I) = Y_i^T \mathcal{C}(E_i, A_i)$ and z is continuous during switching if and only if the same is true of $Y_{\sigma}^T x$. Hence, GUES of the new system (6.10)-(6.12) and the original system (5.3)-(5.4) are equivalent.

PROOF: Consider any solution $x(\cdot)$ of the original system (5.2) and let

$$z(t) = Y_{\sigma(t)}^T x(t). \quad (6.15)$$

Suppose t is a point of continuity of σ and $\sigma(t) = i$. Then $z(t) = Y_i^T x(t)$ and $X_i Y_i^T \dot{x} = A_i x$. Since A_i is invertible, we can multiply both sides of the last equation by $Y_i^T A_i^{-1}$ to obtain $Y_i^T A_i^{-1} X_i Y_i^T \dot{x} = Y_i^T x$, that is,

$$\tilde{E}_{\sigma(t)} \dot{z} = z, \quad (6.16)$$

where $\tilde{E}_i = Y_i^T A_i^{-1} X_i$.

We now claim that there is a matrix T_i such that $x(t) = T_i z(t)$. Since $\sigma(t) = i$, it follows that $x(t)$ must be in the consistency space \mathcal{C}_i of (E_i, A_i) . Since \mathcal{C}_i and the kernel of E_i intersect only at zero and the kernel of Y_i^T is contained in the kernel of E_i , it follows that \mathcal{C}_i and the kernel of Y_i^T intersect only at zero. This implies that the restriction of Y_i^T to \mathcal{C}_i yields a one-to-one map from \mathcal{C}_i onto the subspace $Y_i^T \mathcal{C}_i$. Thus, this map has an inverse map T_i from $Y_i^T \mathcal{C}_i$ to \mathcal{C}_i ; hence $x(t) = T_i z(t)$, that is,

$$x(t) = T_{\sigma(t)} z(t). \quad (6.17)$$

Now suppose σ is discontinuous at t_* and switches from i to j ; then

$$x(t_*^-) = T_i z(t_*^-) \quad \text{and} \quad z(t_*^+) = Y_j^T x(t_*^+)$$

and switching conditions (5.3)-(5.4) imply (6.11)-(6.12).

We now show that the consistency space of (\tilde{E}_i, I) is $Y_i^T \mathcal{C}_i$. Considering k sufficiently large, we have

$$\begin{aligned} \mathcal{C}_i &= \text{Im}((A_i^{-1} E_i)^k), \\ \mathcal{C}(\tilde{E}_i, I) &= \text{Im}(\tilde{E}_i^k) = \text{Im}(\tilde{E}_i^{k+1}). \end{aligned}$$

Now note that

$$Y_i^T (A_i^{-1} E_i)^k = Y_i^T (A_i^{-1} X_i Y_i^T)^k = (Y_i^T A_i^{-1} X_i)^k Y_i^T = \tilde{E}_i^k Y_i^T. \quad (6.18)$$

Since $\tilde{E}_i = Y_i^T A_i^{-1} X_i$, we must have

$$\text{Im}(\tilde{E}_i^k) = \text{Im}(\tilde{E}_i^{k+1}) = \text{Im}(\tilde{E}_i^k Y_i^T A_i^{-1} X_i) \subset \text{Im}(\tilde{E}_i^k Y_i^T) \subset \text{Im}(\tilde{E}_i^k).$$

This implies that $\text{Im}(\tilde{E}_i^k Y_i^T) = \text{Im}(\tilde{E}_i^k) = \mathcal{C}(\tilde{E}_i, I)$. It now follows from (6.18) that $Y_i^T \mathcal{C}_i = Y_i^T \text{Im}((A_i^{-1} E_i)^k) = \text{Im}(\tilde{E}_i^k Y_i^T)$. This yields the desired result that $\mathcal{C}(\tilde{E}_i, I) = Y_i^T \mathcal{C}_i$.

Consider now any continuous solution $z(\cdot)$ of the new descriptor system (6.10)-(6.12), and let $x(t) = T_{\sigma(t)} z(t)$. We will show that $x(\cdot)$ is a solution of the original system. Suppose t is a point of continuity of σ and $\sigma(t) = i$. Then $z(t)$ is in $Y_i^T \mathcal{C}_i$ and $x(t) = T_i z(t)$. Since T_i is the inverse of Y_i^T restricted to \mathcal{C}_i , we see that $x(t) \in \mathcal{C}_i$ and $z(t) = Y_i^T x(t)$. Also

$$x = T_i z = T_i \tilde{E}_i \dot{z} = T_i Y_i^T A_i^{-1} X_i Y_i^T \dot{x} = T_i Y_i^T A_i^{-1} E_i \dot{x}.$$

Since x is in \mathcal{C}_i , we have $A^{-1} E_i x$ in \mathcal{C}_i ; recalling that $T_i Y_i^T$ is the identity operator on \mathcal{C}_i we obtain that $T_i Y_i^T A_i^{-1} E_i \dot{x} = A^{-1} E_i \dot{x}$. Thus $x = A_i^{-1} E_i \dot{x}$. Now suppose σ is discontinuous at t_* and switches from i to j ; then

$$x(t_*^-) = T_i z(t_*^-) \quad \text{and} \quad z(t_*^+) = Y_j^T x(t_*^+),$$

and switching conditions (6.11)-(6.12) imply (5.3)-(5.4). **Q.E.D.**

Comment 4 (Impulse-Free Switching) In later sections we will consider systems for which $Y_\sigma^T x$ is continuous during switching; this is equivalent to the following switching condition. If σ switches from i to j at a point of discontinuity t_* , then,

$$\boxed{Y_i^T x(t_*^-) = Y_j^T x(t_*^+).} \quad (6.19)$$

Since $x(t_*^+)$ must be in $\mathcal{C}_j = \mathcal{C}(E_j, A_j)$, the above switch can only occur at states $x(t_*^-)$ in $\mathcal{C}_i = \mathcal{C}(E_i, A_i)$ for which

$$Y_i^T x(t_*^-) \in Y_j^T \mathcal{C}_j. \quad (6.20)$$

The system cannot switch from an arbitrary state in \mathcal{C}_i unless

$$Y_i^T \mathcal{C}_i \subset Y_j^T \mathcal{C}_j. \quad (6.21)$$

Conditions 6.20 and 6.21 are sufficient to ensure *impulse-free switching* and in this thesis we only consider impulse-free switching. Thus, to be able to arbitrarily switch from one subsystem to another, one would need

$$\boxed{Y_i^T \mathcal{C}_i = Y_j^T \mathcal{C}_j \quad \forall \quad i, j = 1, \dots, N.} \quad (6.22)$$

Otherwise, switching has to be restricted.

However, if (E_j, A_j) is index one and $Y_j \in \mathbb{R}^{r \times n}$ is full column rank, where $r = \text{rank}(E)$, then, switching to this system can occur from any state. To see this, recall that the kernel of Y_j^T and \mathcal{C}_j intersect only at the origin, and since the system (E_j, A_j) is index one, the dimension of \mathcal{C}_j is $r = \text{rank}(E_j)$. Hence the dimension of $Y_j^T \mathcal{C}_j$ is r . Since $Y_j^T \in \mathbb{R}^{r \times n}$, we now see that $Y_j^T \mathcal{C}_j = \mathbb{R}^r$; hence (6.20) is satisfied for any $x(t_*^-)$. This means that switching to an index one system can occur from any state.

Note that, for any system, if $E_i = E$ for all i and Ex is continuous, then $Y^T x$ is continuous for any full rank decomposition and (X, Y) of E .

Before proceeding further, we present the following lemma.

Lemma 20 [118] *For matrices $A_{m \times n}$ and $B_{n \times p}$, the following statements are true.*

$$(i) \quad \text{rank}(AB) = \text{rank}(A) \text{ and } \text{Im}(AB) = \text{Im}(A) \text{ if } \text{rank}(B) = n.$$

$$(ii) \quad \text{rank}(AB) = \text{rank}(B) \text{ and } \text{ker}(AB) = \text{ker}(B) \text{ if } \text{rank}(A) = n.$$

6.2.4 Arbitrary switching

As mentioned earlier, the system can arbitrarily switch from $\sigma = i$ to $\sigma = j$ if

$$Y_i^T \mathcal{C}_i = Y_j^T \mathcal{C}_j \quad \text{for all } i, j = 1, \dots, N. \quad (6.23)$$

In this section, we explore some cases under which this condition is satisfied for index-one descriptor systems. If all the descriptor systems (E_i, A_i) for $i = 1, \dots, N$ are index-one, then the consistency space \mathcal{C}_i is given by

$$\mathcal{C}_i = \text{Im}(A_i^{-1}E_i) \quad \text{and} \quad Y_i^T \mathcal{C}_i = \text{Im}(Y_i^T A_i^{-1} X_i Y_i^T). \quad (6.24)$$

Since (E_i, A_i) is index-one $Y_i^T A_i^{-1} X_i \in \mathbb{R}^{r \times r}$ is non-singular, where $r = \text{rank}(E_i)$, we can apply Lemma 20 to conclude that

$$Y_i^T \mathcal{C}_i = \text{Im}(Y_i^T A_i^{-1} X_i Y_i^T) = \text{Im}(Y_i^T A_i^{-1} X_i) = \mathbb{R}^r. \quad (6.25)$$

Hence (6.22) is satisfied as long as $r = \text{rank}(E_i) = \text{rank}(E_j)$ for any $i, j = 1, \dots, N$. In this thesis, we focus on arbitrary switching under the condition that $r = \text{rank}(E_i) = \text{rank}(E_j)$ for any $i, j = 1, \dots, N$ and $Y_\sigma^T x$ is continuous.

For index-one switched descriptor systems, continuity of $Y_\sigma^T x$ provides the initial conditions for the next mode of operation (see condition 6.19). For switched descriptor systems with different indices (for example, switching between index-one and index-two systems), continuity of $Y_\sigma^T x$ also provides the appropriate time-instant at which impulse free switching is possible, thus leading to state dependent switching. We consider switched descriptor systems with different indices, in the next chapter.

Now, we present a similar reduced order system in the input-output framework for an index-one descriptor system. This reduced ordered system will be used later to derive a KYP-like Lemma for SISO index-one descriptor systems.

Theorem 31 *Consider the following index-one descriptor system*

$$\Sigma_n : \begin{cases} E\dot{x}(t) &= Ax(t) + bu(t); \\ y(t) &= c^T x(t) + du(t) \end{cases} \quad (6.26)$$

with $E, A \in \mathbb{R}^{n \times n}$, $b, c \in \mathbb{R}^n$, $d \in \mathbb{R}$ and $\text{rank}(E) = r < n$. Let $E = XY^T$ be a full rank decomposition of the singular matrix E . Then Σ_n can be written as

$$\Sigma_p : \begin{cases} \dot{z}(t) &= \tilde{A}z(t) + \tilde{b}u(t); \\ y(t) &= \tilde{c}^T z(t) + \tilde{d}u(t) \end{cases} \quad (6.27)$$

with

$$\begin{aligned}\tilde{A} &= (Y^T A^{-1} X)^{-1}, & \tilde{b} &= (Y^T A^{-1} X)^{-1} Y^T A^{-1} b, \\ \tilde{c}^T &= c^T A^{-1} X (Y^T A^{-1} X)^{-1}, & \tilde{d} &= d - c^T A^{-1} b + c^T A^{-1} X (Y^T A^{-1} X)^{-1} Y^T A^{-1} b\end{aligned}$$

and $z(t) = Y^T x(t)$.

PROOF: Consider $E\dot{x}(t) = Ax(t) + bu(t)$, and right-multiply both sides by $Y^T A^{-1}$ to obtain

$$Y^T A^{-1} X Y^T \dot{x}(t) = Y^T \dot{x}(t) + Y^T A^{-1} bu(t).$$

Thus $Y^T A^{-1} X \dot{z}(t) = \dot{z}(t) + Y^T A^{-1} bu(t)$ which further implies

$$\dot{z}(t) = (Y^T A^{-1} X)^{-1} \dot{z}(t) + (Y^T A^{-1} X)^{-1} Y^T A^{-1} bu(t).$$

Now consider $y(t) = c^T x(t) + du(t)$ and substitute $x(t) = A^{-1} E \dot{x}(t) - A^{-1} bu(t)$ to obtain

$$\begin{aligned}y(t) &= c^T (A^{-1} E \dot{x}(t) - A^{-1} bu(t)) + du(t) \\ \Rightarrow y(t) &= c^T A^{-1} X \dot{z}(t) + (d - c^T A^{-1} b) u(t).\end{aligned}$$

Again substitute $\dot{z}(t) = (Y^T A^{-1} X)^{-1} \dot{z}(t) + (Y^T A^{-1} X)^{-1} Y^T A^{-1} bu(t)$ to obtain

$$y(t) = c^T A^{-1} X (Y^T A^{-1} X)^{-1} \dot{z}(t) + (d - c^T A^{-1} b + c^T A^{-1} X (Y^T A^{-1} X)^{-1} Y^T A^{-1} b) u(t).$$

By gathering the terms together, we have

$$\begin{aligned}\dot{z}(t) &= \tilde{A} \dot{z}(t) + \tilde{b} u(t); \\ y(t) &= \tilde{c}^T \dot{z}(t) + \tilde{d} u(t)\end{aligned}$$

as claimed.

6.2.5 Lur'e type switched descriptor systems

In section 6.3 and the next chapter (section 7.1) we obtain some very specific results for systems described by

$$E_{\sigma(t)} \dot{x} = A_{\sigma(t)} x, \quad \sigma(t) \in \{1, 2\}, \quad (6.28)$$

$$\text{where } \text{rank}(A_1^{-1} E_1 - A_2^{-1} E_2) = 1. \quad (6.29)$$

In this case, one can readily show that the corresponding switching system (6.28) can be expressed as a feedback combination of the LTI system

$$\begin{aligned}E\dot{x} &= Ax + bu, \\ y &= c^T x + du,\end{aligned} \quad (6.30)$$

subject to the switching output feedback controller

$$u = -k_{\sigma(t)}y \quad \text{with} \quad k_1 = 0, \quad k_2 = 1, \quad (6.31)$$

where

$$c^T = c_1^T - c_2^T A \quad \text{and} \quad d = -c_2^T b. \quad (6.32)$$

This fact can be observed by choosing any column vector c_2 satisfying $c_2^T b = -d$ and then choosing $c_1^T = c^T + c_2^T A$. Then the output y is given by

$$\begin{aligned} y = c^T x + du &= (c_1^T - c_2^T A)x + (-c_2^T b)u \\ &= c_1^T x - c_2^T (Ax + bu) \\ &= c_1^T x - c_2^T E\dot{x}. \end{aligned}$$

When $u = -y$, we have $E\dot{x} = Ax - by$; further substituting $y = c_1^T x - c_2^T E\dot{x}$, we get

$$\begin{aligned} E\dot{x} &= Ax - b(c_1^T x - c_2^T E\dot{x}) \\ \Rightarrow (E - bc_2^T E)\dot{x} &= (A - bc_1^T)x. \end{aligned}$$

Similarly for $u = 0$, we have $E\dot{x} = Ax$. Now we show that the situation in which

$$A_1 = A, \quad E_1 = E, \quad A_2 = A - bc_1^T, \quad E_2 = E - bc_2^T E \quad (6.33)$$

for some column matrices b, c_1 and c_2 ; the rank condition (6.29) holds. To see this, apply a matrix inversion formula to obtain that

$$A_2^{-1} = (A - bc_1^T)^{-1} = A^{-1} + k^{-1}A^{-1}bc_1^T A^{-1},$$

where

$$k = 1 - c_1^T A^{-1} b = 1 + d - c^T A^{-1} b. \quad (6.34)$$

Then computations show that

$$A_2^{-1} E_2 = A_1^{-1} E_1 - gh^T, \quad \text{where} \quad g = A^{-1} b \quad \text{and} \quad h^T = -k^{-1} c^T A^{-1} E. \quad (6.35)$$

Thus, rank condition (6.29) holds.

We will obtain some very specific results for these switching systems under the assumption that $E\dot{x}$ is continuous during switching. Associated with this switching system are the constituent linear systems which we denote as follows

$$\Sigma_1 : E_1 \dot{x} = A_1 x, \quad \Sigma_2 : E_2 \dot{x} = A_2 x.$$

We consider the case in which both of the above systems are index one in Section 6.3. Section 7.1 considers cases when the systems have mixed index.

6.3 RESULTS FOR INDEX-ONE SYSTEMS

In this section, we consider switching descriptor systems for which each constituent system is an index-one system. First, we obtain a general result. We then use this result and Theorem 29 from [65] to obtain a simple spectral characterization of stability for a special class of Lur'e type switching descriptor systems.

The following general result is a corollary of Lemmas 18 and 19.

Corollary 7 *Consider a switching descriptor system described by (5.2) where $Y_\sigma^T x$ is continuous during switching and (X_i, Y_i) is a decomposition of E_i with $Y_i \in \mathbb{R}^{n \times r}$ for $i = 1, \dots, N$. Suppose that there is a symmetric positive-definite matrix P satisfying*

$$P\tilde{E}_i + \tilde{E}_i^T P < 0 \quad \text{for } i = 1, \dots, N \quad (6.36)$$

where $\tilde{E}_i = Y_i^T A_i^{-1} X_i$. Then the system is **GUES**.

Comment 5 Satisfaction of (6.36) implies that each \tilde{E}_i must be Hurwitz. If E_i is singular and (X_i, Y_i) is a full rank decomposition then (E_i, A_i) must be index-one and stable.

6.3.1 A special class of index-one systems

Here, we consider systems described by

$$E_{\sigma(t)} \dot{x} = A_{\sigma(t)} x, \quad \sigma(t) \in \{1, 2\}, \quad (6.37)$$

where each constituent system is index one and stable. We show that if a simple eigenvalue condition holds then, the system is Quadratically Stable (**QS**), hence it is **GUES**. To achieve this result, we utilize Theorem 29 from [65]; in this chapter, we explore the connection between passivity and common Lyapunov functions to obtain compact conditions for the existence of a common solution to the Lyapunov equation for a pair of matrices. The following result follows from Corollary 7 and Theorem 29.

Theorem 32 *Consider a switching descriptor system described by (6.37), where $Y_\sigma^T x$ is continuous during switching and (X_i, Y_i) is a full rank decomposition of E_i with $Y_i \in \mathbb{R}^{n \times r}$ for $i = 1, 2$. Suppose that the following conditions are satisfied, where $\tilde{E}_i = Y_i^T A_i^{-1} X_i$.*

- (a) (E_1, A_1) and (E_2, A_2) are stable.
- (b) (E_1, A_1) and (E_2, A_2) are index-one.
- (c) $\text{rank}(\tilde{E}_1 - \tilde{E}_2) = 1$
- (d) $\tilde{E}_1 \tilde{E}_2$ has no negative real eigenvalues.

Then the switching descriptor system (6.37) is *GUES*.

PROOF: For each $i \in \{1, 2\}$, the pair (X_i, Y_i) is a full rank decomposition of E_i and (E_i, A_i) is index one and stable; hence \tilde{E}_i is Hurwitz (see Corollary 1.) Since $\text{rank}(\tilde{E}_1 - \tilde{E}_2) = 1$ there exist column matrices g and h such that $\tilde{E}_2 = \tilde{E}_1 - gh^T$. Since $\tilde{E}_1 \tilde{E}_2$ has no negative real eigenvalues, it now follows from Theorem 29 that there exists a symmetric positive-definite matrix P such that

$$P\tilde{E}_i + \tilde{E}_i^T P < 0 \quad \text{for } i = 1, 2. \quad (6.38)$$

Corollary 7 now implies that the switching descriptor system (6.37) is *GUES*. **Q.E.D.**

6.3.2 Lur'e type switching systems

Consider the situation in which

$$A_1 = A, \quad E_1 = E, \quad A_2 = A - bc_1^T, \quad E_2 = E - bc_2^T E \quad (6.39)$$

for some column matrices b, c_1 and c_2 where $d = -c_2^T b \neq -1$ and Ex is continuous during switching. Suppose that (X, Y) is a full rank decomposition of E ; then (X_2, Y) is a full rank decomposition of E_2 , where $X_2 = (I - bc_2^T)X$. Also, continuity of Ex and $Y_\sigma^T x$ are equivalent.

As demonstrated in Section 6.3.2, one can readily show that this system can be expressed as a feedback combination of the LTI system (6.30) subject to the switching feedback (6.31). We will show that hypothesis (c) of the previous lemma holds and hypothesis (d) is equivalent to (d') $A_1^{-1} E_1 A_2^{-1} E_2$ has no negative real eigenvalues.

To show that (c) holds, we recall (6.35) which says that

$$A_2^{-1} E_2 = A_1^{-1} E_1 + k^{-1} A^{-1} bc^T A^{-1} E;$$

hence, pre-multiplication by Y^T results in

$$\tilde{E}_2 Y^T = \tilde{E}_1 Y^T - gh^T Y^T,$$

where

$$\tilde{g} = Y^T A^{-1} b \quad \text{and} \quad h^T = -k^{-1} c^T A^{-1} X. \quad (6.40)$$

Since Y^T is full row rank, we must have

$$\tilde{E}_2 = \tilde{E}_1 - gh^T; \quad (6.41)$$

thus, hypothesis (c) holds. To obtain the equivalency of (d) and (d') we first compute that

$$\tilde{E}_1 \tilde{E}_2 = (Y^T A_1^{-1} X)(Y^T A_2^{-1} X_2) = Y^T (A_1^{-1} E_1 A_2^{-1} X_2)$$

Now, we use the fact that the non-zero eigenvalues of MN and NM coincide for any two matrices M and N of compatible dimensions. This implies that the non-zero eigenvalues of $\tilde{E}_1\tilde{E}_2$ are the same as those of

$$(A_1^{-1}E_1A_2^{-1}X_2)Y^T = A_1^{-1}E_1A_2^{-1}E_2.$$

It now follows that (d) and (d') are equivalent.

6.4 KYP-LIKE LEMMA FOR SISO INDEX-ONE SYSTEMS

An alternative statement of the KYP lemma for *regular SISO* systems is that strict positive realness of $H(s) = d + c^T(sI - A)^{-1}b$ with $d > 0$ is equivalent to the existence of a matrix $P = P^T > 0$ satisfying

$$\begin{aligned} A^T P + PA &< 0 \quad \text{and} \\ \left(A - \frac{bc^T}{d}\right)^T P + P \left(A - \frac{bc^T}{d}\right) &< 0. \end{aligned}$$

From Theorem 7, a regular *SISO* transfer is *SPR* for $d > 0$ if and only if the matrix $(A - \frac{1}{d}bc^T)A$ has no eigenvalues on the closed negative real axis $(-\infty, 0]$. Based on these observations, we can conclude that if A and $A - \frac{bc^T}{d}$ are Hurwitz matrices, then the switched system

$$\dot{x}(t) = A(t)x(t); \quad A(t) \in \mathcal{A} = \left\{A, A - \frac{bc^T}{d}\right\}$$

has the *CQLF* $V(x) = x^T P x$ if and only if the matrix product $A \left(A - \frac{bc^T}{d}\right)$ has no negative real eigenvalues (see Theorem 29).

Now, we recall our earlier result from Theorem 14 stating that

$$\begin{aligned} H(s) &= d + c^T(sE - A)^{-1}b \\ &= \tilde{d} + \tilde{c}^T(sI - \tilde{A})^{-1}\tilde{b} \quad (\text{see Theorems 14 and 31}) \end{aligned}$$

is *SPR* if and only if $\tilde{A}^{-1}(\tilde{A}^{-1} + \tilde{A}^{-1}\tilde{b}(\tilde{d} - \tilde{c}^T\tilde{A}^{-1}\tilde{b})^{-1}\tilde{c}^T\tilde{A}^{-1}) = \tilde{E}_1\tilde{E}_2$ has no negative real eigenvalues, where $\tilde{E}_1 = Y^T A^{-1} X$ and $\tilde{E}_2 = Y^T \left(A - \frac{bc^T}{d}\right)^{-1} X$.

Under these circumstances, we can apply Theorems 29 and 32 to show that there exists a Lyapunov function $V(z) = z^T \tilde{P} z > 0$ for $z \neq 0$ such that

$$\begin{aligned} \tilde{E}_1^T \tilde{P} + \tilde{P} \tilde{E}_1 &< 0 \quad \text{and} \\ \tilde{E}_2^T \tilde{P} + \tilde{P} \tilde{E}_2 &< 0. \end{aligned}$$

hence

$$\begin{aligned} E^T A^{-T} P + P A^{-1} E & \text{ is negative definite on } \mathcal{C}(E, A) \text{ and} \\ E^T \left(A - \frac{bc^T}{d} \right)^{-T} P + P \left(A - \frac{bc^T}{d} \right)^{-1} E & \text{ is negative definite on } \mathcal{C} \left(E, A - \frac{bc^T}{d} \right). \end{aligned}$$

where $P = Y \tilde{P} Y^T$ is positive definite on $\mathcal{C}(E, A)$ and $\mathcal{C} \left(E, A - \frac{bc^T}{d} \right)$. Based on this discussion, we can directly state a **KYP**-like Lemma for **SISO** index-one descriptor systems.

Lemma 21 *Let (E, A) and $\left(E, A - \frac{bc^T}{d} \right)$ be two stable index-one descriptor system pairs. Then the following statements are equivalent:*

- (i) $d + \Re\{c^T(j\omega E - A)^{-1}b\} > 0$ for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$.
- (ii) There exists a matrix $P = P^T$, positive definite on $\mathcal{C}(E, A)$ and $\mathcal{C} \left(E, A - \frac{bc^T}{d} \right)$, such that
 - (a) $E^T A^{-T} P + P A^{-1} E$ is negative definite on $\mathcal{C}(E, A)$ and
 - (b) $E^T \left(A - \frac{bc^T}{d} \right)^{-T} P + P \left(A - \frac{bc^T}{d} \right)^{-1} E$ is negative definite on $\mathcal{C} \left(E, A - \frac{bc^T}{d} \right)$.

6.5 EXAMPLES

Example 1 (Switching between index one descriptor systems)

Consider the switched descriptor system:

$$E\dot{x} = A(t)x, \quad A(t) \in \{A_1, A_2\}$$

with

$$E = \begin{bmatrix} 3 & 7 & 8 \\ 1 & 3 & 1 \\ 3 & 7 & 8 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix},$$

where E is continuous during switching. Note (E, A_1) , (E, A_2) are both stable systems and that the switched system satisfies the assumptions of Lemma 2. Stability of the switched descriptor system follows from the fact that $E A_1^{-1} E A_2^{-1} E$ has no negative real eigenvalues. Note also that a full rank decomposition of E is given by

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 7 \end{bmatrix}. \quad (6.42)$$

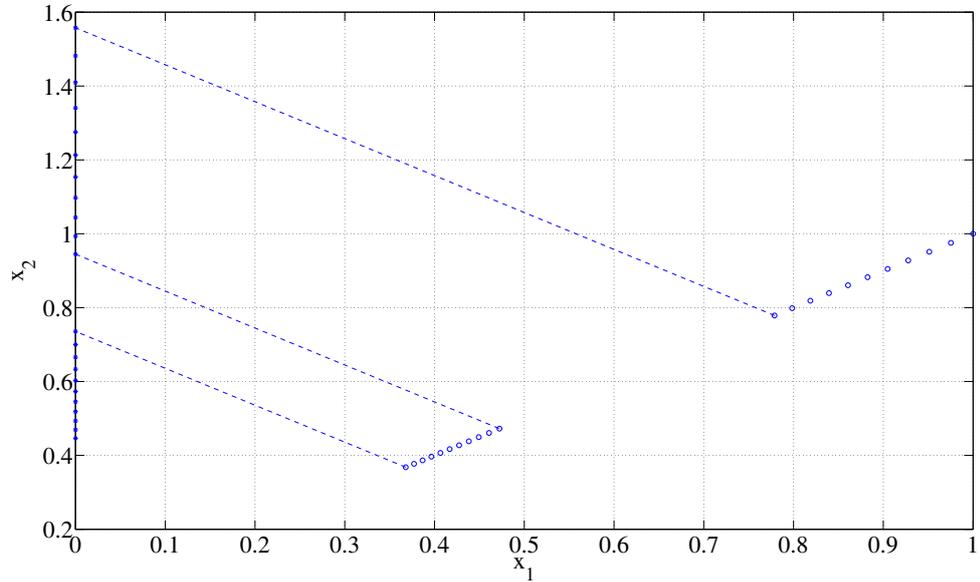


Figure 12: Solution trajectories for (E, A_1) and (E, A_2) on the constraint surfaces $x_1 - x_2 = 0$ and $x_1 = 0$ are represented by “o” and “*”, respectively. Jumps in the states during switching are guided by the assumption that Ex remains continuous during switching.

The reduced order matrices admit a common Lyapunov solution, since both

$$Y^T A_1^{-1} X = \begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix}, \quad Y^T A_2^{-1} X = \begin{bmatrix} -3 & 0 \\ -3 & -2 \end{bmatrix}, \quad (6.43)$$

are stable and both are lower triangular (however, this is only apparent after a full rank decomposition) [119].

Example 2 (Switching between index one descriptor systems)

Consider the following switching descriptor system

$$E\dot{x} = A(t)x; \quad A(t) \in \{A_1, A_2\},$$

where

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Assume further that $E\dot{x}$ is continuous for all t . A full rank decomposition of E is given by

$$X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Both (E, A_1) and (E, A_2) are stable and index-one descriptor systems. Also

$$\text{rank}(Y^T A_1^{-1} X - Y^T A_2^{-1} X) = 1$$

and the eigenvalue of $Y^T A_1^{-1} X Y^T A_2^{-1} X$ is 2. Hence from Theorem 32, the switched descriptor system is GUES about zero for arbitrary switching. A solution to the system is depicted in Figure 12 with $(x_2(0), x_1(0)) = (1, 1)$, and A_1 and A_2 are switched periodically with an arbitrary time period $T = 0.5\text{sec}$.

6.6 CONCLUSIONS

In this chapter, we derived results on the quadratic stability of a class of index-one switched descriptor systems. We obtain, these results by exploiting the relationship between the strict positive realness property of an LTI system and quadratic stability of the switched system. We further showed that some of our ideas also apply to nonlinear Lur'e-like descriptor systems.

SWITCHED DESCRIPTOR SYSTEMS: HIGHER INDEX SYSTEMS AND STATE DEPENDENT SWITCHING

In this chapter, we consider a special case of switching between descriptor systems having different indices. Conditions for stability are obtained for such systems under special state dependent switching conditions.

7.1 INTRODUCTION

In this chapter, we consider classes of switching systems where not all constituent systems are index-one descriptor systems. We deal with two particular scenarios here.

1. switching between index-zero (*regular system*) and index-one descriptor systems.
2. switching between index-one and a particular class of index-two descriptor systems.

Such systems arise in various situations; in particular when the structure of a controller changes in response to an external command.

7.2 PRELIMINARY RESULTS

7.2.1 State Dependent Switching

In this chapter, we are interested in analysing the stability properties a switched system

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x, \quad \sigma(t) \in \{1, \dots, N\}, \quad (7.1)$$

where the matrix pairs (E_i, A_i) may not have the same index for all i . We recall from section 6.2.4 that, if (E_i, A_i) is index one and (X_i, Y_i) is the full rank decomposition of E_i then

$$Y_i^T \mathcal{C}_i = \mathbb{R}^r \quad \text{for all } i, j = 1, \dots, N. \quad (7.2)$$

where $r = \text{rank}(E_i)$. Thus, at a point of discontinuity t_* , if σ switches from i to j we have $Y_i^T x(t_*^-) \in Y_j^T \mathcal{C}_j$ thereby allowing arbitrary impulse-free switching between index-one descriptor systems. For index-two systems

$$Y_i^T \mathcal{C}_i = \text{Im}(Y_i^T A_i^{-1} X_i Y_i^T) = \text{Im}(Y_i^T A_i^{-1} X_i) \subset \mathbb{R}^r, \quad (7.3)$$

because $Y_i^T A_i^{-1} X_i$ is singular for index-two systems. Hence we can always find $x(t_*^-)$ such that $Y_i^T x(t_*^-) \notin Y_j^T \mathcal{C}_j$. Thus we cannot arbitrarily switch to an index-two system. This fact motivates state dependent switching when one of the constituent subsystems (E_i, A_i) has index-two. The state dependent switching rule proposed in this chapter is an assumption that $Y_\sigma^T x$ (or x) is continuous during switching. This assumption of continuity provides the appropriate time-instant t_* at which impulse free switching is possible.

Initially we consider switching between a *regular system* and an index-one system, and then we use this result to analyse switching between index-one and index-two descriptor systems.

The following lemma is useful in our results for switching descriptor systems with constituent systems having different indices.

Lemma 22 *Suppose $A, B \in \mathbb{R}^{n \times n}$ with $\text{rank}(A - B) = \text{rank}(A) - \text{rank}(B)$ and*

$$\begin{aligned} A + A^T &> 0, \\ B + B^T &\geq 0. \end{aligned}$$

Then, the kernels of B and $B + B^T$ are equal.

PROOF: Since $Q_A > 0$, where $Q_A := A + A^T$, we see that $\text{rank}(A) = n$. Let $r := \text{rank}(B)$. Then, by assumption, we have $\text{rank}(A - B) = n - r$. Recall that the nullity of a matrix is the dimension of its kernel. First, we show that the nullity of $Q_B := B + B^T$ is at most $n - r$. So, suppose that $x \neq 0$ is in the kernel of Q_B . Then

$$0 = x^T Q_B x = x^T (A + A^T)x + 2x^T (B - A)x = x^T Q_A x - 2x^T (A - B)x,$$

Since $Q_A > 0$ and $x \neq 0$, we have $x^T Q_A x > 0$; hence $(A - B)x \neq 0$, that is, x is not in the null-space of $A - B$. Thus, the kernel of Q_B and $A - B$ intersect only at zero. Since the rank of $A - B$ is $n - r$, its nullity is r ; hence the nullity of Q_B is at most $n - r$.

We now show that the kernel of Q_B contains the kernel of B . Therefore, suppose that x is in the kernel of B , that is, $Bx = 0$. Then

$$x^T Q_B x = 2x^T Bx = 0.$$

Since $Q_B \geq 0$, it follows that $Q_B x = 0$, that is, x is in the kernel of Q_B . Thus, the kernel of Q_B contains the kernel of B .

Since B has rank r , its nullity is $n - r$. Since we also know that the nullity of Q_B is less than or equal to $n - r$, it now follows that the kernel of Q_B is the same as the kernel of B . **Q.E.D.**

7.3 SWITCHING BETWEEN INDEX-ZERO AND INDEX-ONE SYSTEMS

We first obtain this general result from Lemmas 18 and 22.

Lemma 23 *Consider a switching descriptor system described by (5.2) where x is continuous during switching. Suppose that, for some $N_1 \leq N$, there is a symmetric positive-definite matrix P satisfying*

$$PA_i^{-1}E_i + (A_i^{-1}E_i)^T P < 0, \quad i = 1, \dots, N_1 \quad (7.4)$$

$$PA_j^{-1}E_j + (A_j^{-1}E_j)^T P \leq 0, \quad j = N_1 + 1, \dots, N \quad (7.5)$$

and for each $j \in \{N_1 + 1, \dots, N\}$ there is an index $i_j \in \{1, \dots, N_1\}$ such that

$$\text{rank}(A_{i_j}^{-1}E_{i_j} - A_j^{-1}E_j) = \text{rank}(A_{i_j}^{-1}E_{i_j}) - \text{rank}(A_j^{-1}E_j). \quad (7.6)$$

Then, the system is *GUES*.

PROOF: We prove this result by showing that the hypotheses of Lemma 18 hold. Since P is positive definite, hypothesis (a) holds. Also, the continuity of $x(\cdot)$ implies that hypothesis (c) holds. To see that hypothesis (b) holds, we apply Lemma 22 with $A = -PA_j^{-1}E_j$ and $B = -PA_j^{-1}E_j$ to obtain that the kernel of $Q_j := -PA_j^{-1}E_j - (A_j^{-1}E_j)^T P$ is the same as that of $-PA_j^{-1}E_j$ which also equals the kernel of $A_j^{-1}E_j$; thus Q_j and $A_j^{-1}E_j$ have the same kernel. Since $Q_j \geq 0$ and the kernel of $A_j^{-1}E_j$ and $\mathcal{C}(E_j, A_j)$ intersect only at zero, we conclude that Q_j is positive definite on the consistency space of (E_j, A_j) . Hypothesis (b) now follows by taking into account (7.4). It now follows from Lemma 18 that the switching system (6.37) is *GUES*. **Q.E.D.**

7.3.1 A special class of index-0 and index-one systems

Here we consider systems described by

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x, \quad \sigma(t) \in \{1, 2\} \quad (7.7)$$

where each constituent system is stable, with the first being index zero and the second index one; also the rank of $A_1^{-1}E_1 - A_2^{-1}E_2$ is one. We show that if the matrix $A_1^{-1}E_1A_2^{-1}E_2$ has no negative real eigenvalues, exactly one eigenvalue at zero and some other regularity conditions hold then, the system is *QS*; hence it is stable. To achieve this result, we utilize Theorem 30 from [68]. The following result follows from Lemma 23 and Theorem 30.

Theorem 33 *Consider a switching descriptor system described by (7.7) where x is continuous during switching and suppose that it satisfies the following conditions*

- (a) (E_1, A_1) and (E_2, A_2) are stable.

(b) (E_1, A_1) is index-zero and (E_2, A_2) is index-one.

(c) There exists column matrices g and h such that

$$A_2^{-1}E_2 = A_1^{-1}E_1 - gh^T, \quad (7.8)$$

where $(A_1^{-1}E_1, g)$, $(A_1^{-1}E_1, h)$ are controllable and observable, respectively.

(d) The matrix $A_1^{-1}E_1A_2^{-1}E_2$ has no negative real eigenvalues and exactly one zero eigenvalue.

Then the switching descriptor system (7.7) is globally uniformly exponentially stable about zero.

PROOF: We first show that the hypotheses of Theorem 30 hold with $A = A_1^{-1}E_1$. For $i = 1, 2$, (E_i, A_i) is stable; hence the non-zero eigenvalues of $A_i^{-1}E_i$ have negative real parts. Since (A_1, E_1) is index zero, $A_1^{-1}E_1$ is nonsingular and has no zero eigenvalues. This implies that $A_1^{-1}E_1$ is Hurwitz.

Since $A_1^{-1}E_1A_2^{-1}E_2$ has exactly one eigenvalue at zero, its nullity is one; the non-singularity of $A_1^{-1}E_1A_2^{-1}$ now implies that the nullity of E_2 is one; hence the rank of E_2 and $A_2^{-1}E_2$ is $n - 1$. Since (E_2, A_2) has index one and the nullity of E_2 is one, the matrix $A_2^{-1}E_2$ has a single eigenvalue at zero. Thus, all eigenvalues of $A_2^{-1}E_2$ have negative real part except one which is zero.

Recalling hypotheses (c) and (d) in this theorem, we see that the hypotheses of Theorem 30 hold with $A = A_1^{-1}E_1$. Hence there exists a matrix $P = P^T > 0$ such that

$$PA_1^{-1}E_1 + (A_1^{-1}E_1)^T P < 0, \quad (7.9)$$

$$PA_2^{-1}E_2 + (A_2^{-1}E_2)^T P \leq 0. \quad (7.10)$$

Since $\text{rank}(A_1^{-1}E_1 - A_2^{-1}E_2) = \text{rank}(gh^T) = 1 = \text{rank}(A_1^{-1}E_1) - \text{rank}(A_2^{-1}E_2)$, Lemma 23 now implies that the switching descriptor system (7.7) is globally uniformly exponentially stable about zero. **Q.E.D.**

Comment 6 (Switching) The above result requires x to be continuous during switching. It follows from (7.4) that $A_1^{-1}E_1$ is nonsingular; this implies that the system (E_1, A_1) is a regular system and its consistency space is the whole state space. Hence switching to this system can occur at any state. Except in the trivial case that $A_2^{-1}E_2 = A_1^{-1}E_1$, the matrix E_2 is singular which implies that the index of (E_2, A_2) is at least one; hence the consistency space of this system is not the whole state space. Thus, the switching system does not switch to the second linear system from an arbitrary point in the state space. To switch to the second system, the state must be in the consistency space of that system.

7.3.2 Lur'e type switching systems

Consider the situation in which the system is a Lur'e type system as described in Section 6.2.5. Then hypothesis (c) holds with

$$g = A^{-1}b \quad \text{and} \quad h^T = k^{-1}c^T A^{-1}E \quad \text{where} \quad k = 1 + d - c^T A^{-1}b. \quad (7.11)$$

Assuming (E_1, A_1) is index zero is equivalent to E being non-singular. If (E_2, A_2) is index-one then $E_2 = (I - bc_2^T)E$ is singular; this is equivalent to $d = -c_2^T b = -1$.

Example 3 Switching between index-zero and index-one descriptor systems

Consider a mixed switched system of the form

$$E_{\sigma(t)}\dot{x} = Ax,$$

where $\sigma(t) \in \{1, 2\}$ and x is continuous. For

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix},$$

note that (E_1, A) is a stable *regular system* and the pair (E_2, A) is a stable index one descriptor system. Note also that $A^{-1}E_1 - A^{-1}E_2 = gh^T$, where $g^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $h^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and the pairs $(A^{-1}E_1, g)$ and $(A^{-1}E_1, h)$ are controllable and observable, respectively. The eigenvalues of $A^{-1}E_1A^{-1}E_2$ are $(1.75, 0)$. Hence from Theorem 33, the switched system described above is globally uniformly exponentially stable about zero.

7.4 SWITCHING BETWEEN INDEX ONE AND INDEX TWO SYSTEMS

Now to conclude, we consider switching between index-one and index-two descriptor systems. First we have the following result which is a corollary of Lemmas 19 and 23.

Corollary 8 Consider a switching descriptor system described by (5.2) where $Y_\sigma^T x$ is continuous during switching and (X_i, Y_i) is a decomposition of E_i with $Y_i \in \mathbb{R}^{n \times r}$ for $i = 1, \dots, N$. Suppose that, for some $N_1 \leq N$, there is a symmetric positive-definite matrix P such that the following conditions are satisfied, where $\tilde{E}_i = Y_i^T A_i^{-1} X_i$.

$$P\tilde{E}_i + \tilde{E}_i^T P < 0, \quad i = 1, \dots, N_1 \quad (7.12)$$

$$P\tilde{E}_j + \tilde{E}_j^T P \leq 0, \quad j = N_1 + 1, \dots, N \quad (7.13)$$

and for each $j \in \{N_1 + 1, \dots, N\}$ there is an index $i_j \in \{1, \dots, N_1\}$ such that

$$\text{rank}(\tilde{E}_{i_j} - \tilde{E}_j) = \text{rank}(\tilde{E}_{i_j}) - \text{rank}(\tilde{E}_j). \quad (7.14)$$

Then, the system is globally uniformly exponentially stable about zero.

7.4.1 A special class of index-one and index-two systems

The following result can be obtained from Lemma 19 and Theorem 33.

Theorem 34 Consider a switching descriptor system described by

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x, \quad \sigma(t) \in \{1, 2\}, \quad (7.15)$$

where $Y_{\sigma}^T x$ is continuous during switching and (X_i, Y_i) is a full rank decomposition of E_i with $Y_i \in \mathbb{R}^{n \times r}$ for $i = 1, 2$. Suppose that the following conditions are satisfied where $\tilde{E}_i = Y_i^T A_i^{-1} X$ for $i = 1, 2$.

- (a) (E_1, A_1) and (E_2, A_2) are stable.
- (b) (E_1, A_1) is index one and (E_2, A_2) is index two.
- (c) There exists, vectors g and h such that

$$\tilde{E}_2 = \tilde{E}_1 - gh^T \quad (7.16)$$

with (\tilde{E}_1, g) controllable and (\tilde{E}_1, h) observable.

- (d) $\tilde{E}_1 \tilde{E}_2$ has no negative real eigenvalues and exactly one zero eigenvalue.

Then the switched descriptor system (7.15) is globally uniformly exponentially stable about zero.

PROOF: Recall Lemma 19 on reduced order systems. Since (X_1, Y_1) is a full rank decomposition of E_1 and (A_1, E_1) is stable and index-one, its corresponding reduced order system (\tilde{E}_1, I) is stable and index-zero. Since (X_2, Y_2) is a full rank decomposition of E_2 and (A_2, E_2) is stable and index-two, its corresponding reduced order system (\tilde{E}_2, I) is stable and index one. Theorem 33 now guarantees GUES of the reduced-order switching system. Lemma 19 now implies the same stability properties for the switching system (7.15). **Q.E.D.**

7.4.2 Lur'e type switching systems

Consider the situation in which the system is a Lur'e type system as described in Section 6.2.5 with $d = -c_2^T b = -1$ and Ex is continuous during switching. Suppose (X, Y) is a full rank decomposition of E . Then (X_2, Y) is a decomposition of E_2 , where $X_2 = (I - bc_2^T)X$. This is not a full rank decomposition since X_2 does not have maximum column rank. Also, continuity of Ex and $Y_{\sigma}^T x$ are equivalent.

We will show that hypotheses (c) and (d) of the above lemma are equivalent to the following hypotheses.

(c') The matrices

$$Q_c := \begin{bmatrix} EA^{-1}b & (EA^{-1})^2b & \cdots & (EA^{-1})^rb \end{bmatrix}, \quad (7.17)$$

$$(7.18)$$

$$Q_o := \begin{bmatrix} c^T A^{-1}E \\ c^T (A^{-1}E)^2 \\ \vdots \\ c^T (A^{-1}E)^r \end{bmatrix} \quad (7.19)$$

have maximum rank $r = \text{rank}(E)$, where $c^T = c_1^T - c_2^T A$.

(d') $A_1^{-1}E_1A_2^{-1}E_2$ has no negative real eigenvalues and $n - r + 1$ eigenvalues at zero.

In Section 6.3.2, we have already seen that $\tilde{E}_2 = \tilde{E}_1 - gh^T$, where

$$g = Y^T A^{-1}b \quad \text{and} \quad h^T = -k^{-1}c^T A^{-1}X \quad (7.20)$$

and $k = k = 1 + d - c^T A^{-1}b = -c^T A^{-1}b$.

We now show that controllability of (\tilde{E}_1, g) is equivalent to Q_c having rank r . The pair (\tilde{E}_1, g) is controllable if and only if the controllability matrix

$$\tilde{Q}_c = \begin{bmatrix} g & \tilde{E}g & \cdots & \tilde{E}^{r-1}g \end{bmatrix}$$

has rank r . Since X has full column rank, the above controllability matrix has the same rank as $X\tilde{Q}_c$. Noting that, for any $k = 0, 1, \dots$,

$$X\tilde{E}^k = X(Y^T A^{-1}X)^k = (XY^T A^{-1})^k X = (EA^{-1})^k X,$$

we obtain that

$$X\tilde{E}^k g = (EA^{-1})^k XY^T A^{-1}g = (EA^{-1})^{k+1}g.$$

Hence, $X\tilde{Q}_c = Q_c$ which yields the desired result.

We now show that observability of (\tilde{E}_1, h) is equivalent to Q_o having rank r . Since $h^T = -k^{-1}c^T A^{-1}X$, we see that the pair (\tilde{E}_1, h) is observable if and only if the matrix

$$\tilde{Q}_o = \begin{bmatrix} \hat{h}^T \\ \hat{h}^T \tilde{E} \\ \vdots \\ \hat{h}^T \tilde{E}^r \end{bmatrix}$$

has rank r , where $\hat{h}^T = c^T A^{-1} X$. Since Y^T has full row rank, the above observability matrix has the same rank as $\tilde{Q}_o Y^T$. Noting that, for any $k = 0, 1, \dots$,

$$\tilde{E}^k Y^T = (Y^T A^{-1} X)^k Y^T = Y^T (A^{-1} X Y^T)^k = Y^T (A^{-1} E)^k,$$

we obtain that

$$\hat{h}^T \tilde{E}^k Y^T = c^T A^{-1} X Y^T (A^{-1} E)^k = c^T (A^{-1} E)^{k+1}.$$

Hence $\tilde{Q}_o Y^T = Q_o$. This yields the desired result.

To obtain the equivalency of (d) and (d') , we observe that

$$\begin{aligned} \tilde{E}_1 \tilde{E}_2 &= (Y^T A_1^{-1} X)(Y^T A_2^{-1} X_2) = Y^T (A_1^{-1} E_1 A_2^{-1} X_2) \\ A_1^{-1} E_1 A_2^{-1} E_2 &= (A_1^{-1} E_1 A_2^{-1} X_2) Y^T. \end{aligned}$$

The desired result now follows from the fact that, for any two matrices $M, N \in \mathbb{R}^{n \times r}$, the eigenvalues of MN^T are the eigenvalues of $N^T M$ plus $n - r$ eigenvalues at zero.

7.5 CONCLUSIONS

In this chapter, we provided a state dependent switching rule associated with a simple spectral condition under which switching between index zero and index one or index-one and index-two descriptor systems is [GUES](#). These ideas also extend to nonlinear Lur'e-like descriptor systems.

Part III

DISCRETIZATION

In this part, we explore passivity and mixedness preserving discretization methods for descriptor systems. We also consider the stability preserving properties of diagonal Padé approximations to the matrix exponential. We show that while diagonal Padé approximations preserve quadratic stability when going from continuous-time to discrete-time, the converse is not true.

As part of exploring other properties of diagonal Padé approximations, [120] proved that for linear time-invariant systems, certain types of polyhedral Lyapunov functions are preserved by diagonal Padé approximations, under the assumption that the continuous-time system matrix A_c has distinct eigenvalues. In this part, we show that this result also holds true in the case that A_c has non-trivial Jordan blocks. This work was carried out in collaboration with Prof. Martin Corless¹, Prof. Patrizio Colaneri², Dr. Francesco Rossi³, Selim Solmaz⁴ and Prof. Robert Shorten⁵

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BACKGROUND AND PRELIMINARY RESULTS

In this chapter, we introduce discretization methods used to preserve passivity and Lyapunov stability of regular LTI systems. These methods will be used in later chapters to develop similar methods for LTI descriptor systems.

8.1 INTRODUCTION

In this chapter, we present some existing results on the topics discretization of LTI systems and linear switched systems. Our primary goal is to find suitable discretization methods which preserve the properties of passivity, mixedness and Lyapunov stability of descriptor systems. Our approach to find appropriate discretization methods for descriptor systems is based on the understanding of similar discretization methods for *regular systems*. However, there exist no prior work on the topic of discretization of *regular* mixed systems. Hence we focus primarily on passivity preserving and Lyapunov stability preserving discretization methods in this background chapter.

8.2 PRESERVING PASSIVITY UNDER DISCRETIZATION

Discretization methods for LTI systems can be classified into two broad categories:

1. discretizing the continuous-time transfer functions and
2. discretizing the continuous-time state space model.

Initially, we consider discretization methods for continuous-time transfer functions.

8.2.1 Discretizing the transfer function

In this thesis, we primarily focus on Tustin's Method owing to its passivity preserving nature, as understood from the following Lemma. For $-\pi \leq \theta \leq \pi$ and $-\infty \leq \omega \leq \infty$, we can state the following.

Lemma 24 [69] *Let $H(j\omega)$ be a continuous-time transfer function and assume that $H(j\omega)$ is discretized by the Tustin discretization method with sampling period h . The discrete-time transfer function is given by*

$$G(e^{j\theta}) = H\left(\frac{2e^{j\theta} - 1}{h e^{j\theta} + 1}\right). \quad (8.1)$$

If $H(j\omega)$ is continuous-time positive real, then $G(e^{j\theta})$ is discrete-time positive real.

Now, we consider the following continuous-time state space model:

$$\Sigma_c : \begin{cases} \dot{x}(t) = A_c x(t) + B_c u(t), \\ y(t) = C_c^T x(t) + D_c u(t), \end{cases} \quad (8.2)$$

with the corresponding transfer function

$$H_c(j\omega) = D_c + C_c^T (j\omega I - A_c)^{-1} B_c. \quad (8.3)$$

Tustin's method of numerical approximation for the transfer function $H_c(s)$ results in

$$H_d(j\omega) = H_c \left(\frac{2 e^{j\theta} - 1}{h e^{j\theta} + 1} \right) = D_d + C_d^T (e^{j\theta} I - A_d)^{-1} B_d,$$

where

$$\begin{aligned} A_d &= (I - \frac{h}{2} A_c)^{-1} (I + \frac{h}{2} A_c), & B_d &= \frac{h}{2} \left(I - \frac{h}{2} A_c \right)^{-1} B_c, \\ C_d^T &= 2C_c^T (I - \frac{h}{2} A_c)^{-1}, & D_d &= D_c + \frac{h}{2} C_c^T \left(I - \frac{h}{2} A_c \right)^{-1} B_c. \end{aligned}$$

8.2.2 Discretizing the state space model

Here, we consider discretization of a *regular* continuous-time state space model given by

$$\Sigma_c : \begin{cases} \dot{x}(t) = A_c x(t) + B_c u(t), \\ y(t) = C_c^T x(t) + D_c u(t). \end{cases} \quad (8.4)$$

Assuming the presence of a **ZOH** element on its input with a sampling time $h > 0$, it can be shown that the following discrete-time state space model expresses the evolution of the state x along a discrete time axis:

$$\Sigma_d : \begin{cases} x[(k+1)h] = A_d x[kh] + B_d u[kh], \\ y[kh] = C_d^T x[kh] + D_d u[kh], \end{cases} \quad (8.5)$$

where

$$A_d = e^{A_c h}, \quad (8.6)$$

$$B_d = A_c^{-1} (e^{A_c h} - I) B_c, \quad (8.7)$$

$$C_d = C_c, \quad (8.8)$$

$$D_d = D_c. \quad (8.9)$$

Numerical methods to evaluate the matrix exponential $e^{A_c h}$ are given by Padé approximation methods (see Definition 18). However, the selection of a numerical method to evaluate the matrix exponential does not influence the passivity property. Because passivity is not an intrinsic property of the system, it depends on the choice appropriate input and output variables. Hence we postpone the discussion on Padé approximation methods to the later section.

A passivity preserving method for the state space models has been proposed by [72, 73, 74]. Traditional discretization methods for state space models do not change the output (see 8.5) and they do not preserve passivity. Hence a different output has been defined by [72, 73, 74], such that the discrete-time state space model is passive whenever the continuous-time model is passive. This new output can be defined as

$$y^*(kh) = \int_{kh}^{kh+h} y(\tau) d\tau \quad (8.10)$$

$$= C_d^{T*} x(kh) + D_d^* u(kh). \quad (8.11)$$

then

$$C_d^{T*} = C_c^T A_c^{-1} (e^{A_c h} - I), \quad (8.12)$$

$$D_d^* = C_c^T A_c^{-2} (e^{A_c h} - I - A_c h) B_c + D_c h. \quad (8.13)$$

Theorem 35 [76][75] *Using the output y^* defined above, the following relationships can be stated between the continuous-time system Σ_c , and the discrete-time equivalent Σ_d :*

1. *If Σ_c is passive, then Σ_d is passive.*
2. *If Σ_c is strictly-input passive, then Σ_d is strictly-input passive.*
3. *If Σ_c is strictly-output passive, then Σ_d is strictly-input passive.*

There exists, no equivalent methods for preserving passivity of descriptor systems. This task will be carried out in the next chapter.

8.3 PRESERVING LYAPUNOV STABILITY UNDER DISCRETIZATION

We introduce some new notation in this part of the thesis. The 2-measure of a square matrix X is defined as $\mu_2(X) = \frac{1}{2} \lambda_{\max}(X + X^T)$ and the 2-norm as $\|X\|_2 = \sqrt{\lambda_{\max}(X^T X)}$. Also, letting X_{ij} be the entries of X , we define the ∞ -measure as $\mu_\infty(X) = \max_i (X_{ii} + \sum_{j \neq i} |X_{ij}|)$ and the ∞ -norm as $\|X\|_\infty = \max_i \sum_j |X_{ij}|$. Let $W \in \mathbb{R}^{m \times n}$ be a weight matrix, and consider the Hölder p -vector norms $V_p(x) = \|Wx\|_p = (\sum_{i=1}^m |w_i^T x|^p)^{1/p}$ $1 \leq p \leq \infty$, where w_i is the i th row of the weight matrix W .

Also, we denote the convex-hull operator using $\text{conv}\{\}$. Given a finite set of points x_1, x_2, \dots, x_N , the convex-hull is the convex combination of its points, given by

$$\text{conv}\{x_i\}_{i=1}^N = \{\sum_{i=1}^N \alpha_i x_i \mid \alpha_i \in \mathbb{R}^+ \cup \{0\}, \sum_{i=1}^N \alpha_i = 1\}$$

and the interior of set S is denoted using $\text{int}(S)$.

We begin with a *regular* continuous-time **LTI** system given by

$$\Sigma_c : \dot{x} = A_c x \quad (8.14)$$

and the corresponding discrete-time **LTI** system with sampling time $h > 0$ given by

$$\Sigma_d : x[(k+1)h] = A_d(h)x[kh]; A_d(h) = C(A_c, h). \quad (8.15)$$

A square matrix A_c is said to be *Hurwitz stable* if all of its eigenvalues lie in the **OLHP**. A square matrix A_d is said to be *Schur stable* if all its eigenvalues lie in the open interior of the unit disc. Assume that A_c is Hurwitz stable and A_d is Schur stable. The following Theorems, present the necessary and sufficient conditions under which $V_p(x) = \|Wx\|_p$ is a Lyapunov function for the systems (8.14) and (8.15)

Theorem 36 [121] $V_p(x)$ is a Lyapunov function for the system (8.14) if there exist $W_c \in \mathbb{R}^{m \times n}$, $m \geq n$, $\text{rank}(W_c) = n$, and $Q_c \in \mathbb{R}^{m \times m}$ such that

$$W_c A_c - Q_c W_c = 0, \quad \mu_p(Q_c) < 0. \quad (8.16)$$

Theorem 37 [121] $V_p(x)$ is a Lyapunov function for the system (8.15) if there exist $W_d \in \mathbb{R}^{m \times n}$, $m \geq n$, $\text{rank}(W_d) = n$, and $Q_d \in \mathbb{R}^{m \times m}$ such that

$$W_d A_d - Q_d W_d = 0, \quad \|Q_d\|_p < 1. \quad (8.17)$$

Quadratic Lyapunov functions ($p = 2$): For the special case when $p = 2$, the existence of a Lyapunov function $V_2(x) = \|Wx\|_2 = (\sum_{i=1}^m |w_i^T x|^2)^{1/2}$ enforces the existence of a *quadratic Lyapunov function* $\|Wx\|^2 = x^T P x$, where $P = W^T W$ is termed as *Lyapunov matrix* for the continuous-time case and *Stein matrix* for the discrete-time case. Indeed, in the continuous-time case,

$$A_c^T P + P A_c = W^T (Q_c + Q_c^T) W < 0,$$

whereas in the discrete-time case,

$$A_d^T P A_d - P = W^T (Q_d^T Q_d - I) W < 0.$$

Polyhedral Lyapunov functions ($p = \infty$ or 1): For the case when $p = \infty$, we recall a pair of results for special polyhedral Lyapunov functions from [122], [123], for the continuous-time and discrete-time case, respectively.

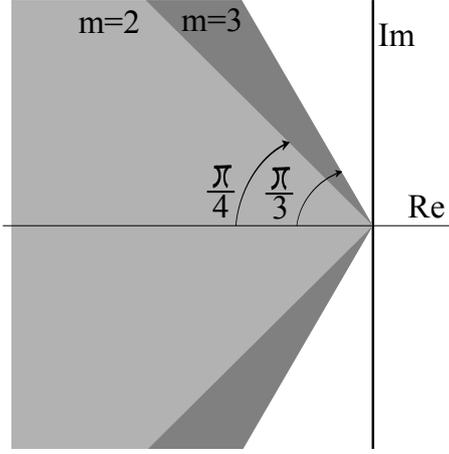


Figure 13: The sectors $\mathcal{S}_c(m)$ for $m = 2$ (angle $\pi/4$), $m = 3$ (angle $\pi/3$).

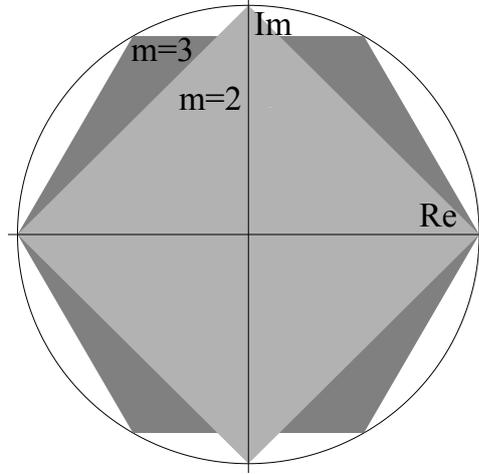


Figure 14: The polygons for $m = 2$ (square), $m = 3$ (hexagon).

Lemma 25 Consider a Hurwitz stable matrix A_c , with distinct eigenvalues, with n_r real and $2n_c$ complex eigenvalues. For each pair of conjugate complex eigenvalue $\lambda_i = \alpha_i \pm j\beta_i$, $i = 1, 2, \dots, n_c$, take an integer m_i such that λ_i lies in the sector $\mathcal{S}_c(m_i)$, where

$$\mathcal{S}_c(m) = \{\lambda = -\alpha + j\beta : \alpha > 0, |\beta| < \frac{\sin(\frac{\pi}{m})}{1 - \cos(\frac{\pi}{m})} \alpha\}. \quad (8.18)$$

Then there exists $W_c \in \mathcal{R}^{N \times n}$ and $Q_c \in \mathcal{R}^{N \times N}$, with $N = \sum_{i=1}^k m_i + n_r$, satisfying (8.16). ■

In Figure 13, the sectors $\mathcal{S}_c(m)$ are drawn for $m = 2$ (angle $\pi/4$), $m = 3$ (angle $\pi/3$).

Lemma 26 Consider a Schur stable matrix A_d , with distinct eigenvalues, with n_r real and $2n_c$ complex eigenvalues. For each pair of conjugate complex eigenvalue $\lambda_i = \sigma_i \pm j\omega_i$, $i = 1, 2, \dots, n_c$, take an integer m_i such that λ_i lies in the interior of the regular polygon $\mathcal{P}_{ol}(m_i)$, where

$$\mathcal{P}_{ol}(m) = \text{int conv} \left\{ e^{j\frac{p\pi}{m}} \right\}_{p=0}^{2m-1}. \quad (8.19)$$

Then there exists $W_d \in \mathcal{R}^{N \times n}$ and $Q_d \in \mathcal{R}^{N \times N}$, with $N = \sum_{i=1}^k m_i + n_r$, satisfying (8.17). ■

In Figure 14 the polygons $\mathcal{P}_{ol}(m)$ are depicted for $m = 2$ (square), $m = 3$ (hexagon). The two Lemmas above have been shown to be valid also in case of multiple eigenvalues. As

shown in [122], matrix W_c can be constructed for distinct eigenvalues as follows. Let T_c be the state-space transformation that puts A_c in its real Jordan form, i.e.

$$T_c A_c T_c^{-1} = \begin{bmatrix} H_{c1} & 0 & \cdots & 0 & 0 \\ 0 & H_{c2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & H_{cn_c} & 0 \\ 0 & 0 & 0 & 0 & R_c \end{bmatrix} \quad \text{where } H_{ci} = \begin{bmatrix} -\alpha_i & \beta_i \\ -\beta_i & -\alpha_i \end{bmatrix}$$

where and R_c is a $n_r \times n_r$ diagonal matrix accounting for the real eigenvalues. Moreover let, for $i = 1, 2, \dots, n_c$:

$$W_{ci} = \begin{bmatrix} 1 & 0 \\ \cos(\frac{\pi}{m_i}) & \sin(\frac{\pi}{m_i}) \\ \cos(\frac{2\pi}{m_i}) & \sin(\frac{2\pi}{m_i}) \\ \vdots & \vdots \\ \cos(\frac{(m_i-1)\pi}{m_i}) & \sin(\frac{(m_i-1)\pi}{m_i}) \end{bmatrix}, \quad Q_{ci} = \begin{bmatrix} x_i & y_i & 0 & 0 & 0 \\ 0 & x_i & y_i & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & x_i & y_i \\ -y_i & 0 & 0 & 0 & x_i \end{bmatrix}$$

where

$$x_i = -\alpha_i - \frac{\beta_i \cos(\frac{\pi}{m_i})}{\sin(\frac{\pi}{m_i})}, \quad y_i = \frac{\beta_i}{\sin(\frac{\pi}{m_i})}$$

Then, it is easy to verify that (8.16) is satisfied with

$$W_c = \begin{bmatrix} W_{c1} & 0 & \cdots & 0 & 0 \\ 0 & W_{c2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & W_{cn_c} & 0 \\ 0 & 0 & \cdots & 0 & I \end{bmatrix} T_c, \quad (8.20)$$

$$Q_c = \begin{bmatrix} Q_{c1} & 0 & \cdots & 0 & 0 \\ 0 & Q_{c2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & Q_{cn_c} & 0 \\ 0 & 0 & \cdots & 0 & R_c \end{bmatrix} \quad (8.21)$$

Notice indeed that $\mu_\infty(Q_c) < 0$ is forced by the assumption on the position of the eigenvalues that is equivalent to $x_i + |y_i| < 0$.

The computation of the polyhedral Lyapunov function $\|W_d x_d\|_\infty$ for the discrete-time system (8.15) follows the same lines and can be found in [123]. Let T_d the state-space transformation that puts A_d in its real Jordan form, i.e.

$$T_d A_d T_d^{-1} = \begin{bmatrix} H_{d1} & 0 & \cdots & 0 & 0 \\ 0 & H_{d2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & H_{dn_c} & 0 \\ 0 & 0 & \cdots & 0 & R_d \end{bmatrix} \quad \text{where} \quad H_{di} = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \quad (8.22)$$

and R_d is a $n_r \times n_r$ diagonal matrix accounting for the real eigenvalues. It can be verified that (8.17) is met by choosing $W_d = W_c T_c^{-1} T_d$ and Q_d as in (13), with R_c replaced by R_d and Q_{ci} replaced by

$$Q_{di} = \begin{bmatrix} z_{1,i} & z_{2,i} & \cdots & z_{m-1,i} & z_{m,i} \\ -z_{m,i} & z_{1,i} & \cdots & z_{m-2,i} & z_{m-1,i} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -z_{3,i} & -z_{4,i} & \cdots & z_{1,i} & z_{2,i} \\ -z_{2,i} & -z_{3,i} & \cdots & -z_{m,i} & z_{1,i} \end{bmatrix}$$

where $z_{j,i}$ are such that $\sum_{j=1}^m |z_{j,i}| < 1$. Notice that $\|Q_d\|_\infty < 1$ is forced by the assumption on the position of the eigenvalues in the regular polygon $\mathcal{P}_{ol}(m_i)$.

8.3.1 Discretization of $\dot{x} = Ax$

Our primary interest in this part of the thesis is to examine the invariance of Lyapunov functions under discretization. Since the closed form solution of (8.14) is given by $x(t) = e^{A_c(t-t_0)}x(t_0)$ for $t \geq t_0$. For a sampling time $h > 0$ and $k = 0, 1, 2, \dots$, if we set $t_0 = kh$ and $t = (k+1)h$, we obtain the discrete-time approximation given by $x[(k+1)h] = e^{A_c h} x[kh]$. Hence, discretization of a continuous-time LTI system involves the calculation of matrix exponential, which may not yield satisfactory results [86]. One must therefore rely on numerical methods that are able to approximate the solution of a differential equation to any desired accuracy. In this thesis, we mainly focus on single step RK methods [87].

One can study the behaviour of Runge-Kutta methods with a simple scalar linear differential equation given by (see Dahlquist Criterion in [87])

$$\dot{y} = \lambda y \quad \text{with} \quad \lambda \in \mathbb{C}. \quad (8.23)$$

The exact solution $y(t) = e^{\lambda(t-t_0)}y_0$ remains bounded as $t \rightarrow \infty$ when $\Re[\lambda] \leq 0$. This observation motivates the following definition.

Definition 17 [87] (A-Stability): Consider the continuous system $\dot{x} = Ax$ and the corresponding discrete time system given by $x[(k+1)h] = C(Ah)x[kh]$, with step size $h > 0$ and $k = 0, 1, 2, \dots$. Then $C(z)$ is A-stable, if it satisfies

$$|C(z)| \leq 1 \text{ for } \Re[z] \leq 0. \quad (8.24)$$

Such A-Stable approximations (e.g. bilinear transform) are widely exploited in systems and control as they map the open left half of the complex plane to the interior of the unit disc. This implies the well known fact that these maps preserve stability of LTI systems as stated formally in the following lemma.

Lemma 27 (Preservation of stability) Suppose that A_c is a Hurwitz stable matrix and, for any sampling time $h > 0$, let $A_d = C(A_c h)$ be an A-Stable approximation of $e^{A_c h}$. Then A_d is Schur stable.

An implicit Runge-Kutta method for (8.23) is of the form

$$y[(k+1)h] = C(h\lambda)y[kh] \quad (8.25)$$

with a rational function $C(z)$ (also called stability function), which is a p^{th} order approximation to the exponential at the origin if

$$C(z) = e^z + \mathcal{O}(z^{p+1}) \text{ as } z \rightarrow 0. \quad (8.26)$$

Thus, it is of interest to study rational functions of given degrees that approximate the exponential function the best. Padé approximations $C_{[L/M]}$ (see Definition 18) with numerator degree L and denominator degree M with highest possible order $p = L + M$ are used to approximate these functions. Also note that Padé approximations with $L \leq M \leq L + 2$ are A-stable.

Definition 18 (Padé Approximations [88], [124], [89]) : The $[L/M]$ Padé approximation to the exponential function e^z is the rational function $C_{L/M}$ defined by

$$C_{L/M}(z) = \frac{Q_L(z)}{Q_M(-z)}, \quad (8.27)$$

where

$$\begin{aligned} Q_L(z) &= \sum_{k=0}^L l_k z^k, & Q_M(z) &= \sum_{k=0}^M m_k z^k, \\ l_k &= \frac{(L+M-k)!L!}{(L+M)!k!(L-k)!}, & m_k &= \frac{(L+M-k)!M!}{(L+M)!k!(M-k)!}. \end{aligned} \quad (8.28)$$

Thus the $[L/M]$ Padé approximation to $e^{A_c h}$, the matrix exponential with sampling time h , is given by

$$C_{[L/M]}(A_c h) = Q_L(A_c h) Q_M^{-1}(-A_c h) \quad (8.29)$$

where $Q_L(A_c h) = \sum_{k=0}^L c_k(A_c h)^k$ and $Q_M(A_c h) = \sum_{k=0}^M c_k(A_c h)^k$

In this thesis, we only consider A-stable Padé approximations. Another important criterion for the selection of an approximation method is its absolute monotonicity. In some situations, the radius of absolute monotonicity is used to select the appropriate step size. The property of absolute monotonicity can be understood through the following Definitions and Theorems.

Definition 19 [125](**Absolute monotonicity**): A function $C(z) : \mathbb{R} \rightarrow \mathbb{R}$ is absolutely monotonic at x if $C(z)$, and all of its derivatives exist and are non-negative at $z = x$.

Theorem 38 [125] For $r > 0$, a polynomial $C(z)$ is absolutely monotonic at $z = -r$ if and only if it is absolutely monotonic on the interval $z \in (-r, 0]$.

Definition 20 [125](**Radius of absolute monotonicity**): The radius of absolute monotonicity r_C of a function $C(z) : \mathbb{R} \rightarrow \mathbb{R}$ is the largest value of $r = r_C$ such that $C(z)$, and all of its derivatives exist and are non-negative for $z \in (-r_C, 0]$.

Several numerical methods have been developed to compute r_C for a given approximation function $C(z)$ [125].

8.3.2 Bilinear transform

The $[1/1]$ -order diagonal Padé approximation $C_{[1/1]}(z)$ (also known as Bilinear Transform or Tustin Transform) is a popular discretization method for control and communication engineers. We now present some of its properties. Consider continuous-time system (8.14) and its discrete-time equivalent (8.15).

Theorem 39 [90] If the discrete time matrix A_d is obtained using $[1/1]$ order Padé approximation given by

$$A_d(h) = C_{[1/1]}(A_c h) = \left(I + A_c \frac{h}{2} \right) \left(I - A_c \frac{h}{2} \right)^{-1}, \quad (8.30)$$

then the following observations can be stated.

1. The Lyapunov function class $V_2(x)$ is preserved during discretization.
2. The Lyapunov function classes $V_\infty(x)$ and $V_1(x)$ are not preserved during discretization.

It should be noted that the converse to statement 1 is also true, i.e., if $V_2(x)$ is a Lyapunov function for the discrete time system (8.15), then $V_2(x)$ is also a Lyapunov function for the continuous-time system (8.14). Statement 2 can be proved using a counterexample; that a given polyhedral Lyapunov function in continuous-time may not be a Lyapunov function for the sampled discrete-time system obtained via the bilinear transformation (with a fixed sampling time). However, it was also proved in [120], that if an eigenvalue λ of a continuous-time matrix belongs to $\mathcal{S}_c(m)$ (defined in (8.18)) then its image under a diagonal Padé transformation (for any order and any sampling time h) belongs to $\mathcal{P}_{ol}(m)$ (defined in (8.19)). This result allows us to conclude that a continuous-time polyhedral Lyapunov function of the form $V_\infty = \|Wx\|_\infty$, is preserved under diagonal Padé approximation if A_c has distinct eigenvalues [120]. In this part we focus on the validity of this result for a more general case, when A_c has non-trivial Jordan blocks.

8.3.3 Discretization of switched systems

Consider a *regular* continuous-time switched system of the form

$$\Sigma_{sc} : \dot{x}(t) = A_c(t)x(t), \quad A_c(t) \in \mathcal{A}_c = \{A_{c1}, \dots, A_{cN}\} \quad (8.31)$$

with its approximate discrete-time counterpart,

$$\Sigma_{sd} : x(k+1) = A_d(k)x(k), \quad A_d(k) \in \mathcal{A}_d = \{A_{d1}, \dots, A_{dN}\} \quad (8.32)$$

Given a finite set of Hurwitz stable matrices \mathcal{A}_c , a matrix P is a Common Lyapunov Matrix (CLM) for \mathcal{A}_c if $A_c^T P + P A_c < 0$ for all A_c in \mathcal{A}_c . In this case, we say that the continuous-time switching system (8.31) is QS with Lyapunov function $V(x) = x^T P x$ and V is a CQLF for \mathcal{A}_c . Given a finite set of Schur stable matrices \mathcal{A}_d a matrix P is a Common Stein Matrix (CSM) for \mathcal{A}_d if $A_d^T P A_d - P < 0$ for all A_d in \mathcal{A}_d . In this case, we say that the discrete-time switching system (8.32) is QS with Lyapunov function $V(x) = x^T P x$ and V is a CQLF for \mathcal{A}_d .

A notable result concerning common quadratic Lyapunov functions is that they are invariant under $[1/1]$ -order Padé approximation. This means that, given a set of Hurwitz stable matrix $\{A_{ci}\}$ and $P = P^T > 0$ satisfying

$$A_{ci}^T P + P A_{ci} < 0,$$

then

$$C_{[1/1]}(A_{ci}h)^T P C_{[1/1]}(A_{ci}h) < P, \quad \forall h > 0.$$

The following corollary is easily deduced from the above discussion.

Corollary 9 *Suppose that $V(x(t)) = x^T(t) P x(t)$ is a CQLF for a finite set of matrices \mathcal{A}_c . Then $V(x(k)) = x^T(k) P x(k)$ is CQLF for any finite set of matrices \mathcal{A}_d , where each A_d in \mathcal{A}_d is a $[1/1]$ -order Padé approximation of $e^{A_c h}$ for some A_c in \mathcal{A}_c and $h > 0$.*

PROOF : If P is a CLM for \mathcal{A}_c then, P is an Lyapunov matrix for every A_c in \mathcal{A}_c . It now follows from Theorem 39, that P is a Stein matrix for every A_d in \mathcal{A}_d . Hence P is a CSM for \mathcal{A}_d . **Q.E.D.**

PRESERVATION OF PASSIVITY AND MIXEDNESS FOR DESCRIPTOR SYSTEMS

In this chapter, we show that Tustin’s method of discretization is also valid for preserving passivity and mixedness of index-one descriptor system transfer functions. We also consider the output averaging method to discretize the descriptor system state space models while preserving passivity. For both cases, we derive the corresponding discrete time state space matrices in terms of the original continuous time matrices.

9.1 INTRODUCTION

In this chapter, we consider discretization methods for descriptor systems which preserve the passivity and mixedness properties. Initially, we consider two different passivity preserving discretization methods:

1. discretizing the continuous-time transfer functions using Tustin transform (see Theorem 24) and
2. discretizing the continuous-time state space model using output averaging method (see Theorem 35).

These well-established methods for *regular systems* are extended for index-one descriptor systems. We further consider the problem of preserving “mixed” property for both *regular* and index-one descriptor systems transfer functions using Tustin transform.

9.2 PRELIMINARY RESULTS

In this section, we present some preliminary results necessary to discretize a continuous-time descriptor system. Initially, we use Lemmas 5, 6, 7 to reduce a descriptor system to an equivalent lower-order *regular* system. This can be achieved by iteratively applying Lemma 19 to achieve order reduction and index reduction while preserving stability of the original descriptor system. This is possible provided that there is a decomposition (\tilde{X}, \tilde{Y}) of \tilde{E} with $\tilde{X}, \tilde{Y} \in \mathbb{R}^{r \times \tilde{r}}$ with $\tilde{r} < r$. Since a square matrix always has a full rank decomposition, one can always iteratively reduce a single linear system (E, A) to a regular system.

9.2.1 Reduction to a regular system

Let

$$E\dot{x} = Ax \quad (9.1)$$

be an index- k descriptor system such that $E = X_1 Y_1^T$. Then we obtain the new reduced index system given by

$$E_1(Y_1^T \dot{x}) = (Y_1^T x), \text{ where } E_1 = Y_1^T A^{-1} X_1. \quad (9.2)$$

From Lemma 7, the reduced-order system (9.2) has an index- $(k-1)$, and we continue with full rank decomposition of E_1 such that $E_1 = X_2 Y_2^T$ and

$$E_2(Y_2^T Y_1^T \dot{x}) = (Y_2^T Y_1^T x), \text{ where } E_2 = Y_2^T X_2. \quad (9.3)$$

We continue this procedure k times such that E_{k-1} has a full rank decomposition of the form $E_{k-1} = X_k Y_k^T$ and

$$E_k(Y_k^T \dots Y_2^T Y_1^T \dot{x}) = (Y_k^T \dots Y_2^T Y_1^T x), \text{ where } E_k = Y_k^T X_k. \quad (9.4)$$

Since $E\dot{x} = Ax$ is an index- k descriptor system, (9.4) is an index-zero descriptor system (regular system) and E_k is non-singular.

In the following, we present some preliminary results.

Lemma 28 *If $w(t) = Y_k^T \dots Y_2^T Y_1^T x(t) = 0$, then $x(t) = 0$.*

PROOF: Consider

$$E_{k-1}(Y_{k-1}^T \dots Y_2^T Y_1^T \dot{x}) = (Y_{k-1}^T \dots Y_2^T Y_1^T x) \text{ where } E_{k-1} = Y_{k-1}^T X_{k-1}. \quad (9.5)$$

Now, we use full rank decomposition to obtain $E_{k-1} = X_k Y_k^T$ and

$$E_k(Y_k^T \dots Y_2^T Y_1^T \dot{x}) = (Y_k^T \dots Y_2^T Y_1^T x) \text{ where } E_k = Y_k^T X_k. \quad (9.6)$$

If \mathcal{C} is the consistency space of $E\dot{x} = Ax$, i.e. $x \in \mathcal{C}$, then Lemma 19 states that $Y_{k-1}^T \dots Y_2^T Y_1^T \mathcal{C}$ is the consistency space of (9.5) and $Y_k^T Y_{k-1}^T \dots Y_2^T Y_1^T \mathcal{C}$ is the consistency space of (9.6). Recall that the consistency space $Y_{k-1}^T \dots Y_2^T Y_1^T \mathcal{C}$ and the kernel of Y_k^T intersect only at origin. Hence if

$$Y_k^T Y_{k-1}^T \dots Y_2^T Y_1^T x(t) = 0, \text{ then } Y_{k-1}^T \dots Y_2^T Y_1^T x(t) = 0. \quad (9.7)$$

Continuing this procedure k times, we obtain $x(t) = 0$. **Q.E.D.**

Lemma 29 $E_k Y_k^T \dots Y_1^T = Y_k^T \dots Y_1^T A^{-1} E$.

PROOF: Consider

$$\begin{aligned}
E_k Y_k^T \dots Y_1^T &= (Y_k^T X_k) Y_k^T \dots Y_1^T \\
&= Y_k^T \underline{X_k Y_k^T} Y_{k-1}^T \dots Y_1^T \\
&= Y_k^T E_{k-1} Y_{k-1}^T \dots Y_1^T \quad (\text{substituting } X_k Y_k^T = E_{k-1}) \\
&= Y_k^T Y_{k-1}^T X_{k-1} Y_{k-1}^T \dots Y_1^T \\
&= Y_k^T Y_{k-1}^T \underline{X_{k-1} Y_{k-1}^T} Y_{k-2}^T \dots Y_1^T \\
&= Y_k^T Y_{k-1}^T E_{k-2} Y_{k-2}^T \dots Y_1^T \quad (\text{substituting } X_{k-1} Y_{k-1}^T = E_{k-2}) \\
&= Y_k^T Y_{k-1}^T Y_{k-2}^T \underline{X_{k-2} Y_{k-2}^T} \dots Y_1^T \\
&\vdots \\
&= Y_k^T Y_{k-1}^T Y_{k-2}^T \dots Y_1^T A^{-1} X_1 Y_1^T \\
&= Y_k^T Y_{k-1}^T Y_{k-2}^T \dots Y_1^T A^{-1} E.
\end{aligned}$$

Q.E.D.

Lemma 30 For a non-negative integer p , $E_k^p Y_k^T \dots Y_1^T = Y_k^T \dots Y_1^T (A^{-1} E)^p$.

PROOF: The proof is obtained by applying Lemma 29 p times.

$$\begin{aligned}
E_k^p Y_k^T \dots Y_1^T &= (Y_k^T X_k)^{p-1} (Y_k^T X_k) Y_k^T \dots Y_1^T \\
&= (Y_k^T X_k)^{p-1} Y_k^T Y_{k-1}^T Y_{k-2}^T \dots Y_1^T A^{-1} E \quad (\text{Applying Lemma 5}) \\
&= (Y_k^T X_k)^{p-2} (Y_k^T X_k) Y_k^T \dots Y_1^T A^{-1} E \\
&= (Y_k^T X_k)^{p-2} Y_k^T \dots Y_1^T (A^{-1} E)^2 \\
&\vdots \\
&= Y_k^T Y_{k-1}^T Y_{k-2}^T \dots Y_1^T (A^{-1} E)^p.
\end{aligned}$$

Q.E.D.

9.2.2 Discretization of $E\dot{x} = Ax$

We use Padé approximation methods to discretize descriptor systems. Conventionally, Padé approximations have been applied to approximate the matrix exponential appearing in the solution for a *regular system*. However, our approach is valid for any proper rational approximation of e^z which is analytic on the closed left half plane. In this section we discretize the descriptor system $E\dot{x} = Ax$ with an initial condition $x_0 = x(0) \in \mathcal{C}$.

$$E_k (Y_k^T \dots Y_2^T Y_1^T \dot{x}) = (Y_k^T \dots Y_2^T Y_1^T x) \text{ where } E_k = Y_k^T X_k \quad (9.8)$$

$$\Rightarrow E_k \dot{w} = w \text{ where } w = Y_k^T \dots Y_2^T Y_1^T x. \quad (9.9)$$

If an $[L/M]$ Padé discretization function is given by $C_{[L/M]}(z)$, then the corresponding discrete-time system would be

$$\begin{aligned} w[(p+1)h] &= C_{[L/M]}(E_k^{-1}h)w[ph] \\ \Rightarrow w[(p+1)h] &= Q_M(-E_k^{-1}h)^{-1}Q_L(E_k^{-1}h)w[ph] \\ \Rightarrow Y_k^T \dots Y_2^T Y_1^T x[(p+1)h] &= \tilde{Q}_M(-E_k h)^{-1}(E_k)^{M-L}\tilde{Q}_L(E_k h)Y_k^T \dots Y_2^T Y_1^T x[ph], \end{aligned}$$

where $\tilde{Q}_L(s) = \sum_{i=0}^L l_{L-i}s^i$ and $\tilde{Q}_M(s) = \sum_{i=0}^M m_{M-i}s^i$ (see Definition 18), hence

$$\tilde{Q}_M(-E_k h)Y_k^T \dots Y_2^T Y_1^T x[(p+1)h] = (E_k)^{M-L}\tilde{Q}_L(E_k h)Y_k^T \dots Y_2^T Y_1^T x[ph].$$

Applying Lemma 29 and 30, we have

$$Y_k^T \dots Y_2^T Y_1^T \tilde{Q}_M(-A^{-1}Eh)x[(p+1)h] = Y_k^T \dots Y_2^T Y_1^T (A^{-1}E)^{M-L}\tilde{Q}_L(A^{-1}Eh)x[ph].$$

Thus

$$Y_k^T \dots Y_2^T Y_1^T (\tilde{Q}_M(-A^{-1}Eh)x[(p+1)h] - (A^{-1}E)^{M-L}\tilde{Q}_L(A^{-1}Eh)x[ph]) = 0,$$

and from Lemma 28, we have

$$\begin{aligned} \tilde{Q}_M(-A^{-1}Eh)x[(p+1)h] - (A^{-1}E)^{M-L}\tilde{Q}_L(A^{-1}Eh)x[ph] &= 0 \\ \Rightarrow x[(p+1)h] &= \tilde{Q}_M(-A^{-1}Eh)^{-1}(A^{-1}E)^{M-L}\tilde{Q}_L(A^{-1}Eh)x[ph]. \end{aligned} \quad (9.10)$$

Thus $\tilde{C}_{[L/M]}(E, A, h) = \tilde{Q}_M(-A^{-1}Eh)^{-1}(A^{-1}E)^{M-L}\tilde{Q}_L(A^{-1}Eh)$.

Remark 2 For descriptor systems of the form $E\dot{x} = Ax + Bu$, we can follow a similar approach and obtain

$$\begin{aligned} x[(p+1)h] &= \tilde{Q}_M(-A^{-1}Eh)^{-1}(A^{-1}E)^{M-L}\tilde{Q}_L(A^{-1}Eh)x[ph] \\ &+ (\tilde{Q}_M(-A^{-1}Eh)^{-1}(A^{-1}E)^{M-L}\tilde{Q}_L(A^{-1}Eh) - I)A^{-1}Bu[ph] \end{aligned} \quad (9.11)$$

Equations (9.10) and (9.11) should not be used for the actual computation of a numerical approximation for a descriptor system with index greater than one. In this chapter, we only focus on the passivity preservation of index-one systems, hence the discretization method presented is still valid. This methodology also provides a useful framework for analysing the Lyapunov function preserving property of different Padé approximations in the later chapters. Another important observation is that equations (9.10) and (9.11) provide a *generalization* of Padé approximations for descriptor systems.

9.3 PASSIVITY PRESERVING DISCRETIZATION METHODS

9.3.1 Discretization of the transfer functions

We begin this sub-section by recalling Theorem 31, according to which the state space model of an index-one descriptor systems given by

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= C^T x(t) + Du(t), \end{aligned} \quad (9.12)$$

where $E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{n \times m}, D \in \mathbb{R}^{m \times m}$ and $\text{rank}(E) = r < n$, can be expressed as

$$\begin{aligned} \dot{z}(t) &= \tilde{A}z(t) + \tilde{B}u(t), \\ y(t) &= \tilde{C}^T z(t) + \tilde{D}u(t), \end{aligned} \quad (9.13)$$

where

$$\begin{aligned} \tilde{A} &= \tilde{E}^{-1} = (Y^T A^{-1} X)^{-1}, & \tilde{B} &= \tilde{E}^{-1} Y^T A^{-1} B, \\ \tilde{C}^T &= C^T A^{-1} X \tilde{E}^{-1}, & \tilde{D} &= (D - C^T A^{-1} B + C^T A^{-1} X \tilde{E}^{-1} Y^T A^{-1} B). \end{aligned}$$

Also note that the matrices $X, Y \in \mathbb{R}^{n \times p}$ correspond to the full rank decomposition of E and $z(t) = Y^T x(t)$.

The modified state space model for an index-one descriptor system is a *regular system*. This fact allows us to apply Theorem 24 to a passive continuous-time modified state space model, thus resulting in a passive discrete time model given by

$$\begin{aligned} z[(k+1)h] &= A_d z[kh] + B_d u[kh], \\ y[kh] &= C_d^T z[kh] + D_d u[kh], \end{aligned} \quad (9.14)$$

where

$$\begin{aligned} A_d &= \left(I - \frac{h}{2}\tilde{A}\right)^{-1} \left(I + \frac{h}{2}\tilde{A}\right), & B_d &= \frac{h}{2} \left(I - \frac{h}{2}\tilde{A}\right)^{-1} \tilde{B}, \\ C_d^T &= 2\tilde{C}^T \left(I - \frac{h}{2}\tilde{A}\right)^{-1}, & D_d &= \tilde{D} + \frac{h}{2}\tilde{C}^T \left(I - \frac{h}{2}\tilde{A}\right)^{-1} \tilde{B}. \end{aligned}$$

To further simplify these matrices, we consider

$$\begin{aligned}
z[(k+1)h] &= A_d z[kh] + B_d u[kh] \\
&= \left(I - \frac{h}{2}\tilde{A}\right)^{-1} \left(I + \frac{h}{2}\tilde{A}\right) z[kh] + \frac{h}{2} \left(I - \frac{h}{2}\tilde{A}\right)^{-1} \tilde{B}u[kh] \\
&= \left(I - \frac{h}{2}\tilde{E}^{-1}\right)^{-1} \left(I + \frac{h}{2}\tilde{E}^{-1}\right) z[kh] + \frac{h}{2} \left(I - \frac{h}{2}\tilde{E}^{-1}\right)^{-1} \tilde{B}u[kh] \\
&= \left(\tilde{E} - \frac{h}{2}I\right)^{-1} \left(\tilde{E} + \frac{h}{2}I\right) z[kh] + \frac{h}{2} \left(\tilde{E} - \frac{h}{2}I\right)^{-1} \tilde{E}\tilde{B}u[kh] \\
\Rightarrow \left(\tilde{E} - \frac{h}{2}I\right) z[(k+1)h] &= \left(\tilde{E} + \frac{h}{2}I\right) z[kh] + \frac{h}{2}\tilde{E}\tilde{B}u[kh].
\end{aligned}$$

Since $z = Y^T x$, we have

$$\left(Y^T A^{-1} X - \frac{h}{2}I\right) Y^T x[(k+1)h] = \left(Y^T A^{-1} X + \frac{h}{2}I\right) Y^T x[kh] + \frac{h}{2} Y^T A^{-1} B u[kh]$$

further leading to

$$Y^T \left(\left(A^{-1}E - \frac{h}{2}I\right) x[(k+1)h] - \left(A^{-1}E + \frac{h}{2}I\right) x[kh] - \frac{h}{2}A^{-1}B u[kh] \right) = 0.$$

Now we recall our earlier discussion on the consistency space of descriptor systems, whereby $\ker(Y^T)$ and $\mathcal{C}(E, A)$ intersect only at the origin. Hence for an index-one descriptor system, if $x \in \mathcal{C}(E, A) = \text{Im}(A^{-1}E)$, then $Y^T x = 0 \Rightarrow x = 0$. Based on this property, we state that

$$\left(A^{-1}E - \frac{h}{2}I\right) x[(k+1)h] = \left(A^{-1}E + \frac{h}{2}I\right) x[kh] + \frac{h}{2}A^{-1}B u[kh]$$

hence

$$x[(k+1)h] = \left(A^{-1}E - \frac{h}{2}I\right)^{-1} \left(A^{-1}E + \frac{h}{2}I\right) x[kh] + \frac{h}{2} \left(A^{-1}E - \frac{h}{2}I\right)^{-1} A^{-1}B u[kh].$$

Similarly, we can simplify C_d^T as

$$\begin{aligned}
C_d^T &= 2C_c^T \left(I - \frac{h}{2}A_c\right)^{-1} = 2C^T A^{-1} X \tilde{E}^{-1} \tilde{E} \left(\tilde{E} - \frac{h}{2}I\right)^{-1} \\
&\Rightarrow C_d^T \left(\tilde{E} - \frac{h}{2}I\right) = 2C^T A^{-1} X,
\end{aligned}$$

now post-multiplying both sides by Y^T we have

$$\begin{aligned} C_d^T \left(Y^T A^{-1} E - \frac{h}{2} Y^T \right) &= 2C^T A^{-1} E \\ \Rightarrow C_d Y^T &= 2C^T A^{-1} E \left(A^{-1} E - \frac{h}{2} I \right)^{-1}. \end{aligned}$$

The discrete-time matrix D_d can be simplified as

$$\begin{aligned} D_d &= D_c + \frac{h}{2} C_c^T \left(I - \frac{h}{2} A_c \right)^{-1} B_c \\ &= D - C^T A^{-1} B + C^T A^{-1} X \tilde{E}^{-1} Y^T A^{-1} B + \frac{h}{2} C^T A^{-1} X \tilde{E}^{-1} \left(I - \frac{h}{2} \tilde{E}^{-1} \right)^{-1} \tilde{E}^{-1} Y^T A^{-1} B \\ &= D - C^T A^{-1} B + C^T A^{-1} X \tilde{E}^{-1} Y^T A^{-1} B + \frac{h}{2} C^T A^{-1} X \tilde{E}^{-1} \left(\tilde{E} - \frac{h}{2} I \right)^{-1} Y^T A^{-1} B \\ &= D - C^T A^{-1} B + C^T A^{-1} X \tilde{E}^{-1} \left(I + \frac{h}{2} \left(\tilde{E} - \frac{h}{2} I \right)^{-1} \right) Y^T A^{-1} B \\ &= D - C^T A^{-1} B + C^T A^{-1} X \tilde{E}^{-1} \left(\tilde{E} - \frac{h}{2} I + \frac{h}{2} \right) \left(\tilde{E} - \frac{h}{2} I \right)^{-1} Y^T A^{-1} B \\ &= D - C^T A^{-1} B + C^T A^{-1} X \left(\tilde{E} - \frac{h}{2} I \right)^{-1} Y^T A^{-1} B = D + C^T \left(\frac{2}{h} E - A \right)^{-1} B. \end{aligned}$$

Based on these calculations, we can state the next theorem.

Theorem 40 *Let $H(j\omega) = D + C^T(j\omega E - A)B$ be a continuous-time transfer function of an index-one descriptor system, and assume that $H(j\omega)$ is discretized by the Tustin discretization method with sampling period h . The discrete-time transfer function is given by*

$$G(e^{j\theta}) = H \left(\frac{2 e^{j\theta} - 1}{h e^{j\theta} + 1} \right) = D_{td} + C_{td}^T (e^{j\theta} I - A_{td}) B_{td},$$

where

$$A_{td} = \left(A^{-1} E - \frac{h}{2} I \right)^{-1} \left(A^{-1} E + \frac{h}{2} I \right), \quad (9.15)$$

$$B_{td} = \frac{h}{2} \left(A^{-1} E - \frac{h}{2} I \right)^{-1} A^{-1} B, \quad (9.16)$$

$$C_{td}^T = 2C^T A^{-1} E \left(A^{-1} E - \frac{h}{2} I \right)^{-1}, \quad (9.17)$$

$$D_{td} = D + C^T \left(\frac{2}{h} E - A \right)^{-1} B. \quad (9.18)$$

If $H(j\omega)$ is continuous-time positive real, then $G(e^{j\theta})$ is discrete-time positive real.

9.3.2 Discretization of the state space model

In this sub-section, we consider a passivity preserving method for the state space model, based on the output averaging method from Theorem 35. The proposed average output is given by

$$y^*(kh) = \int_{kh}^{kh+h} y(\tau) d\tau. \quad (9.19)$$

Now, we consider an index-one descriptor system with a state space model described by (9.12). Index-1 descriptor systems can be modified into a *regular system* given by (9.14). Thus

$$y^*(kh) = \int_{kh}^{kh+h} y(\tau) d\tau \quad (9.20)$$

$$= C_{md}^{T*} z(kh) + D_{md}^* u(kh) \quad (9.21)$$

$$= C_{md}^{T*} Y^T x(kh) + D_{md}^* u(kh) \quad (9.22)$$

$$= C_{ad}^{T*} x(kh) + D_{ad}^* u(kh) \quad (9.23)$$

with C_{md}^{T*} and D_{md}^* matrices for the modified *regular system* (9.13) and C_{ad}^{T*} and D_{ad}^* for the original descriptor system corresponding to averaged output.

For the modified *regular system* (9.13), we can apply Theorem 35 to obtain

$$C_{md}^{T*} = C_c^T \tilde{A}^{-1} (e^{\tilde{A}h} - I).$$

To evaluate $e^{\tilde{A}h}$, we can use Padé approximations from Definition 18 (with order $[L/M]$) leading to

$$\begin{aligned} C_{md}^{T*} &= C^T A^{-1} X \tilde{E}^{-1} \cdot \tilde{E} \cdot (Q_M(\tilde{E}^{-1}) Q_L(\tilde{E}^{-1})^{-1} - I) \\ &= C^T A^{-1} X (Q_L(-\tilde{E}^{-1}h) Q_M(\tilde{E}^{-1}h)^{-1} - I) \\ &= C^T A^{-1} X (\tilde{Q}_L(-\tilde{E}h) (\tilde{E})^{M-L} \tilde{Q}_M(\tilde{E}h)^{-1} - I). \end{aligned} \quad (9.24)$$

where $\tilde{Q}_L(s) = \sum_{i=0}^L l_{L-i} s^i$ and $\tilde{Q}_M(s) = \sum_{i=0}^M m_{M-i} s^i$, hence

$$\begin{aligned} C_{md}^{T*} &= C^T A^{-1} X (\tilde{Q}_L(-\tilde{E}h) (\tilde{E})^{M-L} - \tilde{Q}_M(\tilde{E}h)) \tilde{Q}_M(\tilde{E}h)^{-1} \\ C_{md}^{T*} \tilde{Q}_M(\tilde{E}h) &= C^T A^{-1} X (\tilde{Q}_L(-\tilde{E}h) (\tilde{E})^{M-L} - \tilde{Q}_M(\tilde{E}h)). \end{aligned}$$

Pre-multiplying both sides by Y^T , we have

$$\begin{aligned} C_{md}^{T*} \tilde{Q}_M(\tilde{E}h) Y^T &= C^T A^{-1} X (\tilde{Q}_L(-\tilde{E}h) (\tilde{E})^{M-L} - \tilde{Q}_M(\tilde{E}h)) Y^T \\ C_{md}^{T*} Y^T \tilde{Q}_M(A^{-1} E h) &= C^T A^{-1} X Y^T (\tilde{Q}_L(-A^{-1} E h) (A^{-1} E)^{M-L} - \tilde{Q}_M(A^{-1} E h)) \\ C_{md}^{T*} Y^T &= C^T A^{-1} E (\tilde{Q}_L(-A^{-1} E h) (A^{-1} E)^{M-L} - \tilde{Q}_M(A^{-1} E h)) \tilde{Q}_M(A^{-1} E h)^{-1} \\ \Rightarrow C_{md}^{T*} Y^T = C_{ad}^{T*} &= C^T A^{-1} E (\tilde{Q}_L(-A^{-1} E h) (A^{-1} E)^{M-L} \tilde{Q}_M(A^{-1} E h)^{-1} - I). \end{aligned} \quad (9.25)$$

Similarly, we simplify the matrix D_{md}^* as

$$\begin{aligned}
 D_{md}^* = D_{ad}^* &= C_c^T A_c^{-2} (e^{A_c h} - I - A_c h) B_c + D_c h \\
 &= C^T A^{-1} X \tilde{E}^{-1} \cdot \tilde{E}^2 \cdot (\tilde{Q}_L(-\tilde{E}h)(\tilde{E})^{M-L} \tilde{Q}_M(\tilde{E}h)^{-1} - I - \tilde{E}^{-1}h) \tilde{E}^{-1} Y^T A^{-1} B \\
 &\quad + (D - C^T A^{-1} B + C^T A^{-1} X \tilde{E}^{-1} Y^T A^{-1} B) h \\
 &= C^T A^{-1} X \tilde{E} \cdot (\tilde{Q}_L(-\tilde{E}h)(\tilde{E})^{M-L} \tilde{Q}_M(\tilde{E}h)^{-1} - I) \tilde{E}^{-1} Y^T A^{-1} B \\
 &\quad - (C^T A^{-1} X \tilde{E}^{-1} Y^T A^{-1} B) h + (D - C^T A^{-1} B + C^T A^{-1} X \tilde{E}^{-1} Y^T A^{-1} B) h \\
 &= C^T A^{-1} X (\tilde{Q}_L(-\tilde{E}h)(\tilde{E})^{M-L} \tilde{Q}_M(\tilde{E}h)^{-1} - I) Y^T A^{-1} B + (D - C^T A^{-1} B) h.
 \end{aligned}$$

Substituting C_{md}^{T*} for $C^T A^{-1} X (\tilde{Q}_L(-\tilde{E}h)(\tilde{E})^{M-L} \tilde{Q}_M(\tilde{E}h)^{-1} - I)$ (see equation (9.24)), we get

$$D_{ad}^* = C_{md}^{T*} Y^T A^{-1} B + (D - C^T A^{-1} B) h.$$

Substituting $C^T A^{-1} E (\tilde{Q}_L(A^{-1} E h)(A^{-1} E)^{M-L} \tilde{Q}_M(-A^{-1} E h)^{-1} - I)$ for $C_{md}^{T*} Y^T$ (see equation (9.25)), we get

$$D_{ad}^* = C^T A^{-1} E (\tilde{Q}_L(A^{-1} E h)(A^{-1} E)^{M-L} \tilde{Q}_M(-A^{-1} E h)^{-1} - I) A^{-1} B + (D - C^T A^{-1} B) h.$$

Based on these calculations, we can state the next result.

Theorem 41 Consider an continuous-time index-one descriptor system given by

$$\Sigma_c : \begin{cases} E \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = C^T x(t) + Du(t), \end{cases}$$

where $E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}, D \in \mathbb{R}^{m \times m}$ and $\text{rank}(E) = p < n$. Assume that there exists a ZOH element on its input with sampling time $h > 0$ and the output is obtained through equation (9.19). The the corresponding discrete time system is given by

$$\Sigma_{ad} \begin{cases} x[(k+1)h] = A_{ad} x[ph] + B_{ad} u[kh], \\ y^*[kh] = C_{ad}^{T*} x[kh] + D_{ad}^* u[kh], \end{cases}$$

where

$$A_{ad} = \tilde{Q}_M(-A^{-1} E h)^{-1} (A^{-1} E)^{M-L} \tilde{Q}_L(A^{-1} E h), \quad (9.26)$$

$$B_{ad} = (\tilde{Q}_M(-A^{-1} E h)^{-1} (A^{-1} E)^{M-L} \tilde{Q}_L(A^{-1} E h) - I) A^{-1} B, \quad (9.27)$$

$$C_{ad}^{T*} = C^T A^{-1} E (\tilde{Q}_L(-A^{-1} E h)(A^{-1} E)^{M-L} \tilde{Q}_M(A^{-1} E h)^{-1} - I), \quad (9.28)$$

$$\begin{aligned}
 D_{ad}^* &= C^T A^{-1} E (\tilde{Q}_L(A^{-1} E h)(A^{-1} E)^{M-L} \tilde{Q}_M(-A^{-1} E h)^{-1} - I) A^{-1} B \\
 &\quad + (D - C^T A^{-1} B) h, \quad (9.29)
 \end{aligned}$$

$$= C_{ad}^{T*} A^{-1} B + (D - C^T A^{-1} B) h, \quad (9.30)$$

and $\tilde{Q}_M(-A^{-1} E h)^{-1} (A^{-1} E)^{M-L} \tilde{Q}_L(A^{-1} E h)$ is a generalized rational A-stable approximation of the matrix exponential of order $[L/M]$. Then Σ_{ad} is passive if Σ_c is passive.

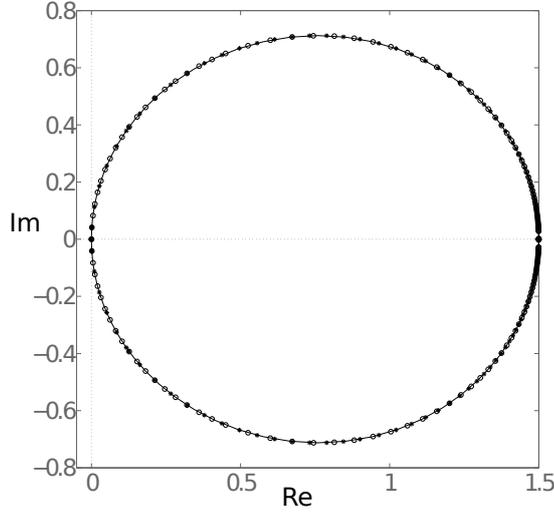


Figure 15: Nyquist plots for the passive continuous-time descriptor system (“—”) and the corresponding discrete-time transfer functions G_1 (“*”) and G_2 (“o”) obtained through Tustin’s method and output averaging method, respectively.

Example 4 To illustrate our results we consider a passive, continuous-time index-one descriptor system of the form (3.1), where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and $d = 0$. Then

$$H(j\omega) = d + c^T(j\omega E - A)^{-1}b = \frac{2(j\omega) + 3}{(j\omega)^2 + 3(j\omega) + 2}.$$

We consider both methods of discretizations using Theorems 40 and 41 and obtain the following discrete-time transfer functions.

$$G_1(e^{j\theta}) = \frac{5e^{j2\theta} + 6e^{j\theta} + 1}{6e^{j2\theta} + 2e^{j\theta}} \quad \left\{ \begin{array}{l} \text{Tustin's method with sampling time } h = 2 \end{array} \right.$$

$$G_2(e^{j\theta}) = \frac{0.5833e^{j2\theta} + 0.5e^{j\theta} - 0.08333}{e^{j2\theta} - 0.3333e^{j\theta}} \quad \left\{ \begin{array}{l} \text{Output averaging method} \\ \text{with sampling time } h = 2 \\ \text{and first order generalized} \\ \text{Padé approximation.} \end{array} \right.$$

It can be further observed from the Nyquist plots of $G_1(e^{j\theta})$ and $G_2(e^{j\theta})$, that both methods preserve passivity.

9.4 MIXEDNESS PRESERVING DISCRETIZATION METHODS

In this section, we obtain the conditions necessary to preserve mixedness property of an LTI transfer function (*regular system* or index-one descriptor system). These conditions are based on preserving the frequency points at which the system transfer function makes a transition from passivity to the small gain property or vice versa. This approach motivates an eigenvalue-based test that completely characterises the “mixed” property in discrete time while providing the transition frequencies. Such a test is an independently valuable tool that can be used to test mixedness of any given linear shift invariant discrete-time system. However, we begin with the definitions of a mixed system in discrete time.

9.4.1 Discrete-time “mixed” systems

For $-\pi \leq \theta \leq \pi$, $-\infty \leq \omega \leq \infty$ and a sampling interval $h > 0$, assume that $\bar{a}, \bar{b} \in \mathbb{R}_+$ and $\bar{a} \leq \bar{b} \leq \pi$, where \bar{a} and \bar{b} are in radians.

Definition 21 [126, Section 10.1.3] *A discrete-time system with proper, real-rational transfer function matrix $G(z)$ is said to be input-output stable if all of the poles of $G(z)$ lie inside the unit circle on the complex plane.*

Definition 22 *An input-output stable, discrete-time system with square, proper, real-rational transfer function matrix $M(z)$ is said to be input and output strictly positive over $[-\bar{b}, -\bar{a}] \cup [\bar{a}, \bar{b}]$ if there exist real numbers $k, l > 0$ such that*

$$-kM^*(e^{j\theta})M(e^{j\theta}) + M^*(e^{j\theta}) + M(e^{j\theta}) - lI \geq 0$$

for all $\theta \in [-\bar{b}, -\bar{a}] \cup [\bar{a}, \bar{b}]$.

Definition 23 *For an input-output stable, discrete-time system with proper, real-rational transfer function matrix $M(z)$, define the system gain over $[-\bar{b}, -\bar{a}] \cup [\bar{a}, \bar{b}]$ as*

$$\varepsilon := \min\{\bar{\varepsilon} \in \mathbb{R}_+ : -M^*(e^{j\theta})M(e^{j\theta}) + \bar{\varepsilon}^2 I \geq 0 \text{ for all } \theta \in [-\bar{b}, -\bar{a}] \cup [\bar{a}, \bar{b}]\}.$$

The system is said to have a gain of less than one over $[-\bar{b}, -\bar{a}] \cup [\bar{a}, \bar{b}]$ if $\varepsilon < 1$.

We define a “mixed” discrete-time system analogous to the continuous-time case.

Definition 24 *An input-output stable, discrete-time system with square, proper, real-rational transfer function matrix $M(z)$ is said to be “mixed” if, for each $\theta \in [-\pi, \pi]$, either of the following hold:*

- (i) there exist $k, l > 0$ such that $-kM^*(e^{j\theta})M(e^{j\theta}) + M^*(e^{j\theta}) + M(e^{j\theta}) - lI \geq 0$;
- (ii) there exists $\varepsilon < 1$ such that $-M^*(e^{j\theta})M(e^{j\theta}) + \varepsilon^2 I \geq 0$.

9.4.2 A symplectic matrix-based test for “mixed” discrete-time systems

Suppose that we are given an arbitrary, causal, linear, shift-invariant system that is described by the equations

$$\begin{aligned} x[(k+1)h] &= Ax[kh] + Bu[kh], \quad x(0) = x_0, \\ y[kh] &= Cx[kh] + Du[kh], \end{aligned}$$

where $x[kh] \in \mathbb{R}^n$, $u[kh] \in \mathbb{R}^m$, $y[kh] \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$ and $D \in \mathbb{R}^{m \times m}$ with A stable.¹ Furthermore, suppose that A is non-singular. Denoting $M(z) := C(zI - A)^{-1}B + D$ and $M^*(z) := [M(z^{-1})]^T$ gives

$$M^*(z) = \left[\begin{array}{c|c} A^{-T} & -A^{-T}C^T \\ \hline B^T A^{-T} & D^T - B^T A^{-T} C^T \end{array} \right] \quad (9.31)$$

from [110, Section 21.4].² Let $G_1(e^{j\theta}) := -kM^*(e^{j\theta})M(e^{j\theta}) + M^*(e^{j\theta}) + M(e^{j\theta}) - lI$ and $G_2(e^{j\theta}) := -M^*(e^{j\theta})M(e^{j\theta}) + \varepsilon^2 I$. Consider the following two results.

Lemma 31 *Suppose that $k, l \in \mathbb{R}$ and consider $G_1(e^{j\theta})$ as defined above. Let $Y := I - kD$ and suppose that $X_1 := -kD^T D + D^T + D - lI$ and $\tilde{X}_1 := X_1 - B^T A^{-T} C^T Y$ are invertible. For some $\theta_0 \in \mathbb{R}$, the matrix $G_1(e^{j\theta_0})$ has a zero eigenvalue if and only if the matrix S_1 has an eigenvalue on the unit disc at the point $e^{j\theta_0}$, where*

$$S_1 := \begin{pmatrix} E_1 + U_1 E_1^{-T} V_1 & -U_1 E_1^{-T} \\ -E_1^{-T} V_1 & E_1^{-T} \end{pmatrix}$$

and $E_1 := A - B X_1^{-1} Y^T C$, $U_1 := -B X_1^{-1} B^T$, $V_1 := k C^T C + C^T Y X_1^{-1} Y^T C$.

PROOF: Given that

$$\begin{aligned} & \begin{bmatrix} (e^{j\theta_0} I - A)^{-1} & 0 \\ -k(e^{j\theta_0} I - A^{-T})^{-1} A^{-T} C^T C (e^{j\theta_0} I - A)^{-1} & (e^{j\theta_0} I - A^{-T})^{-1} \end{bmatrix} \\ &= \left[e^{j\theta_0} I - \begin{pmatrix} A & 0 \\ -k A^{-T} C^T C & A^{-T} \end{pmatrix} \right]^{-1}, \quad (9.32) \end{aligned}$$

¹ A dynamical system is said to be stable in discrete-time if $\rho(A) < 1$ [110, Section 21.1], [113, Section 5.7.1].

² The notation on the right-hand side of (9.31) denotes a state-space realisation.

note that $G_1(e^{j\theta_0}) = -k[-B^T A^{-T}(e^{j\theta_0}I - A^{-T})^{-1}A^{-T}C^T + D^T - B^T A^{-T}C^T][C(e^{j\theta_0}I - A)^{-1}B + D] - B^T A^{-T}(e^{j\theta_0}I - A^{-T})^{-1}A^{-T}C^T + D^T - B^T A^{-T}C^T + C(e^{j\theta_0}I - A)^{-1}B + D - I = \bar{C}(e^{j\theta_0}I - \bar{A})^{-1}\bar{B} + \tilde{X}_1$, where

$$\bar{A} := \begin{pmatrix} A & 0 \\ -kA^{-T}C^T C & A^{-T} \end{pmatrix}, \quad \bar{B} := \begin{pmatrix} B \\ A^{-T}C^T Y \end{pmatrix}$$

and

$$\bar{C} := \begin{pmatrix} Y^T C + kB^T A^{-T}C^T C & -B^T A^{-T} \end{pmatrix},$$

using [127, Lemma 3]. Then, in the manner of [128, Lemma 1],

$$\begin{aligned} \det(G_1(e^{j\theta_0})) &= \det(\bar{C}(e^{j\theta_0}I - \bar{A})^{-1}\bar{B} + \tilde{X}_1) \\ &= \det(\tilde{X}_1) \det(I + \tilde{X}_1^{-1}\bar{C}(e^{j\theta_0}I - \bar{A})^{-1}\bar{B}) \\ &= \det(\tilde{X}_1) \det(I + (e^{j\theta_0}I - \bar{A})^{-1}\bar{B}\tilde{X}_1^{-1}\bar{C}) \quad (\text{Sylvester's Determinant Theorem}) \\ &= \det(\tilde{X}_1) \det((e^{j\theta_0}I - \bar{A})^{-1}) \det(e^{j\theta_0}I - \bar{A} + \bar{B}\tilde{X}_1^{-1}\bar{C}) \\ &= \det(\tilde{X}_1) \det((e^{j\theta_0}I - A)^{-1}) \det((e^{j\theta_0}I - A^{-T})^{-1}) \det(e^{j\theta_0}I - \tilde{H}_1), \end{aligned}$$

where $\tilde{H}_1 := \bar{A} - \bar{B}\tilde{X}_1^{-1}\bar{C}$. Since A is stable, then $\det(e^{j\theta_0}I - A) \neq 0$ for any $\theta_0 \in \mathbb{R}$; and $e^{j\theta_0}I - A$ is invertible and so $\det((e^{j\theta_0}I - A)^{-1}) \neq 0$. Similarly, $\det((e^{j\theta_0}I - A^{-T})^{-1}) \neq 0$ noting that

$$(-1)^n \det(e^{j\theta_0}I) \det(e^{-j\theta_0}I - A) \det(A^{-1}) = \det(e^{j\theta_0}I - A^{-1}) = \det(e^{j\theta_0}I - A^{-T})$$

from [112, Equation 6.1.4]. Thus, $G_1(e^{j\theta_0})$ has a zero eigenvalue if and only if $\det(e^{j\theta_0}I - \tilde{H}_1) = 0$, i.e. \tilde{H}_1 has an eigenvalue on the unit disc at the point $e^{j\theta_0}$. Finally, $\tilde{H}_1 = S_1$ via matrix inversion identities [110, Section 2.3].

Lemma 32 *Suppose that $\varepsilon \in \mathbb{R} \setminus \{0\}$ and consider $G_2(e^{j\theta})$ as defined at the beginning of Section 9.4.2. Suppose that $-DD^T + \varepsilon^2 I$, $X_2 := -D^T D + \varepsilon^2 I$ and $\tilde{X}_2 := X_2 + B^T A^{-T} C^T D$ are invertible. For some $\theta_0 \in \mathbb{R}$, the matrix $G_2(e^{j\theta_0})$ has a zero eigenvalue if and only if the matrix S_2 has an eigenvalue on the unit disc at the point $e^{j\theta_0}$, where*

$$S_2 := \begin{pmatrix} E_2 + U_2 E_2^{-T} V_2 & -U_2 E_2^{-T} \\ -E_2^{-T} V_2 & E_2^{-T} \end{pmatrix}$$

and $E_2 := A + B X_2^{-1} D^T C$, $U_2 := -B X_2^{-1} B^T$, $V_2 := \varepsilon^2 C^T (-DD^T + \varepsilon^2 I)^{-1} C$.

PROOF: Given (9.32) with $k = 1$, note that $G_2(e^{j\theta_0}) = -[-B^T A^{-T}(e^{j\theta_0}I - A^{-T})^{-1}A^{-T}C^T + D^T - B^T A^{-T}C^T][C(e^{j\theta_0}I - A)^{-1}B + D] + \varepsilon^2 I = \bar{C}(e^{j\theta_0}I - \bar{A})^{-1}\bar{B} + \tilde{X}_2$, where

$$\bar{A} := \begin{pmatrix} A & 0 \\ -A^{-T}C^T C & A^{-T} \end{pmatrix}, \quad \bar{B} := \begin{pmatrix} B \\ -A^{-T}C^T D \end{pmatrix}$$

and

$$\bar{C} := \begin{pmatrix} -D^T C + B^T A^{-T} C^T C & -B^T A^{-T} \end{pmatrix},$$

from [127, Lemma 3]. Then, in the manner of [128, Lemma 1] and similarly to the proof of Lemma 31, $\det(G_2(e^{j\theta})) = \det(\tilde{X}_2) \det((e^{j\theta}I - A)^{-1}) \det((e^{j\theta}I - A^{-T})^{-1}) \det(e^{j\theta}I - \tilde{H}_2)$, where $\tilde{H}_2 := \tilde{A} - \tilde{B}\tilde{X}_2^{-1}\bar{C}$. The remainder of the proof follows in the manner of the proof of Lemma 31.

Let $\tilde{G}_1(e^{j\theta}) := G_1(e^{j\theta})$, where $k = l = 0$. Similarly, let $\tilde{G}_2(e^{j\theta}) := G_2(e^{j\theta})$, where $\varepsilon = 1$. Consider Lemmas 31 and 32. Set

$$\begin{aligned} \Theta_p &:= \{\theta \in [-\pi, \pi] : S_1 \text{ has an eigenvalue on the unit disc at } e^{j\theta}\}, \\ \Theta_s &:= \{\theta \in [-\pi, \pi] : S_2 \text{ has an eigenvalue on the unit disc at } e^{j\theta}\}. \end{aligned}$$

Suppose that we divide the interval $-\pi$ to π into smaller intervals, where any elements of Θ_p and Θ_s are, respectively, set as open interval endpoints, as follows:

$$\begin{aligned} \text{Division Group 1} &:= (-\pi, \theta_{p_1}), (\theta_{p_1}, \theta_{p_2}), \dots, (\theta_{p_{\bar{n}-1}}, \theta_{p_{\bar{n}}}), (\theta_{p_{\bar{n}}}, \pi) \\ \text{Division Group 2} &:= (-\pi, \theta_{s_1}), (\theta_{s_1}, \theta_{s_2}), \dots, (\theta_{s_{\bar{m}-1}}, \theta_{s_{\bar{m}}}), (\theta_{s_{\bar{m}}}, \pi) \end{aligned}$$

where \bar{n} = number of elements in Θ_p ; \bar{m} = number of elements in Θ_s ; $\theta_{p_1}, \theta_{p_2}, \dots, \theta_{p_{\bar{n}}}$ denote the elements of Θ_p listed in increasing order; and $\theta_{s_1}, \theta_{s_2}, \dots, \theta_{s_{\bar{m}}}$ denote the elements of Θ_s listed in increasing order. If Θ_p is empty, then $\bar{n} = 0$ and Division Group 1 consists of the single interval $[-\pi, \pi]$; similarly, if Θ_s is empty, then $\bar{m} = 0$ and Division Group 2 consists of the single interval $[-\pi, \pi]$. If $\theta_{p_1} = -\pi$ and $\theta_{p_{\bar{n}}} = \pi$, then Division Group 1 becomes $(-\pi, \theta_{p_2}), (\theta_{p_2}, \theta_{p_3}), \dots, (\theta_{p_{\bar{n}-1}}, \pi)$. Similarly, if $\theta_{s_1} = -\pi$ and $\theta_{s_{\bar{m}}} = \pi$, then Division Group 2 becomes $(-\pi, \theta_{s_2}), (\theta_{s_2}, \theta_{s_3}), \dots, (\theta_{s_{\bar{m}-1}}, \pi)$.

Suppose that we check the sign definiteness of $\tilde{G}_1(e^{j\theta})$ over each of the individual intervals in Division Group 1 and the sign definiteness of $\tilde{G}_2(e^{j\theta})$ over each of the individual intervals in Division Group 2. Note that checking the sign definiteness over any of these intervals can be achieved by checking the sign definiteness at a single θ from within the interval, e.g. at the interval midpoint. Let I_{S_1} denote the set of θ belonging to the intervals over which $\tilde{G}_1(e^{j\theta}) > 0$, and I_{S_2} denote the set of θ belonging to the intervals over which $\tilde{G}_2(e^{j\theta}) > 0$. We have the following result.

Theorem 42 *The following two statements are equivalent:*

- (i) *a discrete-time system with transfer function matrix as described at the beginning of Section 9.4.2, is a “mixed” system;*
- (ii) $I_{S_1} \cup I_{S_2} = \{\theta \in \mathbb{R} : -\pi \leq \theta \leq \pi\}$.

PROOF: (i) \Rightarrow (ii) By Definition 24, a “mixed” discrete-time system is such that $\{\theta \in \mathbb{R} : -\pi \leq \theta \leq \pi \text{ and } G_1(e^{j\theta}) \geq 0 \text{ for } k > 0 \text{ and } l > 0\} \cup \{\theta \in \mathbb{R} : -\pi \leq \theta \leq \pi \text{ and } G_2(e^{j\theta}) \geq 0 \text{ for } \varepsilon < 1\} = \{\theta \in \mathbb{R} : -\pi \leq \theta \leq \pi\}$. Since the existence of $k > 0$ and $l > 0$ such that $G_1(e^{j\theta}) \geq 0$ for any $\theta \in \mathbb{R}$ implies that $\tilde{G}_1(e^{j\theta}) > 0$, and the existence of $\varepsilon < 1$ such that $G_2(e^{j\theta}) \geq 0$ for any $\theta \in \mathbb{R}$ implies that $\tilde{G}_2(e^{j\theta}) > 0$, then $\{\theta \in \mathbb{R} : -\pi \leq \theta \leq \pi \text{ and } \tilde{G}_1(e^{j\theta}) > 0\} \cup \{\theta \in \mathbb{R} : -\pi \leq \theta \leq \pi \text{ and } \tilde{G}_2(e^{j\theta}) > 0\} = \{\theta \in \mathbb{R} : -\pi \leq \theta \leq \pi\}$ which leads to (ii).

(ii) \Rightarrow (i) Consider those intervals in Division Group 1 over which $\tilde{G}_1(e^{j\theta}) > 0$. For illustration purposes, suppose that one of the intervals has open endpoints. Denote this interval by $(\theta_{p_\alpha}, \theta_{p_\beta})$ and observe that $\tilde{G}_1(e^{j\theta}) \geq 0$ over $[\theta_{p_\alpha}, \theta_{p_\beta}]$. Observe that an infinitesimal increase in k and l yields $G_1(e^{j\theta}) \geq 0$ over $(\theta_{p_\alpha}, \theta_{p_\beta})$. Dealing with closed interval endpoints is straightforward since $\tilde{G}_1(e^{j\theta_0}) > 0$ at $\theta_0 \in \mathbb{R}$ implies that there exist $k > 0$ and $l > 0$ such that $G_1(e^{j\theta_0}) \geq 0$ at $\theta_0 \in \mathbb{R}$.

Now, consider those intervals in Division Group 2 over which $\tilde{G}_2(e^{j\theta}) > 0$. For illustration purposes, suppose that one of the intervals has open endpoints. Denote this interval by $(\theta_{s_\alpha}, \theta_{s_\beta})$ and observe that $\tilde{G}_2(e^{j\theta}) \geq 0$ over $[\theta_{s_\alpha}, \theta_{s_\beta}]$. Observe that an infinitesimal decrease in ε yields $G_2(e^{j\theta}) \geq 0$ over $(\theta_{s_\alpha}, \theta_{s_\beta})$. Dealing with closed interval endpoints is, again, straightforward since $\tilde{G}_2(e^{j\theta_0}) > 0$ at $\theta_0 \in \mathbb{R}$ implies that there exists $\varepsilon < 1$ such that $G_2(e^{j\theta_0}) \geq 0$ at $\theta_0 \in \mathbb{R}$.

This leads to (i).

Note that Theorem 42 can be applied to square, multi-input, multi-output systems where graphical methods for determining “mixedness” might be impractical or unavailable.

9.4.3 Preserving mixedness under discretization

Now we recall Lemma 3, 4 from Section 2.3.1 of Chapter 2. Consider a stable continuous-time system transfer function $M(s) = D + C^T(sI - A)^{-1}B \in \mathcal{RH}_\infty$ and let $M_1(j\omega) = M(j\omega) + M(j\omega)^*$ (where $k = l = 0$). Similarly, let $M_2(j\omega) := -M(j\omega)^*M(j\omega) + I$ (where $\varepsilon = 1$). Consider Lemmas 3 and 4. Set

$$\Omega_p := \{\omega \in [-\infty, \infty] : N_1 \text{ has an eigenvalue on the imaginary axis}\}$$

$$\Omega_s := \{\omega \in [-\infty, \infty] : N_2 \text{ has an eigenvalue on the imaginary axis}\}.$$

Suppose that we divide the real axis $-\infty$ to ∞ into smaller intervals, where any elements of Ω_p and Ω_s are set as open interval endpoints, as follows:

$$\text{Division group 1} := (-\infty, \omega_{p_1}), (\omega_{p_1}, \omega_{p_2}), \dots, (\omega_{p_{n-1}}, \omega_{p_n}), (\omega_{p_n}, \infty),$$

$$\text{Division group 2} := (-\infty, \omega_{s_1}), (\omega_{s_1}, \omega_{s_2}), \dots, (\omega_{s_{m-1}}, \omega_{s_m}), (\omega_{s_m}, \infty),$$

where n = number of elements in Ω_p ; m = number of elements in Ω_s ; $\omega_{p_1}, \omega_{p_2}, \dots, \omega_{p_n}$ denote the elements of Ω_p listed in increasing order; and $\omega_{s_1}, \omega_{s_2}, \dots, \omega_{s_m}$ denote the elements

of Ω_s listed in increasing order.

Let I_{N_1} denote the set of ω belonging to the intervals over which $M_1(j\omega) > 0$, and I_{N_2} denote the set of ω belonging to the intervals over which $M_2(j\omega) > 0$. If $I_{N_1} \cup I_{N_2} = \{\theta \in \mathbb{R} : -\infty \leq \omega \leq \infty\}$ then $M(s)$ is a “mixed” system.

Upon discretization, from Lemmas 31 and 32 we will have

$$\begin{aligned}\Theta_p &:= \{\theta \in [-\pi, \pi] : S_1 \text{ has an eigenvalue on the unit disc at } e^{j\theta}\} \\ \Theta_s &:= \{\theta \in [-\pi, \pi] : S_2 \text{ has an eigenvalue on the unit disc at } e^{j\theta}\},\end{aligned}$$

and

$$\begin{aligned}\text{Division Group 1} &:= (-\pi, \theta_{p_1}), (\theta_{p_1}, \theta_{p_2}), \dots, (\theta_{p_{\bar{n}-1}}, \theta_{p_{\bar{n}}}), (\theta_{p_{\bar{n}}}, \pi), \\ \text{Division Group 2} &:= (-\pi, \theta_{s_1}), (\theta_{s_1}, \theta_{s_2}), \dots, (\theta_{s_{\bar{m}-1}}, \theta_{s_{\bar{m}}}), (\theta_{s_{\bar{m}}}, \pi),\end{aligned}$$

where \bar{n} = number of elements in Θ_p ; \bar{m} = number of elements in Θ_s ; $\theta_{p_1}, \theta_{p_2}, \dots, \theta_{p_{\bar{n}}}$ denote the elements of Θ_p listed in increasing order; and $\theta_{s_1}, \theta_{s_2}, \dots, \theta_{s_{\bar{m}}}$ denote the elements of Θ_s listed in increasing order.

Considering the continuous time transition frequencies Ω_p and Ω_s and the discrete time transition frequencies Θ_p, Θ_s , it is obvious that “mixedness” of a continuous time transfer function $M(s)$ is preserved if :

1. For every ω_{p_i} and ω_{s_i} there should be a corresponding θ_{p_i} and θ_{s_i} respectively. Thus $n = \bar{n}$ and $m = \bar{m}$.
2. If $\omega_{p_i} \leq \omega_{p_j}$ ($\omega_{s_i} \leq \omega_{s_j}$), then we should have $\theta_{p_i} \leq \theta_{p_j}$ ($\theta_{s_i} \leq \theta_{s_j}$). Hence if there exists a function $f : \omega \rightarrow \theta$, then f must be monotonic function.

Now we show that Tustin’s method given by $j\omega = \frac{2}{h} \frac{e^{j\theta} - 1}{e^{j\theta} + 1}$, satisfies these criteria. If $M(s)$ is the continuous time transfer function, then corresponding discrete time transfer function is given by $G(e^{j\theta}) = M(\frac{2}{h} \frac{e^{j\theta} - 1}{e^{j\theta} + 1})$. Now we check if this discretization method satisfies each of the conditions sufficient for preserving “mixedness”.

Condition 1: Transition frequencies Ω_p and Ω_s are obtained from imaginary eigenvalues of N_1 and N_2 . Let us obtain Ω_p from the equation $\det[M_1(j\omega)] = 0 \iff \det(j\omega I - N_1) = 0$. For Tustin’s approximation $G(e^{j\theta})$, Θ_p is obtained by solving $\det[G_1(e^{j\theta})] = \det[M_1(\frac{2}{h} \frac{e^{j\theta} - 1}{e^{j\theta} + 1})] = 0$, i.e.,

$$\begin{aligned}\det\left(\frac{2}{h} \frac{e^{j\theta} - 1}{e^{j\theta} + 1} I - N_1\right) &= 0 \\ \iff \det\left(e^{j\theta} I - (I - N_1 h/2)^{-1} (I + N_1 h/2)\right) &= 0,\end{aligned}$$

hence we can see that for the Tustin method, $S_1 = (I - N_1 h/2)^{-1}(I + N_1 h/2)$. Since $(I - N_1 h/2)^{-1}(I + N_1 h/2)$ is the bilinear transform of N_1 , all the imaginary eigenvalues of N_1 will be mapped onto the unit circle. Similarly, we can prove that $S_2 = (I - N_2 h/2)^{-1}(I + N_2 h/2)$. Thus the first condition is satisfied.

Condition 2: If $j\omega = \frac{2}{h} \frac{e^{j\theta} - 1}{e^{j\theta} + 1}$, then $\omega h/2 = \tan(\theta/2) \Rightarrow \theta = 2\tan^{-1}(\omega h/2)$. It can be observed that $2\tan^{-1}(\omega h/2)$ is a strictly monotonic function (increasing). Hence Condition 2 is also satisfied.

Thus Tustin's method is a suitable candidate for discretizing mixed **LTI** transfer functions. Since index-one transfer function can be modified into a *regular system* transfer function, the above discussion is equally valid for index-one systems.

9.5 CONCLUSIONS

In this chapter, we considered passivity preserving discretization of **LTI** descriptor systems. We did so by extending the existing methods for discretizing *regular systems* while preserving passivity. We also showed that Tustin's method of approximating continuous-time transfer functions is equally effective for preserving mixedness.

PRESERVATION OF QUADRATIC STABILITY FOR SWITCHED
LINEAR SYSTEMS

In this chapter, we consider the stability preserving properties of diagonal Padé approximations to the matrix exponential. We show that while diagonal Padé approximations preserve quadratic stability when going from continuous-time to discrete-time, the converse is not true. We discuss the implications of this result for discretizing switched linear systems. We also show that for continuous-time switched systems which are exponentially stable, but not quadratically stable, a Padé approximation may not preserve stability. Finally, we show that diagonal Padé approximations for continuous time descriptor systems also preserve quadratic stability.

10.1 INTRODUCTION

The $[1/1]$ -order Padé approximation (or bilinear map) is known not only to preserve stability (A-stable), but also preserve quadratic Lyapunov functions. That is, a positive definite matrix P satisfying $A_c^*P + PA_c < 0$ will also satisfy $A_d^*PA_d - P < 0$, where A_d is the mapping of A_c under the bilinear transform [90] with some sampling time h [129]. This makes it extremely useful when transforming a continuous-time switching system

$$\Sigma_{sc} : \dot{x}(t) = A_c(t)x(t), \quad A_c(t) \in \mathcal{A}_c = \{A_{c1}, \dots, A_{cN}\} \quad (10.1)$$

into its approximate discrete-time counterpart¹,

$$\Sigma_{sd} : x(k+1) = A_d(k)x(k), \quad A_d(k) \in \mathcal{A}_d = \{A_{d1}, \dots, A_{dN}\}, \quad (10.2)$$

because the existence of a common positive definite matrix P satisfying $A_c^*P + PA_c < 0$ for all $A_c \in \mathcal{A}_c$ implies that the same P satisfies $A_d^*PA_d - P < 0$ for all $A_d \in \mathcal{A}_d$. Thus quadratic stability of the continuous-time switching system implies quadratic stability of the discrete-time counterpart. This property is useful in obtaining results in discrete-time from their continuous-time counterparts [90], and in providing a robust method to obtain a stable discrete-time switching system from a continuous-time one.

¹ Discretization error is zero, only at sampling instants.

Our objective in this chapter is to determine whether this property is preserved by higher-order (more accurate) diagonal Padé approximants. From the point of view of discretization, lower-order approximants are not always satisfactory, and one often chooses higher-order Padé approximations in real applications. Later, we present an example of an exponentially stable continuous-time switching system for which a discretisation based on a $[1/1]$ -order Padé approximation is unstable, but, discretizations based on second-order approximations are stable for any sampling time. Also, it is well known that the $[1/1]$ -order Padé approximation (the bilinear approximation) can map a negative real eigenvalue to a negative eigenvalue if the sampling time is large. In such situations, while stability is preserved, qualitative behavior is not preserved even for LTI systems; a non-oscillatory continuous mode is transformed into an oscillatory discrete-time mode. In this context, we establish the following facts concerning general diagonal Padé approximations:

- (i) Consider an LTI system $\Sigma_c : \dot{x} = A_c x$ and let $\Sigma_d : x(k+1) = A_d x(k)$ be any discrete-time system obtained from Σ_c using any diagonal Padé approximation and any sampling time. If V is any quadratic Lyapunov function for Σ_c then, V is a quadratic Lyapunov function for Σ_d .
- (ii) The converse of the statement in (i) is only true for first-order Padé approximations.
- (iii) Consider a switched system $\Sigma_{sc} : \dot{x} = A_{sc}(t)x$, $A_{sc}(t) \in \{A_{c1}, \dots, A_{cn}\}$ and let $\Sigma_d : x(k+1) = A_{sd}(k)x(k)$, $A_{sd}(k) \in \{A_{d1}, \dots, A_{dn}\}$ be a discrete-time switched system obtained from Σ_{sc} using any diagonal Padé approximations and any sampling times. If V is any quadratic Lyapunov function for Σ_{sc} then, V is a quadratic Lyapunov function for Σ_{sd} .
- (iv) The converse of the statement in (iii) is only true for first-order Padé approximations.
- (v) Consider an exponentially stable switched system $\Sigma_{sc} : \dot{x} = A_{sc}(t)x$, $A_{sc}(t) \in \{A_{c1}, \dots, A_{cn}\}$. Let $\Sigma_d : x(k+1) = A_{sd}(k)x(k)$, $A_{sd}(k) \in \{A_{d1}, \dots, A_{dn}\}$ be a discrete-time switched system obtained from Σ_{sc} using a p th order diagonal Padé approximation. Then, Σ_{ds} may be unstable, even when $p = 1$.

10.2 PRELIMINARY RESULTS

The following definitions and results are useful in developing the main result, Theorem 43, which is given in Section 10.3.

As we shall see, bilinear transforms play a key role in studying general diagonal Padé approximations. Next, we present a *complex* version of this map that inherits some of the properties of the real bilinear map given in Theorem 39.

Lemma 33 (The complex bilinear transform) *Let A_c be a Hurwitz stable matrix and for any complex number λ with $\Re(\lambda) > 0$, define the matrix*

$$A_d = (\lambda I + A_c)(\lambda^* I - A_c)^{-1}. \quad (10.3)$$

Then P is a Lyapunov matrix for A_c if and only if P is a Stein matrix for A_d .

PROOF : Consider any matrix $P = P^* > 0$. When A_d is given by (10.3), the Stein inequality $A_d^* P A_d - P < 0$ can be expressed as

$$(\lambda^* I - A_c)^{-*} (\lambda I + A_c)^* P (\lambda I + A_c) (\lambda^* I - A_c)^{-1} - P < 0.$$

Post-multiplication by $\lambda^* I - A_c$ and pre-multiplication by $(\lambda^* I - A_c)^*$ results in the following equivalent inequality

$$(\lambda I + A_c)^* P (\lambda I + A_c) - (\lambda^* I - A_c)^* P (\lambda^* I - A_c) < 0,$$

which simplifies to

$$(\lambda + \lambda^*) (P A_c + A_c^* P) < 0.$$

Since $\lambda + \lambda^* > 0$, this last inequality is equivalent to the Lyapunov inequality $P A_c + A_c^* P < 0$. Thus P is a Lyapunov matrix for A_c if and only if it is a Stein matrix for A_d .

The main result that we shall prove in this chapter concerns common Stein matrices for discrete-time systems.

Lemma 34 *If P is a CSM for A_1, \dots, A_m , then P is a Stein matrix for the matrix product $\prod_{i=1}^m A_i$.*

PROOF : Suppose that P is a common Stein matrix for two matrices A_1 and A_2 , that is,

$$A_1^* P A_1 < P \quad \text{and} \quad A_2^* P A_2 < P. \quad (10.4)$$

Pre-multiply the first inequality by A_2^* and post-multiply it by A_2 and use the second inequality to obtain

$$A_2^* A_1^* P A_1 A_2 \leq A_2^* P A_2 < P, \quad (10.5)$$

that is, $(A_1 A_2)^* P (A_1 A_2) < P$, which implies that P is a Stein matrix for the product $A_1 A_2$. This shows that the statement of the lemma is true for $m = 2$. Now assume that it is true for $m = k$, and then let $M_k = \prod_{i=1}^k A_i$. Since $M_{k+1} = M_k A_{k+1}$, it follows from the result for two matrices that P is a Stein matrix for M_{k+1} . Hence, by induction, the proposed lemma is true for all m . So it can be concluded that if all the constituent matrices of a product have a CSM P , then P is a Stein matrix for the product.

10.3 HIGHER ORDER DIAGONAL PADÉ APPROXIMATIONS

We now present the main result of this chapter: Theorem 43. A main consequence of this result is that common quadratic Lyapunov functions are preserved by all diagonal Padé discretizations for all sampling times. Thus, quadratic stability is preserved under all diagonal Padé discretizations of a quadratically stable continuous-time switched system. This result is stated formally in Corollary 10.

Theorem 43 *Suppose that A_c is a Hurwitz stable matrix and A_d is any $[L/L]$ -order Padé approximation to $e^{A_c h}$ for any $h > 0$. If P is a Lyapunov matrix for A_c then, P is a Stein matrix for A_d .*

PROOF: Consider any matrix P which is a Lyapunov matrix for A_c . Recall that $A_d = Q_L(A_c h)Q_L^{-1}(-A_c h)$. Since the coefficients of the polynomial Q_p are real,

$$Q_L(sh) = kh^L \prod_{j=1}^n (\alpha_j + s) \prod_{i=1}^m (\lambda_i + s)(\lambda_i^* + s)$$

for some $k \neq 0$, where $2m + n = L$, the real numbers $-\alpha_j$, $j = 1, \dots, n$ are the real zeros of Q_L and the complex numbers $-\lambda_i$, $-\lambda_i^*$, $i = 1, \dots, m$ are the non-real zeros of Q_L . Since all the zeros of Q_p have negative real parts ([88][124]), we must have $\alpha_j > 0$ for all j and $\Re(\lambda_i) > 0$ for all i . It now follows that A_d can be expressed as

$$A_d = \left(\prod_{j=1}^n (\alpha_j I + A_c) \right) \left(\prod_{i=1}^m (\lambda_i I + A_c)(\lambda_i^* I + A_c) \right) \left(\prod_{i=1}^m (\lambda_i I - A_c)(\lambda_i^* I - A_c) \right)^{-1} \left(\prod_{j=1}^n (\alpha_j I - A_c) \right)^{-1}$$

which, due to commutativity of the factors, can be expressed as

$$A_d = \left(\prod_{j=1}^n (\alpha_j I + A_c)(\alpha_j^* I - A_c)^{-1} \right) \left(\prod_{i=1}^m (\lambda_i I + A_c)(\lambda_i^* I - A_c)^{-1} \right) \left(\prod_{i=1}^m (\lambda_i^* I + A_c)(\lambda_i I - A_c)^{-1} \right).$$

Hence A_d is a product of bilinear terms of the form $(\lambda I + A_c)(\lambda^* I - A_c)^{-1}$, where $\Re(\lambda) > 0$. Since P is a Lyapunov matrix for A_c , it follows from Lemma 33 that P is a Stein matrix for each of the bilinear terms. Thus A_d is a product of a bunch of matrices each of which have P as a Stein matrix. It now follows from Lemma 34 that P is a Stein matrix for A_d .

In other words, the theorem states that if A_d is a diagonal Padé approximation of $e^{A_c h}$ for any $h > 0$, then a Lyapunov matrix for A_c is also a Stein matrix for A_d . Lemma 39 tells us that the converse of this statement is true for $L = 1$. However, the converse of this statement is not necessarily true for $L \geq 2$; that is, for $L \geq 2$, a Stein matrix for A_d is not necessarily a Lyapunov matrix for A_c , and in general \mathcal{L}_{A_c} is strictly contained in \mathcal{S}_{A_d} . This is demonstrated in the following example.

Example 5 Consider the Hurwitz stable matrix:

$$A_c = \begin{bmatrix} 1.56 & -100 \\ 0.1 & -4.44 \end{bmatrix}$$

Now consider the matrix A_d obtained under the $[2/2]$ -order diagonal Padé approximation of $e^{A_c h}$ with the discrete time step $h = 2$:

$$A_d = \begin{bmatrix} -0.039 & 0.4205 \\ -0.0004 & -0.0138 \end{bmatrix}.$$

The matrix

$$P = \begin{bmatrix} 2.3294 & -0.0138 \\ -0.0138 & 2.7492 \end{bmatrix}$$

is a Stein matrix for A_d but is not a Lyapunov matrix for A_c .

The following corollary is easily deduced from the main theorem. This is probably the most useful result in the chapter. It says that quadratic stability is preserved under all diagonal Padé discretizations of a quadratically stable continuous-time switched system.

Corollary 10 *Suppose that $P = P^* > 0$ is a CLM for a finite set of matrices \mathcal{A}_c . Then P is CSM for any finite set of matrices \mathcal{A}_d , where each A_d in \mathcal{A}_d is a diagonal Padé approximation of $e^{A_c h}$ of any order for some A_c in \mathcal{A}_c and $h > 0$.*

PROOF : If P is a CLM for \mathcal{A}_c then, P is an Lyapunov matrix for every A_c in \mathcal{A}_c . It now follows from Theorem 43, that P is a Stein matrix for every A_d in \mathcal{A}_d . Hence P is a CSM for \mathcal{A}_d .

The last corollary shows that diagonal Padé approximations preserve quadratic stability for switching systems. Thus, quadratic stability of a continuous-time switching system implies quadratic stability of the corresponding discrete-time switching system obtained via a diagonal Padé discretization. However, it is very important to note that the corollary does not imply the converse. In fact converse is not true in general as the following example illustrates.

Example 6 Consider the Hurwitz stable matrices:

$$A_{c1} = \begin{bmatrix} 1.56 & -100 \\ 0.1 & -4.44 \end{bmatrix}, \quad A_{c2} = \begin{bmatrix} -1 & 0 \\ 0 & -0.1 \end{bmatrix}.$$

Since the matrix product $A_{c1}A_{c2}$ has negative real eigenvalues it follows that there is no CLM [130] for $\{A_{c1}, A_{c2}\}$. Now consider the matrices A_{d1}, A_{d2} obtained under the $[2/2]$ -order diagonal Padé approximation of $e^{A_{ci}h}$ with the discrete time step $h = 2$:

$$A_{d1} = \begin{bmatrix} -0.039 & 0.4205 \\ -0.0004 & -0.0138 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.1429 & 0 \\ 0 & 0.8187 \end{bmatrix}.$$

These matrices have a CSM

$$P_d = \begin{bmatrix} 2.3294 & -0.0138 \\ -0.0138 & 2.7492 \end{bmatrix}.$$

Comment 7 Example 1, together with Corollary 1, illustrate the following facts. Let \mathcal{A}_c be a finite set of Hurwitz stable matrices and \mathcal{A}_d the corresponding finite set of Schur stable matrices obtained under diagonal Padé approximations for fixed L and h . If P is a CLM for \mathcal{A}_c , then P is a CSM for \mathcal{A}_d . However, as the example demonstrates, the existence of a CSM for \mathcal{A}_d does not imply the existence of a CLM for \mathcal{A}_c .

10.4 A CONVERSE RESULT

We have seen that if P is a Lyapunov matrix for A_c , then for any positive integer L , P is a Stein matrix for the $[L/L]$ -order Padé approximation of $e^{A_c h}$ for **all** $h > 0$ that is,

$$A_d(h)^* P A_d(h) - P < 0 \quad \text{for all } h > 0,$$

where $A_d(h)$ is a diagonal Padé approximation (of any fixed order) to $e^{A_c h}$. The next lemma tells us that in order to achieve a converse result, we need the following additional condition to hold,

$$\lim_{h \rightarrow 0} \frac{A_d(h)^* P A_d(h) - P}{h} < 0. \quad (10.6)$$

Lemma 35 *Suppose that, for all $h > 0$, the matrix $A_d(h)$ is a Padé approximation (of any fixed order) to $e^{A_c h}$. Then P is a Lyapunov matrix for A_c if and only if P is a Stein matrix for $A_d(h)$ for all $h > 0$ and (10.6) holds.*

PROOF: In view of our previous results, we can prove this result if we show that

$$\lim_{h \rightarrow 0} \frac{A_d(h)^* P A_d(h) - P}{h} = P A_c + A_c^* P. \quad (10.7)$$

To demonstrate this limit, first recall that $A_d(h) = Q_L(A_c h) Q_L(-A_c h)^{-1}$ and

$$Q_L(A_c h) = I + \frac{1}{2}(A_c h) + h^2 D_L(A_c h),$$

where D_L is a polynomial. Hence

$$\lim_{h \rightarrow 0} Q_L(-A_c h) = I$$

and

$$\lim_{h \rightarrow 0} \frac{Q_L(A_c h)^* P Q_L(A_c h) - Q_L(-A_c h)^* P Q_L(-A_c h)}{h} = P A_c + A_c^* P$$

Since

$$A_d(h)^* P A_d(h) - P = Q_L(-A_c h)^{-*} [Q_L(h A_c)^* P Q_L(h A_c) - Q_L(-A_c h)^* P Q_L(-A_c h)] Q_L(-A_c h)^{-1}$$

we obtain the desired result (10.7).

10.5 IMPLICATIONS OF MAIN RESULT

The starting point for our work was the recently published paper [90]. One of the main results of that paper was the fact that the bilinear transform preserves quadratic stability when applied to continuous-time switched systems. We have shown that this property also holds for general diagonal Padé approximations (although the converse statement is not true). This is an important observation due to the fact that while the bilinear transform is stability preserving, it is not always a good approximation to the matrix exponential. Our result says that “more accurate” approximations are also stability preserving when going from continuous-time to discrete-time.

Two potential applications of this result are immediate. First, stable discrete-time LTI systems can be obtained from their continuous-time counterparts in a manner akin to that described in [90]. Secondly, our results provide a method to discretize quadratically stable linear switched system in a manner that preserves stability. That is, given a quadratically stable switched linear systems, a discrete-time counterpart obtained using diagonal Padé approximations to the matrix exponential, will also be quadratically stable. Since this property is true for all orders of approximation, and for all sampling times, our main result says that *quadratic stability is robustly preserved under diagonal Padé discretizations of any order*.

In the context of the previous comment, it is important to realize that the robust stability preserving property of diagonal Padé approximations is a unique feature of quadratically stable systems. It was recently shown that non-quadratic Lyapunov functions may not be preserved under the bilinear transform with sampling time $h = 2$. This fact was first demonstrated in [90], where it was proven that unlike quadratic Lyapunov functions, ∞ -norm and 1-norm type Lyapunov functions are not necessarily preserved under the bilinear mapping with $h = 2$. In fact, the situation may be worse as the following example illustrates.

Example 7 Consider a continuous-time switching system described by (10.1) with $\mathcal{A}_c = \{A_{c1}, A_{c2}, A_{c3}\}$ where

$$A_{c1} = \begin{bmatrix} -19.00 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -0.10 \end{bmatrix}, \quad A_{c2} = \begin{bmatrix} -19 & 0 & 0 \\ -10 & -9 & 0 \\ -18.75 & 0 & -0.10 \end{bmatrix}, \quad A_{c3} = \begin{bmatrix} -19.00 & 0 & 18.75 \\ 0 & -9 & 8.75 \\ 0 & 0 & -0.10 \end{bmatrix}.$$

Using the ideas in [131] (also see Theorem 44 in the next section) it can be shown that this continuous-time switching system is globally exponentially stable. It follows from the results of Dayawansa and Martin [52] that this switching system has a Lyapunov function (though this is not necessarily quadratic). Now, consider a discrete-time approximation to the above system. We assume that switching is restricted to only occur at multiples of the sampling time $h = 0.25$. Using the $[1/1]$ -order Padé approximation, we obtain a discrete-time switching system described by (10.2) with $\mathcal{A}_d = \{A_{d1}, A_{d2}, A_{d3}\}$ where

$$A_{di} = \left(I - \frac{1}{8}A_{ci}\right)^{-1} \left(I + \frac{1}{8}A_{ci}\right), \quad i = 1, 2, 3.$$

that is,

$$A_{d1} \approx \begin{bmatrix} -0.40 & 0 & 0 \\ 0 & -0.06 & 0 \\ 0 & 0 & 0.98 \end{bmatrix}, \quad A_{d2} \approx \begin{bmatrix} -0.40 & 0 & 0 \\ -0.35 & -0.06 & 0 \\ -1.37 & 0 & 0.98 \end{bmatrix}, \quad A_{d3} \approx \begin{bmatrix} -0.40 & 0 & 1.37 \\ 0 & -0.06 & 1.01 \\ 0 & 0 & 0.98 \end{bmatrix}.$$

We now claim that the discrete-time switching system is unstable. To see this, we simply consider the incremental switching sequence $A_{d3} \rightarrow A_{d2} \rightarrow A_{d1}$; then the dynamics of the system evolve according to the product

$$A_d = A_{d1}A_{d2}A_{d3}.$$

Since the eigenvalues of A_d are approximately $\{-0.002, -0.060, -1.035\}$, then with one eigenvalue outside the unit disc, this switching sequence, repeated periodically results in an unstable system.

Clearly, by selecting a smaller sampling time one obtains a better approximation to the continuous-time system. However, selecting an appropriate sampling time is difficult for switched systems since the sampling time is usually related to solution growth rates. While this is simple to calculate for an LTI system, bounds on the solution growth rates are usually very difficult to calculate for a switched system. On the other hand, were the original system quadratically stable, then our main result implies that stability can never be lost by a bad or unlucky choice of sampling time.

A further comment on the counter example

Example 7 in the previous section indicates that our main result and its corollary do not, in general, extend to switched systems which are exponentially stable, but do not have a

quadratic Lyapunov function. An interesting question therefore to ask is how one discretizes a general, exponentially stable, switching system. In this section we give a preliminary result in this direction. Specifically, we take a closer look at Example 7, and ask the question as to how one might discretize the system in the example so that exponential stability is preserved irrespective of the choice of sampling time. Our results can be summarised as follows:

- (i) Even-ordered Padé discretizations preserve exponential stability for the system class illustrated by Example 7. This is true for any even ordered approximation, and for any sampling time.
- (ii) Odd-ordered Padé discretizations preserve exponential stability provided the sampling time is smaller than a computable bound.

The above items say that even-ordered Padé discretizations preserve stability in a robust manner; odd-ordered ones do not. Example 7 is an example of a switching system of the form (10.1) where every matrix A_c in \mathcal{A}_c has real negative eigenvalues and every pair of matrices in \mathcal{A}_c have $n - 1$ common eigenvectors (namely all such matrix pairs are pairwise triangularizable). It is shown in [131] that such systems are exponentially stable. This result follows from the following theorem in [131] which we give here to aid our discussion.

Theorem 44 [131] *Suppose $\mathcal{V} = \{v_1, \dots, v_{n+1}\}$ is a set of vectors in \mathbb{R}^n with the property that any subset of n vectors is linearly independent. Let*

$$\mathcal{M} = \{M_i : i = 0, 1, \dots, n\},$$

where $M_0 = [v_1 \ \dots \ v_n]$ and

$$M_i = [v_1 \ \dots \ v_{n+1} \ v_{i+1} \ \dots \ v_n] \quad \text{for} \quad i = 1, 2, \dots, n, \quad (10.8)$$

that is, M_i is obtained by replacing the i -th column in M_0 with the vector v_{n+1} . Let \mathcal{A}_c be any finite subset of the following set of matrices:

$$\{MDM^{-1} : M \in \mathcal{M} \text{ and } D \text{ is diagonal negative definite} \} \quad (10.9)$$

Then the continuous-time switching system (10.1) is globally exponentially stable.

Recently, a discrete-time version of this result was obtained in [132]. Namely, a discrete-time switching system is exponentially stable if every pair of matrices in \mathcal{A}_d share $n - 1$ common eigenvectors, and if all eigenvalues are real, inside the unit circle, and positive [132] (i.e. there is no oscillatory behavior). In both the discrete-time case and the continuous-time case, the same type of Lyapunov function is used to prove stability. Since Padé approximations are eigenvector preserving, it immediately follows that any approximations that map real negative eigenvalues to positive ones, will, by invoking the above result, preserve exponential stability.

Using the above observations we obtain our next result. To describe this result, consider any positive integer L and let

$$\bar{\alpha}_L = \begin{cases} \text{largest real zero of } Q_L \\ -\infty \text{ if } Q_L \text{ has no real zeros} \end{cases}$$

Since all real zeros of Q_L must be negative, we must have $\bar{\alpha}_L < 0$. When p is odd, Q_L must have at least one real zero; hence $\bar{\alpha}_L$ is finite. When L is even, we show later than Q_L does not have any real zeros; hence $\bar{\alpha}_L = -\infty$ for even L . To illustrate,

$$Q_1(s) = 1 + \frac{1}{2}s, \quad Q_2(s) = 1 + \frac{1}{2}s + \frac{1}{12}s^2;$$

hence

$$\bar{\alpha}_1 = -2, \quad \bar{\alpha}_2 = -\infty.$$

Theorem 45 *Suppose that \mathcal{A}_c is a set of matrices satisfying the hypotheses of Theorem 44 and let*

$$\underline{\alpha} = \min\{\alpha : \alpha \text{ is an eigenvalue of } A_c \text{ and } A_c \in \mathcal{A}_c\}.$$

Consider any positive integer L and define

$$\bar{h}_L = \begin{cases} \bar{\alpha}_L / \underline{\alpha} & \text{if } Q_L \text{ has a real zero} \\ \infty & \text{if } Q_L \text{ has no real zeros} \end{cases} \quad (10.10)$$

Let \mathcal{A}_d be any finite subset of

$$\{C_{[L/L]}(hA_c) : A_c \in \mathcal{A}_c \text{ and } 0 < h < \bar{h}_L\}$$

Then the discrete-time switching system (10.2) is globally exponentially stable.

PROOF: We first show that all the eigenvalues of the matrices in \mathcal{A}_d must be positive, real and less than one. So, consider any matrix A_d in \mathcal{A}_d . This matrix can be expressed as $A_d = C_{[L/L]}(A_c h)$ where A_c is in \mathcal{A}_c and $h < \bar{\alpha}_L / \underline{\alpha}$. From the description of \mathcal{A}_c we have $A_c = MDM^{-1}$ where D is diagonal with negative diagonal elements, $\alpha_1, \dots, \alpha_n$. Consider any $i = 1, \dots, n$. Since α_i is an eigenvalue of A_c , it follows from the definition of $\underline{\alpha}$ that $\alpha_i \geq \underline{\alpha}$; hence $h\alpha_i \geq h\underline{\alpha}$. Recalling the requirement that $h < \bar{\alpha}_L / \underline{\alpha}$ and noting that $\underline{\alpha} < 0$ we must have $h\underline{\alpha} > \bar{\alpha}_L$; hence

$$h\alpha_i > \bar{\alpha}_L$$

Since $Q_L(s) \neq 0$ for $s > \bar{\alpha}_L$ where $\bar{\alpha}_L < 0$ and $Q_L(0) = 1 > 0$, it follows from the continuity of Q_L that $Q_L(s) > 0$ for $s > \bar{\alpha}_L$; hence $Q_L(h\alpha_i) > 0$. Since $-h\alpha_i > 0$, we also have

$Q_L(-h\alpha_i) > 0$. Hence $C_{[L/L]}(h\alpha_i) = Q_L(h\alpha_i)/Q_L(-h\alpha_i) > 0$. Since $h\alpha_i < 0$ and $C_{[L/L]}$ maps the open left half plane into the open unit disk, we must also have $C_{[L/L]}(h\alpha_i) < 1$. Since $A_d = C_{[L/L]}(A_c h)$ and $A_c = MDM^{-1}$, we have

$$A_d = M\Lambda M^{-1}$$

where Λ is diagonal with diagonal elements

$$\Lambda_{ii} = C_{[L/L]}(h\alpha_i), \quad i = 1, \dots, p$$

Hence $C_{[L/L]}(h\alpha_1), \dots, C_{[L/L]}(h\alpha_L)$, are the eigenvalues of A_d and these eigenvalues are positive, real and less than one.

We will now show that

$$\mathcal{A}_d = \{e^{\tilde{A}_c} : \tilde{A}_c \in \tilde{\mathcal{A}}_c\} \quad (10.11)$$

where $\tilde{\mathcal{A}}_c$ is a set of matrices which satisfy the hypotheses of Theorem 44. This will imply that the continuous-time switching system

$$\dot{x} = \tilde{A}_c(t)x(t) \quad \tilde{A}_c(t) \in \tilde{\mathcal{A}}_c \quad (10.12)$$

is globally exponentially stable. Relationship (10.11) tells us that the state of the discrete-time system (10.2) corresponds to the state at $t = 0, 1, 2, \dots$ of the continuous-time system (10.12) switching at these times; this will imply that the discrete-time switching system is globally exponentially stable. To achieve the above goal, consider any $i = 1, \dots, L$ and we let $\tilde{\alpha}_i = \ln[C_{[L/L]}(h\alpha_i)]$. Then $\tilde{\alpha}_i$ is negative real and

$$C_{[L/L]}(h\alpha_i) = e^{\tilde{\alpha}_i}. \quad (10.13)$$

Now consider $\tilde{A}_c = M\tilde{D}M^{-1}$ where \tilde{D} is the diagonal matrix with negative diagonal elements $\tilde{\alpha}_1, \dots, \tilde{\alpha}_L$. Since $\tilde{A}_c = M\tilde{D}M^{-1}$ we also have $e^{\tilde{A}_c} = M\tilde{\Lambda}M^{-1}$ where $\tilde{\Lambda}$ is diagonal with diagonal elements

$$\tilde{\Lambda}_{ii} = e^{\tilde{\alpha}_i}, \quad i = 1, \dots, L.$$

It follows from (10.13) that $\tilde{\Lambda} = \Lambda$; hence

$$A_d = e^{\tilde{A}_c}.$$

Since \mathcal{A}_c is a finite set of matrices satisfying the hypotheses of Theorem 44, it now follows that \mathcal{A}_d can be expressed as (10.11) where $\tilde{\mathcal{A}}_c$ is a finite set of matrices satisfying the hypotheses of Theorem 44. As explained above this now implies that the discrete-time switching system is globally exponentially stable.

Note that $\underline{\alpha}$ is the most negative eigenvalue of the matrices in \mathcal{A}_c . In the example of the previous section, $\underline{\alpha} = -19$ whereas $\bar{\alpha}_L = \bar{\alpha}_1 = -2$; hence $\bar{h}_L = -2 / -19 = 0.1053$. In this example, $h = 0.25 > \bar{h}_L$ and so the hypotheses of the above theorem are not satisfied. In Example 7, it can be easily verified that, had we discretized with $h < 0.1053$, the corresponding discrete-time switching would have been exponentially stable.

Before proceeding to the next result, we briefly digress to show that for L even, the polynomial Q_L has no real zeros (hence $\bar{h}_L = \infty$ whenever L is even). This conclusion is evident from the following theorem. A general rational approximation $R(z)$, is a approximation to e^z of order ‘ q ’, if $e^z - R(z) = Cz^{q+1} + \mathcal{O}(z^{q+2})$ with $C \neq 0$. Theorem 46 provides the maximum attainable order of such rational approximations under some conditions.

Theorem 46 [133] *Suppose that a rational approximation to the exponential function is given by $R(z) = P_K(z)/Q_J(z)$, where the subscripts K and J denote the orders of the polynomials P_K and Q_J respectively. Let Q_J have only M different complex zeros. If in addition Q_J has a real zero then, the order q of R satisfies*

$$q \leq K + M + 1.$$

If Q_J has no real zeros then,

$$q \leq K + M.$$

A Padé approximation P_K/Q_J is a special case of the rational approximations considered in the above theorem and its order is $q = J + K$ [133], where K and J denote the orders of the polynomials P_K and Q_J . Hence, if Q_J has only M different complex zeros and at least one real zero, it must satisfy $J + K \leq K + M + 1$, that is,

$$J \leq M + 1.$$

If Q_J had a real zero when J is even, it must have two real zeros and, since Q_J has at least M complex zeros, this yields the contradiction that $J \geq M + 2$. Hence, for a Padé approximation P_K/Q_J with J even, Q_J has no real zeros.

Comment 8 The above results tell us that for even order Padé approximations we have $\bar{h}_L = \infty$. This yields the next result.

Theorem 47 *Suppose that \mathcal{A}_c is a finite set of matrices satisfying the hypotheses of Theorem 44 and L is any even positive integer. Then, for any sampling time, the discrete-time switching system (10.2) obtained under the $[L/L]$ -order diagonal Padé approximation is globally exponentially stable.*

The key point in the proof of the last theorem is that even ordered Padé polynomials do not have real zeros. It immediately follows that stability is preserved for any choice of sampling interval. Odd ordered Padé polynomials, on the other hand, have some real zeros, and these zeros can cause difficulties in ensuring that negative real eigenvalues map to positive ones. To preserve stability in this case one must select a sampling time that is small enough. To illustrate this point let us consider again Example 7. We assume that switching is restricted to only occur at multiples of the sampling time $h = 1$ (which is chosen to illustrate the assertions in Theorem 45). As can be seen from Table 1, the first two odd order approximations lead to

Order (L)	$\lambda_{\max}(A_{d1}A_{d2}A_{d3})$	Comment
1	2.5819	Unstable
2	0.5957	Stable
3	1.0710	Unstable
4	0.6539	Stable

Table 1: Stability of some even and odd approximations for Example 7

an unstable discrete time switching system.

Comment 9 The results of this section indicate that the selection of stable Padé discretizations is guided strongly by the knowledge of the Lyapunov function for the original switched system. This suggests the following interesting open question. Namely, to determine if in choosing a discretization method for exponentially stable continuous-time switched systems, knowledge of a Lyapunov function for the original continuous-time system is required.

10.6 QUADRATIC STABILITY AND PADÉ APPROXIMATIONS FOR DESCRIPTOR SYSTEMS

In this section, we extend the concept of quadratic stability to descriptor systems having index greater than one. Then we show that the generalized Padé approximations for descriptor systems introduced earlier also preserve quadratic Lyapunov functions. We begin this section by recalling some basics of Lyapunov stability theory and the order reduction (and index reduction) procedure for general descriptor system of index $k > 0$ (see equation 9.1) from chapter 9.

We say that a scalar valued function V is a *Lyapunov function* for the reduced-order LTI system (9.6) if for all non-zero $w = Y_k^T \dots Y_2^T Y_1^T x \in Y_k^T \dots Y_2^T Y_1^T \mathcal{C}$ we have $V(w) > 0$ and $\dot{V}(w) < 0$; V is a *quadratic Lyapunov function* if V is a Lyapunov function and V can be written as $V(w) = w^T P w$ for some symmetric matrix P ; in this case, we say that P is a

Lyapunov matrix for system (9.6). The existence of a Lyapunov matrix P guarantees that the system is GUES, that is, there are constants $\alpha, \beta > 0$ such that every solution satisfies

$$\|w(t)\| \leq \beta e^{-\alpha(t-t_0)} \|w(t_0)\| \quad \text{for } t \geq t_0. \quad (10.14)$$

From Lemma 19 we can conclude that GUES of (9.6) is equivalent to GUES of (9.1). Suppose P is a Lyapunov matrix for the descriptor system (9.6). Then, the requirement that $V(w) > 0$ all for non-zero w in $Y_k^T \dots Y_2^T Y_1^T \mathcal{C}$ is equivalent to P being positive-definite on $Y_k^T \dots Y_2^T Y_1^T \mathcal{C}$, that is, $w^T P w > 0$ for all w in $Y_k^T \dots Y_2^T Y_1^T \mathcal{C}$. Now note that

$$\begin{aligned} \dot{V} &= 2\dot{w}^T P w \\ &= 2\dot{w}^T P E_k \dot{w} \\ &= -\dot{w}^T Q \dot{w} \end{aligned}$$

where Q is given by

$$\boxed{P E_k + E_k^T P + Q = 0.} \quad (10.15)$$

Consider the descriptor systems (9.1) and (9.6) where x is in \mathcal{C} and w is in $Y_k^T \dots Y_2^T Y_1^T \mathcal{C}$, thus

$$\dot{V} = -\dot{w}^T Q \dot{w} = -(Y_k^T \dots Y_1^T \dot{x})^T Q (Y_k^T \dots Y_1^T \dot{x})$$

and the requirement that $\dot{V}(w) < 0$ for all non-zero w in $Y_k^T \dots Y_1^T \mathcal{C}$ is equivalent to $Q_0 = Y_1 \dots Y_k Q Y_k^T \dots Y_1^T$ being positive-definite on \mathcal{C} . This is equivalent to the requirement that $Y_1 \dots Y_k (P E_k + E_k^T P) Y_k^T \dots Y_1^T$ is negative-definite on \mathcal{C} . Since $E_k Y_k^T \dots Y_1^T = Y_k^T \dots Y_1^T A^{-1} E$ and if we let $P_0 = Y_1 \dots Y_k P Y_k^T \dots Y_1^T$ and $Q_0 = Y_1 \dots Y_k Q Y_k^T \dots Y_1^T$, we have the following conclusion.

A symmetric matrix P_0 is a Lyapunov matrix for descriptor system (9.1) if and only if the matrices P_0 and Q_0 defined by

$$\boxed{P_0(A^{-1}E) + (A^{-1}E)^T P_0 + Q_0 = 0} \quad (10.16)$$

are positive-definite on the consistency space of the system.

Based on this discussion, we propose the following result.

Theorem 48 *Suppose that (E, A) is a continuous time stable descriptor pair and the corresponding $[L/L]$ -order diagonal Padé approximation is*

$$\begin{aligned} x[(p+1)h] &= \tilde{C}_{[L/L]}(E, A, h)x[ph] \\ &= \tilde{Q}_L(-A^{-1}Eh)^{-1} \tilde{Q}_L(A^{-1}Eh)x[ph], \quad h > 0. \end{aligned} \quad (10.17)$$

If P_0 is a Lyapunov matrix for (E, A) then, P_0 is Lyapunov matrix for (10.17).

PROOF: A symmetric matrix P_0 is a Lyapunov matrix for descriptor system (E, A) if and only if

$$P_0(A^{-1}E) + (A^{-1}E)^T P_0 \quad (10.18)$$

is negative-definite on the consistency space \mathcal{C} of the system (E, A) . From our earlier discussion (see equations (10.15) and (10.16)), negative definiteness of $P_0(A^{-1}E) + (A^{-1}E)^T P_0$ on \mathcal{C} is equivalent to negative definiteness of $E_k^T P + P E_k$ on $\mathcal{C}_k = Y_k^T \dots Y_2^T Y_1^T \mathcal{C}$. This is also equivalent to negative definiteness of $E_k^{-T} P + P E_k^{-1}$ on \mathcal{C}_k .

Diagonal Padé approximations preserve the quadratic Lyapunov function [134], hence if there exists a symmetric matrix P , positive definite on $\mathcal{C}_k = Y_k^T \dots Y_2^T Y_1^T \mathcal{C}$ such that

$$E_k^{-T} P + P E_k^{-1} \quad (10.19)$$

is negative-definite on \mathcal{C}_k . Then

$$C_{[L/L]}(E_k^{-1}h)^T P C_{[L/L]}(E_k^{-1}h) - P \quad (10.20)$$

is also negative-definite on \mathcal{C}_k . Hence,

$$\begin{aligned} & Y_1 \dots Y_k [Q_L(E_k^{-1}h)^T Q_L(-E_k^{-1}h)^{-T} P Q_L(-E_k^{-1}h)^{-1} Q_L(E_k^{-1}h) - P] Y_k^T \dots Y_1^T \\ = & Y_1 \dots Y_k [\tilde{Q}_L(E_k h)^T \tilde{Q}_L(-E_k h)^{-T} P \tilde{Q}_L(-E_k h)^{-1} \tilde{Q}_L(E_k h) - P] Y_k^T \dots Y_1^T \\ = & Y_1 \dots Y_k [\tilde{Q}_L(E_k h)^T P \tilde{Q}_L(E_k h) - \tilde{Q}_L(-E_k h)^T P \tilde{Q}_L(-E_k h)] Y_k^T \dots Y_1^T \\ & \text{(applying Lemmas 29 and 30)} \\ = & \tilde{Q}_L(A^{-1}Eh)^T P_0 \tilde{Q}_L(A^{-1}Eh) - \tilde{Q}_L(-A^{-1}Eh)^T P_0 \tilde{Q}_L(-A^{-1}Eh) \\ = & \tilde{Q}_L(-A^{-1}Eh)^{-T} \tilde{Q}_L(A^{-1}Eh)^T P_0 \tilde{Q}_L(A^{-1}Eh) \tilde{Q}_L(-A^{-1}Eh)^{-1} - P_0 \end{aligned}$$

is negative definite on \mathcal{C} .

10.7 CONCLUSIONS

In this chapter, we have shown that diagonal Padé approximations to the matrix exponential preserves quadratic Lyapunov functions between continuous-time and discrete-time switched systems. We have also shown that the converse is not true. Namely, it does not follow that the original continuous-time system is quadratically stable even if the discrete-time system has a quadratic Lyapunov function. Furthermore, it is easily seen that such approximations do not (in general) preserve stability when used to discretize switched systems that are stable (but not quadratically stable). Our results suggest a number of interesting research directions. An immediate question concerns discretization methods that preserve other types of stability. Since general Padé approximations can be thought of as products of complex bilinear transforms, an immediate question in this direction concerns the equivalent map

for other types of Lyapunov functions. For example, given a continuous-time system with some Lyapunov function, what are the mappings from continuous-time to discrete-time that preserve the Lyapunov functions. A natural extension of this question concerns whether discretization methods can be developed for exponentially stable switched and nonlinear systems but which do not have a quadratic Lyapunov function. We will focus on some of these questions in our next chapter. Finally, we also extend our results on preserving quadratic Lyapunov functions for *regular systems* to descriptor systems.

In this chapter, we consider preservation of polyhedral Lyapunov functions for stable continuous-time linear systems under discretization. This problem is motivated by the non-conservative nature of polyhedral Lyapunov functions. It has been shown in [120], that a continuous-time system and its diagonal Padé discretization (of any order and sampling) always share at least one common piecewise linear (polyhedral) Lyapunov function if the continuous time matrix A_c has distinct eigenvalues. We show that this result is also valid for the case when A_c consists of non-trivial Jordan blocks.

11.1 INTRODUCTION

A fundamental issue for the discretization of switched systems is that quadratic stability is only a sufficient condition for exponential stability. A more complete characterisation of exponential stability requires the study of piecewise-linear and piecewise quadratic Lyapunov functions [135]. Thus, for switched systems, the notion of stability preservation goes beyond the notion of quadratic Lyapunov functions, and requires the study of more elaborate Lyapunov functions. This observation motivates the use of polyhedral Lyapunov functions. Such functions are known to be non-conservative in the analysis of stability under arbitrary switching for polytopic and switched systems. In other words, given a switching system that is stable under arbitrary switching, a polyhedral Lyapunov function for this system always exists, while a quadratic Lyapunov function does not exist in general. Our results on diagonal Padé approximations have motivated an interesting result in [120] for polyhedral Lyapunov functions, which states that a stable continuous-time system and its Padé discretized version of order $p \geq 1$ (for any sampling time) always share such a function. This does not contradict previous results in [90], where it was proven, through a counterexample, that a given polyhedral Lyapunov function in continuous-time may not be a Lyapunov function for the sampled discrete-time system obtained via the particular class of bilinear transformations with fixed sampling time. However, this result was proved under the assumption that the continuous-time system matrix A_c has distinct eigenvalues [120]. This result follows by making explicit use of the fact that the diagonal Padé approximation preserves the Jordan structure of a matrix A_c if the matrix has distinct eigenvalues. Unfortunately, this fact no longer holds when A_c has non-trivial Jordan blocks, and the purpose of this chapter is therefore to extend the results of [120] to the case of non-trivial Jordan blocks.

In this chapter, we extend the results in [120] to the case where the system matrix has non-trivial Jordan blocks. As we shall see, this extension is non-trivial and requires developing substantial technical content that deviates significantly from the arguments in [120].

11.2 PROBLEM STATEMENT

Consider a continuous-time linear time-invariant (LTI) system

$$\dot{x}(t) = A_c x(t), \quad (11.1)$$

where the matrix $A_c \in \mathbb{R}^{m \times m}$ is Hurwitz. We are interested in the discrete time approximation

$$x[(k+1)h] = A_d x[kh] \quad (11.2)$$

obtained via the diagonal Padé approximation of order $[n/n]$ with sampling time h . We are primarily interested in the preservation of a polyhedral Lyapunov function $V(x) = \|Wx\|_\infty$ under diagonal Padé approximation. Now we present the result from [120], where the authors proved the following fundamental result:

Theorem 49 [120] *Consider a Hurwitz stable matrix A_c of dimension N and its diagonal Padé discretization A_d of order n . Assume that all eigenvalues of A_c are distinct. Let N_r be the number of real negative eigenvalues, and $2N_c$ be the number of pairs of complex eigenvalues $\sigma_i \pm j\tau_i$, $i = 1, 2, \dots, N_c$. For each pair of complex eigenvalues, let k_i be an integer greater than one such that $\sigma_i \pm j\tau_i$ belongs to the sector $\mathcal{S}_c(k_i) := \{\lambda = \sigma + j\tau : \sigma < 0, |\tau| < \frac{\sin(\frac{\pi}{k_i})}{1 - \cos(\frac{\pi}{k_i})} |\sigma|\}$. Then there exists a $W \in \mathcal{R}^{N' \times N}$, with $N' = \sum_{i=1}^k k_i + N_r$ with*

$$W = \begin{pmatrix} W_1 & 0 & \cdots & 0 & 0 \\ 0 & W_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & W_{N_c} & 0 \\ 0 & 0 & \cdots & 0 & I \end{pmatrix} T_c, \quad (11.3)$$

where

$$W_i = \begin{pmatrix} 1 & 0 \\ \cos(\frac{\pi}{k_i}) & \sin(\frac{\pi}{k_i}) \\ \cos(\frac{2\pi}{k_i}) & \sin(\frac{2\pi}{k_i}) \\ \vdots & \vdots \\ \cos(\frac{(k_i-1)\pi}{k_i}) & \sin(\frac{(k_i-1)\pi}{k_i}) \end{pmatrix}, \quad (11.4)$$

and T_c is the Modal matrix for A_c , such that $V(x) := \|Wx\|_\infty$ is a Lyapunov function both for A_c and A_d .

In this chapter, we prove the same result when A_c has non-trivial Jordan blocks. Given A_c a square matrix, we consider its real Jordan form $J_c = T_c^{-1}A_cT_c$:

$$J_c = \begin{pmatrix} J_c^0 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & J_c^1 & \dots & \mathbf{0} \\ \dots & & & \\ \mathbf{0} & \dots & \mathbf{0} & J_c^l \end{pmatrix}, \quad (11.5)$$

where J_c^1, \dots, J_c^l are all the blocks either of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} \quad (11.6)$$

with $\lambda < 0$ (real eigenvalues), or of the form:

$$\begin{pmatrix} \Lambda & I & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \Lambda & I & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \Lambda & I \\ \mathbf{0} & \mathbf{0} & \dots & \dots & \Lambda \end{pmatrix} \quad (11.7)$$

where $\Lambda = \begin{pmatrix} \sigma & \tau \\ -\tau & \sigma \end{pmatrix}$, $\sigma < 0$, $\tau > 0$, I is the identity matrix of dimension 2 and $\mathbf{0}$ is the null matrix of dimension 2. The first block J_c^0 has the following structure

$$J_c^0 = \begin{pmatrix} \lambda_1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & \dots & \lambda_{n_0} & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \Lambda_1 & \dots & \dots & 0 & 0 \\ 0 & \dots & 0 & & & & 0 & 0 \\ \vdots & & & & & & & \\ 0 & \dots & 0 & 0 & 0 & \dots & \Lambda_{m_0} & \\ 0 & \dots & 0 & 0 & 0 & \dots & & \end{pmatrix}$$

with $\Lambda_i = \begin{pmatrix} \sigma_i & \tau_i \\ -\tau_i & \sigma_i \end{pmatrix}$. In other words, J_c^0 contains the real eigenvalues Λ_i (eventually coinciding) such that the corresponding line and column in the real Jordan form are 0 except on

the diagonal itself, and the complex eigenvalues blocks Λ_i such that the corresponding lines and columns are 0 except on the block itself. We now state our main result:

Theorem 50 *Let A_c be a Hurwitz matrix. Then, there exists a matrix W such that for all $h > 0$ and any order n of approximation, the systems 11.1 and 11.2 with $A_d = P_{[n/n]}(A_ch)$ share the polyhedral Lyapunov function $\|Wx\|_\infty$. We have $W = \tilde{W}T_c$ where T_c is the modal matrix for A_c and the precise structure of \tilde{W} is given in Lemmas 36 and 37.*

We shall prove this theorem in Section 11.5. The proof is based on the study of each block of the matrix J_c independently. For this reason, we first study the two following special cases: the **real case**, in which A_c is given by 11.6; and the **complex case**, in which A_c is given 11.7. These cases are presented in the two following sections.

11.3 THE REAL CASE

In this section, we consider A_c of the form 11.6. We denote its dimension with m . Then

$$A_d = \begin{pmatrix} f_0 & f_1 & f_2 & \dots & f_{m-1} \\ 0 & f_0 & f_1 & \dots & f_{m-2} \\ 0 & 0 & f_0 & \dots & f_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f_0 \end{pmatrix}, \tag{11.8}$$

with $f_i := P_{[n/n]}^{(i)}(\lambda h) \frac{h^i}{i!}$. The index (i) denotes the i -th derivative of $P_{[n/n]}(x)$ with respect to x .

The formula 11.8 can be easily proved by writing $P_{[n/n]}(x) = \sum_{i=0}^\infty a_i x^i$ and studying the expression of the powers A_c^i ; see [136] for details. As a consequence, terms on the upper diagonal have series expressions coinciding with derivatives of the series $\sum_{i=0}^\infty a_i x^i$. The convergence of the series for A_c Hurwitz is given by the fact that Padé approximation and its derivatives have poles in the right-half plane only [137].

We now prove the following lemma.

Lemma 36 *Consider the Hurwitz matrix A_c of the form 11.6 and denote its dimension with m . Then, there exists a positive $\alpha > -\frac{1}{\lambda}$ such that for all $h > 0$ and any order n of approximation, the matrices A_c and $A_d = P_{[n/n]}(A_ch)$ share the common Lyapunov function*

$$V(x) = \|Dx\|_\infty. \tag{11.9}$$

with $D = \text{diag}\{1, \alpha, \dots, \alpha^{m-1}\}$.

PROOF : We first prove that (11.9) is a Lyapunov function for A_c . Since D is invertible, we transform $DA_c = Q_c D$ to

$$Q_c = DA_c D^{-1} = \begin{pmatrix} \lambda & \frac{1}{\alpha} & 0 & \dots & 0 \\ 0 & \lambda & \frac{1}{\alpha} & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda & \frac{1}{\alpha} \\ 0 & 0 & \dots & \dots & \lambda \end{pmatrix} \quad (11.10)$$

Under the condition $\alpha > -\frac{1}{\lambda}$, we have that $\mu_\infty(Q_c) = \lambda + \frac{1}{\alpha} < 0$. Thus $V(x) = \|Dx\|_\infty$ is a Lyapunov function for A_c . Since $A_d = P_{[n/n]}(A_c h)$ is given by 11.8, we can compute $Q_d = DP_{[n/n]}(A_c h)D^{-1}$, that is

$$Q_d = \begin{pmatrix} f_0 & \frac{f_1}{\alpha} & \frac{f_2}{\alpha^2} & \dots & \frac{f_{m-1}}{\alpha^{m-1}} \\ 0 & f_0 & \frac{f_1}{\alpha} & \dots & \frac{f_{m-2}}{\alpha^{m-2}} \\ 0 & 0 & f_0 & \dots & \frac{f_{m-3}}{\alpha^{m-3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f_0 \end{pmatrix} \quad (11.11)$$

To prove $\|Q_d\|_\infty < 1$, we need to prove that

$$|f_0| + \frac{|f_1|}{\alpha} + \frac{|f_2|}{\alpha^2} + \dots + \frac{|f_{m-1}|}{\alpha^{m-1}} < 1 \quad (11.12)$$

Take now any $\bar{h} > 0$. We now compute α such that 11.12 is satisfied for all $h < \bar{h}$. For each $i = 1, \dots, m-1$, compute $M^i = \max_{h \in [0, \bar{h}]} |P_{[n/n]}^{(i)}(\lambda h)|$ and $M = \max_{i=1, \dots, m-1} M^i$. Remark that each M^i is finite, since the derivatives $P_{[n/n]}^{(i)}$ are always finite for non-positive numbers, since Padé approximations and their derivatives have poles with real part that is strictly positive. Hence, M exists, since it is the maximum over a finite set. Then, we bound 11.12 with $|P_{[n/n]}(\lambda h)| + M \left(\frac{h}{\alpha} + \dots + \frac{h^{m-1}}{(m-1)! \alpha^{m-1}} \right) < 1$. Since $\lambda < 0$, we have that $|P_{[n/n]}(\lambda h)| < 1$, hence we can always find α such that $\left(\frac{\bar{h}}{\alpha} + \dots + \frac{\bar{h}^{m-1}}{(m-1)! \alpha^{m-1}} \right) < \frac{1 - |P_{[n/n]}(\lambda h)|}{M}$. This latter fact follows from the fact that $|P_{[n/n]}(\lambda h)|$ is always bounded away from 1. Thus, for all $h < \bar{h}$, condition 11.12 are verified.

We now study the limiting case as $h \rightarrow \infty$. First, define the new variable¹ $x := -\frac{1}{\lambda h}$ and the function $g_0(x) := f_0 = P_{[n/n]}(-1/x)$, that is defined for $x \geq 0$. In particular, at $x = 0$ we have $g_0(0) = \lim_{h \rightarrow \infty} f_0 = \pm 1$, since $|P_{[n/n]}(\infty)| = 1$. Moreover, $|P_{[n/n]}| < 1$ for all $h > 0$, that implies $|g_0(x)| < 1$ for $x > 0$. Its Taylor expansion in 0 (for $x > 0$ only) is thus $g_0(x) =$

¹ This process coincides with the Taylor expansion of $P_{[n/n]}$ at ∞ .

$d_0 + d_1x + o(x)$ with $|d_0| = 1$ and $d_0d_1 < 0$. By substitution, we have $f_0 = d_0 + \frac{d_1}{\lambda h} + o(1/h)$. Differentiating this series i times with respect to h , we have

$$P_{[n/n]}^{(i)}(\lambda h) = (-1)^i \frac{d_1^i i!}{\lambda^{i+1} h^{i+1}} + o(1/h^{i+1}), \quad (11.13)$$

thus $|f_i| = \frac{|d_1|^i}{|\lambda|^{i+1} h} + o(1/h)$. Since $\beta = |\lambda|\alpha > 1$, we have $\frac{|f_i|}{\alpha^i} = \frac{|d_1|^i}{\beta^{i-1} |\lambda|^2 \alpha h} + o(1/h) \leq \frac{|d_1|^i}{|\lambda|^2 \alpha h} + o(1/h)$ for all $i \geq 1$. Thus

$$\|Q_d\|_\infty = |f_0| + \frac{|f_1|}{\alpha} + \frac{|f_2|}{\alpha^2} + \dots + \frac{|f_{m-1}|}{\alpha^{m-1}} \leq 1 - \frac{|d_1|}{|\lambda|h} + \frac{|d_1|}{|\lambda|^2 \alpha h} (m-1) + o(1/h).$$

This means that there exists $h^* > 0$ such that $\|Q_d\|_\infty \leq 1 - \frac{|d_1|}{|\lambda|h} + \frac{|d_1|}{|\lambda|^2 \alpha h} (m-1)$ for all $h \geq h^*$. Then estimate (11.12) with

$$1 - \frac{|d_1|}{|\lambda|h} + \frac{|d_1|}{|\lambda|^2 \alpha h} (m-1) < 1 \quad (11.14)$$

that is true by choosing $\alpha > \frac{m-1}{|\lambda|}$.

We now merge the two cases. Using the first part, choose $\bar{h} = h^*$ and find α_1 so that (11.12) holds for all $h < \bar{h}$ and $\alpha \geq \alpha_1$. Using the limit case, choose $\alpha_2 > \frac{m-1}{|\lambda|}$ so that (11.12) holds for all $h \geq \bar{h}$ and $\alpha \geq \alpha_2$. Choose now $\alpha^* = \max\{\alpha_1, \alpha_2\}$ and observe that 11.12 holds for all h .

Remark 3 *As apparent from the proof of the previous theorem, there exists a value $\bar{\alpha}$ of α that ensures that $V(x) = \|Dx\|_\infty$ is a Lyapunov function for both A_c in 11.6 and A_d in 11.8, for all $\alpha > \bar{\alpha}$. Taking again into account the fact that $\alpha|\lambda| > 1$, an upper bound value of $\bar{\alpha}$ can be found from 11.12, i.e.*

$$\frac{1}{|\lambda|} < \bar{\alpha} \leq \sup_{h>0} \frac{\sum_{i=1}^{m-1} |f_i| |\lambda|^{i-1}}{(1 - |f_0|)}$$

11.3.1 Explicit computation of \bar{h} for the case of real Jordan blocks

Since diagonal Padé approximations are A-stable, we have $|P_{[n/n]}(\lambda h)| < 1$. However, for (11.9) to be a Lyapunov function for the discrete time system we need to verify $\|Q_d\|_\infty < 1$, that is

$$\sum_{j=0}^i \left| \frac{P_{[n/n]}^{(j)}(\lambda h)}{j!} \left(\frac{h}{\alpha}\right)^j \right| < 1 \quad \text{for all } i = 1, \dots, m-1. \quad (11.15)$$

2 For $d_0 = 1$, one needs $d_1 < 0$ to have g decreasing; and the opposite for $d_0 = -1$. More precisely, g_0 decreasing with $d_0 = 1$ implies $d_1 < 0$ or $d_1 = d_2 = 0$ and $d_3 < 0$, or $d_1 = d_2 = d_3 = d_4 = 0$ and $d_5 < 0, \dots$ For simplicity of notation, we study the case $d_1 \neq 0$; the same proof can be adapted to the other cases too.

Assumption: Assume that $P_{[n/n]}(z)$ is absolutely monotonic over $(-r_n, 0]$ such that $\lambda h < r_n$.

We recall that absolute monotonicity of $P_{[n/n]}(z)$ means that all derivatives $P_{[n/n]}^j(z)$ are positive. Thus, for $\lambda h > -r_n$, we have that the series 11.15 have all positive terms. We estimate all these series with

$$\sum_{j=0}^{\infty} \frac{P_{[n/n]}^j(\lambda h)}{j!} \left(\frac{h}{\alpha}\right)^j = P_{[n/n]} \left(\lambda h + \frac{h}{\alpha}\right) \leq |P_{[n/n]} \left(\lambda h + \frac{h}{\alpha}\right)|. \quad (11.16)$$

Since $\lambda h + \frac{h}{\alpha} < 0$ for our choice of α , we have $|P_{[n/n]}(\lambda h + \frac{h}{\alpha})| < 1$. The condition on \bar{h} is thus reduced to $\bar{h} < \frac{r_n}{|\lambda|}$. Some values of r_n , as well as an algorithm for their computation, are given in [138].

11.4 THE COMPLEX CASE

In this section, we consider A_c of the form 11.7. We denote its dimension with $2m$. Then

$$A_d = \begin{pmatrix} F_0 & F_1 & F_2 & \dots & F_{m-1} \\ 0 & F_0 & F_1 & \dots & F_{m-2} \\ 0 & 0 & F_0 & \dots & F_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & F_0 \end{pmatrix} \quad (11.17)$$

with $F_i := P_{[n/n]}^{(i)}(\lambda h) \frac{h^i}{i!}$. As in the previous Section, derivative notation should be interpreted as the rational functions that are derivatives of the rational function $P_{[n/n]}(x)$. The proof of this formula is as for the real case. The only detail to be careful with is that, in this case, the product of matrices only involves λ and I , for which the product is commutative.

Let k be a natural number such that $\sigma + j\tau \in \mathcal{S}_c(k)$. Let

$$\tilde{W} = \begin{pmatrix} 1 & 0 \\ \cos(\frac{\pi}{k}) & \sin(\frac{\pi}{k}) \\ \cos(\frac{2\pi}{k}) & \sin(\frac{2\pi}{k}) \\ \vdots & \vdots \\ \cos(\frac{(k-1)\pi}{k}) & \sin(\frac{(k-1)\pi}{k}) \end{pmatrix}.$$

This matrix defines a Lyapunov function $\|\tilde{W}x\|_{\infty}$ both for the block $\lambda = \begin{pmatrix} \sigma & \tau \\ -\tau & \sigma \end{pmatrix}$ and $P_{[n/n]}(\lambda h)$ for all $h > 0$, as proved in [122]. We now use this fact to compute the Lyapunov function for A_c and A_d , and consequently prove the following lemma.

Lemma 37 Consider the Hurwitz matrix A_c of the form 11.7 and denote its dimension with $2m$. Then there exists an $\alpha > \frac{1}{-\sigma - \tau \frac{1 - \cos(\frac{\pi}{k})}{\sin(\frac{\pi}{k})}}$ such that for all $h > 0$ and order n of approximation, the matrices A_c and $A_d = P_{[n/n]}(A_c h)$ share the common Lyapunov function

$$V(x) = \|Wx\|_\infty \quad (11.18)$$

with

$$W = \begin{pmatrix} \tilde{W} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \tilde{W}\alpha & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{W}\alpha^2 & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \tilde{W}\alpha^{m-1} \end{pmatrix} \quad (11.19)$$

Proof : We first prove that (11.18) is a Lyapunov function for A_c . We already know that there exists a certain \tilde{Q}_c with $\mu_\infty(\tilde{Q}_c) < 0$ satisfying $\tilde{W}\lambda = \tilde{Q}_c\tilde{W}$. Moreover, $\mu_\infty(\tilde{Q}_c) = |\sigma| - \frac{|\tau|\cos(\frac{\pi}{k})}{\sin(\frac{\pi}{k})} + \frac{|\tau|}{\sin(\frac{\pi}{k})} < 0$. See details in [120, 122]. Thus $WA_c = Q_cW$ is satisfied, with

$$Q_c = \begin{pmatrix} \tilde{Q}_c & I/\alpha & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \tilde{Q}_c & I/\alpha & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \tilde{Q}_c \end{pmatrix} \quad (11.20)$$

We have $\mu_\infty(Q_c) = \mu_\infty(\tilde{Q}_c) + \frac{1}{\alpha} < 0$ due to the condition on α . Remark that such α exists, since $\sigma + j\tau \in \mathcal{S}_c(k)$ is equivalent to $\frac{1}{-\sigma - \tau \frac{1 - \cos(\frac{\pi}{k})}{\sin(\frac{\pi}{k})}} > 0$. Thus $V(x) = \|Wx\|_\infty$ is a Lyapunov function for A_c .

Compute now $A_d = P_{[n/n]}(A_c h)$, that is given by 11.17. We have to find Q_d satisfying $WA_d = Q_dW$ and $\|Q_d\|_\infty < 1$. As a candidate, we look for

$$Q_d := \begin{pmatrix} Q_0 & Q_1/\alpha & Q_2/\alpha^2 & \dots & Q_{m-1}/\alpha^{m-1} \\ \mathbf{0} & Q_0 & Q_1/\alpha & \dots & Q_{m-2}/\alpha^{m-2} \\ \mathbf{0} & \mathbf{0} & Q_0 & \dots & Q_{m-3}/\alpha^{m-3} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & Q_0 \end{pmatrix} \quad (11.21)$$

with Q_0, Q_1, \dots, Q_{m-1} to be found. The explicit computation of $WA_d = Q_dW$ gives the following conditions

$$\tilde{W}F_0 = Q_0\tilde{W}, \quad \tilde{W}F_i = Q_i\tilde{W}, \quad i = 1, \dots, m-1. \quad (11.22)$$

Since $\Lambda = \begin{pmatrix} \sigma & \tau \\ -\tau & \sigma \end{pmatrix}$, then the eigenvalues of $F_0 = P_{[n/n]}(\lambda h)$ lie in $\mathcal{P}_{ol}(k) := \text{int conv} \left\{ e^{j \frac{p\pi}{m}} \right\}_{p=0}^{2m-1}$, as we proved in [120]. As a consequence, for each $h > 0$ there exists Q_0 such that $\tilde{W}F_0 = Q_0\tilde{W}$ and $\|Q_0\|_\infty < 1$, see [123].

For each other F_i , observe that its entries are all bounded functions of $h > 0$, and consequently its eigenvalues are bounded too. Thus, one can choose a $\rho > 1$ sufficiently big to have the eigenvalues of $\frac{F_i}{\rho_i}$ as small as wished. In particular, one can always have the eigenvalues of $\frac{F_i}{\rho_i}$ with norm less than R_k , the radius of a ball centered in 0 and completely contained in $\mathcal{P}_{ol}(k)$. As a consequence, there exists \tilde{Q}_i satisfying $\tilde{W}\frac{F_i}{\rho_i} = \tilde{Q}_i\tilde{W}$ and $\|\tilde{Q}_i\|_\infty < 1$; see again [123]. Then the conditions in 11.22 are all verified by taking $Q_i = \tilde{Q}_i\rho_i$.

Hence, recalling that $\alpha > \frac{1}{\mu_\infty(Q_c)}$ we have

$$\begin{aligned} \|Q_d\|_\infty &\leq \|Q_0\|_\infty + \|Q_1/\alpha\|_\infty + \dots + \|Q_{m-1}/\alpha^{m-1}\|_\infty \\ &\leq \|Q_0\|_\infty + \frac{1}{\alpha} \sum_{i=1}^{m-1} \|Q_i\|_\infty \mu_\infty(Q_c)^{i-1} \end{aligned}$$

Similarly to the real case, one has to study the limit case $h \rightarrow \infty$. By developing the ∞ -norm of the Q_i around ∞ , one finds expressions similar to f_i in the real case, and the result follows. Notice in fact that $\|Q_0\|_\infty$, as a function of $h > 0$, can be written as $\|Q_0\|_\infty = 1 - \phi(h)$ with ϕ a strictly positive function of $h > 0$. In conclusion $\|Q_d\|_\infty < 1$ if

$$\alpha > \sup_{h>0} \frac{\sum_{i=1}^{m-1} \|Q_i\|_\infty \mu_\infty(Q_c)^{i-1}}{1 - \|Q_0\|_\infty}$$

Remark 4 Also for the case of multiple complex eigenvalues, we can conclude that there exists a value $\bar{\alpha}$ of α that ensures that $V(x) = \|W\|_\infty$ is a Lyapunov function for both A_c in 11.7 and A_d in 11.17, for all $\alpha > \bar{\alpha}$. An upper bound value of $\bar{\alpha}$ can be found computed in the following way, i.e.

$$\frac{1}{|\mu_\infty(Q_c)|} < \bar{\alpha} \leq \sup_{h>0} \frac{\sum_{i=1}^{m-1} \|Q_i\|_\infty \mu_\infty(Q_c)^{i-1}}{1 - \|Q_0\|_\infty}$$

11.4.1 Explicit computation of \bar{h} for the case of complex Jordan blocks

From 11.17 we have $A_d = P_{[n/n]}(A_c h)$. We now have to find Q_d satisfying $WA_d = Q_d W$. Since $\Lambda = \begin{pmatrix} \sigma & \tau \\ -\tau & \sigma \end{pmatrix}$, then $P_{[n/n]}(\Lambda h) = \begin{pmatrix} \sigma_0 & \tau_0 \\ -\tau_0 & \sigma_0 \end{pmatrix}$ for some σ_0, τ_0 . Similarly, all derivatives

have the same structure, that is $P_{[n/n]}^{(i)}(\Lambda h) = \begin{pmatrix} \sigma_i & \tau_i \\ -\tau_i & \sigma_i \end{pmatrix}$ for some σ_i, τ_i . Define now the following matrices:

$$B^i := \sum_{p=0}^i \frac{F_p}{\alpha^p}. \quad (11.23)$$

Due to the previous observation, we know that $B^i = \begin{pmatrix} \tilde{\sigma}_i & \tilde{\tau}_i \\ -\tilde{\tau}_i & \tilde{\sigma}_i \end{pmatrix}$, with $\tilde{\sigma}_i, \tilde{\tau}_i$ depending on h and α . We now find a condition on h such that the eigenvalues $\tilde{\sigma}_i \pm j\tilde{\tau}_i$ of matrices B^0, B^1, \dots, B^{m-1} lie in $\mathcal{P}_{ol}(k) := \text{intconv} \left\{ e^{j\frac{p\pi}{m}} \right\}_{p=0}^{2m-1}$. This would ensure that \tilde{W} is a polyhedral Lyapunov matrix for each B^i , hence ensuring that $\|Wx\|_\infty$ is a polyhedral Lyapunov function for A_d . Since the spectral radius of any square matrix $\rho(A) \leq \|A\|_2$, we evaluate the bound on the following matrix norm of

$$\|B^i\|_2 = \left\| \sum_{j=0}^i P_{[n/n]}^j(\Lambda h) \frac{1}{j!} \left(\frac{h}{\alpha} \right)^j \right\|_2$$

Let $\beta = \left(\frac{2|\sigma|}{1 - \cos(\frac{\pi}{k})} \right)$ and consider $\|\Lambda h + (\beta h)I\|_2$

$$\begin{aligned} &= \sqrt{(\sigma h)^2 \left(\frac{1 + \cos(\frac{\pi}{k})}{1 - \cos(\frac{\pi}{k})} \right)^2 + (\tau h)^2} \\ &< \sqrt{(\sigma h)^2 \left(\frac{1 + \cos(\frac{\pi}{k})}{1 - \cos(\frac{\pi}{k})} \right)^2 + (\sigma h)^2 \left(\frac{\sin(\frac{\pi}{k})}{1 - \cos(\frac{\pi}{k})} \right)^2} \\ &= \frac{|\sigma h|}{1 - \cos(\frac{\pi}{k})} \sqrt{2 + 2\cos(\frac{\pi}{k})} \leq \frac{2|\sigma h|}{1 - \cos(\frac{\pi}{k})} = \beta h. \end{aligned}$$

Since $\|\Lambda h + (\beta h)I\|_2 < \beta h$, we can use Taylor series to expand $P_{[n/n]}(\Lambda h)$ as

$$P_{[n/n]}(\Lambda h) = \sum_{j=0}^{\infty} P_{[n/n]}^j(-\beta h) \frac{1}{j!} (\Lambda h + (\beta h)I)^j \quad (11.24)$$

Then from triangular inequality we have

$$\|P_{[n/n]}(\Lambda h)\|_2 \leq \sum_{j=0}^{\infty} \|P_{[n/n]}^j(-\beta h) \frac{1}{j!} (\Lambda h + (\beta h)I)^j\|_2.$$

Assume that $P_{[n/n]}(z)$ is absolutely monotonic over $(-r_n, 0]$ such that $\beta h < r_n$.

Then $P_{[n/n]}^j(-\beta h) \geq 0$ for $j \geq 0$ and

$$\|P_{[n/n]}(\Lambda h)\|_2 \leq \sum_{j=0}^{\infty} P_{[n/n]}^j(-\beta h) \frac{1}{j!} \|(\Lambda h + (\beta h)I)^j\|_2$$

If we let $\|\Lambda h + (\beta h)I\|_2 = \gamma h$, then

$$\begin{aligned} \|P_{[n/n]}(\Lambda h)\|_2 &\leq \sum_{j=0}^{\infty} P_{[n/n]}^j(-\beta h) \frac{1}{j!} (\gamma h)^j \\ &= P_{[n/n]}(-\beta h + \gamma h) \end{aligned}$$

Similarly it can be proved that

$$\|P_{[n/n]}^j(\Lambda h)\|_2 \leq P_{[n/n]}^j(-\beta h + \gamma h) \quad \forall j \geq 0. \quad (11.25)$$

Since $-\beta + \gamma$

$$\begin{aligned} &= -\frac{2|\sigma|}{1 - \cos\left(\frac{\pi}{k}\right)} + \sqrt{(\sigma)^2 \left(\frac{1 + \cos\left(\frac{\pi}{k}\right)}{1 - \cos\left(\frac{\pi}{k}\right)}\right)^2 + (\tau)^2} \\ &< -\frac{2|\sigma|}{1 - \cos\left(\frac{\pi}{k}\right)} + \frac{|\sigma|}{1 - \cos\left(\frac{\pi}{k}\right)} \sqrt{2 + 2\cos\left(\frac{\pi}{k}\right)} < 0, \end{aligned}$$

from Assumption 1 $P_{[n/n]}^j(-\beta h + \gamma h) \geq 0 \quad \forall j \geq 0$.

Now consider

$$\|B^i\|_2 = \left\| \sum_{j=0}^i P_{[n/n]}^j(\Lambda h) \frac{1}{j!} \left(\frac{h}{\alpha}\right)^j \right\|_2,$$

from triangular inequality and observation (11.25) we have

$$\begin{aligned} \|B^i\|_2 &\leq \sum_{j=0}^i \|P_{[n/n]}^j(\Lambda h) \frac{1}{j!} \left(\frac{h}{\alpha}\right)^j\|_2 \\ &\leq \sum_{j=0}^i P_{[n/n]}^j(-\beta h + \gamma h) \frac{1}{(j)!} \left(\frac{h}{\alpha}\right)^j. \end{aligned}$$

As the partial sum $\sum_{j=0}^i P_{[n/n]}^j(-\beta h + \gamma h) \frac{1}{(j)!} \left(\frac{h}{\alpha}\right)^j$ consists of non-negative terms, we have

$$\begin{aligned} &\sum_{j=0}^i P_{[n/n]}^j(-\beta h + \gamma h) \frac{1}{(j)!} \left(\frac{h}{\alpha}\right)^j \\ &\leq \sum_{j=0}^{\infty} P_{[n/n]}^j(-\beta h + \gamma h) \frac{1}{(j)!} \left(\frac{h}{\alpha}\right)^j \\ &= P_{[n/n]}(-\beta h + \gamma h + \frac{h}{\alpha}). \end{aligned}$$

Since $P_{[n/n]}(z)$ maps $\mathcal{S}_c(k)$ into $\mathcal{P}_{ol}(k)$ [120], we know that $P_{[n/n]}(-\beta + \gamma + \frac{1}{\alpha}) \in \mathcal{P}_{ol}(k)$ if $-\beta h + \gamma h + \frac{h}{\alpha} < 0$. However, $-\beta + \gamma + \frac{1}{\alpha} =$

$$\begin{aligned} & -\frac{2|\sigma|}{1 - \cos\left(\frac{\pi}{k}\right)} + \sqrt{(\sigma)^2 \left(\frac{1 + \cos\left(\frac{\pi}{k}\right)}{1 - \cos\left(\frac{\pi}{k}\right)}\right)^2 + (\tau)^2} + \frac{1}{\alpha} \\ > & -\frac{2|\sigma|}{1 - \cos\left(\frac{\pi}{k}\right)} + \sqrt{(\tau)^2 \left(\frac{1 + \cos\left(\frac{\pi}{k}\right)}{\sin\left(\frac{\pi}{k}\right)}\right)^2 + (\tau)^2} + \frac{1}{\alpha} \\ = & -\frac{2|\sigma|}{1 - \cos\left(\frac{\pi}{k}\right)} + \frac{|\tau|}{\sin\left(\frac{\pi}{k}\right)} \sqrt{2 + 2\cos\left(\frac{\pi}{k}\right)} + \frac{1}{\alpha} \end{aligned}$$

Hence we need $-\frac{2|\sigma|}{1 - \cos\left(\frac{\pi}{k}\right)} + \frac{|\tau|}{\sin\left(\frac{\pi}{k}\right)} \sqrt{2 + 2\cos\left(\frac{\pi}{k}\right)} + \frac{1}{\alpha} < 0$, leading to

$$\alpha > \frac{1}{\left(\frac{2}{1 - \cos\left(\frac{\pi}{k}\right)}\right) \left(|\sigma| - |\tau| \frac{1 - \cos\left(\frac{\pi}{k}\right)}{\sin\left(\frac{\pi}{k}\right)} \sqrt{\frac{1 + \cos\left(\frac{\pi}{k}\right)}{2}}\right)}.$$

However, it can be observed that

$$\begin{aligned} & \frac{1}{\left(\frac{2}{1 - \cos\left(\frac{\pi}{k}\right)}\right) \left(|\sigma| - |\tau| \frac{1 - \cos\left(\frac{\pi}{k}\right)}{\sin\left(\frac{\pi}{k}\right)} \sqrt{\frac{1 + \cos\left(\frac{\pi}{k}\right)}{2}}\right)} \\ < & \frac{1}{\left(|\sigma| - |\tau| \frac{1 - \cos\left(\frac{\pi}{k}\right)}{\sin\left(\frac{\pi}{k}\right)} \sqrt{\frac{1 + \cos\left(\frac{\pi}{k}\right)}{2}}\right)} \\ < & \frac{1}{|\sigma| - |\tau| \frac{1 - \cos\left(\frac{\pi}{k}\right)}{\sin\left(\frac{\pi}{k}\right)}} \end{aligned}$$

From the Lemma 4 we know that $\alpha > \frac{1}{|\sigma| - |\tau| \frac{1 - \cos\left(\frac{\pi}{k}\right)}{\sin\left(\frac{\pi}{k}\right)}}$, hence $-\beta h + \gamma h + \frac{h}{\alpha} < 0$ and we

have $P_{[n/n]}(-\beta h + \gamma h + \frac{h}{\alpha}) \in \mathcal{P}_{ol}(k)$. Hence the eigenvalues $\tilde{\sigma}_i \pm j\tilde{\tau}_i$ lie inside $\mathcal{P}_{ol}(k)$ if Assumption 1 is satisfied. Thus we select \bar{h} such that $\bar{h} < \frac{r_n}{|\beta|}$.

11.5 PROOF OF MAIN THEOREM

In this section, we now prove Theorem 50. We use Lemmas 36 and 37, as well as the results from the paper [120], given by Theorem 49. The basic idea is to show that we can deal with each Jordan block independently.

Take A_c a Hurwitz matrix, and $J_c = T_c^{-1}A_cT_c$ its real Jordan form 11.5. The fundamental observation for the following is that $A_d = P_{[n/n]}(A_ch) = T_c^{-1}P_{[n/n]}(J_ch)T_c$ with

$$P_{[n/n]}(J_c h) = \begin{pmatrix} P_{[n/n]}(J_c^0 h) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & P_{[n/n]}(J_c^1 h) & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & P_{[n/n]}(J_c^l h) \end{pmatrix}.$$

This is a standard property of the Padé approximation, since it is a rational function of matrices. As already remarked, $P_{[n/n]}(J_c h)$ is not the real Jordan form of A_d , since $P_{[n/n]}(J_c^i h)$ are not real or complex blocks for $i > 0$. We now define W, Q_c, Q_d satisfying

$$WA_c = Q_c W \quad WA_d = Q_d W, \quad (11.26)$$

$$\mu(Q_c)_\infty < 0, \quad \|Q_d\|_\infty < 1, \quad (11.27)$$

that ensures that $V(x) = \|Wx\|_\infty$ is a Lyapunov function both for 11.1 and 11.2 with $A_d = P_{[n/n]}(A_c h)$. First of all, we find W^i, Q_c^i, Q_d^i for each J_c^i . For the block J_c^0 , use Theorem 49, that gives W^0 and the corresponding Q_c^0, Q_d^0 . For blocks J_c^i , either use Lemma 36 for the real case or Lemma 37 for the complex case, that give W^i and the corresponding Q_c^i and Q_d^i . Define

$$\tilde{W} = \begin{pmatrix} W^0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & W^1 & \mathbf{0} & \dots & \mathbf{0} \\ \dots & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & W^l \end{pmatrix}$$

and $W = \tilde{W}T_c$. We prove that W defines a Lyapunov function $V(x) = \|Wx\|_\infty$ both for 11.1 and 11.2 with $A_d = P_{[n/n]}(A_c h)$. It is sufficient to find Q_c and Q_d satisfying 11.26-11.27. By direct computation, one can prove that

$$Q_c = \begin{pmatrix} Q_c^0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & Q_c^1 & \mathbf{0} & \dots & \mathbf{0} \\ \dots & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & Q_c^l \end{pmatrix} \quad (11.28)$$

and

$$Q_d = \begin{pmatrix} Q_d^0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & Q_d^1 & \mathbf{0} & \dots & \mathbf{0} \\ \dots & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & Q_d^l \end{pmatrix} \quad (11.29)$$

satisfy these conditions, since

$$\begin{aligned} WA_c &= \tilde{W}T_cA_c = \tilde{W}J_cT_c = \\ &= \begin{pmatrix} W^0J_c^0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & W^1J_c^1 & \mathbf{0} & \dots & \mathbf{0} \\ \dots & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & W^lJ_c^l \end{pmatrix} T_c = \\ &= Q_c\tilde{W}T_c = Q_cW \end{aligned}$$

The same holds for WA_d . Moreover, $\mu(Q_c) = \max_{i=0,\dots,l} \mu(Q_c^i) < 0$ and $\|Q_d\|_\infty = \max_{i=0,\dots,l} \|Q_d^i\|_\infty < 1$.

11.6 EXAMPLES

Example 1: In this example we illustrate the result indicated in Lemma 36 using a numerical example. In particular, we wish to show by construction the existence of a Lyapunov function that is preserved by diagonal Padé approximations of any step size and order. To this end, consider a Hurwitz matrix A_c of the form 11.6 with $\lambda = -3$ and $m = 3$. Then, it is easily verified that a Lyapunov function for the continuous time matrix A_c given by

$$V(x) = \|Dx\|_\infty.$$

with $D = \text{diag}\{1, \alpha, \alpha^2\}$ and $\alpha > -\frac{1}{(-3)}$. Now we consider 1^{st} order diagonal Padé approximation $A_d = P_{[1/1]}(A_ch)$ for e^{A_ch} and plot $\|Q_d\|_\infty = \|DA_dD^{-1}\|_\infty$ w.r.t to h and α .

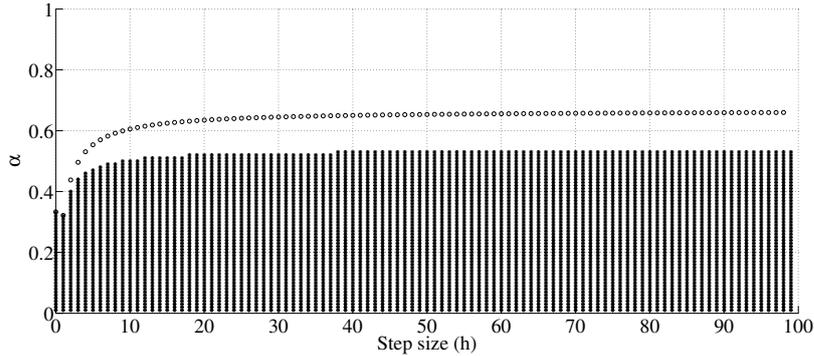


Figure 16: Plot showing values of h and α where $\|Q_d(h, \alpha)\|_\infty > 1$ using “*” and $\sum_{i=1}^{m-1} |f_i| |\lambda|^{i-1} / (1 - |f_0|)$ w.r.t h using “o”.

It can be observed from Figure 16 that there exists a finite limiting value of α , defining the boundary of the infeasible values of α as $h \rightarrow \infty$. We denote this value of α as $\bar{\alpha}$ and any

Lyapunov function $V(x) = \|Dx\|_\infty$ with $\alpha > \bar{\alpha}$ will be preserved during discretization using diagonal Padé approximation with any step size h and order n . A similar bound was proposed

$$\sum_{i=1}^{m-1} |f_i| |\lambda|^{i-1}$$

in Remark 3. To compare these two bounds, we plot $\frac{\sum_{i=1}^{m-1} |f_i| |\lambda|^{i-1}}{(1-|f_0|)}$ w.r.t h (using “o”) in Figure 16. It can be observed that the bound on $\bar{\alpha}$, proposed in Remark 3 is accurate but clearly more conservative. We also plot the boundary of numerically approximated infeasible values of α for different orders of diagonal Padé approximations in Figure 17.

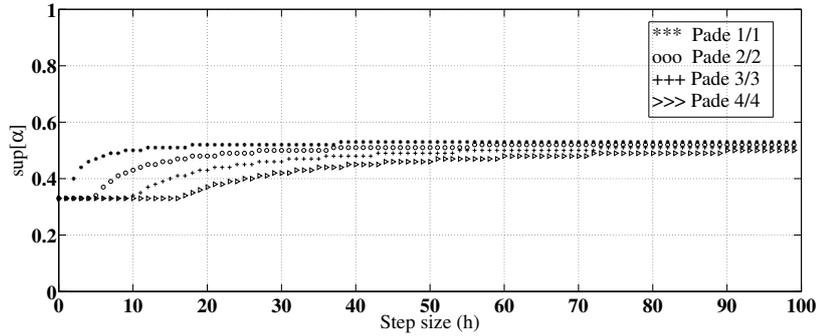


Figure 17: Plot showing boundary of infeasible values of α for different orders of diagonal Padé approximations.

Example 2: In some situations it is of interest to first define the Lyapunov function by fixing α . In such situations the pertinent problem then becomes one of estimating \bar{h} for preserving the Lyapunov function. We now show how this can be achieved for matrices with real Jordan blocks using 1st order diagonal Padé approximations. Consider a Hurwitz matrix A_c and the Lyapunov function $V(x)$ as defined in Example 1. Let us choose $\alpha = \alpha^* = 0.34 < \bar{\alpha}$ (from Example 1 $\bar{\alpha}$ can be approximately estimated as 0.53). If the goal of discretization is to preserve this given Lyapunov function, then we need to find values of h such that $\|Q_d(\alpha^*, h)\|_\infty < 1$. Hence we plot $\|Q_d(\alpha^*, h)\|_\infty$ w.r.t. h in Figure 18. It can be observed that $\|Q_d(\alpha^*, h)\|_\infty$ decreases monotonically for a certain range of step sizes $(0, \bar{h})$ and then starts to increase again. Our goal is to numerically evaluate this upper bound \bar{h} , which guarantees the preservation of Lyapunov function if $h < \bar{h}$. Note that while this can always be done numerically, sometimes we can find an algebraic bound on h . To see this consider

$$\begin{aligned} \|Q_d(\alpha^*, h)\|_\infty &= \|DP_{[1/1]}(A_c h)D^{-1}\|_\infty \\ &= \sum_{j=0}^2 \left| \frac{P_{[1/1]}^{(j)}(\lambda h)}{j!} \left(\frac{h}{\alpha^*}\right)^j \right| \end{aligned} \tag{11.30}$$

In the case of odd-ordered Padé approximations, we know that $P_{[n/n]}(x)$ is absolutely monotonic for $x \in (-r_n, 0]$, for a certain r_n depending on n . We recall that absolute monotonicity

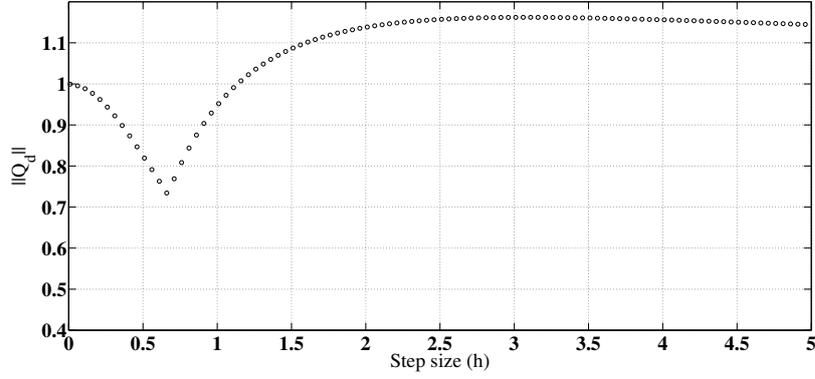


Figure 18: Plot showing $\|Q_d(\alpha^*, h)\|_\infty$ (“o”) w.r.t. h .

means that all derivatives are positive. For $P_{[1/1]}(x)$ we know that $r_1 = 2$ [138], hence if we choose h such that $\lambda h > -2$, then the series 11.30 has all positive terms and then we can estimate 11.30 with

$$\sum_{j=0}^{\infty} \frac{P_{[1/1]}^j(\lambda h)}{j!} \left(\frac{h}{\alpha^*}\right)^j = P_{[1/1]}(\lambda h + \frac{h}{\alpha^*}) \leq |P_{[1/1]}(\lambda h + \frac{h}{\alpha^*})|. \tag{11.31}$$

Since $\lambda h + \frac{h}{\alpha^*} < 0$ for our choice of α , we have $|P_{[1/1]}(\lambda h + \frac{h}{\alpha^*})| < 1$. Hence $\bar{h} < \frac{r_1}{|\lambda|} = 2/3$. For complex jordan blocks of form (11.7) \bar{h} can be evaluated in a similar manner as

$$\bar{h} < r_n \frac{1 - \cos(\frac{\pi}{k})}{2|\sigma|}.$$

Some values of r_n , as well as an algorithm for their computation, are given in [138].

11.7 CONCLUSIONS

In this chapter, we have considered the problem of preservation of certain types polyhedral Lyapunov functions under discretisation. In particular, we have shown that the results described in [120] on polyhedral Lyapunov functions extends to the case of linear systems with non-trivial Jordan structures. We have shown that the concept of absolute monotonicity plays an important role in preserving polyhedral Lyapunov functions.

CONCLUSIONS

In this thesis we considered stability properties of certain special classes of linear descriptor systems. In the first part of the thesis we considered the linear time invariant case, where we focused on passivity and a generalization of passivity and small gain theorems called “mixed” property. An important bottleneck for control design based on the above properties is their verification. Hence we develop easily verifiable and compact spectral conditions to check for passivity and mixedness of [SISO](#) and [MIMO](#) descriptor systems in the first part of the thesis. To obtain our results, we use only elementary concepts from linear algebra and the existing results for *regular systems*. This construction results in a test that involves only the evaluation of the eigenvalues of a matrix which is determined in an elementary manner from E, A, B, C, D ; while avoiding generalized eigenvalue calculation. Apart from developing tests to check the mixed property of a descriptor system, we also provided a proof based on classical Nyquist stability techniques for the stability of simple feedback-loops, consisting of two [LTI](#) “mixed” systems. This proof corrects an oversight in [38] and [39], where, the system output signals were assumed to be bounded *a priori*. Importantly, these results paved the way to obtaining new sufficient conditions for the stability of large-scale interconnections of “mixed” systems.

In the second part we considered the stability analysis of switched descriptor systems. We begin our analysis by proposing an alternate generalized Lyapunov equation for descriptor systems based on an equivalent reduced-order (also reduced index) *regular system*. Corresponding to the new generalized Lyapunov equation, we also propose alternate sufficient conditions for stability of switched descriptor systems. We also derive a [KYP](#)-like Lemma for a special class of descriptor systems called index one systems. This [KYP](#)-like Lemma allows us to generate necessary and sufficient conditions for the existence of a common quadratic Lyapunov function for a special class of switched descriptor systems. Here, we show that if a simple eigenvalue condition holds, then the switched descriptor system is exponentially stable for arbitrary switching. In this part we also provide a state dependent switching rule associated with a simple spectral condition under which switching between index zero and index one or between index one and index two descriptor systems is exponentially stable.

In the final part of the thesis we considered the problem of discretization. Our approach towards discretization was based on preserving certain stability properties of the continuous-time system. In the first two parts of the thesis we considered the properties: *passivity* and *mixedness* for linear time invariant systems and *Lyapunov stability* for linear switched systems. In the final part we explored the discretization methods that preserve the afore-

mentioned properties. We show that when discretizing a transfer function, passivity and mixedness of index-one descriptor systems can be preserved using Tustin's approximation. We also showed that when discretizing a state space model, the output averaging method can be used to preserve passivity of index-one descriptor systems.

Finally, we focused on the Lyapunov function preserving discretization methods. We proved that diagonal Padé approximations preserve quadratic Lyapunov functions and common quadratic Lyapunov functions irrespective of their order and sampling size h . We also showed that the converse is not true. We further explored the conditions under which diagonal Padé approximations preserve polyhedral Lyapunov functions and showed that there always exists a polyhedral Lyapunov function that can be preserved using diagonal Padé approximations. Also, we derived generalized Padé approximations for descriptor systems and showed that numerical methods with diagonal Padé approximations as their stability functions preserve generalized quadratic Lyapunov functions.

FUTURE DIRECTIONS

In this chapter we discuss some general ideas for possible extensions of the work presented in this thesis. We present these ideas with respect to the different properties analysed in this thesis.

Passivity: Mixed property is a generalization of passivity and small gain theorems, however, another interesting generalization of passivity can be proposed based on the similarity of conditions between passivity and quadratic stability. A similar idea has been proposed by [139]. To understand this new generalization we recall that a proper transfer matrix $H(s)$ is passive if

$$H(j\omega)^* + H(j\omega) > 0 \quad \forall \omega \in [-\infty, \infty]. \quad (13.1)$$

Motivated by the similarity of the above equation with the Lyapunov inequality $A^T P + PA < 0$, we propose a less conservative generalization of passivity by modifying the condition for passivity, whereby there exists a positive definite matrix P such that

$$H(j\omega)^* P + PH(j\omega) > 0 \quad \forall \omega \in [-\infty, \infty]. \quad (13.2)$$

Condition (13.1) is more conservative because it imposes the restriction that P can only be an identity matrix. An immediate observation regarding this new generalization of passivity is that, it does not impose any additional constraints on the interconnection structure while maintaining stability of the interconnected system. Hence it would be interesting to further explore the implications of such a generalization on the individual subsystems of an interconnection and the interconnection structure itself.

“Mixed” Property: An alternate version of **KYP** Lemma, shows that passivity of a **SISO** transfer function is equivalent to the existence of a common quadratic Lyapunov function for a pair of **LTI** systems formed using the matrices from the original system transfer function. The frequency dependent nature of “mixed” property provides the motivation for a similar **KYP**-like Lemma for “mixed” systems. Generalized **KYP** Lemma proposed by [140] may be used to obtain similar results for **SISO** “mixed” systems.

Switched Descriptor Systems: Theorem 44 shows that a special *regular* switched system where all the constituent sub-systems share at least $n - 1$ common eigenvectors is globally attractive. Similar idea may be used to construct a globally attractive switched descriptor system, where all the constituent sub-systems share certain number of eigenvectors (or generalized eigenvectors). Another possible extension for descriptor systems is the formulation

of a polyhedral Lyapunov function an index -1 descriptor system using the corresponding reduced order descriptor system $Y^T A^{-1} X (Y^T \dot{x}) = Y^T x$; given by $V(x) = \|WY^T x\|_\infty$ and

$$WY^T(Y^T A^{-1} X) = QWY^T; \quad \mu_\infty(Q) < 0. \quad (13.3)$$

The matrix W is full rank matrix and (X, Y) is the full rank decomposition of E such that $\text{rank}(W) = \text{rank}(X) = \text{rank}(Y^T) = \text{rank}(E)$. Further development in this direction may lead to several new results concerning common polyhedral Lyapunov functions for switched descriptor systems.

Discretization: We showed that Tustin's method of discretizing the transfer functions, preserves the mixed property of a *regular* and index-1 descriptor system. However, the problem of discretizing the state space model of a mixed system is still an open question and remains an interesting problem for future research. Also our results on preserving quadratic stability using diagonal Padé approximations suggest a number of interesting research directions. An immediate question concerns discretization methods that preserve other types of stability. Since general Padé approximations can be interpreted as products of complex bilinear transforms, an immediate question in this direction concerns the equivalent map for other types of Lyapunov functions. Namely, given a continuous-time system with some Lyapunov functions, what are the mappings from continuous-time to discrete-time that preserve the Lyapunov functions. A natural extension of this question concerns whether discretization methods can be developed for exponentially stable switched and nonlinear systems but that do not have a quadratic Lyapunov function.

BIBLIOGRAPHY

- [1] M. Günther and U. Feldmann. CAD based electric circuit modeling in industry. part I: Mathematical structure and index of network equations. *Mathematics and computers in simulation*, 2000. 1
- [2] PC Müller. Descriptor systems: pros and cons of system modelling by differential-algebraic equations. *Mathematics and computers in simulation*, 53(4):273–279, 2000. 1, 3
- [3] D. Estevez Schwarz and C. Tischendorf. Structural analysis of electric circuits and consequences for MNA. *International Journal of Circuit Theory and Applications*, 28(2):131–162, 2000. 2
- [4] D.N. Stengel, DG Luenberger, RE Larson, and TB Cline. Descriptor-variable approach to modeling and optimization of large-scale systems. final report, march 1976–february 1979. Technical report, Systems Control, Inc., Palo Alto, CA (USA), 1979. 3
- [5] K.E. Brenan, S.L. Campbell, S.L.V. Campbell, and L.R. Petzold. *Numerical solution of initial-value problems in differential-algebraic equations*, volume 14. Society for Industrial Mathematics, 1996. 3
- [6] S.L.V. Campbell and SL Campbell. *Singular systems of differential equations*, volume 1. Pitman London, 1980. 3
- [7] S.L.V. Campbell and SL Campbell. *Singular systems of differential equations*, volume 2. Pitman London, 1982. 3
- [8] R. Sincovec, A. Erisman, Yip E., and M. Epton. Analysis of descriptor systems using numerical algorithms. *IEEE Transactions on Automatic Control*, 26(1):139–147, 1981. 3
- [9] G. Verghese, BC Levy, and T. Kailath. A generalized state-space for singular systems. *IEEE Transactions on Automatic Control*, 26(4):811–831, 1981. 3
- [10] H.K. Khalil and JW Grizzle. *Nonlinear systems*. Macmillan Publishing Company New York, 1992. 3, 27
- [11] J. Bao, P.L. Lee, and B.E. Ydstie. *Process control: the passive systems approach*. Springer-Verlag, 2007. 3
- [12] B.D.O. Anderson and S. Vongpanitlerd. *Network analysis and synthesis: a modern systems theory approach*. Prentice Hall, 1973. 4, 5, 27

- [13] J.C. Willems. Dissipative dynamical systems part ii: Linear systems with quadratic supply rates. *Archive for Rational Mechanics and Analysis*, 45(5):352–393, 1972. 4
- [14] M. Vidyasagar. *Input-output analysis of large-scale interconnected systems: decomposition, well-posedness, and stability*. Springer-Verlag, 1981. 4
- [15] P. Moylan and D. Hill. Stability criteria for large-scale systems. *IEEE Transactions on Automatic Control*, 23(2):143–149, 1978. 4, 59, 67, 68
- [16] R.W. Freund and F. Jarre. An extension of the positive real lemma to descriptor systems. *Optimization methods and software*, 19(1):69–87, 2004. 5, 38, 50
- [17] N. Wong. An efficient passivity test for descriptor systems via canonical projector techniques. In *Design Automation Conference, 2009. DAC'09. 46th ACM/IEEE*, pages 957–962. IEEE, 2009. 5, 6, 39, 50
- [18] Z. Bai and R.W. Freund. Eigenvalue-based characterization and test for positive realness of scalar transfer functions. *Automatic Control, IEEE Transactions on*, 45(12):2396–2402, 2000. 5
- [19] W. Gao and Y. Zhou. Eigenvalue-based algorithms for testing positive realness of siso systems. *Automatic Control, IEEE Transactions on*, 48(11):2051–2055, 2003. 5
- [20] D. Henrion. Linear matrix inequalities for robust strictly positive real design. *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, 49(7):1017–1020, 2002. 5
- [21] R. Shorten and C. King. Spectral conditions for positive realness of single-input-single-output systems. *Automatic Control, IEEE Transactions on*, 49(10):1875–1879, 2004. 5, 6, 29, 30
- [22] Schroeder C. and Stykel T. Passivation of LTI systems. Technical report, DFG Research Center Matheon, Tech. Rep. 368-2007, 2007. 5, 30
- [23] S. Grivet-Talocia. Passivity enforcement via perturbation of hamiltonian matrices. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 51(9):1755–1769, 2004. 5, 30
- [24] K. Zhou and J.C. Doyle. *Essentials of robust control*, volume 104. Prentice Hall Upper Saddle River, NJ, 1998. 5
- [25] P. Benner, R. Byers, V. Mehrmann, and H. Xu. A robust numerical method for the γ -iteration in \mathcal{H}_θ control. *Linear algebra and its applications*, 425(2-3):548–570, 2007. 5

- [26] R.N. Shorten, P. Curran, K. Wulff, and E. Zeheb. A note on spectral conditions for positive realness of transfer function matrices. *Automatic Control, IEEE Transactions on*, 53(5):1258–1261, 2008. 5, 6, 31
- [27] M. Corless and R. Shorten. On a class of generalized eigenvalue problems and equivalent eigenvalue problems that arise in systems and control theory. *Automatica*, 47(3):431–442, 2011. 5, 6
- [28] M. Corless and R. Shorten. On the characterization of strict positive realness for general matrix transfer functions. *Automatic Control, IEEE Transactions on*, 55(8):1899–1904, 2010. 5, 6
- [29] R.C. Li and R. Li. Test positive realness of a general transfer function matrix. Technical report, Dept. Math., Univ. of Kentucky, Lexington, Tech. Rep. 2000-20, 2004. 5
- [30] P. Benner and D. Chu. A new test for passivity of descriptor systems. *Oberwolfach reports, Mathematisches Forschungsinstitut Oberwolfach, Tech. Rep.*, 11(2005):266, 2005. 5
- [31] D. Chu and R.C.E. Tan. Algebraic characterizations for positive realness of descriptor systems. *SIAM Journal on Matrix Analysis and Applications*, 30:197, 2008. 5
- [32] H. Weiss, Q. Wang, and JL Speyer. System characterization of positive real conditions. *Automatic Control, IEEE Transactions on*, 39(3):540–544, 1994. 5
- [33] P. Van Dooren. The computation of kronecker’s canonical form of a singular pencil. *Linear Algebra and its Applications*, 27:103–140, 1979. 5
- [34] N. Wong and C.K. Chu. A fast passivity test for stable descriptor systems via skew-hamiltonian/hamiltonian matrix pencil transformations. *Circuits and Systems I: Regular Papers, IEEE Transactions on*, 55(2):635–643, 2008. 5
- [35] Z. Zhang and N. Wong. Passivity test of immittance descriptor systems based on generalized hamiltonian methods. *Circuits and Systems II: Express Briefs, IEEE Transactions on*, 57(1):61–65, 2010. 5
- [36] R. März. Canonical projectors for linear differential algebraic equations. *Computers & Mathematics with Applications*, 31(4-5):121–135, 1996. 6
- [37] R. März. Projectors for matrix pencils. Technical report, Humboldt-Univ. zu Berlin, Berlin, Germany, Tech. Rep. 04-24, 2004. 6
- [38] W.M. Griggs, B.D.O. Anderson, and A. Lanzon. A “mixed” small gain and passivity theorem for an interconnection of linear time-invariant systems. In *Proceedings of the European Control Conference 2007*, pages 2410–2416, 2007. 6, 7, 16, 19, 31, 59, 62, 171

- [39] W.M. Griggs, B.D.O. Anderson, and A. Lanzon. A “mixed” small gain and passivity theorem in the frequency domain. *Systems & control letters*, 56(9-10):596–602, 2007. [6](#), [7](#), [16](#), [19](#), [59](#), [62](#), [171](#)
- [40] C. Rohrs, L. Valavani, M. Athans, and G. Stein. Robustness of continuous-time adaptive control algorithms in the presence of unmodeled dynamics. *Automatic Control, IEEE Transactions on*, 30(9):881–889, 1985. [6](#)
- [41] B. Anderson, R.R. Bitmead, C.R. Johnson Jr, P.V. Kokotovic, R.L. Kosut, I.M.Y. Mareels, L. Praly, and B.D. Riedle. *Stability of adaptive systems: Passivity and averaging analysis*. MIT press, 1986. [6](#)
- [42] IMY Mareels, BDO Anderson, RR Bitmead, M. Bodson, and SS Sastry. Revisiting the mit rule for adaptive control. In *Proceedings of the 2nd IFAC Workshop on Adaptive Systems in Control and Signal Processing*, pages 161–166, 1986. [6](#)
- [43] B. Anderson. Failures of adaptive control theory and their resolution. *Communications in Information & Systems*, 5(1):1–20, 2005. [6](#)
- [44] W.M. Griggs, B. Anderson, and R.N. Shorten. A test for determining systems with “mixed” small gain and passivity properties. *Systems and Control Letters*, 60(7):479, 2011. [6](#), [16](#), [31](#), [32](#), [33](#)
- [45] F. Vasca and L. Iannelli. *Dynamics and Control of Switched Electronic Systems: Advanced Perspectives for Modeling, Simulation and Control of Power Converters*. Springer, 2012. [8](#)
- [46] G. Zhai, R. Kou, J. Imae, and T. Kobayashi. Stability analysis and design for switched descriptor systems. *International Journal of Control, Automation and Systems*, 7(3):349–355, 2009. [8](#), [10](#), [11](#), [80](#)
- [47] G. Zhai, X. Xu, J. Imae, and T. Kobayashi. Qualitative analysis of switched discrete-time descriptor systems. *International Journal of Control, Automation and Systems*, 7(4):512–519, 2009. [8](#), [10](#)
- [48] G. Zhai and X. Xu. A unified approach to analysis of switched linear descriptor systems under arbitrary switching. In *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, pages 3897–3902, 2009. [8](#), [10](#)
- [49] G. Zhai and X. Xu. Commutation condition for stability analysis of switched linear descriptor systems. In *2010 IEEE International Symposium on Intelligent Control (ISIC)*, pages 2088–2093, 2010. [8](#)
- [50] D. Liberzon and S. Trenn. On stability of linear switched differential algebraic equations. In *Decision and Control, 2009 held jointly with the 2009 28th Chinese*

- Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, pages 2156–2161. IEEE, 2009. 8, 9
- [51] S. Trenn. *Distributional differential algebraic equations*. PhD thesis, Universitätsbibliothek Ilmenau, 2009. 8, 9, 26, 78
- [52] W.P. Dayawansa and C.F. Martin. A converse lyapunov theorem for a class of dynamical systems which undergo switching. *Automatic Control, IEEE Transactions on*, 44(4):751–760, 1999. 9, 14, 146
- [53] S. Trenn and F. Wirth. A converse lyapunov theorem for switched daes. *Proceedings in Applied Mathematics and Mechanics*, 8:5, 2012. 9
- [54] I. Masubuchi, Y. Kamitane, A. Ohara, and N. Suda. H_∞ control for descriptor systems: A matrix inequalities approach. *Automatica*, 33(4):669–673, 1997. 10, 79
- [55] FL Lewis. Preliminary notes on optimal control for singular systems. In *Proceedings of the 24th IEEE Conference on Decision & Control: December 11-13, 1985, Bonaventure Hotel & Spa, Fort Lauderdale, Florida*, volume 1, page 266, 1985. 10, 79
- [56] J.Y. Ishihara and M.H. Terra. On the lyapunov theorem for singular systems. *IEEE Transactions on Automatic Control*, 47(11):1926–1930, 2002. 10, 79
- [57] K. Takaba, N. Morihira, and T. Katayama. A generalized lyapunov theorem for descriptor system. *Systems & Control Letters*, 24(1):49–51, 1995. 10, 79
- [58] T. Stykel. *Analysis and numerical solution of generalized Lyapunov equations*. PhD thesis, Technische Universität Berlin, Universitätsbibliothek (Diss.-Stelle), 2002. 10, 23, 79
- [59] D.H. Owens and D.L.J. Debeljkovic. Consistency and liapunov stability of linear descriptor systems: A geometric analysis. *IMA Journal of Mathematical Control and Information*, 2(2):139–151, 1985. 10, 21, 24, 79, 80
- [60] Y. Mori, T. Mori, and Y. Kuroe. A solution to the common lyapunov function problem for continuous-time systems. In *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on*, volume 4, pages 3530–3531. IEEE, 1997. 10
- [61] RN Shorten and KS Narendra. On the stability and existence of common lyapunov functions for stable linear switching systems. In *Decision and Control, 1998. Proceedings of the 37th IEEE Conference on*, volume 4, pages 3723–3724. IEEE, 1998. 10
- [62] K.S. Narendra and J. Balakrishnan. A common lyapunov function for stable lti systems with commuting a-matrices. *Automatic Control, IEEE Transactions on*, 39(12):2469–2471, 1994. 10, 80

- [63] R.A. Horn and C.R. Johnson. *Matrix analysis*. Cambridge Univ Pr, 1990. [10](#), [44](#), [52](#)
- [64] D. Liberzon, S. Trenn, and F. Wirth. Commutativity and asymptotic stability for linear switched daes. In *50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC)*, pages 417–422. IEEE, 2011. [10](#)
- [65] R.N. Shorten and K.S. Narendra. On common quadratic lyapunov functions for pairs of stable lti systems whose system matrices are in companion form. *Automatic Control, IEEE Transactions on*, 48(4):618–621, 2003. [10](#), [81](#), [92](#)
- [66] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM Review-The Flagship Journal of the Society for Industrial and Applied Mathematics*, 49(4):545–592, 2007. [10](#), [77](#), [81](#)
- [67] X. Mu, J. Wei, and R. Ma. Stability of linear switched differential algebraic equations with stable and unstable subsystems. *International Journal of Systems Science*, 2012. [11](#)
- [68] R. Shorten, M. Corless, K. Wulff, S. Klinge, and R. Middleton. Quadratic stability and singular siso switching systems. *Automatic Control, IEEE Transactions on*, 54(11):2714–2718, 2009. [12](#), [81](#), [82](#), [101](#)
- [69] J. Jiang. Preservations of positive realness under discretizations. *Journal of the Franklin Institute*, 330(4):721–734, 1993. [12](#), [13](#), [109](#)
- [70] M. De la Sen. Preserving positive realness through discretization. *Positivity*, 6(1):31–45, 2002. [12](#)
- [71] J.B. Hoagg, S.L. Lacy, R.S. Erwin, and D.S. Bernstein. First-order-hold sampling of positive real systems and subspace identification of positive real models. In *American Control Conference, 2004. Proceedings of the 2004*, volume 1, pages 861–866. IEEE, 2004. [12](#), [13](#)
- [72] S. Stramigioli, C. Secchi, A.J. van der Schaft, and C. Fantuzzi. A novel theory for sampled data system passivity. In *Intelligent Robots and Systems, 2002. IEEE/RSJ International Conference on*, volume 2, pages 1936–1941. IEEE, 2002. [13](#), [111](#)
- [73] J.H. Ryu, Y.S. Kim, and B. Hannaford. Sampled-and continuous-time passivity and stability of virtual environments. *Robotics, IEEE Transactions on*, 20(4):772–776, 2004. [13](#), [111](#)
- [74] S. Stramigioli, C. Secchi, A.J. vanderSchaft, and C. Fantuzzi. Sampled data systems passivity and discrete port-hamiltonian systems. *Robotics, IEEE Transactions on*, 21(4):574–587, 2005. [13](#), [111](#)

- [75] R. Costa-Castelló and E. Fossas. On preserving passivity in sampled-data linear systems. In *American Control Conference, 2006*, pages 6–pp. IEEE, 2006. 13, 111
- [76] N. Kottenstette and P.J. Antsaklis. Stable digital control networks for continuous passive plants subject to delays and data dropouts. In *Decision and Control, 2007 46th IEEE Conference on*, pages 4433–4440. IEEE, 2007. 13, 111
- [77] A. Dontchev and F. Lempio. Difference methods for differential inclusions: a survey. *SIAM review*, pages 263–294, 1992. 14
- [78] G. Grammel. Towards fully discretized differential inclusions. *Set-Valued Analysis*, 11(1):1–8, 2003. 14
- [79] L. Grüne and PE Kloeden. Higher order numerical approximation of switching systems. *Systems & control letters*, 55(9):746–754, 2006. 14
- [80] R.G. Sanfelice and A.R. Teel. Dynamical properties of hybrid systems simulators. *Automatica*, 46(2):239–248, 2010. 14
- [81] A. Kastner-Maresch and F. Lempio. Difference methods with selection strategies for differential inclusions. *Numerical functional analysis and optimization*, 14(5-6):555–572, 1993. 14
- [82] AR Humphries and AM Stuart. Runge-kutta methods for dissipative and gradient dynamical systems. *SIAM journal on numerical analysis*, pages 1452–1485, 1994. 14
- [83] A. Pietrus and V.M. Veliov. On the discretization of switched linear systems. *Systems & Control Letters*, 58(6):395–399, 2009. 14
- [84] A. Zappavigna, P. Colaneri, S. Kirkland, and R. Shorten. Essentially negative news about positive systems. *Linear Algebra and its Applications*, 436(9):3425–3442, 2012. 14
- [85] AK Baum and V. Mehrmann. Numerical integration of positive linear differential-algebraic systems. *Submitted to Linear Algebra Applications*, 2012. 14
- [86] C. Moler and C. Van Loan. Nineteen dubious ways to compute the exponential of a matrix. *SIAM review*, pages 801–836, 1978. 15, 115
- [87] E. Hairer and G. Wanner. *Solving ordinary differential equations II: Stiff and differential-algebraic problems*. Springer, 2010. 15, 115, 116
- [88] G. Birkhoff and RS Varga. Discretization errors for well-set cauchy problems. i. *J. Math. Phys.*, 44:1–23, 1965. 15, 116, 142
- [89] G.A. Baker and P.R. Graves-Morris. *Padé approximants*, volume 59. Cambridge Univ Pr, 1996. 15, 116

- [90] T. Mori, TV Nguyen, Y. Mori, and H. Kokame. Preservation of Lyapunov functions under bilinear mapping. *Automatica*, 42(6):1055–1058, 2006. [15](#), [117](#), [139](#), [145](#), [155](#)
- [91] W.M. Griggs. *Stability Results for Feedback Control Systems*. PhD thesis, The Australian National University, Australia, 2007. [16](#)
- [92] K. Weierstrass. Über ein die homogenen funktionen zweiten grades betreffendes theorem, nebst anwendung desselben auf die theorie der kleinen schwingungen. *Monatsh. Akad. der Wissensch.*, pages 207–220, 1858. [23](#)
- [93] L. Kronecker and Deutsche Akademie der Wissenschaften zu Berlin. *Algebraische reduction der schaaren bilinearer formen*. 1890. [23](#)
- [94] S.L. Campbell and C.W. Gear. The index of general nonlinear daes. *Numerische Mathematik*, 72(2):173–196, 1995. [23](#)
- [95] C.W. Gear. Differential-algebraic equation index transformations. *SIAM Journal on Scientific and Statistical Computing*, 9:39, 1988. [23](#)
- [96] K.S. Narendra and J.H. Taylor. *Frequency domain criteria for absolute stability*. Academic Press New York, 1973. [24](#)
- [97] T. Kailath. *Linear systems*, volume 1. Prentice-Hall Englewood Cliffs, NJ, 1980. [24](#)
- [98] L. Dai. *Singular control systems*. Lecture Notes in Control and Information Sciences, 118, 1989. [24](#)
- [99] P. Kunkel and V.L. Mehrmann. *Differential-algebraic equations: analysis and numerical solution*. European Mathematical Society, 2006. [25](#), [26](#)
- [100] T. Geerts. Solvability conditions, consistency, and weak consistency for linear differential-algebraic equations and time-invariant singular systems: the general case. *Linear algebra and its applications*, 181:111–130, 1993. [26](#)
- [101] B. Brogliato. *Dissipative systems analysis and control: theory and applications*. Springer Verlag, 2007. [26](#), [27](#), [28](#), [60](#)
- [102] H.J. Maquez. *Nonlinear control systems*. Wiley-Interscience, 2003. [27](#), [28](#)
- [103] M. Corless and R. Shorten. A correct characterization of strict positive realness for mimo systems. In *American Control Conference, 2009. ACC'09.*, pages 469–475. IEEE, 2009. [28](#), [29](#)
- [104] P. Triverio, S. Grivet-Talocia, M.S. Nakhla, F.G. Canavero, and R. Achar. Stability, causality, and passivity in electrical interconnect models. *Advanced Packaging, IEEE Transactions on*, 30(4):795–808, 2007. [31](#), [32](#)

- [105] E. Zeheb, R. Shorten, and S.S.K. Sajja. Strict positive realness of descriptor systems in state space. *International Journal of Control (see errata online)*, 83(9):1799–1809, 2010. [43](#)
- [106] R.N. Shorten, P. Curran, K. Wulff, and E. Zeheb. A note on spectral conditions for positive realness of transfer function matrices. *Automatic Control, IEEE Transactions on*, 53(5):1258–1261, 2008. [43](#), [48](#), [55](#)
- [107] R. Shorten, M. Corless, K. Wulff, S. Klinge, and R. Middleton. Quadratic stability and singular siso switching systems. *Automatic Control, IEEE Transactions on*, 54(11):2714–2718, 2009. [43](#), [44](#), [46](#), [47](#)
- [108] M. Corless and R. Shorten. On a class of generalized eigenvalue problems and equivalent eigenvalue problems that arise in systems and control theory. *Automatica*, 47(3):431–442, 2011. [43](#), [44](#)
- [109] C.A. Desoer and M. Vidyasagar. *Feedback systems: input-output properties*, volume 55. Society for Industrial Mathematics, 2009. [60](#)
- [110] K. Zhou, J.C. Doyle, K. Glover, et al. *Robust and optimal control*, volume 40. Prentice Hall Upper Saddle River, NJ, 1996. [61](#), [63](#), [65](#), [66](#), [132](#), [133](#)
- [111] S. Skogestad and I. Postlethwaite. *Multivariable feedback control: analysis and design*, volume 2. Wiley, 2007. [61](#)
- [112] CD Meyer. Matrix analysis and applied linear algebra. *SIAM, Philadelphia*, 2000. [62](#), [67](#), [133](#)
- [113] S. Barnett. Introduction to mathematical control theory(book). *Oxford, Clarendon Press*, 1975. [68](#), [132](#)
- [114] R.E. Kalman. Lyapunov functions for the problem of lur’e in automatic control. *Proceedings of the National Academy of Sciences of the United States of America*, 49(2):201, 1963. [80](#)
- [115] KR Meyer. On the existence of lyapunov function for the problem of lur’e. *Society for Industrial Mathematics*, 1965. [80](#)
- [116] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15. Society for Industrial Mathematics, 1994. [80](#)
- [117] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. A geometrical criterion for the stability of certain non-linear non-autonomous systems. *IEEE Transactions on Circuit Theory*, 11(3):406–407, 1964. [81](#)
- [118] CD Meyer. Matrix analysis and applied linear algebra. *SIAM, Philadelphia*, 2000. [88](#)

- [119] Y. Mori, T. Mori, and Y. Kuroe. A solution to the common lyapunov function problem for continuous-time systems. In *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on*, volume 4, pages 3530–3531. IEEE, 1997. [96](#)
- [120] F. Rossi, P. Colaneri, and R. Shorten. Padé discretization for linear systems with polyhedral lyapunov functions. *Automatic Control, IEEE Transactions on*, 56(11):2717–2722, 2011. [107](#), [118](#), [155](#), [156](#), [162](#), [163](#), [166](#), [170](#)
- [121] H. Kiendl, J. Adamy, and P. Stelzner. Vector norms as lyapunov functions for linear systems. *Automatic Control, IEEE Transactions on*, 37(6):839–842, 1992. [112](#)
- [122] A. Polanski. On infinity norms as lyapunov functions for linear systems. *Automatic Control, IEEE Transactions on*, 40(7):1270–1274, 1995. [112](#), [114](#), [161](#), [162](#)
- [123] F.J. Christophersen and M. Morari. Further results on “infinity norms as lyapunov functions for linear systems”. *Automatic Control, IEEE Transactions on*, 52(3):547–553, 2007. [112](#), [115](#), [163](#)
- [124] R.C. Ward. Numerical computation of the matrix exponential with accuracy estimate. *SIAM Journal on Numerical Analysis*, pages 600–610, 1977. [116](#), [142](#)
- [125] JA Van de Griend and J. Kraaijevanger. Absolute monotonicity of rational functions occurring in the numerical solution of initial value problems. *Numerische Mathematik*, 49(4):413–424, 1986. [117](#)
- [126] W.S. Levine. *The control handbook*. CRC, 1996. [131](#)
- [127] M. Corless and R. Shorten. On a class of generalized eigenvalue problems and equivalent eigenvalue problems that arise in systems and control theory. *Automatica*, 47(3):431–442, 2011. [133](#), [134](#)
- [128] R.N. Shorten, P. Curran, K. Wulff, and E. Zeheb. A note on spectral conditions for positive realness of transfer function matrices. *Automatic Control, IEEE Transactions on*, 53(5):1258–1261, 2008. [133](#), [134](#)
- [129] T. Ando. Set of matrices with common lyapunov solution. *Archiv der Mathematik*, 77(1):76–84, 2001. [139](#)
- [130] R.N. Shorten and K.S. Narendra. Necessary and sufficient conditions for the existence of a common quadratic lyapunov function for a finite number of stable second order linear time-invariant systems. *International Journal of Adaptive control and Signal processing*, 16(10):709–728, 2002. [144](#)
- [131] R. Shorten and F.O. Cairbre. A new methodology for the stability analysis of pairwise triangularizable and related switching systems. *IMA journal of applied mathematics*, 67(5):441–457, 2002. [146](#), [147](#)

- [132] S. Solmaz, R. Shorten, K. Wulff, and F. Ó Cairbre. A design methodology for switched discrete time linear systems with applications to automotive roll dynamics control. *Automatica*, 44(9):2358–2363, 2008. [147](#)
- [133] G. Wanner, E. Hairer, and S.P. Nørsett. Order stars and stability theorems. *BIT Numerical Mathematics*, 18(4):475–489, 1978. [150](#)
- [134] R. Shorten, S. Corless, M. and Sajja, and Solmaz S. On padé approximations and the preservation of quadratic stability for switched linear systems. *Systems & Control Letters*, 60(9):683–689, 2011. [153](#)
- [135] F. Blanchini and S. Miani. *Set-theoretic methods in control*. Springer, 2008. [155](#)
- [136] P. Lancaster and M. Tismenetsky. *The theory of matrices: with applications*. Academic Pr, 1985. [158](#)
- [137] G. Birkhoff and RS Varga. Discretization errors for well-set cauchy problems. i. *J. Math. Phys.*, 44:1–23, 1965. [158](#)
- [138] JA Van de Griend and J. Kraaijevanger. Absolute monotonicity of rational functions occurring in the numerical solution of initial value problems. *Numerische Mathematik*, 49(4):413–424, 1986. [161](#), [170](#)
- [139] S. Boyd and Q. Yang. Structured and simultaneous lyapunov functions for system stability problems. *International Journal of Control*, 49(6):2215–2240, 1989. [173](#)
- [140] T. Iwasaki and S. Hara. Generalized kyp lemma: unified frequency domain inequalities with design applications. *Automatic Control, IEEE Transactions on*, 50(1):41–59, 2005. [173](#)