

**A STUDY OF CONCEPT IMAGES  
AND CONCEPT DEFINITIONS  
RELATED TO METRIC SPACES**

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# Declaration

I, the undersigned, hereby declare that this thesis entitled, 'A Study of Concept Images and Concept Definitions related to Metric Spaces' is my own work, and that all the published work I have consulted or quoted has been acknowledged in the references.

I also confirm that this thesis is submitted for the examination for the degree Doctor of Philosophy at the National University of Ireland Maynooth. It has not been submitted to any other university or degree-giving institution.

I, the undersigned, agree that the Library of the National University of Ireland Maynooth may lend or copy this thesis upon request.

Safia F. Hamza.

Date: 25/10/2012

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# Abstract

The aim of this thesis is to discover some of the concept images held by students concerning metric spaces. I was specifically interested in the concept of an open set in a metric space. This work also addresses the topics of distance and open balls, which are both essential when considering open sets. Also, I sought to investigate the relationships between students' concept images and the concept definition, and the uses students make of their concept images and the concept definition when working on problems. Furthermore, this thesis explores some of the attitudes of students to mathematical definitions, and their understanding of the role definitions play in mathematics.

A set of questions was designed in order to gain insight into students' understanding of the concept of open set, and their opinions on mathematical definitions. The questions were used in a written questionnaire, which was completed in class by 16 students who were enrolled in a metric spaces course and in individual interviews with eleven students.

From the results of the study I was able to categorise the concept images of an open set into five categories: those based on the definition; those based on boundary points; those based on a union of open balls; those based on open sets in Euclidean space; and those based on visualisation. I also categorised the concept images of distance into four categories: concept image based on the measurement of similarity; those based on comparison/difference between points; those based on physical distance; and those based on the formal definition. I was able to uncover some influences on the formation of concept images, and some misconceptions related the concept of an open set. The study also discovered that these students generally understand the role of definitions in advanced mathematics; however they often consider and use their conceptions in the place of definitions.

# Chapter 1

## Introduction

This thesis contributes to the work concerning students' understanding of advanced mathematical concepts. These are abstract concepts and are usually considered as difficult for many students. The thesis focuses on the nature of students' knowledge about the area of Topology in particular about metric spaces, and how a student may develop an understanding of certain topics in this area. The main mathematical concept under consideration is the open set concept in a metric space and I will also consider the topics of distance and open balls which are basic for the concept of open set.

In this Chapter, I will state the thesis' research questions, I will briefly outline what is meant by the terms 'concept definition' and 'concept image' associated with a mathematical concept, then I will review the literature related to my study, and finally I will outline the construction of this thesis.

### 1.1 Research Questions

The goal of the study was to explore students' understanding of the concepts of open set and the related concepts of distance and open balls in a metric space in detail, and to make deductions from the observations of the students' thinking. Based on these goals our study was guided by the following research questions:

- 1- Do students understand the role and the use of definitions in mathematics?
- 2- What definitions and images do students have about open sets and distance in metric spaces?
- 3- Which definition and images do students use while working on problems?
- 4- Are students consistent in the use of their concept definition and concept image? And also, is there consistency between students' conceptions and the formal definition?

In order to get access to data on my research questions, a questionnaire and interviews were used. These consisted of a variety of questions designed to cover all

the possible aspects of our intended concept and to discover the possible conceptions students might hold. These instruments were designed specially for this study. They were administered to a group of students who were taking a first course in metric topology. Sixteen students completed the questionnaire and eleven students were interviewed. The analysis which I will present in this thesis is based on the types of written and oral arguments given by students to justify their answers to the questions asked in both the written questionnaire and the interviews.

In explaining how students develop their understanding of a mathematical concept, it is important to distinguish between students' concept image and the concept definition. By a student's concept image I mean everything a student has in his/her mind about a mathematical concept. This might include conceptions, pictures, impressions, experiences, and even a student's own definitions. While, by the concept definition I mean the formal definition of a mathematical concept. The distinction between concept definition and concept image is described in detail in the work of Tall and Vinner (1981) which is one of the most important contributions to this area of mathematics education in explaining how students understand mathematical concepts. More explanations on students' concept image and concept definition will be provided in the literature review later.

To date, there has been quite a lot of research on concept images related to concepts in Calculus, Linear Algebra and Introductory Analysis. However, very little is known about concept image in advanced areas of Mathematics. This thesis aims to shed light on this issue. The issue is seen as important in the area of mathematics education. Selden and Selden (1998) laid out a list of research questions regarding the teaching and learning of mathematics at undergraduate level, which they considered to be important. Among them were questions related to how understanding could be fostered and questions on the nature of mathematical definitions and how students use them.

## **1.2 Literature Review**

This section gives a general overview of the work that has been done previously in mathematics education in relation to the topics of this thesis. I will consider the relevant literature in the following sections.

### **1.2.1 Advanced Mathematical Thinking**

Much of mathematics education research has recently focused on studying the ways that students learn advanced mathematical concepts and on identifying some of the most general elements affecting their advanced mathematical thinking. Concerning the term ‘Advanced Mathematical Thinking’, one might question initially whether the term advanced refers to the mathematics or to the thinking or both (Tall 1988 and Selden and Selden 2005). The term advanced mathematical thinking is used in my study here to refer to the thinking employed in advanced mathematics (that is in abstract topics). Selden and Selden (2005) discussed some different perspectives on the idea of advanced mathematical thinking. For example, one of the perspectives discussed was Edwards, Dubinsky and McDonald’s (2005) definition of advanced mathematical thinking:

Advanced Mathematical thinking is thinking that requires deductive and rigorous reasoning about mathematical notions that are not entirely accessible to us through our five senses. (p. 17-18)

Thinking in advanced mathematics requires students to change the habits that were used in elementary mathematics. This is a very important point as the transition from computational mathematics to abstract mathematics is experienced as a very difficult process for many students. Tall (1991) pointed to the need for advanced mathematical thinking, and reported that

The move from elementary to advanced mathematical thinking involves a significant transition: that from *describing to defining*, from *convincing to proving* in logical manner based on definition. (p. 20)

### **1.2.2 Concept Definition and Concept Image**

Students encounter a wide range of information during their learning of mathematics, and during the development of a mathematical concept students naturally will rely on their previous knowledge. Therefore, much of the work in this area has aimed at describing in detail, students’ conceptions and their developments of mathematical concepts. Tall and Vinner (1981) discussed the processes involved in students’ learning of mathematics. They explained the distinction between the aspect of reasoning in advanced mathematics which is based on formal definitions (*concept*

*definition*), and the aspect of reasoning that students use which is based on their conception of a concept (*concept image*). The term concept definition is used to indicate a mathematical definition and they stated it as:

a form of words used to specify that concept (Tall and Vinner 1981, p. 152).

The term concept image is used to mean all that an individual has in his/her mind about a concept, and this would include mental pictures, experiences and impressions that are associated with it, and they defined the concept image as:

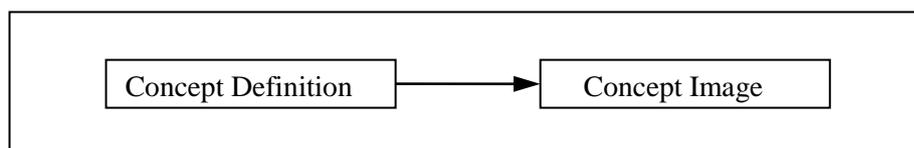
the total cognitive structure that associated with the concept, which includes all the mental pictures and associated properties and processes. (Tall and Vinner 1981, p. 152).

They explained also that a concept image is not a static item in memory; it builds and is reconstructed over time as individuals meet new stimuli. They also used the term *evoked concept image* to describe the part of a concept image which is evoked by the concept name at a specific time. They defined it as:

the portion of the concept image which is activated at a particular time. (Tall and Vinner 1981, p. 152)

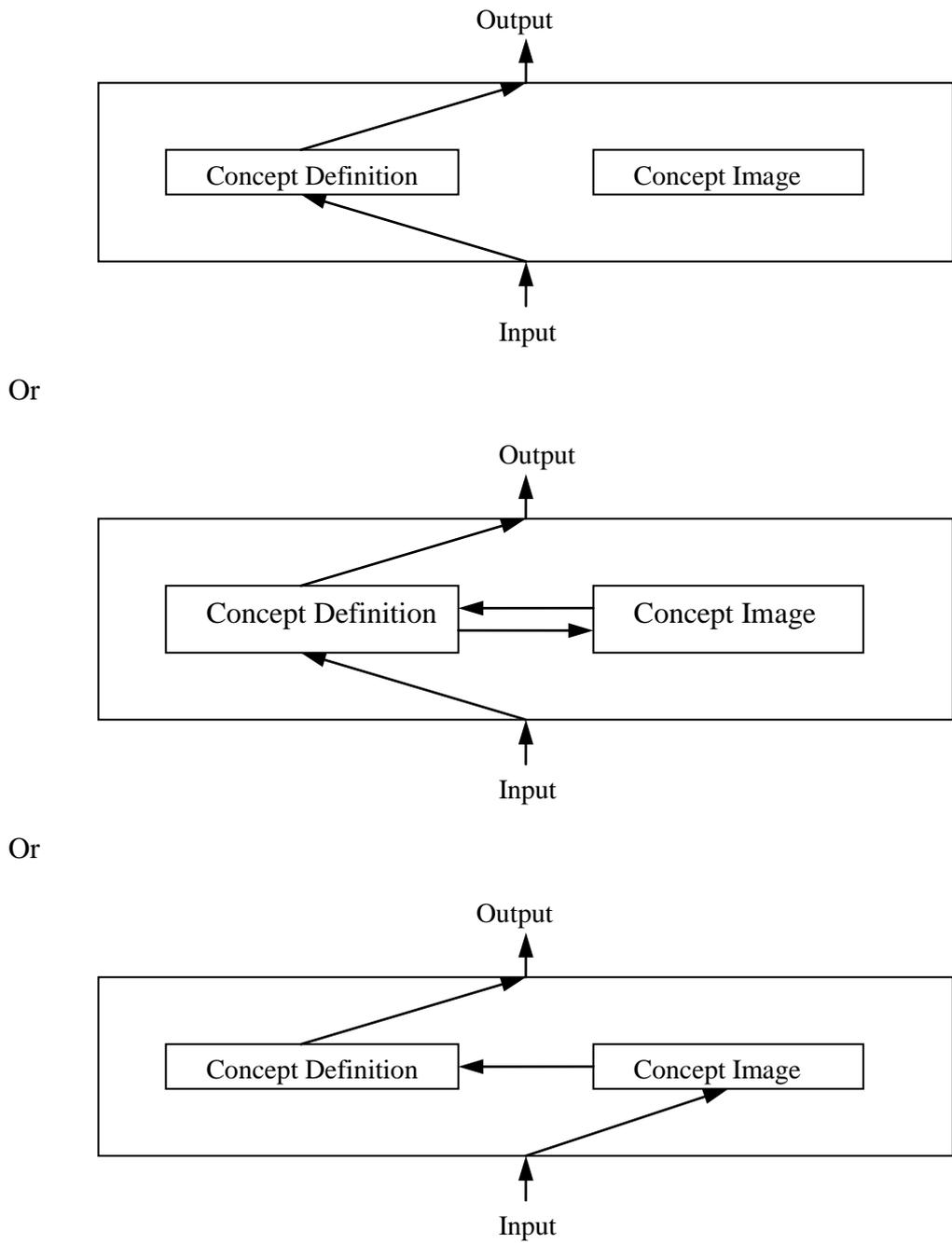
Tall and Vinner used their construct to describe some possible factors that might be the source of secondary school and university students' cognitive conflict in relation to the concepts of limit of sequences; limit of functions; and continuous functions.

Based on the above idea, Vinner (1991) addressed the reason for students' misconceptions and discussed the interplay between the concept image and concept definition during the process of concept formation. He reported that, many mathematicians presumed that the concept image of their students will be only formed from the given concept definition in the development of a formal concept, as in Figure 1.1.



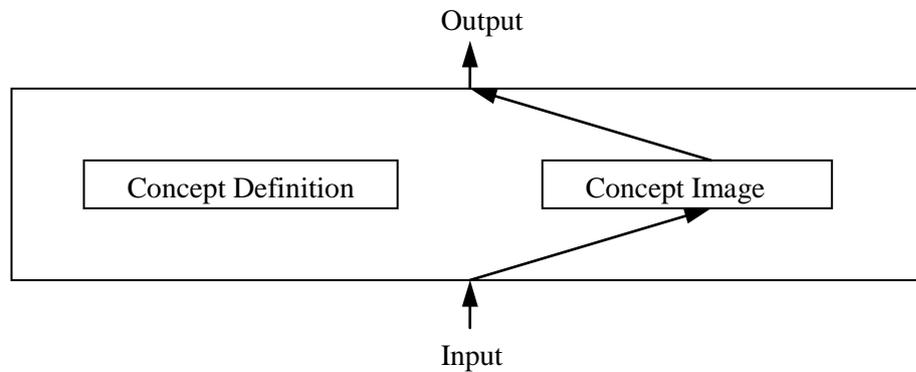
**Figure 1.1: The formal development of a concept  
(Taken from Vinner 1991, p. 71)**

Vinner also stated that ‘when a problem is posed to you in a technical context, you are not supposed to formulate your solution before consulting the concept definition’. So it is important that when students are working on tasks, no matter if the concept image interplays with the concept definition or not, students should use the concept definition as a final stage to answer the tasks, and in one of the three ways shown in Figure 1.2.



**Figure 1.2: Reasoning consulting the concept definition**  
 (Taken from Vinner 1991, p. 71-72)

However Vinner found that students often based their solution solely on their concept image, as illustrated in Figure 1.3 which Vinner called the ‘intuitive response’.



**Figure 1.3: Reasoning consulting the concept image**  
(Taken from Vinner 1991, p. 73)

The students might not be aware that their concept image is not necessarily consistent with its concept definition at all times, and that it might contain contradictory aspects.

Rösken and Rolka (2007) examined some German secondary school students’ conceptual learning concerning the notion of the definite integral. They designed a questionnaire about the integral concept in order to explore students’ concept images. From the results they found that definitions play a marginal role in students’ learning of the concept and they rely mainly on their concept images which are based on intuition when reasoning about concepts.

### **1.2.3 The Role and Features of Definitions**

Using the ideas of Tall and Vinner, many other studies have been carried out to analyse students’ understanding in advanced mathematics. As mathematics is a theoretical system, definitions have an important role in the acquisition of its concepts (in particular for abstract concepts). However, the use and the formation of mathematical definitions is different from those definitions in real life where individuals acquire concepts depending mainly on their conception without need for proper definitions (Vinner 1991).

Edwards and Ward (2004) investigated students’ understanding of the content of definitions in mathematics in order to explore their awareness of the role that formal

definitions play in mathematics. Edwards is a researcher in undergraduate mathematics education and Ward is a mathematics lecturer at an undergraduate institution. In a meeting at a summer institute, Edwards explained to Ward the findings of her PhD thesis (in 1997) on students' understanding and use of definitions in real analysis, where she found that some students have difficulty with tasks involving, for example, definitions of limit and continuity. Ward intuitively assumed that these words like limit and continuous bring with them a number of different associations from non-mathematical use and from early encounters in elementary mathematics. Ward also presumed that students would have less trouble with definitions in abstract algebra, as students have not encountered the words involved in there in their elementary mathematics courses. Together Edwards and Ward investigated undergraduate mathematics majors' understanding and use of definitions in an introductory abstract algebra course which was taught by Ward and observed by Edwards. They used the same methodology that Edwards used in her real analysis study and the data were collected from some written class assignments and interviews. Ward was surprised that his students had difficulties similar to those in Edwards's study, and both reported three surprises that arise from their two studies on undergraduate mathematics majors (Edwards's study on real analysis students and Ward's study on his abstract algebra students):

Surprise 1: Many students do not categorize mathematical definitions the way mathematicians do. (Edwards and Ward 2004, p. 415)

Edwards and Ward explained that mathematical definitions are stipulated definitions which release their term from all possible connotations that the term might take from non-technical use and create the use of it, whilst most of everyday life definitions are definitions which are extracted from an actual use (practice) of their term and so they describe the usage of it. They found that students' failure to appreciate mathematical definitions as stipulated definitions could be a reason for their misuse of definitions.

Surprise 2: Many students do not use definitions the way mathematicians do, *even when the students can correctly state and explain the definitions.* (Edwards and Ward 2004, p. 416)

Most mathematicians would expect that students' responses will be based solely on the concept image only if they do not know or do not understand the definitions. Edwards and Ward found that some students were able to state and explain the

definitions and could reason from definitions if the definitions did not conflict with their conception, but when their concept image conflicted with the concept definitions, they only relied on their concept image.

Surprise 3: Many students do not use definitions the way mathematicians do, *even in the apparent absence of any other course of action.* (Edwards and Ward 2004, p. 417)

They found that even if students have no previous experience with some concepts, some of them might bring other inappropriate conceptions to be used in abstract tasks that can be solved only by using the definitions.

In addition, Przenioslo (2004) investigated and assessed a number of Polish university students' concept images of the limits of functions concept. The participants in the study were students who had just commenced their university studies in mathematics and students who had finished analysis courses. Her choice of students from different levels was for the purpose of discovering the variance between concept images which are formed in secondary schools and the concept images that formed in university mathematical studies courses. She designed a set of mathematical problems related to the concept of limit, and to avoid automatic answers the problems were rather simple but not quite routine. She used several research instruments in her study (e.g. written tests; interviews; observations; group discussions; students' notes, and informal conversations with teachers and students). In the analysis, Przenioslo (2004) found that students have various conceptions related to the notion of the limit concept (e.g. one conception was based on the idea of neighbourhoods and this was the most efficient conception in problem solving; and another conception was based on the behavior of points on a graph and this conception quite frequently led to incorrect solutions). She also observed that many concept images of students who completed their analysis course seemed to be formed perhaps in secondary school, as the same concept images were observed in this group as with students who were commencing their studies. She also realised that her students showed an unawareness of the role of definitions in mathematics, they often treated their associations as a definition or, in particular as parts of the definition. She documented that:

The phrase 'by definition' did not mean for those using it that something results from the definition but it was a piece of their own definition of limit. (p. 116)

Przenioslo (2004) noticed that some students used the phrase ‘by definition’ when explaining their conceptions.

As mentioned before, there is no doubt about the crucial importance of definitions in the acquisition of mathematical concepts. Therefore how students deal with and think of mathematical definitions is an important subject of study. Zaslavsky and Shir (2005) looked at the ways that four senior-high school students think of the general notion of mathematical definitions and specific concept definitions using individual and group activities. The students were asked whether a number of statements related to different mathematical concepts could be considered as possible definitions for such concepts. Data analysis was based on students’ responses to questionnaires and transcriptions of videotaped discussions. The authors listed the roles of definitions considered by mathematics community:

- Introducing the objects of theory and capturing the essence of a concept by conveying its characterized properties.
- Constituting fundamental components for concept formation.
- Establishing the foundation for proofs and problem solving.
- Creating uniformity in the meaning of concepts. (Zaslavsky and Shir 2005, p. 317)

They reported that the students in their study agreed with the mathematics community on the importance of these roles. That is, students mentioned some of the roles of definitions in mathematics such as: classification of example and non-example of a concept; proving and problem solving, and understanding the meaning of mathematical concepts. Zaslavsky and Shir (2005) also observed that the students use three criteria when considering statements: mathematical (if the statements’ condition is both necessary and sufficient for the intended concept); communicative (if the statements are clear and understandable); and figurative (if the statements are using the properties obvious from generic diagrams). Moreover, the results of their study pointed to two types of reasoning used by students when justifying the given statements: example-based reasoning and definition-based reasoning. They found that most of the examples used in their study were counterexamples, and students used them to support their disagreements with a statement as a possible definition of a particular concept. They found that a very small number of students used examples

when agreeing with a statement. The other type of reasoning (i.e. that which is based on definitions) was used by students both to agree and to disagree with a statement.

Van Dormolen and Zaslavsky (2003) addressed different aspects of mathematical definitions considering the imperative and the preferable features of a mathematical definition. The imperative features which are considered to be necessary for a mathematical definition are: hierarchy (in that, a mathematical definition should be described in terms of a well-known general concept); existence (that is, an instance of a concept should be given); equivalence (statements that represent a concept should be equivalent); and axiomatization (a definitions should be given in deductive manner). The optional features of a mathematical definition that Van Dormolen and Zaslavsky (2003) considered are: minimality (a definition's statement should only contain the necessary properties for the existence of a concept); elegance; and degenerations (in which a definition allows for unexpected instances of the concept, in some cases one might alter the definition to exclude these instances).

Leikin & Zazkis (2010) considered the use and understanding of definitions in mathematics. They used example generation tasks (that is generating several example of definitions of certain mathematical concepts) to examine prospective teachers' content knowledge about defining mathematical concepts. They analysed teachers' example spaces based on their correctness and richness and found that teachers' knowledge of definitions varies in different areas of mathematics. Leikin & Zazkis (2010) pointed to the meaning of definitions in mathematics, and to the desirable characteristics of mathematical definitions. For example, in accordance to the link of [abstractmath.org](http://abstractmath.org) they reported that (note that the following is quoted from Leikin & Zazkis (2010)):

A definition of a concept in mathematics has properties that are different from definitions in other subjects, including the following:

- (1) The definition contains a list of properties of the concept which are necessary and sufficient conditions of the concept.
- (2) Any example of a concept must fit all the requirements of its definition (not only most of them), as the definition determines the set of necessary conditions.

- (3) Every mathematical object that fits all the requirements of the definition is an example of the concept because the definition contains sufficient conditions of the concept.
- (4) Every correct statement about the concept follows logically from its definition.
- (5) Definitions are crisp, not fuzzy. This means that any object either is or is not an example of the concept.
- (6) Definitions tend to be minimal: they provide a small amount of structural information and properties that are sufficient to determine the concept.
- (7) Usually, much is known about the concept in addition to what is in the definition. For example, equivalent to the definition statements are theorems once the definition is given.
- (8) The same concept can have definitions that appear to be vastly different and it may be difficult to prove that they describe the same concept. Sometimes proving the equivalence of definitions of the same concept may be complex, e.g. a conic section given in Euclidean geometry, analytic geometry and algebra.
- (9) Mathematics texts use special wording in definitions. (Leikin & Zazkis 2010, p. 452-453)

In line with the previous study, Zandieh & Rasmussen (2010) enhanced the use of defining activity as a powerful tool in addressing students' use of definitions. They developed the (DMA) framework (Defining as a Mathematical Activity) which pointed to the role that defining can play in students' growth in understanding mathematics (that is from less formal to more formal way of reasoning). The framework combines the pedagogical theory of Realistic Mathematics Education (RME) and Tall and Vinner's distinction between concept image and concept definition. It links four levels of activity from RME with the notions of using and creating concept images and concept definitions:

Situational activity involves using a concept image to create a concept definition, Referential activity focuses on using a concept definition to create a concept image, General activity involves creating new concept images and concept definitions, and Formal activity focuses on using established concept

images and concept definitions to serve other mathematical goals. (Zandieh & Rasmussen 2010, p. 60)

This framework is the result of a retrospective account of a significant teaching experiment carried out in an undergraduate geometry course, and the analysis showed that the movement from less formal to more formal reasoning enabled students to make stronger links between their concept images and concept definitions.

Alcock and Simpson (2009) summarised the outcomes of a number of studies which examined students' difficulty with the understanding of mathematical concepts. They discussed students' neglect of the role that definitions play in advanced mathematics and described how they leaned on their concept images that they had formed from their previous experience. They also reported on the role of pre-existing concept images in students' reasoning about a concept and how everyday use of mathematical terms can also affect students' concept image especially the ones provided by opposites (as in everyday use things cannot be both increasing and decreasing or both open and closed).

When Cornu (1991) addressed some didactic aspects of the notion of limits, he found that this notion is associated with a variety of conceptions. He explained that prior to learning most mathematical concepts, students previously have some ideas on the subject, images and intuitions, which come from everyday life, and which are related to the used terms. Cornu called these kinds of conceptions of a concept which take place before any formal learning '*spontaneous conceptions*'. As students hold these everyday life conceptions for a long time, they are slow to disappear when students are introduced to the mathematical concept which uses the everyday term formally, and the students mix and adapt the new knowledge to shape their concept image. Concerning the concept of limit, he reported that words such as 'tend to' and 'limit' have meanings in students' minds before encountering the formal one, and that students keep using these meanings when they meet the concept formally. For example the term 'tend towards' could mean:

- to approach (eventually staying away from it)
- to approach ... without reaching it
- to approach ... just reaching it
- to resemble (without any variation, such as "this blue tends towards violet").

(Cornu 1991, p. 154)

He also explained that the term ‘limit’ is mostly deemed as ‘impossible limit’ and also could mean:

- an impossible limit which is reachable,
- an impossible limit which is impossible to reach,
- a point which one approaches, without reaching it,
- ...
- a maximum or minimum,
- ...
- the end, the finish. (Cornu 1991, p. 154-155)

### 1.2.4 Mathematical Language

Mathematical language is known to be affected by real life language. Mason and Pimm (1984) and Rowland (2001) have discussed the effect of everyday life terms on logical statements in mathematics. Both studies mentioned, in particular, the ambiguous use of the term ‘any’ with mathematical quantifiers. It is sometimes used in place of ‘all’ and sometimes in place of ‘some’. Mason and Pimm (1984) reported that:

Mathematicians tend to use ‘any’ to mean ‘every’, and occasionally their meaning conflicts with ordinary usage. (p.281)

Therefore, in everyday life, the term ‘any’ is frequently used to mean ‘some’ and it is rarely used in place of ‘every’. On the contrary in mathematics, such a term is often used to refer to ‘every’. Rowland (2001) also pointed to the same term that:

the fact is that the quantifier *any*, despite widespread use in mathematics at all levels, is irredeemably ambiguous, and may in turn be intended to mean ‘every’ (this is the universal quantifier *for all*) or ‘some’ (the existential quantifier *there exist*). (p.185)

### 1.2.5 The Role of Intuition

Some studies such as Fischbein (1987); Tirosh (1991) and Rösken and Rolka (2007) have mentioned the presence and constancy of alternative conceptions, such as intuitions and preconceptions, which are not in line with formal definitions.

When I tried to search for the meaning of the term ‘intuition’ I found that there are a variety of meanings associated to this term. For example, Semadeni (2008) explained that:

When a person produces sound mathematical reasoning and arrives at plausible conclusions without referring to precise definitions or known theorems, mathematicians say that this person “has intuition” in the given domain. (p. 2)

By this, any concept image that is not based on a formal definition or on a formal statement is considered to be intuition.

Fischbein (1987) tried to suggest a common definition for the term ‘intuition’. He discussed much of the work that has been done on the field of intuition, and reported on many meanings related to the term ‘intuition’. He observed that each of the meanings seems to be associated with particular terms such as: understanding; guess; common sense; belief; and insight. As a common definition of the term ‘intuition’, Fischbein (1987) suggested the following:

Intuitive knowledge is immediate knowledge; that is, a form of cognition which seem to present itself to a person as being self-evident. (p. 6)

Fischbein (1987) also considered students’ individual concept formation and explained how learning in mathematics is affected by intuitive models. He stated that:

It is very well known that concepts and formal statements are very often associated, in a person’s mind, with some particular instances. What is usually neglected is the fact that such particular instances may become, for that person, universal representatives of the respective concepts and statements and then acquire the heuristic attributes of models. (p. 149-150)

Tirosh (1991) discussed students’ intuitions regarding the comparison of infinite sets and indicated that students adapted the same intuitive criteria to compare infinite sets as finite sets. Tirosh stated that:

In fact, there is evidence, both in the science and mathematics education literature, that contradictory intuitions may be a main obstacle to acquiring formal knowledge (Fischbein & Gazit, 1984; Stavy, Eisen & Yakobi, 1987). Moreover, inadequate intuitive beliefs often continue to affect student’s choice of solutions to problems even after formal instruction of the relevant theory. (Tirosh 1991, p.205)

### **1.2.6 Visualisation**

In dealing with the concept image, it is known that visual images (e.g., imaging, drawing pictures or using technological tools) have significant effects on mathematical thinking which should be appreciated, specially in some topics such as continuity, differentiation etc. which are known to be graphical concepts. Many studies have explored how students deal with problems, and have found that students could use visual reasoning or non-visual reasoning when arguing about problems. For example, Alcock and Simpson (2004) examined the ways used, by eighteen first year students, who use visual reasoning, when solving problems about convergence of sequences and series, during semi-structured interviews. They found that all students, who used visual images in their reasoning about real analysis, were able to view mathematical constructs as objects to be compared; were able to deduce quick conclusions about sets of mathematical objects; and also were confident about their conclusions. Alcock and Simpson (2004) observed that students who used visual images successfully made a strong connection between visual and formal representations while those who used visual images unsuccessfully were not comfortable with formal representations. Thus they agreed with the significance of using the sophisticated visual images to explore subtleties of formal analytic concepts.

### **1.2.7 Previous Experience**

Previous experience also has an influence on students' learning of mathematical concepts. Vinner & Dreyfus (1989) studied some students' images of the function concept. The subjects in their study were first year college students and some junior high (secondary) mathematics teachers who had been introduced to the concept before, and who were asked to complete a questionnaire. The questionnaire consisted of questions designed to explore students' thinking about the concept. In it, students were asked whether certain graphs were the graphs of functions; whether a function with certain properties existed; and also to define a function. In the study, they found that, more than half of the students were able to give the definition of a function, however about half of these students did not use the definition when working on tasks but rather they relied on their concept images (formed through early experience) about the concept which led them to give incorrect answers. They argued that the

inconsistency between concept image and concept definition could cause trouble for developing students' correct understanding of mathematical concepts.

In addition, Vinner (1991) addressed students' thoughts of the concept of tangent. His students were taking a calculus course and were given definitions of tangent as a limit of secants or in terms of a line with a common point to the graph of function with the slope equal to the derivative at that point. However, he found that many students have the concept image of a tangent to a circle which was formed from their previous experience with the concept.

McGowen and Tall (2010) also addressed the role of early experience (met-before) on the learning of mathematics. They described that 'The term met-before applies to all current knowledge that arises through previous experience, both positive and negative.' They explained that previous experience could be supportive (in which the old ideas can make sense in the new concepts) and could be problematic (in which the old ideas cannot work in the new concepts). They pointed out that mathematicians should not only consider the positive old experience which is seen to be required for new learning but also should address the negative one that could hinder the learning process.

### **1.2.8 Studies on Students' Concept Images**

Much work has been concerned with exploring students' understandings of and difficulties with mathematical concepts and has shown that some difficulties could occur when students may depend only on their concept images in their reasoning about concepts and these concept images might be mathematically incomplete or incorrect and hence are at odds with formal definitions.

Nordlander and Nordlander (2011) tried to look beneath Swedish students' concept images of the concept of complex numbers. They designed a questionnaire to attain their aim, and the subjects for the study were chosen on purpose to be engineering students at tertiary level rather than mathematics majors. This choice was seen to be of benefit to determine the most common associations with the concept. Based on the answers provided by the students, this study explored some concept images which illustrated students' adoption of the notion of complex numbers; it also showed some misconceptions related to the concept; and moreover, discovered that students have difficulties with absorbing the basic properties of complex number.

As a contribution to the study of students' conception of mathematical concepts, Bingolbali and Monaghan (2008) investigated students' concept image of the derivative. They observed that many of the learning theories that developed using the construct of Tall and Vinner (1981), of the concept image and concept definition, were cognitive theories of learning (i.e. focusing on the individual mind and the cognitive aspects). They also argued that the construct could be also used to study the social theories of learning (social context). Bingolbali and Monaghan studied first year Mechanical Engineering and Mathematics students' thoughts of the derivative, in particular the rate of change and tangent notions. They used many methods to collect data: qualitative (questionnaires and interviews); quantitative (pre-; post-; and delayed post-test) and ethnographic (lesson observations and informal debates). The authors reported that students bring all manner of ideas to bear when working on mathematical tasks. From the results they showed that students' development of their concept image is affected by the teaching practices and by their departmental affiliations. They found that mathematics students' concept images formed in the direction of the tangent notion of derivative, whilst mechanical engineering students' concept images were formed in the direction of the rate of change notion of derivative. This seemed to be as a result of the differing emphasis placed on these interpretations in the different types of courses given to mathematics majors and engineers.

On the same lines, Maull and Berry (2000) examined how engineering students understand mathematical concepts, and found that engineering students gradually developed their concept images in ways that made engineering sense for them, in ways which differ from those of mathematics students.

A lot of interest has been given to abstract topics and other research projects have studied students' difficulties with such topics. For example, Maracci (2008) studied graduate and undergraduate students' understanding and possible difficulties concerning specific notions of vector space theory (especially the notion of linear combinations), and he used clinical interviews to observe 15 students. In order to get more insight into students' understanding and gain the most important elements for understanding students' difficulties, he adopted two different theoretical frameworks in data analysis: Fischbein's theory of tacit intuitive models and Sfard's process-object duality theory. Maracci explained that 'Tacit models constitute an implicit dimension of the individual's knowledge of which the individual her/himself is not aware. Such dimension influences all the processes of knowledge constructing and

developing such as the processes of problem solving and discovering' (Maracci 2008, p. 270); and also according to Sfard's theory 'the emergence of *structural conception* of a mathematical object on the one hand provides information, which may be more easily processed and manipulated by the individual than that provided by an operational conception, and on the other hand allows the reorganization of the operational knowledge itself' (Maracci 2008, P. 270). He explained that using more than one approach in data analysis can contribute to getting deeper understanding of students' problems with mathematical concepts, and he found that both theories which he used are useful in describing his students' difficulties. He described a possible intuitive model, where students used one approach to draw their general conclusion and those students seem to be not aware of the limited domain of such an approach. He also explained that students had trouble with reorganising linear combinations and seemed to conceive linear combinations as a process (via sum and scalar multiplication given vectors) instead of as objects (vectors obtained from linear combinations).

Wawro et al. (2011) examined how first-year undergraduate students understand the basic ideas of a topic in a linear algebra course (in this case the notion of subspace), and how they coordinate their concept images with the formal definitions. To achieve their intent the authors observed eight students, who were enrolled in a first year honours calculus course, using semi-structured interviews, and the questions were phrased in a way so as to prompt students to describe their own ways of thinking. In the analysis Wawro et al. used the idea of Tall and Vinner (1981) to distinguish students' responses. Using grounded theory analysis, they found that students possess a variety of concept images of the subspace notion in linear algebra; students interpreted the formal definition in terms consistent with their rich concept image; and they also identified some students' recognition for the need of the formal definition in their situation. In addition, they reported on how definitions, in the field of linear algebra, can be a useful tool in developing overall intuitions about mathematical concepts.

In line with the work mentioned, my study is meant to contribute towards an improvement in the learning and teaching of advanced mathematical concepts. It describes students' concept images of the concepts open sets and distance in metric spaces in relation to concept definition, and to how inconsistency between the two can cause difficulties for students to develop mathematically correct understanding of the

concept. We try to discover the ways students understand the fundamental ideas of the concepts and how those understandings are connected to the idea of the formal definitions.

### **1.2.9 Students' Consistency**

Much work has concentrated on investigating students' concept images in relation to concept definitions, and some of it has shown that inconsistency between concept images and formal concept definitions can cause trouble to students who are learning mathematical concepts (Tall & Vinner 1981, Vinner & Dreyfus 1989, and Rösken and Rolka 2007). Other studies have shown that students often interpret formal definitions in ways consistent with their concept images (Wawro et al. 2011).

Alcock and Simpson (2011) examined 187 first year mathematics and mathematics-related degree students' classification of real sequences, before and after students had been introduced to definitions of increasing and decreasing sequences during a real analysis course.

They explained that classification of objects in mathematics is logically simpler than classification in real life because only one criterion would be used in mathematics for the classification opposed to real life where many criteria might be used. Therefore, Alcock and Simpson (2011) considered whether students have an idea of “*concept consistency*”, that is ‘whether he or she understands that there should be a single mechanism for judging whether a mathematical object is a member of a set (whether or not that mechanism is a formal definition (P. 94))’.

To evaluate if students possess this concept consistency, they studied students' consistency in classifying mathematical objects. In both cases, that is before and after students were introduced to formal definitions, the result showed that a large number of students did not seem to employ a single mechanism when classifying at all, and so the result of their study suggested students' possible lack of what Alcock and Simpson called ‘concept consistency’.

### **1.2.10 Improving Learning**

As I mentioned, most of the studies on students' understanding of mathematical concepts have focused on how an inconsistency between concept definition and concept image can cause difficulty in students' development of concepts.

Research findings serve to understand the difficulty that students encounter in their learning processes in mathematics and they also serve to improve the teaching of mathematics. Fischbein (1987) stated that:

The basic didactical approach which, in our opinion, may help the student to cope with such situations is to make him aware of his own intuitive constraints and of the sources of the mental contradictions...’ (p.189)

Using this statement, in which teaching could help students to manage their intuitive ideas, some studies have supported the use of research results to improve the learning and teaching of mathematics. For example, Tsamir (2001) addressed the comparison of infinite sets as an example of the conflict between formal and intuitive interpretations and she used the results that came from some studies (e.g. Duval (1983) and Tirosh & Tsamir (1996)) to design the ‘*It’s the Same Task*’ (IST) activity and observe the effect of this activity on students’ ways of comparing infinite sets. The findings of both studies, Duval (1983) and Tirosh & Tsamir (1996), showed that many students gave different and contradictory answers to the same comparison of infinite sets task when the task was presented in different ways. Therefore, and based on Fischbein’s (1987) argument above, Tsamir (2001) found that these findings could be used to promote students’ awareness of their intuitive responses. The IST research-based activity applied the cognitive conflict teaching approach as a didactic tool. It consisted of three phrases in which students had to think of different representations that are known to trigger inconsistent answers to the same comparison of infinite sets task. The activity was conducted in individual interviews with fifteen secondary school students who were identified by their teachers as talented in mathematics. The activity prompted students to recognise the issue of making contradictory responses to the same task and to avoid this issue by using the unique formal criterion, one-to-one correspondence, in comparing infinite sets.

Furthermore, when Sierpiska (1987) examined some aspects related to the notion of the limit of sequence in high school students, she found that students have strong intuitive attitudes concerning mathematical knowledge. To cope with this obstacle, she argued that:

The student will have to rise above his convictions, to analyse from outside the means he had used to solve problems in order to formulate the hypotheses he

had admitted so far, and become aware of the possible rival hypotheses.  
(Sierpiska 1987, p.374)

In mathematics, descriptions, such as graphic, numeric and algebraic representations, are commonly used to explain a concept. These representations individually might have limitations that do not reflect all aspects of formal definitions.

Giraldo et al. (2003) discusses the difference between a formal definition of a concept and a description which refers to some properties of a concept. They argued about the positive use of cognitive conflict or in their terminology a theoretical-computational conflict (that is any pedagogical situation with an apparent contradiction between the mathematical theory and a computational representation of a given concept. p. 445), when teaching some concepts (e.g. derivative and limit) to enrich students' concept images. They pointed out that a suitable use of an example that shows the limitations of any description of a concept in comparison with the formal theory could have positive role in improving students' concept images.

### **1.2.11 History of the Concept of Metric Space**

Kreyszig (1997) asserts that general topology has its roots in classical analysis. At the end of the 19<sup>th</sup> and start of the 20<sup>th</sup> centuries, mathematicians like Poincaré, Lebesgue and Hadamard began to generalise concepts from real and complex analysis. In 1906 the French mathematician M. Fréchet made an important contribution to topology when he introduced the notion of a metric space in his PhD thesis titled *Sur quelques points du calcul fonctionnel* which was written under the direction of Hadamard. He also carried out major work on general topology and functional analysis. In his thesis, he developed the theory of an abstract set (which is defined by its elements, axioms, and relationships), so that the notions of limits and continuity can be studied in other mathematical situations (other than the real and complex ones). In particular, by the idea of axiomatic abstract space, Fréchet introduced the concept of a metric space, however the name metric space was later created by Hausdorff. Kreyszig in the Handbook of the History of General Topology (1997) reported on the interaction between topology and functional analysis and discussed the creation of the idea of metric space, and he documented that:

As most important concept he [Fréchet] introduced *metric space*, which he called *class (E) of elements* (suggested by écart, Jordan's term for distance,

which he adopted). He denoted the distance from  $A$  to  $B$  by  $(A, B)$  and chose the very same axioms that we use, writing them in the form

$$(A, B) \geq 0$$

$(A, B)$  n'est nul que si  $A$  et  $B$  sont identiques.

$$(A, B) \leq (A, C) + (C, B)$$

$$(A, B) = (B, A). \text{ (Kreyszig 1997, p. 369)}$$

This useful idea of a metric space was an important step in the development of topology. The first definition of a topological space was given by F. Riesz in 1906, he concentrated on generalising the continuum and called his spaces 'mathematical continuums' as opposed to physical continuums (Kreyszig 1997). His axioms dealt with generalising the notion of accumulation points. Later, Hausdorff formulated the now familiar definition of a topology using neighbourhoods.

### **1.3 Outline of the Thesis**

This thesis consists of six chapters, of which this chapter is the introduction and contains the literature review.

Chapter 2 outlines the methodology used in my study. It explains how qualitative research is a useful tool with which to address my research questions. The chapter describes how the questions were designed for the questionnaire and the interviews, and how the students were chosen for the study. It also explains how the study was conducted and how the analysis was carried out.

Chapter 3 addresses the first two of the research questions. It discusses the analysis of responses to questions designed to reveal students' appreciation of the role of definitions in mathematics. It also contains the analysis of answers to questions designed to reveal the most dominant conceptions and definitions that students have about the concepts of open sets and distance in a metric space.

Chapter 4 concerns the third of the research questions stated above. It presents the analysis of students' answers to the task questions given in the questionnaire and in the interviews as well. There were six problems about the concept of an open set and all will be analysed in this chapter. The chapter explains the ways students approached the problems and gives more insight into students' thoughts concerning open sets and

the relevant concepts. The analysis allows me to further explore the concept images of the students.

Chapter 5 concerns the outcomes of Chapters 3 and 4, and addresses the fourth of the research questions. It outlines the categories of students' concept images of open sets, the categories of students' concept images of the distance in a metric space, the categories of misconceptions related to open sets, and it also addresses the influences on students' concept images.

Finally, Chapter 6 represents the discussion and the conclusion deduced from the other chapters. Using the results of the other chapters, Chapter 6 discusses the findings related to my research questions, and relates them to the mathematics education literature. It also discusses the implications of my findings and possible further research.

# Chapter 2

## Methodology

This study was carried out at the Department of Mathematics and Statistics at NUI Maynooth. The subjects in the study were students who came from a third year module on metric spaces either in Arts (three year degree) or in Science (four year degree). Most of the students were in the penultimate year of a mathematics degree program. Some students were taking a Higher Diploma in Mathematics. These students had a degree in a numerate subject and the Higher Diploma course would bring them to honours degree standard in mathematics. Most of the students had encountered open sets before in previous courses, but usually only in the case of the real line. In the metric spaces course the students had been introduced to the formal definition of an open set and had dealt with theorems and lemmas using that formal definition.

### 2.1 Qualitative Research

In general, qualitative research is one of the methods that are used by educational researchers to analyse data, these methods are described in the Research Methods in Education Hand Book by The Open University (2001). This method services to investigate deeply the different ways and patterns in which subjects experience, make sense of, and think about concepts. Qualitative research is a useful method due to its main features which are stated in the Research Methods in Education Hand Book by The Open University (2001) as:

- A focus on natural settings.
- An interest in meanings, perspectives and understandings.
- An emphasis on process.
- Inductive analysis and grounded theory. (p.49)

Therefore, qualitative research is appropriate for the research questions that I am studying because it examines key issues in real situations, so that a subject's behaviour and attitudes can be observed as it naturally occurs. Also in qualitative

research, individuals are able to use their own words when giving information. It helps to understand the meaning associated with subjects' attitudes, and also to interpret their understanding. Moreover, it is concerned with exploring in depth how the understanding develops and how individuals make sense of their experience. Also, the data collected using qualitative research is useful for exploratory research as the analysis can be used to produce a theory that was not set in advance.

Typically, some of the most common methods used to collect data from individuals, in qualitative research, are questionnaires and interviews or task-based interviews.

A questionnaire as research instrument is defined to be a series of written questions for the purpose of gaining information from the written answers provided by the respondents (McKnight et al, 2000), and is considered to be economical with time. This approach of acquiring information from individuals has some advantages and disadvantages associated with it, and the Research Methods in Education Hand Book by The Open University (2001) outlines the main ones. The main advantages are:

- Questionnaires do not take much time to administer, so are useful for a large sample.
- Everyone is asked the same questions.
- Can be designed so that analysis is relatively simple. (p.185)

The main disadvantages are:

- Responses rate may be low and you could get a biased sample.
- Danger of differing interpretations of the same questions – respondents cannot ask for explanations.
- People's preferred responses may not be allowed for in your questionnaire. (p.185)

In addition to the advantages of questionnaires listed above, mathematics education research questionnaires can also be conducted during courses and during class times, and they are useful as they can be employed to seek information involving background experience and perceptions about specific objectives.

Interviews are also one of the research instruments used in qualitative research. They usually consist of a series of questions asked to obtain information from the respondents, and in this instrument the questions and the responses are more verbal than written. This instrument also has advantages and disadvantages connected with it,

and the main ones are also listed in the Research Methods in Education Hand Book by The Open University (2001). The main advantages are:

- Does not run the risk (as with questionnaires) of low response rate.
- Allow you to probe particular issues in depth.
- Likely to generate a lot of information. (p.184)

And the main disadvantages are:

- Takes time to administer.
- Respondents will be affected by their perceptions of you and your research, and what responses they feel are appropriate.
- Takes time to write up and analyse. (p.184)

Interviews as a method of collecting data are useful to examine, in depth, subjects' thoughts on a situation.

The task-based interview is one of the possible types of interviews. They are used to observe closely and naturally how subjects interact with and think about problems, as described by Goldin (1997). That is, subjects are given a problem to solve during the interview, they have to explain out loud their work and thinking, and the interviewer can ask probing questions in order to understand the process of the subjects' thoughts (McKnight et al, 2000).

Goldin (1997) outlined the most important principles of the design of task-based interview which are:

- 1- *Accessibility*. Interview tasks should embody mathematical ideas and structures appropriate for the subjects being interviewed. Subjects must be able to represent task configurations, conditions, and goals internally and, where appropriate, externally.
- 2- *Rich representational structure*. Mathematical tasks should embody meaningful semantic structures capable of being represented imagistically, formal symbolic structures capable of notational representation, and opportunities to connect these. Tasks should also suggest or entail strategies of some complexity and involve planning and executive-control-level representation. Opportunities should be included for self-reflection and retrospection.

- 3- *Free problem solving.* Subjects should engage in free problem solving wherever possible to allow an observation of spontaneous behaviors and reasons for spontaneous choice. Providing premature guidance results in a loss of information. This principle may mean some sacrifice of the speed with which the subject understands the problem or progresses through it.
- 4- *Explicit criteria.* Major contingencies should be addressed in the interview design as explicitly and clearly as possible. These contingencies should distinguish “correct” and “incorrect” responses (but rarely) with structured questions designed to give subjects opportunities to self-correct in any contingency. This is an important key to the replicability and generalizability of task-based interview methodology.
- 5- *Interaction with the learning environment.* Various external representational capabilities should be provided, which permits interaction with a rich, observable learning or problem-solving environment and allow inferences about problem solvers’ internal representations. (Goldin 1997, p. 61-62)

To make our study more beneficial, data was collected using both questionnaires and interviews. The structure of the scripts for the questionnaires and the interviews were made based on Goldin’s (1997) principles, that is the questionnaire’s script and the interviews were designed to investigate a certain mathematical idea (in this case open sets and distance) which had been experienced by all of our students. The questions were designed so that students would be expected to be able to give responses. For the purpose of studying the concept of open set in depth, questions were written clearly and formally. They were varied in nature, so that the questionnaire and each interview consisted of a sequence of questions. There were questions about students’ definitions and images of the concept of open set, to which all students were expected to respond. The students were then asked to solve a sequence of mathematical problems which increased in difficulty, but all of which were reasonable considering the background of the students. In order to explore students’ strategies, they were not forced to give specific answers and they were free to use their own words and ways to answer the questions. All students’ responses were accepted, that is correct and incorrect answers were treated equally. In the questionnaire’s script there were blank spaces after each question for students to write down their answers.

In the interviews, students were sometimes asked unprepared questions which emerged from their responses in order to get more deeply into their thinking. From time to time they also were reminded to say out loud what they thought and to explain what they were doing. Students also were given a pen and pages to use while working.

The interviews which I administered were audio recorded using a small voice recorder. Making audio recordings of the interviews' conversations is a very useful method in order to be able to make transcriptions for data collection, and there are some practical points to be considered when doing so (see Research Methods in Education Handbook by The Open University 2001, p. 252 for an overview).

## **2.2 Design of Questions**

As I mentioned before, the design of the questions we used in the questionnaire and the interviews was based on Goldin's (1997) principles. In addition, in the early days of my study we conducted a pilot study. We administered and analysed this pilot study before we set the research questions of our study. In the following sections I will describe briefly the work of the pilot study, and then I will describe separately and in detail the design of the questionnaire and the interviews.

### **2.2.1 Pilot Study**

In autumn 2009 we investigated students' understanding of the infinity concept. We designed a questionnaire to get insight into students' concept images concerning the concept and it was given to three different groups of students at NUI Maynooth. Thirty five students volunteered to take part in that study. The questionnaire was conducted during the students' classes and completed by them in 20 minutes. The method which we used in analysing our data was qualitative and the study showed some of concept images that students hold about the cardinality of infinite sets (for more detail about the pilot study see Hamza and O'Shea (2011)).

From this pilot study, I gained experience in designing the types of questions best suited to gaining insight into students' concept definition and concept image. The experience of analysing the data was also beneficial.

### 2.2.2 Questionnaire

Because of the pilot study mentioned in the previous section, we had experience of designing a questionnaire for the main study in this thesis. As a first step we designed the script of the questionnaire. Based on my research questions I started by devising many questions related to the topics of open set and distance in a metric space. We discussed the questions very carefully and we chose some of them to be included in the questionnaire. In all, six task problems were designed to be used in the study. Four of them were used in the questionnaire and some of these plus the other two were used in the interviews. The responses to the problems will be analysed in Chapter 4. In order to avoid confusion, I will use the numbering given there to label the task problems here rather than the numbering given in the questionnaire. The questionnaire can be found in Appendix 1. In the questionnaire we tried to avoid terms that were unfamiliar to the students, and to write the questions in a simple way in order for them to be understandable. Please note that the language, terminology and notation used in these questions was identical to that used by the lecturer in the metric spaces course.

A pilot test was conducted for the questionnaire with two students who were not part of the class. These students had met the topics in question in their previous studies. The pilot was conducted to see how long the questionnaire would take and to iron out any problems or possible misinterpretations of our questions. Eventually the questionnaire's script consisted of six questions which tested topics concerning distance, open balls and open sets. The questions which were chosen for the questionnaire are the following:

1. (i) Define the term: Open set in a metric space.  
(ii) How would you explain the idea of an open set in a metric space to a friend of yours?
2. (i) Define the term: distance in a metric space.  
(ii) How would you explain the idea of distance in a metric space to a friend of yours?

Problem 1. Consider the metric space  $(\mathbf{R}, d)$  where  $d$  is the standard metric, and let  $A = [0, 2)$ . Is the set  $A$ :

Open                      Closed                      Both                      Neither

Please explain your answer!



according to the most important fundamental principles outlined by Goldin (1997). That is, each interview was based on the content of the mathematical topics of open sets and distance in a metric space and consisted of a series of questions.

All the questions which I asked in the interviews were designed at the same time as the questionnaire was written. Some of the questions that we asked in interviews were used in the questionnaire to observe how students answer these questions; and other questions were designed specially for the interviews for the purpose of further exploring students' thinking about topics that are essential for our concepts. Some of the questions that I asked were not designed in advance but emerged from students' answers to other questions. I asked these questions to look deeper into students' approaches. At the beginning of the interviews students were asked general questions at a level which all were expected to answer, and then the questions became more difficult. The later questions were designed to challenge students.

Each interview consisted of three stages and each stage included some questions relating to a certain aim. In the first stage the students were asked general questions about definitions in mathematics and about how they dealt with them. The second stage consisted of questions revealing students' ideas about the concepts of open set and distance in a metric space. Then the third stage contained tasks to show if students were able to use their conceptions in solving problems.

There was no pilot test for the interviews, but when we finished the 1<sup>st</sup> and 2<sup>nd</sup> interviews we listened to the recordings carefully before I continued to do the other interviews. We did this to see if there were any problems. No problem with the format of the interview emerged.

The general questions that were asked at the beginning of the interviews included the following:

1. Do you like mathematical definitions?
2. What do you think is the point of definitions in mathematics?
3. When you are presented with a new definition in mathematics, what do you do?
4. Do you memorise definitions?
5. Do you try to relate the definition to material you know already?
6. Do you try to understand the reason the definition was made?
7. Do you use pictures in your mind to understand definitions?

8. Do you refer to definitions when you read or are working on problems?
9. When you try to understand a definition, do you focus on every single word? Or do you think some parts are more important than others?
10. If you are asked to state a definition, do you state it as in your notes or as you understand it?
11. Is it easy for you to understand mathematical definitions?
12. Do you find definitions in maths are different from definitions in other subjects?

These questions are not particularly concerned with the concept of open set in metric spaces, but they were very important for us to get information about students' experiences with definitions in mathematics. These questions also could be very helpful for us to understand later how students regard definitions that are related to the open set concept.

In the second part of the interviews, students were reminded that the given questions were not a test and they were encouraged to talk freely and were made to feel comfortable. At this stage I had to make sure that there was a pen and a piece of paper in front of each student in case they might need them. I began this part by asking the following open-ended questions which were also used in the questionnaire for the purpose of probing students' concept definitions and images related to the open set and distance topics:

1. (i) Define the term *open set* in metric space.  
(ii) How would you explain the idea of an open set in a metric space to a friend of yours?
2. (i) Define the term *distance* in a metric space.  
(ii) How would you explain the idea of distance in a metric space to a friend of yours?

At this stage, students also had to state whether they agreed or disagreed with some statements about open sets and to justify their choice. On occasion students were reminded to say out loud what they thought. The statements are:

- (a) A set is open if all its points are centre of open balls.
- (b) A set is open if it lacks its ends.
- (c) If a set is not open then it is closed.

- (d) A set is open if all its points are near to each other.
- (e) A set is open if all its points are similar.
- (f) A set is open if its complement is closed.

Students were given these statements to explore further their conceptions of the concept of open set, so that we could identify some of the features of open sets which are connected to students' concept images. The statements were not given in formal mathematical language. This was done in order to see how students would react to these ideas. This is similar to the methods used by Zaslavsky and Shir (2005). The format 'A set is x if y is true' is the way definitions are given in some textbooks (see for example Simmons (1963), Munkres (1975) and Roseman (1999)). Statement (a) was given to explore if students have a conception based on the formal definition and whether they are aware of the importance of open balls within open sets. Statement (b) was asked to examine if the early experience of openness (with  $\mathbf{R}$  or open intervals in  $\mathbf{R}$ ) would affect students' concept image. Statement (c) was asked to see if the intuitive use of the words open and closed in real life would influence students' thinking of open sets in a metric space. Statement (d) was given to students to explore whether the nearness of points within a set and also the lecturer's explanation of metrics as measures of similarity have an effect on students' concept image. We asked Statement (f) to find out if students have thought of an open set using the definition of closed sets.

The final stage of the interview was designed to challenge the students. Each student was given two or three different problems to solve. Each problem was written on a sheet and the sheets were given to the students to comment on and to answer them and the students were asked to explain what they did. The interview questions can be found in Appendix 2. I will use here the numbering given in Chapter 4 to list the task problems used in the interviews. The problems which were given in the interviews were the following:

Problem 2. Consider the metric space  $(\mathbf{Z}, d_{\mathbf{Z}})$  where  $d_{\mathbf{Z}}$  is the standard metric inherited from  $\mathbf{R}$ , and let  $B = \{m-1, m, m+1\}$ . Is B an open ball?

Yes

No

- If yes, please specify the centre and the radius of the ball.
- If no, please explain your answer.
- Can you find an open ball C which is a subset of B?

Yes No

Explain your answer!

Problem 3. Let  $Y = [0, 2]$  and consider the metric space  $(Y, d_Y)$  where  $d_Y$  is the standard metric on  $Y$  inherited from  $\mathbf{R}$ . Let  $A = [0, 2)$ . Is the subset  $A$ :

Open Closed Both Neither

Explain!

Problem 4. Let  $(a, b)$  be an interval in  $\mathbf{R}$  and  $S = (a, b) \times \{0\}$ , and let  $d$  be the standard metric on  $\mathbf{R}^2$ . As a subset of  $(\mathbf{R}^2, d)$  is  $S$ :

Open Closed Both Neither

Please explain your answer!

Problem 5. Let  $X$  be the set of all real sequences. Define:

$$d(\{a_k\}, \{b_k\}) = \begin{cases} 0 & \text{if } a_k = b_k \text{ for any } k \in N \\ 1/k & \text{if } k = \min_{n \in N} \{n : a_n \neq b_n\} \end{cases}$$

-Can you describe this metric in words? Or

- What do you think this metric measures?

-Let  $\{0\} = \{0, 0, 0, \dots\}$ , if  $d(\{a_n\}, \{0\}) = 1$  what can you say about  $\{a_n\}$ ?

-What is  $B(\{0\}, 1)$ ? Or -What is  $B(\{0\}, 1/2)$ ?

-Is the set of sequences  $\{\{a_n\} : a_1 = 0 \text{ or } 1\}$  open? Or

-Is the set of sequences  $\{\{a_n\} : a_1 = 0\}$  open?

Problems 2 and 3 from this section of the interview were also given in the questionnaire. The reason for giving these problems was explained in the previous section, and in addition they were given in the interview to observe closely and find out how students approached these problems.

Problems 4 and 5 were designed specially for the interviews. Problem 4 was devised to reveal if students were aware of all aspects of the definition of an open set and also to reveal the intuitive ways they used to approach the problem too. Problem 5 was the most challenging one used in the interviews. It investigates students' ability to think of a metric space that is not familiar, and one for which it is difficult to visualise its elements. It also examines how students use the given definition to solve the given tasks. The metric space in this problem consists of sequences, and thus the elements of

an open set in this setting were difficult to visualise but it still can be approached using the formal definitions.

## 2.3 Administration of the Study

In this section I will describe how the subjects (students) were chosen, how the questionnaire was administered, and how the interview was structured.

The metric spaces course took place in Autumn 2010. The course ran for twelve weeks and there were two lectures each week. The syllabus for this course is: Metric spaces: definitions and examples, convergence and continuity in metric spaces; uniform continuity; pointwise and uniform convergence, open and closed sets; basic properties; continuity in terms of open sets; limit points; closure; interior and boundary, completeness and compactness.

After eight weeks, the lecturer asked the whole class to volunteer for the study and to fill in a questionnaire. The students were informed that the study was optional, that is filling in the questionnaire was not compulsory, they were also informed that the survey was about topics in Topology and that the study was anonymous.

There were sixteen students in the class that day and everybody who was in the class agreed to fill in the questionnaire. The questionnaire was given to the students in their class under supervision of their lecturer. Students were only asked to indicate their groups (e.g. 2<sup>nd</sup> Arts, Higher Diploma, etc.) and not their names, and the survey was completed by them in 20 minutes at the end of one of their lecture hours.

A couple of days after completing the questionnaire, all students in the class who were doing the metric spaces course were also asked to participate in the interview process. They were also informed that participation was not compulsory and that the study was anonymous. A sheet of paper was passed around the class by the lecturer and any student who was happy to do an interview had to write down his/her name and contact details (phone number or e-mail address). Eleven students agreed to volunteer for the interviews and were contacted via the contact details which they had given. All the students who volunteered for the interviews took part in the questionnaire except for one student who was not in the class on the day of conducting the questionnaire.

The interviews were carried out individually according to the students' convenience. Ten interviews were conducted in the last two weeks of the course in Autumn 2010 and one interview was administered early in Spring 2011. Students were asked if they would agree to audio recording during the interview. All students were happy to record the interviews and each interview lasted about 30- 40 minutes, one student asked to listen to the audio recording before he left. To facilitate analysis, the interviews were audio taped as we explained and also were transcribed fully by me, and inscriptions which were made by students while working on the tasks were also collected.

After the student had agreed to the audio recording, I began to introduce myself and explain my research study. Then for a few minutes I chatted with the student about his/her study to put the student at ease and let the conversation start between me and the student. I then asked the questions detailed in section 2.2.3

## 2.4 Analysis

As I said before, in our study a questionnaire and task-based interviews were designed to discover students' reasoning concerning the open sets and distance concepts. The analysis which I present is based on students' answers to the questions which I asked in the questionnaire and the interviews.

The reliability of the data that are used for qualitative research can be enhanced when more than one researcher is involved in the analysis. McKnight et al (2000) stated that:

*'It is desirable to have colleagues, perhaps a team of researchers, involved in data collections, analysis, and conclusion-drawing. Input from other researchers can be an invaluable tool for double-checking that the results obtained from the research are in fact credible, reasonable, and supported by data not just wishful thinking on the part of the research.'* (p. 71)

To enhance the reliability of my study, my supervisor and I went through the process of the data analysis separately. We discussed our findings in regular meetings over the course of two years. We both agreed with the analysis that I will present here.

At the beginning of the data analysis I assigned a reference to each script of the questionnaire (e.g. A, B, C etc) and a reference to each student in the interview (e.g. 1<sup>st</sup>, 2<sup>nd</sup>, etc) as students were informed that the study is anonymous and the data would be treated confidentially. We began by analysing the interviews first and thereafter we analysed the questionnaire's scripts.

### **2.4.1 Transcription**

In analysing the interviews, I started by transcribing the audiotape of each interview. Each transcript begins with the first question related to the study. I tried to transcribe exactly the words as I heard them including even the discourse markers (e.g. um, you know, etc.) and I listened to the tapes many times. Any word or sentence that I could not understand I put it in brackets in order to check later. Because English is not my first language, I asked my colleagues who are English speakers to listen to the audiotapes to check if I made mistakes and also to recognise the missing words. My supervisor also had to listen to the audiotapes for further revision to the transcripts.

After we both approved the transcription of the data collected, we read the transcripts as the next step of the analysis process. We read each transcript many times to study them deeply and to become familiar with students' responses.

### **2.4.2 Coding**

The analysis carried out here followed the principles of Grounded Theory. In this type of analysis researchers usually do not begin with a theory for which they try to produce evidence. Instead, they usually generate a theory from the data (Strauss and Corbin, 1990). The theory emerges from a rigorous process of coding.

Miles and Huberman (1994) define the coding process as 'To view a set of field notes, transcribed or synthesised, and to dissect them meaningfully, while keeping the relation between the parts intact...'. (p. 56)

The first step in this coding process is for the researchers to familiarise themselves with the transcripts. My advisor and I did this by reading through each transcript many times. We marked any interesting points and kept track of any themes as they started to emerge. We discussed each student's transcript in detail, before moving on to the next phase of coding. In this phase, we analysed the transcripts question by question. We did this in order to focus on the issues raised by each task.

At first, we looked at the questions that aimed to discover students' definitions and ideas about open sets and distance in a metric space. We coded and kept notes of the ideas and features that emerged from each student's answers, and then we gathered all the ideas which arose from all the students' answers and we coded them and then we classified them in categories.

Secondly, we looked at the solutions to the tasks that were given to the students and started to sort correct and incorrect answers given by each student. Thereafter we tried to code and recognise all the ideas and strategies that students used to answer the problems and to see if their definitions and ideas did or did not help them in their responses. We also wanted to see if any other new ideas about the concept might emerge. In order to investigate the relationship between students' concept definitions and concept image we made tables or matrices (McKnight et al. 2000, p.81) to visually represent the data.

After that, we looked at students' answers to the questions about definitions in mathematics generally in order to see if the students were comfortable with them and to study how they treated them. This data was coded using the same process described above (McKnight et al. 2000)

As a final stage in analysing the interviews, we went through the procedure of the analysis many times, and we gathered all the information that we coded and all the classifications that we sorted, and we kept the results we obtained. We discussed each student in detail over a long period of time in order to be certain of the data collected.

After we finished analysing the interviews, we began to analyse the questionnaire scripts. We started by reading each script returned by students. In analysing the questionnaires we tried to follow the procedure that we used in analysing the interviews. So we commenced this stage by analysing the first two questions in the scripts so as to identify and code the features students might have in their conceptions of open sets and distance in metric spaces, and we classified them into categories. Then we looked at the answers to the given problems and we collected correct, incorrect answers and unattempted questions that are left blank in groups. Afterwards, we investigated each one; so we looked at the group of incorrect answers in an attempt to figure out the conceptions which did not help students to answer the problems. Then I looked at the group of correct answers to see which ideas were helpful for the students and to see also if there are other new ideas which might be revealed. I did not

include the questions which are left blank within the correct or incorrect group; because it might be that the student found the question hard to answer or had no time to answer that question. We repeated this process of analysing the questionnaires many times until we were satisfied.

We discussed each student in the interviews and the questionnaire in detail over a long period of time and we compared the categories created, and we revised the findings attained many times to enhance the solidity of the study.

I will present the result of the analysis in the following three chapters.

## Chapter 3

# Students' Conceptions of Definitions

### 3.1 Introduction

Definitions play a fundamental role in mathematics (Vinner 1991; Edward and Ward 2004). They prescribe the meaning of mathematical concepts in a very specific manner. Many students are not aware of this role and do not differentiate between informal statements that describe a mathematical concept and a formal definition that prescribe what is (and is not) an example of that concept (Alcock and Simpson 2009). Students often do not pay close attention to the formal definitions and do not use them in their mathematical thinking, instead they rely on the image they have in their mind (Przenioslo 2004).

In this chapter I will firstly look at students' conceptions of mathematical definitions in general, and then I will address how students understand the definitions of the topics of open sets and distance in a metric space.

### 3.2 Students' Opinions about Mathematical Definitions

To get information about students' points of view on definitions, I asked the students some general questions about definitions in mathematics in the interviews. The given questions were not mathematical questions, but we designed them in order to observe students' attitudes to mathematical definitions and also to find out if there is consistency between their responses to these general questions and their work on the given tasks. The following are the general questions about definitions that I asked and some information about students' answers to these questions.

#### ***3.2.1 Do you like mathematical definitions?***

This question was asked in order to get a general impression of students' attitudes to definitions. All but the first three students were asked this question. Most students

mentioned that they like mathematics in general but some (4<sup>th</sup>, 6<sup>th</sup> and 8<sup>th</sup>) said they were not happy with definitions. One student said that:

*Definitions, I suppose I'm not great at, sorry, I'm not great at learning things off, but I love maths. (4<sup>th</sup>)*

Another one commented that:

*I like maths, most of its part. Definitions in stuff, no; I think there is a lot to learn in someone of the subject that involved lots of definitions. When ever you don't have to use words, I think it's kinda almost better, you know. (8<sup>th</sup>)*

These three students mentioned memorising definitions when answering this question. Another student seemed to like getting an intuitive idea about a mathematical concept instead of working with the given definitions. This student argued that:

*Um, not particularly no! I don't know, I'd like kind of intuitive ideas and getting things like that, but beyond that I don't really like putting in the work for kind of methodical things, if you know what I mean, I like the new ideas when they come. (7<sup>th</sup>)*

Three students (5<sup>th</sup>, 10<sup>th</sup> and 11<sup>th</sup>) spoke about trying to understand or absorb the meaning of a definition. One of them said:

*Yeah, when I have time to, you know, digest them and absorb them, because definitions need to be absorbed properly. (10<sup>th</sup>)*

The 9<sup>th</sup> student had a positive impression about definitions and did not complain about them. She seemed to know however that many students are not happy with definitions in mathematics. She said that:

*I love them, yeah; I'm one of the few!*

We conclude that many of these students like mathematics generally but some are not happy with its definitions. One reason given for this was that some students do not like memorisation, others said that they do not like the wordy parts of definitions and prefer the intuitive ideas. Other students did not have a problem with definitions and spoke about trying to understand the content of definitions.

### **3.2.2 What do you think is the point of definitions in mathematics?**

This question was given to all students. It was given to see if students are aware of the role that definitions play in mathematics. Most of the students apparently knew that definitions are very important in mathematics. Two of them (5<sup>th</sup> and 10<sup>th</sup>) spoke of

their importance when introducing a new topic. For example, the 5<sup>th</sup> student answered that:

*If you're doing a whole new concept, the definition is going to tell you what is going on, what it is.*

The majority of students (2<sup>nd</sup>, 6<sup>th</sup>, 7<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) mentioned the fact that a definition provides an accurate unambiguous description of a concept. One of them commented that:

*There is no ambiguity over what's being talked about, they're the sort of the bit that's set in stone. (11<sup>th</sup>)*

And another said:

*It is there to say what the stuff is, if you don't know what you're using; it'd be kind of pointless, like nailing jelly to the ceiling. (6<sup>th</sup>)*

Many of these students spoke of the importance of having a solid base given by definitions in order to do mathematics. The 4<sup>th</sup> student echoed this:

*They're everything; you wouldn't be able to do any thing with maths, if you don't understand the definitions.*

Also, another of them said:

*Just it is kind of very clear base of what something is, like you have to be very accurate, especially maths very accurate opinion base of what something is; you can refer back to it when you are doing further things. (2<sup>nd</sup>)*

The 7<sup>th</sup> student explained how his understanding of the importance of definitions improved over time. He said:

*At the start I thought well, they're very kind of pointless; they don't see much of it. But then as you go on, you see much is based around the exact definition and if you have a bit missing, then the whole concept can be wrong.*

So we can see that these students seem to be aware of the role of definitions in mathematics as they explained that definitions tell us exactly what a concept is in mathematics. They also knew that definitions are the base for doing further mathematics.

One student (1<sup>st</sup>) thought that definitions must be known in order to solve problems as he mentioned in his interview. He said:

*I suppose, you need to know definition to answer problems.*

The 3<sup>rd</sup> student seemed to confuse definitions with facts or theorems. He claimed that:

*That is the fact you know, that is somebody prove it already or somebody knows these things so, yeah very important yeah. (3<sup>rd</sup>)*

This student considered that definitions are just facts that have been proven previously by other people.

In general, these students seem to appreciate the important role that definitions play in mathematics. They seem to see them as the fundamental building blocks of the subject.

### **3.2.3 When you are presented with a new definition in mathematics, what do you do?**

All students were asked this question. This question was asked to explore students' reactions to new mathematical definitions. In the analysis of this question we found that the students gave different answers to explain their reactions. Some of these students are mentioned here more than once as they indicated that they would handle a new definition in more than one way.

Some students would understand intuitively what the definition means (7<sup>th</sup>, 9<sup>th</sup>, 10<sup>th</sup> and 11<sup>th</sup>). As an example of that, one of these students (11<sup>th</sup>) argued:

*my first reaction is usually to try and see if I can get intuitively what it means, um, and try and, rather than going through word by word very carefully, and try and, just try absorbing what it is very carefully.*

Another student explained that he usually focuses on the important terms in a definition. This student said that:

*Well, I take it down obviously yeah, and uh reading through it just highlights the key things really, key steps like on it. Look for the logical steps in it yeah. (2<sup>nd</sup>)*

Some other students (1<sup>st</sup> and 9<sup>th</sup>) like to use examples to understand definitions; one of them (9<sup>th</sup>) mentioned that:

*I read through it and translate any other, any other notations into English in my head and um, then try to understand it and try to think of examples, I suppose.*

Also, the 9<sup>th</sup> student and other students (4<sup>th</sup> and 5<sup>th</sup>) prefer to put a new definition into their own words and work on it in their own way. The 4<sup>th</sup> student indicated that:

*I suppose I just try in take it out and um... think about it and then, like say if it's in complicated maths terms, you just try in put it into your own sort of things.*

And the 5<sup>th</sup> student also commented that:

*So sometimes I'll just have to go maybe and write it out again just to see exactly what's going on, and see if I understand every single thing is going on in it and then it's ok, and I'd have to try to work it my own way as well.*

In addition to other ways of dealing with new definitions, a few students (3<sup>rd</sup>, 6<sup>th</sup>, 8<sup>th</sup> and 10<sup>th</sup>) said that they like to memorise them. The 6<sup>th</sup> student in this group mentioned that:

*I go back later at the general, the two or three weeks we have to the exam, and I write out all the definitions and proofs of some theorems, and then I learn them off using memorisation techniques.*

The 10<sup>th</sup> student also explained that:

*well, try to understand them first and then if they if I need to learn them, or I mean uh, yeah, learn them then and try also to apply them. So understand them, learn them and then apply them.*

This student (10<sup>th</sup>) speaks about applying the definitions, as did the 8<sup>th</sup> student. That student felt that the best way to learn a definition was to use it. She answered that:

*I would try to learn it, but it is easier to get it into your head whenever you do the homework. Like you can sit down and learn it, but it won't go into your head unless you do homework, you are forced to use it in examples.*

A different student (7<sup>th</sup>) mentioned that he did not try to memorise definitions and said:

*I don't learn anything off, and let's say I understand it. It's the very first thing I have to do is to understand something.*

One student (1<sup>st</sup>) liked to use visualisation to understand definitions, this student said:

*I suppose, I try to visualise the meaning of the definition like what it looks like.*

Moreover, some students (2<sup>nd</sup>, 5<sup>th</sup> and 6<sup>th</sup>) also would read a definition or write it out again as a way of dealing with it. For example, the 6<sup>th</sup> student claimed that:

*Sometimes I read over it, depends on how much time I have during the week, but I normally rewrite out my notes.*

In summary, we have seen students have different strategies in relation to new definitions. Some try to understand the underlying concept in some way by getting an intuitive feel for it; by visualisation; by using it, or by stating it in their own words.

Some students said that they memorise the definitions. This question will be explored further in the next section.

### **3.2.4 Do you memorise definitions?**

This question was given to all students except the 1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup> and 7<sup>th</sup> students. Some students in the previous sections mentioned memorisation when speaking about definitions. This question was specially given to find out if students would use the technique of memorisation with mathematical definitions. All students who were asked this question agreed that they do memorise definitions but in different situations. Some students (4<sup>th</sup> and 8<sup>th</sup>) would sometimes memorise definitions if they are long or difficult to understand. One of these students (4<sup>th</sup>) explained that:

*I suppose if it's complicated I would um, not generally.*

And the 8<sup>th</sup> student said she would memorise definitions that were long or very 'wordy'.

Some others (5<sup>th</sup> and 11<sup>th</sup>) memorise them only for the exams because definitions are usually asked there. The 11<sup>th</sup> student argued that:

*only for um, if it's an exam and definitions will be asked, I'll memorise them, but in general for learning maths, I would try, I would Rather not, I would try just trying get the idea of it rather than exact wording.*

Some students (5<sup>th</sup> and 9<sup>th</sup>) mentioned that using definitions when they do homework leads them to memorise definitions without having to make a special effort to do so. The 9<sup>th</sup> student said:

*sometimes, but it's more that when I use them, cause I use them so much when I'm doing assignments and stuff, that I write them out every time I use them, so I end up knowing them without having to memorise them.*

While other students (6<sup>th</sup> and 10<sup>th</sup>), think that memorising is something important in mathematics and there are lots of things to be memorised. The 6<sup>th</sup> student indicated that:

*Not really, but you have, kinda have to be able to do this stuff yeah. so it goes into your head.*

The interviewer [I] asked if she was speaking about examinations, and she replied 'no so it goes into your head. So you can actually understand what you're trying to solve, because if you don't have the definitions, you can't really do the uh questions'.

The 7<sup>th</sup> student had already indicated that he did not memorise definitions, and when asked to elaborate he said:

*Normally, if you do have a very good idea, it'll kind of come easily in your head, and as when you write it, you get it more formal. So, once you'd get that sort of gist, the idea behind it, it usually comes anyway!*

So we see that most of the students memorise definitions, especially complicated or long ones. Some students become familiar with the definition through coursework and some students remember the idea and can construct the definition.

### **3.2.5 Do you try to relate the definition to material you know already?**

Eight students were asked this question (4<sup>th</sup>, 5<sup>th</sup>, 6<sup>th</sup>, 7<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup>, 10<sup>th</sup> and 11<sup>th</sup>). It was asked to see if students try to make connections between mathematical concepts when first presented with a concept definition. All of these students said that they usually link mathematical materials with each other. Some said that the linking depends on the kinds of materials. One of them said:

*If I can yeah, often, you know, a definition would use other definitions in it. (9<sup>th</sup>)*

Another student explained that:

*It depends, yeah well, a lot of the material I've learnt before, it would be useful, like properly follows on from it, but if like sometimes it could be just completely different, and they mightn't have anything to do with it, I think. (8<sup>th</sup>)*

It seems for these students that, some new definitions involve material or definitions that they learnt and the latter would be useful and therefore they would link them. But some definitions might be completely new and different from previous materials that students know and so they cannot relate them.

One student in this group commented that:

*I've only started doing that this year. I haven't do it before this year, because uh like especially in [lecturer's] course, you have to relate back to previous modules and the definitions in them so. (6<sup>th</sup>)*

It seems that, this student did not link or search for old mathematical materials when dealing with definitions, but she started looking to previous material when one of her courses required her to do so. It may be that as the material becomes more advanced, that students feel the need to make connections.

Two students (4<sup>th</sup> and 7<sup>th</sup>) said that because they do not memorise definitions but try to understand them, they end up making connections naturally. The 4<sup>th</sup> student said:

*Usually, because I don't learn off definitions, so usually have it in my head what it is around when I try to put it back in that language.*

Also, the 7<sup>th</sup> student commented that:

*When I can yeah! Everything I try to visualise and try to like I said understand them, and see what it is.*

In general students said that they would like to relate definitions with old material that they had learned. It seems that whether they do this or not depends on the type of material in question.

### **3.2.6 Do you try to understand the reason the definition was made?**

This question was given to seven students (4<sup>th</sup>, 6<sup>th</sup>, 7<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup>, 10<sup>th</sup> and 11<sup>th</sup>). It was given to explore further students' thoughts on definitions in mathematics. The students gave a variety of answers to this question. For example, some students (8<sup>th</sup> and 11<sup>th</sup>) said they would try to understand how definitions work rather than why they were made. The 8<sup>th</sup> student said that:

*Um, probably more how it can be used in examples like whenever you do a home work and how it can be used to show things.*

This student would try to understand how a definition could be used in examples and proofs.

The 11<sup>th</sup> student mentioned that:

*If I can, that would be secondary. I'll first, try to work out you know what is it telling me and if that's ok, then I might move on to well, why would they look up that in the first place.*

For this student the reason the definition was made would not be one of his priorities and he would try to understand first what the definition tells him.

The 6<sup>th</sup> student commented that:

*Some of the time, because sometimes I don't get it, so I ask some who does understand it to explain it to me.*

This student apparently meant that when she sometimes did not understand definitions she asked for help from her friends and from tutors. It may be that they told her why the definition was required.

Other students (4<sup>th</sup> and 7<sup>th</sup>) said that it depends on the definition. One of them argued that:

*If I was given a few definitions for one thing, then I would try to understand why particular one is used, but when you only given one definition, it's just kind of only you take that, is the definition. (7<sup>th</sup>)*

This student said that he would try to understand why a specific definition is used only if he was given more than a definition for one concept, but if there was only one definition for a concept then he just accepts it. It is possible that this student interpreted the question as asking why the definition was used rather than asking why it was made.

The 9<sup>th</sup> student was the only one who said that she always looks for the reason for the definition. She said:

*Yeah because, I won't understand it, or I won't be able to use it properly if I don't know why it works.*

So the students in this group do not systematically ask why a definition is needed, they usually just accept the definition and try to understand or use it. It seems that they only occasionally try to figure out why a definition is made and this is sometimes triggered by not understanding a definition or having multiple definitions of a concept.

### ***3.2.7 Do you use pictures in your mind to understand definitions?***

This question was given to all students except the 1<sup>st</sup> student. We asked this question to see if imagination and pictures are used by students to understand concepts. Students gave different responses to this question. Some students (2<sup>nd</sup>, 3<sup>rd</sup> and 11<sup>th</sup>) agreed that pictures are helpful to understand concepts. One of them (2<sup>nd</sup>) mentioned that:

*Yeah, pictures are very helpful.*

When I asked this student 'if the topic does not help to construct a picture, what do you do?' he answered that:

*It is the more difficult to understand the definition and then much more difficult if there's no pictures like. You can try to draw it like, but in sometimes that is not possible any way.*

For this student pictures seem to help when understanding concepts and a concept is more difficult for him if there is no picture.

Some other students (6<sup>th</sup>, 7<sup>th</sup>, 8<sup>th</sup> and 9<sup>th</sup>) said that they sometimes use pictures. The 6<sup>th</sup> student said that she uses visualisation techniques to memorise definitions. The 8<sup>th</sup> student said it was not always possible to construct a picture. The 7<sup>th</sup> student said that he does not draw pictures but does use mental images. And the 9<sup>th</sup> student explained that:

*It depends, because sometimes you can have a picture in your head and it's wrong, because like in one of my subjects, metric spaces..., if you try and imagine a definition of something in metric space, you'll imagine it in real space but it could look like something different in a different kind of, that's just an example. (9<sup>th</sup>)*

This student seemed to be aware that pictures not always helpful. She provided an example and elaborated that some pictures are misleading as they sometimes do not work for all cases of a concept.

The 1<sup>st</sup> student was not asked this question directly but he had previously said that he liked to visualise but found it difficult for abstract subjects like rings and fields.

Other students (4<sup>th</sup>, 5<sup>th</sup> and 10<sup>th</sup>) mentioned that they do not use pictures in order to understand definitions. The 4<sup>th</sup> student commented that:

*I'm not very good at putting pictures in my head of things. Um that why I'm not as confident with definitions in rings and fields and geometry and stuff like that, I just try to make sense of it to be honest.*

This student admitted that, she does not use pictures in her head when trying to make sense of a definition. It seems that her lack of visualisation made some mathematical areas like geometry and abstract algebra more difficult for her. So these three students would like to make sense of definition without using pictures in order to understand them.

The 7<sup>th</sup> and the 10<sup>th</sup> students mentioned the lecturer when answering this question. The 7<sup>th</sup> student said '*I wouldn't attend to draw things, um, unless say lecturer draws for me*', and the 10<sup>th</sup> student also explained that he does not use pictures very much but he would to do so if the lecturer gives a picture or example it makes the concept easier to understand and to remember.

In general the students in this study seemed to value visualisation, they found it easier to do in some subjects rather than others. Three students (out of eleven) did not

like to use pictures and one student was aware of the possible pitfalls when using images.

### **3.2.8 Do you refer to definitions when you read or are working on problems?**

All but the 7<sup>th</sup> student were asked this question. It was given to discover if students use definitions when solving problems. Almost all of our students except for the 10<sup>th</sup> and 11<sup>th</sup> students confirmed that they always use definitions while working on problems. The 1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup> and 5<sup>th</sup> students did not elaborate but just stated that they used definitions when doing problems. For the 4<sup>th</sup>, 6<sup>th</sup> and 9<sup>th</sup> students, the definition seems to be the starting point for their problem solutions. The 9<sup>th</sup> student said:

*Well if I was working on problem and say I was given like the same like that some set is compact. Before I'd even start thinking of how to prove something with it, I'd write out what we know about the set is compact and because I've written out all the things I know about this set, now I can use all these things that I know so.*

Similarly the 6<sup>th</sup> student said:

*Well, like when I'm doing my homework I go back into my notes and rewrite the definitions that I need for questions and then I start doing the questions.*

We have already seen that students learn definitions when using them for assignments and for many of these students, it seems that using definitions in this context is helpful to get familiar with definitions, as they always have to write or read the required definitions for problems. This is again evident in the 8<sup>th</sup> student's response to this question when she explained that:

*Yeah, well, that's how I get it into my head. It's whenever if you get homework and you see it'll force. Like a compact space or closed metric space or something, I'll go back and look at definitions, write out, just to get it into my head. (8<sup>th</sup>)*

Two students (10<sup>th</sup> and 11<sup>th</sup>) did not emphasise using definitions when doing problems. The 10<sup>th</sup> student said:

*I think yeah, because some of the problems or the examples or the exercises, they are definitely very much related to the definitions.*

For this student, it seems that if a problem is related to some definitions he would use them in his work, but it sounds like for him that, not all problems are related to definitions. The 11<sup>th</sup> student commented that:

*Yes if I'm stuck I will, I will refer back to definitions, it wouldn't be the first, my first inclination would be to just go ahead and try it out, but if it's not work.*

This seems to suggest that this student preferred to work on problems without using definitions, and if he experienced difficulty he would then try to use definitions. So he seemed to rely on his understanding of concepts in solving problems rather than referring to the definitions immediately.

In general all students use definitions when dealing with problems, most of them said they always do this and some of them said they sometimes do this. However, most of these students did not show this behaviour when they worked on the task problems that were given to them in the interview as we will show later in Chapter 4.

### **3.2.9 When you try to understand a definition, do you focus on every single word? Or do you think some parts are more important than others?**

This question was given to all interviewed students. It was asked to see if students pay attention to all parts of a definition and to get more insight into how students tackle mathematical definitions.

The answers to the question were varied. Some students (2<sup>nd</sup> and 7<sup>th</sup>) preferred to get the idea behind the definition in order to understand it. One student answered that:

*The broader idea nearly than every exact word, the idea more than the actual words yeah. But if I was to it write down I write down the exact wording. (2<sup>nd</sup>)*

Other students (1<sup>st</sup>, 3<sup>rd</sup>, 8<sup>th</sup> and 11<sup>th</sup>) would like to focus on the important part of a definition. The 1<sup>st</sup> student said:

*Probably, I omit some words; I just focus on the important part of it like.*

This student seemed to ignore some terms of a definition, but later when I said to him that 'so that means you don't think that every word is important, some of them just extra' he replied that 'I suppose, every word is important like!'

The 11<sup>th</sup> student explained that:

*I would tend to focus most on the sort of, symbolic aspect, the number of ends if you know what I mean. So I would have a tendency to do, you know, the let  $\varepsilon >$*

*0, f(x) continuous..., I just skip, what's the edit to the end you know. It doesn't always work, I often have to go back, but my main focus is usually on the end.*

The 11<sup>th</sup> student would prefer to concentrate on the main terms in a definition or what comes at the end in order to understand it, he sometimes neglects the beginning of definitions but he admitted that this way of dealing with definitions does not always work.

These four students seemed to focus on what they consider to be the most important parts of definitions to understand them.

Other students (4<sup>th</sup> and 9<sup>th</sup>) said that there are some main points in a definition but students must be careful of missing things out, as every part in a definition is needed.

The 4<sup>th</sup> student said:

*Usually in a definition every bit is important and you can't really leave out any thing. So em if it's a long enough a definition, reduce certain things um, I don't know, every thing is important, you have to focus on every thing.*

For this student every term in a definition is essential. She tried to explain how we could deal with long ones as by breaking them into pieces but still everything is needed.

The 9<sup>th</sup> student also answered that:

*Sometimes there are more important parts, but you know you have to be careful if you gonna leave something out because sometimes it is very important. Like for all  $x$  you know, that's pretty important rather just for some  $x$ . It would make a big difference where is it, it would only be the matter one word see!*

Two other students (5<sup>th</sup> and 6<sup>th</sup>) explained that they tackle a definition gradually. They start by looking at the main points in a definition and then they try to look at the rest of it. The 6<sup>th</sup> student explained that:

*Usually break it up to the most important parts and you remember them, and then you go back and try to remember the rest of the things. It's like you put it out in bullet points. It's like you put it out in bullet points and you'd remember them and you need to go back and try to remember the correct wording and that's how I do it.*

The 5<sup>th</sup> student said:

*I can't just go through piece by piece and making sure understand every piece and then see how every thing relate to each other. I'd kind to think all important*

*more than just parts of it. I'd make sure that I've understand every thing as a whole. Because it only really works when it's all together like I think it would all be important.*

This student seems to appreciate that every part of a definition is usually important.

The last student was the 10<sup>th</sup>. For this student not all definitions are the same. He said:

*It depends on the definitions ok, some of the definitions, really they're easy to understand and you don't need to focus on each word or I mean each sentence or phrase of definitions, and some of the definitions are short ones so and clear, some of them, they are long and unclear, so you have to be careful with them yeah!*

This student mentioned that there are some definitions which are easy and short and so there is no need to be focus on every single term. But for the long and unclear ones, he might have to do so.

We see that more than half of the students are happy to concentrate on what they think are the important parts of a definition. Four students seem to think that every part of the definition is important.

### ***3.2.10 If you are asked to state a definition, do you state it as in your notes or as you understand it?***

We gave this question to all students. This question was given to see what students use when asked for the statement of a definition. Three students (3<sup>rd</sup>, 7<sup>th</sup> and 11<sup>th</sup>) would like to state definitions as they understand them. One student said:

*I think my understanding it. Because if I understand I remember it, if I can't understand it it's hard to remember that definition. (3<sup>rd</sup>)*

This student indicated that he would like to state definitions as he understood them as he will not remember them if could not understand them. And the other two students also would like to use their own words if they were asked to state a definition. The 7<sup>th</sup> student had already said that he did not memorise definitions. In answer to this question he said:

*Like I said usually understand, then and as you started to understand it like you know what the main points are. If it doesn't matter probably won't put them down exactly as was given to me no.*

All the other eight students would like to state definitions as written in their notes if they were asked to do so. Some of these students commented that:

*probably state from my notes, because it's safer, , because it's always gonna be the..., like for exam or something else that's gonna be the one that they're looking for. (5<sup>th</sup>)*

And,

*No as notes, because um, I think that's how I learn things. I think I have to have it word for word are. I like to make sure it's exactly right rather than in case I forget something or do something wrong. (8<sup>th</sup>)*

The 2<sup>nd</sup> student's answer was very similar to this. For these students, it is apparently better for them to state definitions the way they are given as it is safer. The 10<sup>th</sup> student said that you risk losing marks if you use your own words. And, the 5<sup>th</sup> student mentioned that the way the definitions given is the one that [lecturers] are looking for (especially in exams).

The 4<sup>th</sup> and 6<sup>th</sup> students sometimes state definitions exactly as in the notes and sometimes in their own words. The 6<sup>th</sup> student is also concerned about the risk of doing this:

*sometimes I try to state it out of my notes, and then sometimes when I have migraines during the exams, I'll end up to state it in my own words, which I know, not what they're looking for.*

The 4<sup>th</sup> student is more confident that her words are close enough to mathematical language. She said:

*Well, for um one of my subjects, we have to learn it off the way he state it. But generally because I've been using this maths um, the language is their language as well, so it's usually ok!*

We see that some students prefer to give definitions in their own words but the majority want to give the lecturer's definition, perhaps because of fear of being wrong. This fear seems to relate to losing marks in examinations rather than not giving or using an accurate or adequate definition.

### **3.2.11 Is it easy for you to understand mathematical definitions?**

All students were asked this question. It was given to explore students' difficulties in understanding definitions in mathematics. Students gave various answers to this

question. For many students (1<sup>st</sup>, 3<sup>rd</sup>, 4<sup>th</sup>, 9<sup>th</sup> and 10<sup>th</sup>) definitions are not easy and they explained that that depends on the topic. The 1<sup>st</sup> and 4<sup>th</sup> students mentioned pictures and visualisation in their answers. One of them said:

*It depends on the subject, but um some of them is hard to visualise, what it means, because it is abstract for rings and fields. (1<sup>st</sup>)*

This student seemed to find difficulty in understanding definitions in abstract topics like rings and fields, and his reason was these topics are difficult to visualise. This was similar to the 4<sup>th</sup> student's answer. The 10<sup>th</sup> student also spoke of difficulties with abstraction. So for these students some definitions are not easy because of problems with visualisation.

The 3<sup>rd</sup>, 9<sup>th</sup> and 10<sup>th</sup> students mentioned examples in their answers. For the 3<sup>rd</sup> student it seemed hard to understand definitions directly and examples were very helpful to understand a definition. He said:

*It is better use the examples like to understand the definitions, but not straightaway like.*

Another student said:

*Depends on the definition, Um, usually yeah, like if there's something that I'm struggling with, if there's an example I can compare it to, or if it's used in a theorem or something like that, becomes clear, the more it's used the more clear gets, if it's a particularly difficult one. (9<sup>th</sup>)*

For these students, it seems that definitions are easier to understand if they can relate them to an example. The 9<sup>th</sup> student also implies that seeing the definition in use helps with understanding. So if definitions are used in examples or theorems, that would be more clear for them and so easier to understand.

For the other students (2<sup>nd</sup>, 5<sup>th</sup>, 6<sup>th</sup>, 7<sup>th</sup>, 8<sup>th</sup> and 11<sup>th</sup>) definitions generally are easy enough to understand. The 2<sup>nd</sup> student explained that:

*Um, easy enough, I'd say yeah, not too bad like it's not, it's easier if somebody explain it in words sometimes, yeah.*

For this student definitions are relatively easy but they are easier if they are described in words different from mathematical notation. While for the other students in this group definitions are generally easy to understand. One of them commented that:

*Generally it's ok, but sometimes I just have to read over them few times and write them out again, but generally I don't have much problems with them, the definitions. (5<sup>th</sup>)*

This student usually seemed to have no problem with definitions but sometimes they require an effort, she had to go over them many times in order to understand them.

The 8<sup>th</sup>, 10<sup>th</sup> and 11<sup>th</sup> students also spoke of having to spend time trying to understand definitions. The 11<sup>th</sup> student in answering the question said:

*Um, not tremendously, it is not very easy, I can usually get there, but it takes work.*

The answer of the 6<sup>th</sup> student was different from the others. She claimed that:

*Yes, it's easier to understand mathematical definitions than than the English ones.*

This student has a problem with English, and thus mathematical definitions are easier for her than English definitions.

It seems that the majority of students have some trouble when trying to understand definitions, at least initially. Abstract topics seem to cause the most difficulty. The students employ different strategies like using examples, visualisation, and putting definitions in different words. They all recognise the need to work and persist in order to understand.

### ***3.2.12 Do you find definitions in maths are different from definitions in other subjects?***

Six students (5<sup>th</sup>, 6<sup>th</sup>, 7<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) were asked this question. It was given to discover students' thoughts on mathematical definitions in comparison to definitions in other subjects.

All but the 8<sup>th</sup> student said that definitions in mathematics differ from definitions in other subjects. Four students (5<sup>th</sup>, 6<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) pointed out that mathematical definitions are really specific and tell us exactly what a concept is. One of them said that:

*A little in that; there's no room for any ambiguity at all in maths' definitions and either it's you get or you don't, if it's not there it's not, you can in other subjects is more room for interpretation I think that's not maths. (11<sup>th</sup>)*

The 5<sup>th</sup> student was similar to this. The 9<sup>th</sup> student explained that:

*I don't think so really, because it's still defining something, I think that in maths it always, it's very specific, and it covers all possibilities, where sometimes in subjects like physics or statistics, it would be more vague or just kinda roughly explained rather than given a proper definition. (9<sup>th</sup>)*

These students explained how definitions are precise in mathematics and there is no ambiguity about what they define as they cover all aspects of a concept, whilst, for them, definitions in other subjects are not precise. However the 9<sup>th</sup> student began her answer saying that definitions in any subject are used to define something. This argument is similar the 6<sup>th</sup> student's additional comment on the question about the point of definitions in mathematics when she said 'in all like science subjects, because you need a definition to start off from to do any thing'. For the 6<sup>th</sup> student definitions in mathematics seem to be similar to definitions in physics but different from other subjects. She said that:

*You've to be more rigorous in mathematics than would be in chemistry, but it is kind of the same in physics as well.*

This student also indicated that definitions in mathematics are more precise than other subjects but for her they seem to be similar in physics.

A different student mentioned that definitions in mathematics are similar to other subjects' definitions. This student commented that:

*I thought they would be different until you in, like analysis and metric spaces, they are very like definitions you might learnt in, you know another subjects like business or, you know, there is basic, just you have to learn it, you know! (8<sup>th</sup>)*

This student was thinking that mathematical definitions differ from other definitions, but it seems that she regarded all definitions in all subjects as similar when she found many definitions in the metric spaces course which have to be learnt like definitions in other subjects.

The 7<sup>th</sup> student, in answer to this question explained that, in general, definitions that require more work are harder. He said:

*It depends on which you put on more work into. I was expecting in this year like a little bit more working, and then was bit easier. So it's a lot easier to just kind of sit down and do physics there, and then because this usually you know methods the whole way, you just follow them. Where is maths, is very much you have to think for yourself and understand it before you can do it and things like that.*

This student seemed to find that definitions are more difficult if they take more work for him. Also he apparently found that definitions in physics are easier than those in mathematics because in physics they describe methods and so there is no need to think about what to do while in mathematics, definitions need to be thought about and understood in order to do further things.

From the analysis of this question we can summarise that most of the students felt that definitions in mathematics give an accurate description of a concept, so they are more rigorous than definitions in other subjects. Some students added that definitions in all subjects are given to define something; others added that definitions in mathematics require more thinking than other subjects. Only one student seems to say that definitions in all subjects are the same as they have to be learned.

### **3.2.13 Summary**

From the analysis of students' point of view about mathematical definitions in general we summarise that, the students seem in general to understand the significance of definitions in mathematics. That is they seem to realise that definitions are the building blocks of the subject which provide it with a solid base. They also seem to appreciate the different role that definitions play in mathematics as opposed to other subjects. Almost all of them said that they referred to definitions when working on problems. I will revisit this topic later.

Students did report having difficulty understanding definitions; almost all of them said that this required effort. About half of the students said that understanding definitions was more difficult in abstract subjects or in situations where visualisation was not natural.

Based on students' answers we found students tackle new definitions in different ways, such as: using intuition to find the meaning of definitions; focusing on the key parts of definitions; using examples to understand them; putting definitions into different words, visualising the meaning of definitions; and sometimes memorising them. We also observed that most of students memorise definitions for examinations in particular long or complicated ones. About asking for the statement of a definition, we notice that a few students like to use their own words but most of students prefer to give the lecturer's definition which might be due to the fear of being wrong. They

seemed to answer this question solely in relation to examinations and not in relation to using definitions themselves.

In addition, more than half of the students said that they focus on the important parts of definitions rather than on each word, though some of them were aware of the problems with doing this.

We noticed also that students generally seem to try to link definitions with previous material that they had learnt. Students do not usually ask why a definition is needed and just accept it, but rather they apparently only try to understand how a definition is used.

We conclude also that students seem to value visualisation and they could use it more easily in some subjects rather than others and some seem to be aware of possible pitfalls when using pictures. All students agreed that they use definitions when working on problems, most of them always and some of them sometimes, but many of them did not show this behavior in their problem solving.

### **3.3 Students' Definitions and Conceptions of Open Sets and Metrics**

The main mathematical concept under consideration in this study is the open set concept. Students learn about different types of open sets and distance during the metric spaces course and also during other previous courses. All of these students have taken a first course in analysis on the real line, and the only open sets encountered in these courses would have been open intervals in  $\mathbf{R}$ . So, it was possible for us to ask students different types of questions involving this concept. Students in the questionnaire and the interviews were given two questions whose purpose was to explore their definitions and concept images about the topics of open sets and distance in a metric space: (Note that distance was the term used by the lecturer in the course).

1. (i) Define the term: Open set in metric space?  
(ii) How would you explain the idea of open set in metric space to a friend of yours?
2. (i) Define the term: distance in metric space?  
(ii) How would you explain the idea of distance in a metric space to a friend of yours?

Moreover, the interviewed students were given a number of statements about the concept of an open set and were asked to determine whether they would agree or disagree with them and to explain their choice.

In this part of the chapter I am going to analyse responses to each of the two questions above and then I am going to look at the given statements. The analysis which I carry out here is based on the types of arguments given by students in the questionnaire and the interviews to justify their answers. Using these questions and statements I tried to investigate students' definitions of open sets to reveal how students regard open sets and also to discover the dominant concept images that students used most frequently.

### **3.3.1 Students' Definitions of an Open Set**

In our study we intended mainly to explore how students think of the open set concept and to discover the common images that students hold about this concept. To obtain some information about students' conceptions, we asked the students in the interviews and the questionnaire the first question stated above which consisted of two parts: that is the question '*How do you define the term open set in a metric space?*'; and also the quite open question '*How would you explain the idea of open set in metric space to a friend of yours?*'.

The formal definition of an open set which was given to the students in the course is: '*A subset  $U \subset (X, d)$  is an open set if  $\forall x \in U \exists \varepsilon(x) > 0$  and an open ball  $B(x, \varepsilon(x))$  in  $(X, d)$  s. t.  $B(x, \varepsilon(x)) \subset U$* ' namely, '*A subset  $U$  is open in  $(X, d)$  if every point in the subset is the centre of an open ball in  $(X, d)$  which belongs to the subset  $U$* '.

Here, first I will consider the students' responses to the two parts of the question in the interviews. I will then address the responses to the same question in the questionnaire.

#### **3.3.1.1 The Interviews**

All the students gave answers to these questions except the 10<sup>th</sup> student who refused to answer mathematical questions about the course. From the analysis of these questions we found that the students described their personal notions of open set in different ways. Some students gave definitions and ideas that come from the given

definition in the notes, others gave their general idea about the concept, and others talked about their own favourite ideas regarding this concept. We classify all of the students' definitions and ideas into three categories: (1) the formal definitions, the students in this category are 1<sup>st</sup>, 4<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>, their definitions were very close to the formal definition but they omitted some aspects of it; (2) definitions based on the boundary points, the students in this category are 2<sup>nd</sup>, 5<sup>th</sup>, 7<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>, these students' idea is that if the boundary points of a set are not included in it then that set is open; (3) definitions based on the union of open balls, this category consists of the students 3<sup>rd</sup>, 4<sup>th</sup>, 5<sup>th</sup>, 6<sup>th</sup> and 11<sup>th</sup>, for these students an open set is just a union or collection of open balls. Some of the students above appear twice in different categories, as these students' answers showed evidence for all the categories which include them. We will represent these categories and illustrate each student's definition and idea within them as follows:

### 3.3.1.1.1 The Formal Definitions

As indicated above five students (1<sup>st</sup>, 4<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) fall into this category. These students seem to understand the idea of the formal definition of an open set. However, none of the five students stated the formal definition exactly as given to them in the course.

The 1<sup>st</sup> student's definition of open set when he answered the first question was '*the set is open if for any point in the set you can draw an open ball around it which is contained in the set*' and regarding the second question he commented that '*Probably say the same thing that I said with the definition*'. He seemed to know the full idea of the formal definition but he did not clearly mention the part of the definition which stated that the open ball in question should be defined using the metric  $d$  on  $X$ .

When asked for the definition of an open set the 4<sup>th</sup> student spoke about the union of open balls. However she realised that this was a theorem. In the explanation of the term open set to a friend she said that '*open set is that, there's an open ball contained in it, an every point has an open ball in that open set*'. Note that this student also did not point out that the open ball is defined by the metric  $d$  on  $X$ . In addition, the 8<sup>th</sup> student's definition of an open set was '*An open set is any set where you can set an open ball of radius  $\varepsilon$  around any element in the set, I think*'. So she is similar to the other students above and she said that she would probably say the same thing in her

explanation to a friend. She added that she would ask the friend to picture a ball or union of balls.

The 9<sup>th</sup> student, when asked for a definition said *'the official definition is you can take any open ball around any point and it's still completely contained in the set'*. This student also tried to state the formal definition of the open set but the sentence *'you can take any open ball around any point'* which she gave is not correct. In her definition, all the open balls about any point need to be considered and that is not the correct definition of an open set, the correct sentence should be *'you can take at least one open ball around any point'* so that only one open ball of  $(X, d)$  about any point in the mentioned set is needed. This student was aware of the difference between open balls in a metric space and in a subspace, as will be shown later in her work on a problem which was designed to reveal such misconceptions, but she did not state the difference clearly in her definition of an open set. She also referred back to a definition of open sets which she had seen in a previous courses, i.e. that is open sets do not contain their boundary. This was the idea that she said she would use to explain the concept to a friend.

The 11<sup>th</sup> student also seemed to have some aspects of the formal definition. When this asked for a definition of an open set, he responded by understanding an open set as a union of open balls, and he elaborated on his idea by commenting *'in that every point is the centre of an open ball'*. It is apparently that, in the definition, this student focused only on the part where every point must be a centre of an open ball which matched his idea of an open set that is as a union of open balls.

It seems that the students in this group have the idea of the formal definition of an open set and they seem to be not aware of the importance of the part that they omitted in their definitions. As we wrote before, the formal definition, given in the course, of an open set is *'A subset  $U \subset (X, d)$  is an open set if  $\forall x \in U \exists \epsilon(x)$  and an open ball  $B(x, \epsilon(x))$  in  $(X, d)$  s. t.  $B(x, \epsilon(x)) \subset U$ ' namely, *'A subset  $U$  is open in  $(X, d)$  if every point in the subset is the centre of an open ball in  $(X, d)$  which belongs to the subset  $U$ '*. However the students in this category omitted the underlined part of the definition and they defined an open set as *'A subset  $U$  is open in  $(X, d)$  if every point in the subset must be the centre of an open ball which belongs to the subset  $U$ '*. These students' misconception of the definition might be due to their experience of dealing with metric spaces and their subsets when there is no ambiguity about the metric in*

question. We will see later that this point is important and can lead to confusion when for example you consider the  $x$ -axis in  $\mathbf{R}^2$  with the standard metric. Students sometimes think of open balls around points on the  $x$ -axis as intervals, because they use the standard metric on  $\mathbf{R}$  instead of that on  $\mathbf{R}^2$ . Every phrase of the formal definition is essential and students often do not consider that.

### 3.3.1.1.2 Definitions Based on the Boundary Points

This category includes the students: 2<sup>nd</sup>, 5<sup>th</sup>, 7<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>. The students' idea of an open set is one which does not contain its boundary or end points. This conception was obvious in those students' definitions or explanations. The 2<sup>nd</sup> student stated in his definition of an open set that *'open set something which doesn't have a clear boundary, you can get as close as you like but never get the actual end of the set'*. So this student's conception of open set is clearly based on the boundary points of a set and when explaining to a friend he said the same thing.

The 5<sup>th</sup> student spoke about unions of open balls when asked for a definition but when she tried to explain the open set to a friend she commented that *'I probably start off with the one in  $R$  and just show them like that is an open set or just you don't include the end points, it kind to show them a basic example like that.'* So she would show friends a basic open set in  $\mathbf{R}$  and show them that the end points are not contained in that set.

The 7<sup>th</sup> student admitted that he had forgotten the exact definition and what he gave is his general idea, his words were different but he seems to have the same idea as the 2<sup>nd</sup> student. He said that *'we can say what the general idea, the open set is basically, it isn't like say straight edges, is kinda fuzz out, because it doesn't contain border elements'* and in his explanation to a friend he also said *'so it's kind of fades off infinitesimally close to boundary, but it never quite gets out, fuzzy at the edges'*. For this student an open set is fuzzy at the edges as every point near the boundary is included in the set but the boundary points are not reachable.

We classified the definition of the 9<sup>th</sup> student in the formal definition category, but for her it is easier to think of open sets using the boundary idea, and in her explanation to a friend she used this idea, she said that *'we got a definition for last year which is just a set that doesn't contain its boundary. So it's kinda easiest to think of it in that way, I probably explain it kinda like that, that you know, if you go shorter and shorter*

*distance so you know, no matter how close you get, you'll never quite get there'* so for her it is easier to think of open ball as a set that does not contain its boundary. Also the 11<sup>th</sup> student clarified an open set to a friend by *'take your collection of things, and take the outline of it and the outline is not part of this, so if it goes close as you like to this invisible line, but it can't to touch it'* so he used this idea of open set in his explanation.

We noticed that the students in this group seemed to revert to their old idea of an open set (which is a set that does not include its boundary points) when asked to explain the notion to a friend, as for them it might be the easiest way to think of open sets.

Of the students in this group we found that the 5<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup> students used this boundary idea when they tried to explain an open set to friends which probably for them it is the simplest idea. And so this might be an evidence for the role that the first encounter with a concept plays in some students' conceptions.

### **3.3.1.1.3 Definitions Based on the Union of Open Balls**

The students who were classified in this category are the 3<sup>rd</sup>, 4<sup>th</sup>, 5<sup>th</sup>, 6<sup>th</sup> and 11<sup>th</sup> students. These students think of an open set as a collection of open balls and they all gave this conception as a definition when asked for one. The 3<sup>rd</sup> student stated in his definition of an open set that *'open set, it's the union of open balls'* and he used the same idea when he tried to explain the idea to a friend. Recall that the 4<sup>th</sup> student was included in the formal definition group due to her explanation to a friend, but as definition of an open set she stated that *'An open set is a union of open balls'*. She said she was not sure if that is a definition or a theorem, and she was the only student in this group who tried to make this distinction. The 5<sup>th</sup> and 6<sup>th</sup> students were not very different from the previous ones; they both stated in their definitions of open set that *'open set, it's a collection of open balls'* and the 6<sup>th</sup> student used the same idea in her explanation to a friend. However the 5<sup>th</sup> student said that when explaining to a friend she would start with open sets in  $\mathbf{R}$  and spoke about sets that do not have end points. Also, the 11<sup>th</sup> student gave this idea of an open set when was asked for a definition; he said that *'Open sets are, I would understand them as union of open balls, um in that every point is the centre of an open ball'*, we can see in this answer that he also uses the definition idea.

It is clear that all those students' definitions are based on the idea of the union of open balls.

### 3.3.1.2 The Questionnaire

In the questionnaire most of the students gave answers to the two parts of the first question mentioned above which related to the students' conceptions of the open set concept. In our analysis we found that students wrote various responses to these questions. Three students (C, D and M) did not answer any one of the questions above; also Student B did not answer the question about the definition of an open set, and Student I left the question about explaining the idea of an open set to a friend blank (no response).

We sorted the other students' responses to the questions into four groups: (1) the formal definitions; (2) definitions based on the boundary points; (3) definitions based on the union of open balls ideas; (4) unclear definitions. We will examine each group in detail:

#### 3.3.1.2.1 The Formal Definitions

We divide the students within this group into two parts. The first one includes the students who gave their definitions or ideas in words, and the second one consists of the students who used mathematical notation in their definitions or explanations.

There were four students (B, E, J and K) in the first part of the group. Of these students, student E's answers to the both questions were the same and he/she wrote that *'A set is open if for every point in the set an open ball of radius  $\varepsilon > 0$  can be drawn around it which is entirely inside the set'*. The other three students used the definition when they tried to explain an open set to friends but they defined the open set using different words. One of those students (K) explained that *'U is an open set in a metric space if every element of U, has a ball, whose radius depends on that element, which is entirely contained in U'*.

So these students tried to state their ideas of the definition of an open set but as I mentioned in the interviews section above, these students also did not indicate that the open ball has to be defined using the given metric.

In the second part of this group we found that six students defined an open set by using mathematical notation, however, none of them state the exact formal definition.

Of these students, the students J, K, L and O showed a correct use of the notation. For example Student J wrote ‘A set  $U \subset (X, d)$  is open if,  $\forall x \in U \exists \varepsilon(x) > 0$  s. t.

$B(x, \varepsilon(x)) \subset U$ .’ All four of these students defined an open set by using notation similar to this but did not mention that  $B(x, \varepsilon(x))$  is in  $(X, d)$  in their definitions.

The other students (I and P) seemed to make an incorrect use of the mathematical notation within their definitions. For instance, Student I defined an open set by ‘ $A \subset (X, d)$  is open if  $\forall a \in A, \forall \varepsilon > 0 \exists B(a, \varepsilon) \subset X$ ’. In this student’s definition the subset A of the metric space X is open if all the open balls about any point in A are contained in X, and that is not the correct definition of an open set. The other one (P) wrote: ‘An  $(X, d)$  set is open if  $\forall x \in X, x \in B(X, d)$ ’. This student is confusing the metric space  $(X, d)$  with the ball  $B(x, r)$ . It is not clear that they understand what is meant by an open set.

We realised that none of the students in the second part of this group gave the exact formal definition of an open set which was given in this course and that is similar to the answers of the students in the first part of the same group.

### **3.3.1.2.2 Definitions Based on the Boundary Points**

The answers of five of the students (A, F, G, L and N) to the given questions were categorised under this group. Students F and N give their definitions of an open set by concentrating on the boundary points. Student F defined that ‘A set that is not bounded by any points’. This student also used this idea in explaining to a friend and wrote that ‘A set that is not restricted by a certain region’, so he/she explained an open set as an unrestricted set. The other three students used the boundary points’ idea when they answered the question about explaining an open set to a friend. Student A explained that ‘A set which does not contain its boundary but contain points very close to it, so called edge’.

From the students’ arguments we found that those students hold an idea of open set as a set which is not bounded or does not contain its boundary.

### **3.3.1.2.3 Definitions Based on the Union of Open Balls**

Many students understand an open set as a union of open balls. There were five of these in the questionnaire. Of those five students, the students A, G and H used this idea clearly in their definitions of an open set. One of these students commented that

*‘An open set is a union of open balls, where an open ball about a point of radius  $r$  is given by:  $B_{(a)}r = \{ x \mid d(a, x) < r \}$ , (H)’. Student H and other students (N and O) used this idea also in their explanations to friends where one of the answers was ‘I would ask them to imagine a ball of some radius  $\varepsilon$  about a point. Now every element of the open set  $U$  is at the centre of an open ball of some radius  $\varepsilon$ . All of these open balls combined creates the open set. (O)’. So we can see that this student’s explanation is about an open set as a combination or collection of open balls.*

### **3.3.1.2.4 Unclear Definitions**

There was only one student (P) in this group. This student tried to explain his/her conception of open set to a friend by using an idea which is different from the ones above. This student commented that: *‘Open sets contain all elements with in the set which is less or equal to the radius of the open ball’*. So this student gave an unclear definition of open set and it seemed he/she does not know what an open set means.

### **3.3.1.3 Conclusion**

From the results of the interviews and the questionnaire above we conclude that students hold different conceptions relating to the concept of open set. Some students’ conceptions are based on the formal definitions, that is some students give their understanding of the formal definition; some students’ conceptions are based on the boundary idea, this means some students focus on the boundary points of a set and if these boundary points are not included in the set then they conclude that that set is open; some students’ conceptions of open sets are based on the collection of open balls; and some have unclear conceptions.

Definitions and Ideas	Students
Formal definition	1 <sup>st</sup> & 8 <sup>th</sup>
Boundary idea	2 <sup>nd</sup> & 7 <sup>th</sup>
Unions of open balls	3 <sup>rd</sup> & 6 <sup>th</sup>
Formal definition & unions of open balls	4 <sup>th</sup>
Formal definition & Boundary idea	9 <sup>th</sup>
Boundary idea & unions of open balls	5 <sup>th</sup> & 11 <sup>th</sup>

**Table 3.1: Responses from the Interviews**

We also found that some students use the same type of conception of open sets in their definitions and explanations to friends, and others use different types of conceptions in their definitions and explanations. Tables 3.1 and 3.2 illustrate the types of conceptions of open sets that students used in their definitions and the explanations of the concept.

Definitions and Ideas	Students
Formal definition	B & E & I & J & K & P
Boundary idea	F
Unions of open balls	H
Formal definition & unions of open balls	O
Formal definition & Boundary idea	L
Boundary idea & unions of open balls	A & G & N

**Table 3.2: Responses from the Questionnaire**

### 3.3.2 Students' Conceptions of Metrics

The notion of metric (distance) is a fundamental to define open balls and so to define open sets. In everyday life the notion 'distance' refers to a physical length between two points. The distance in mathematics is a generalisation of the notion of physical distance; it defines a certain way of describing how elements of a set are close to or far apart from each other. In the metric spaces course, the formal definition of a metric was given to students as follows '*Let  $X$  be a nonempty set. A metric on  $X$  is a function  $d : X \times X \rightarrow \mathbf{R}$  satisfying: (i)  $d(x, y) \geq 0 \quad \forall x, y \in X$  and  $d(x, y) = 0 \Leftrightarrow x = y$ . (ii)  $d(x, y) = d(y, x) \quad \forall x, y \in X$ . (iii)  $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in X$ .*' So a metric is a function that measures the distance between two points of a set and behaves according to a certain set of conditions.

The lecturer in this course wanted to emphasise to students that there are many different ways of putting a metric or measuring distance on a set. To illustrate this point he introduced the notion of similarity, and he explained to the class that when working with metric spaces it can be helpful to think of 'distance' as a measure of similarity with smaller distances indicating greater similarity. The example he gave concerned the set of bananas. He said that there were many ways of deciding whether

two bananas were similar, for example you could consider weight, colour, curvature etc. Each of these criteria gives rise to a different metric on the set of bananas and a different notion of what it means for two bananas to be ‘similar’. He emphasised that similarity can mean various things depending on the context. He also showed them many examples to illustrate different ways of measuring similarity or dissimilarity of elements of different metric spaces.

To gain information about students’ understanding of the notion of metric we asked the students in the questionnaire and in the interviews the following questions ‘*How do you define the term distance in a metric space?*’ and ‘*How would you explain the term distance in a metric space to a friend of yours?*’. Note that distance was the term used by lecturer in the course. Using these open questions we aimed to explore the influence of the definition and the explanation of the metric on students’ conception of such a concept.

### **3.3.2.1 The Interviews**

All the students in the interviews answered the above questions including the 10<sup>th</sup> student who was only asked the question ‘*What is your idea about distance in metric spaces?*’ which is similar to the above questions. Recall that this student refused to attempt any of the tasks in the interview, and his answer to this question is included within the answers students gave to both of the questions above.

We looked at each question separately. The findings of the analysis of responses to the first question showed that students have many definitions related to the notion of distance in metric spaces, and we classify students’ definitions into the following categories: (a) idea based on the formal definition of the distance, this category includes only the 7<sup>th</sup> student; (b) idea of distance as a measure of similarity, the students in this group are 1<sup>st</sup>, 2<sup>nd</sup>, 4<sup>th</sup>, 6<sup>th</sup>, 7<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>; (c) idea of distance as a comparison between two points, this contains the students 8<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>; (d) ideas of distance as a difference between two elements, the students who were grouped in here are 5<sup>th</sup>, 6<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>; (e) ideas of distance as if not like physical distance, this group consists of the students 1<sup>st</sup>, 5<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup> and 10<sup>th</sup>; and (g) ideas of distance as a real number, this category includes only the 3<sup>rd</sup> student. We can notice that many students are included in more than one category. This is a result of students’ speaking about the

different conceptions they have. We will show and present an example for each category as follows:

### **3.3.2.1.1 Ideas Based on the Formal Definition**

As mentioned above only one student (7<sup>th</sup>) was included in this group. This student said that:

*A metric is a function from ordered pairs, say it's on the set  $X$ , it is  $X \times X$  to  $\mathbf{R}$ , such that three axioms are satisfied. One is being that  $d(x, y) \geq 0$ , and equal to 0 only if the two things are the same. The second one is that  $d(x, y) = d(y, x)$  for all  $x$  and  $y$  in your metric, another way is that order it doesn't matter. And the third one being the triangle inequality,  $d(x, z) \leq d(x, y) + d(y, z)$ , for all  $x, y, z$  in your set.*

This student apparently understands that the question is about the definition of distance in metric space and he gave the formal definition.

### **3.3.2.1.2 Ideas of Distance as a Measure of Similarity**

Many students (1<sup>st</sup>, 2<sup>nd</sup>, 4<sup>th</sup>, 6<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) thought that the distance is a way of measuring similarity between two elements. For example some of them like the 1<sup>st</sup> student commented that:

*It is a measure of similarity..., it is better to think of it as a measure of similarity between two things!*

And the 11<sup>th</sup> student also commented that:

*Well, distance can mean different things in the metric space, but it is usually a measure of how similar something is to something else based on certain criteria...*

All those students defined a distance as a way of measuring similarity between two elements in a set and their answers were all in the same vein.

### **3.3.2.1.3 Ideas of Distance as a Comparison**

We found that some students (8<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) thought of a distance as a way of comparing two points. The 8<sup>th</sup> student explained that:

*It's like um comparison rather than actual distance. So I thought that was a good way of explaining it. So it was like comparing two elements rather than looking at the distance between them.*

These students defined a distance as a means of comparing elements of a set.

#### **3.3.2.1.4 Ideas of Distance as a Difference**

The students in this category are the 2<sup>nd</sup>, 5<sup>th</sup>, 6<sup>th</sup> and 9<sup>th</sup> students. They thought of a distance as a difference between two points in a set. The 9<sup>th</sup> student said that:

*[The lecturer] gave us really good way of thinking of it, that if you think of it of how different two points are, because you can mix it up if it is like in real space, you can mix it up with just physical distance. So think of it as how similar things are how different things are.*

Those students referred to the distance in metric spaces in their definitions as a difference between elements of a set, in addition to other ideas. It seems that this notion is closely related to the similarity conception. For example, students often mentioned both notions together. The 6<sup>th</sup> student said:

*Distance is either similarity or difference of the points.*

#### **3.3.2.1.5 Ideas of a Distance Different from a Physical Distance**

Many students (1<sup>st</sup>, 5<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup> and 10<sup>th</sup>) indicated that the distance in metric spaces is not like physical/usual distance. The 10<sup>th</sup> student explained that:

*Well, distance in metric space not like the distance that we have in everyday life and the distance in metric space mean something different. The lecturer gave an example, if you want to go from Dublin to say Sligo you may can't go I mean directly, you have to go through another place, so I think this is the definition or the notion of distance in metric space.*

All the five students explained in their definitions that the distance in metric spaces is different from the usual distance in real life and they often tried to illustrate their idea with examples.

#### **3.3.2.1.6 Ideas of Distance as Real Number**

The 3<sup>rd</sup> student was the only one in this group. He thought of distance as real number and commented that:

*Distance uh bigger or equal than 0. Yeah I think it's a real number or something like.*

So this student's conception of a distance is as a real number that is greater or equal to zero. He seems to think of a distance in a formal way but does not give a definition.

From the first question, about the distance in a metric space, we conclude that, most of the students gave answers to the question according to their understanding of the distance which is as a measure of similarity or dissimilarity between elements of sets; one student understands the distance as a non negative real number, and only one student apparently realised that the first question is about a definition so he gave the formal definition of the distance. The students seem to be influenced by the explanation of their lecturer that distance is a measure of similarity or difference and five of them mentioned the lecture when answering this question. Many of the students volunteered that distance in a metric space is not physical distance.

The results of the analysis of the second question, which was about explaining the notion of a distance in metric spaces to a friend showed that, most of the students (4<sup>th</sup>, 5<sup>th</sup>, 7<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) would explain the distance as a measure of similarity or as a difference between two things. Common answers were:

*I probably explain it like that, it's not just strictly the distance between them, is just uh difference between them. (5<sup>th</sup>)*

And,

*I would say, you'd pick criterion and you'd say whether it's how close their location is, or how similar they're in size. It's a measure of how similar, how different they're. (11<sup>th</sup>)*

Those students seem to be happy with the explanation that is given to them by their lecturer and so their conceptions of a distance seem to build on that.

The 2<sup>nd</sup> student would explain the distance to a friend as:

*I quite say just the distance for to explain to friend, just distance between two things to friend.*

This student might find that it is easier to explain distance to a friend as just the physical distance between two things. It might be that his idea as a measure of similarity would need more explanation for a friend.

The 3<sup>rd</sup> student would like to explain the distance as a physical distance. He said that:

*Uh... distance just same as physical distance, yeah.*

So this student seems to feel that it is easier to explain the distance as a physical distance. While in answer to the first question he only said distance is greater than or equal to 0 and it is a real number

As we indicated earlier, the 10<sup>th</sup> student was only asked the question ‘*What is your idea about distance in metric spaces?*’, and recall that he answered:

*Well, distance in metric space not like the distance that we have in every day’s life. The distance in metric space means something different, for example, if you want to go from Dublin to say Sligo, you may can’t go I mean directly you know, you have to go through another place, so I think this is the definition or the notion of distance in metric space.*

This student also seems to be influenced by the lecturer’s explanation and examples about the distance in metric spaces and so he understood that it is different from the notion of distance in real life and he used one of the examples that are given in the course to illustrate his understanding.

Some other students (6<sup>th</sup> and 9<sup>th</sup>) also used examples in their explanations of distance to friends. For example the 6<sup>th</sup> student commented that:

*Well, uh, the discrete metric is where, if the points are equal is zero and where the points aren’t equal is one. Also, the standard metric is just the norm space in the Euclidean, like the Euclidean norm.*

This student cited only some examples of different distances in her explanation to a friend and did not use her idea of a similarity or difference between elements of a set which she gave when asked for a definition.

The 1<sup>st</sup>, 7<sup>th</sup> and 8<sup>th</sup> students were not asked this question. It was an oversight that I did not ask the 1<sup>st</sup> student this question. In the case of the other two students, they spoke clearly about both the definition of the distance in metric spaces and about their ideas about it when answering the first question related to the distance, so I did not ask them this question again.

From the second question we deduce that, the students used different methods to explain to friends which reflect their understanding of a distance. Many students again used their idea of a distance which is as measure of similarity or dissimilarity between objects to explain to friends; others used examples to illustrate their conceptions; some

students would explain it as just distance between points or as physical distance and some students would confirm that it is not like a real life distance.

### 3.3.2.2 The Questionnaire

All the students in the questionnaire gave answers to both of these questions above except for Student B who answered only the second question about the explanation to a friend. In the questionnaire we also analysed each question separately.

The results of the first question ‘*How do you define the term distance in a metric space?*’ showed that students defined the distance in a metric space in different ways. The students’ ideas in the questionnaire were not far from students’ ideas in the interviews.

Many students (A, G, J, L and O) defined distance as a measure of similarity or as a measure of difference between two elements of a set. For example, some of these students answered that:

*Distance in metric spaces can be thought of as measuring similarity. (J)*

And,

*This is a measure of the difference between two elements in a set. (L)*

These students apparently understood how distance is a measure of similarity or dissimilarity between the elements of a set, so they understood the distance as it was explained to them.

Other students (E, F and M) thought that the similarity is measured between functions or metric spaces or metrics and they wrote that:

*It is a measurement of similarity of two functions. (E)*

This student seems to be influenced by the given examples in the course, where his/her answer is might be true if the points of a space are functions. The other two students answered that,

*Distance in metric space doesn't have to be physical distance, it's how similar metric spaces are. (F)*

And that:

*Distance in a metric space is not measured in the usual way of distance; it measures the difference between two metrics. (M)*

These three students seem to be confused and have mixed up distance between elements and something else. They might be also influenced by examples of different

metric spaces, and also they were shown how different metrics can be defined on a set. The students might not realise the subtle mistakes in their answers which showed us their misunderstanding the notion of the distance.

Students D and I, tried to give the definition of the distance. One of them answered that:

$(X, d)$  a metric space,  $d: (X \times X) \rightarrow \mathbf{R}$ , where  $d(a, b) = \text{distance between } a \text{ and } b$ . (I)

This student defined the distance in metric space as a function but his/her definition was not complete (full definition). The student did not list the conditions which must be satisfied by the function  $d$  and might think that what he/she wrote is sufficient to define the distance. However he/she seems to understand that this question is about the definition and answered according to that.

The other one wrote that:

$(X, d)$ .  $d(x, y) \geq 0$   
 $d(x, y) = 0 \Leftrightarrow x = y$   
 $d(x, y) = d(y, x)$   
 $d(x, z) \leq d(x, y) + d(y, z); \forall x, y, z \in X$ . (D)

This student apparently knew that the questions asked for the definition of the distance in a metric space so he/she represented the metric space by  $(X, d)$  and listed the conditions that must hold for the distance  $d$ , but he/she did not define  $d$  in the definition. Those two students (D and I) were the only students who attempted to answer by giving the formal definition of the distance in a metric space.

One student defined the distance as a non-negative number and he answered that:

*Distance  $\geq 0$  between two points.* (N)

This student did not show much explanation but he/she seemed to understand distance between two points as a number that is non negative.

Other students (C, H and P) defined the distance as a metric in metric spaces. One of them answered that:

*Distance in metric space depends on the metric  $\Rightarrow$  the metric is the measure of distance in a metric space.* (H)

An answer like this seems to have no sense but these students seem to differentiate between the words 'distance' and 'metric' in a metric spaces.

One student (K) wrote that '*I don't know*' in his/her answer. This student maybe forgot the definition of the distance in metric space because he/she did give an answer to the second question.

From the results of the question '*How do you define the term distance in a metric space?*' we deduce that students gave different answers to this question. Many students have the idea that a distance is a measure of similarity or dissimilarity but they thought of the measurement differently, some of them understand correctly that the measurement is between elements of a set; some others think that the measurement is between functions which could be true, or between metric spaces or metrics. Also some students think of a distance as a non negative number; and some others consider a distance and a metric in a metric space as being different.

The second question was '*How do you explain the term distance in metric space to a friend of yours?*'. The outcomes of this question in the questionnaire are quite similar to the outcomes of the first question. Many students (A, B, D,G, H, K, and O) explained a distance as measure of similarity between two points/things, and one of them wrote that:

*How similar or close two elements in a set are, or "distance" between them. (A)*

Student I could be added to the seven students. He/she explained that:

*Distance is a way of representing the similarity between two objects with a single number.(I)*

This student seemed to understand distance as a measure of similarity between two things and also he/she explained that distance is a number which represents that similarity.

Two students (C and E) thought of a distance as a measure of similarity between functions which could be true in the case of a space which contains functions. One of them explained that:

*Different metric spaces measure completely different things. One could examine the difference and another the similarity of functions. (E)*

This student assumed that some metric spaces measure the similarity and others measure the difference while in any metric space we could measure the similarity and the dissimilarity.

The answer of Student F to this question was the same as his answer to the first question and he wrote that:

*You can take two metric spaces and see what they have in common and what they don't have in common and from this information determine the distance between them.*

This student thought incorrectly that the distance is how similar or different two metric spaces are and that was the same idea he/she tried to give in the answer to the first question.

The student P answered this question by:

*Distance in metric space is just the relation between one point and another.*

This student defined the distance as a metric in metric space in the first question, but here he/she explained that the distance is a relation between two points. So he/she seems to understand that the distance is just a function of two points in a metric space.

One student has the idea of distance as a real life distance. He/she explained that:

*Distance in a metric space is like distance in real life, it can be defined in many ways. E.g. Distance as the crow flies is diff to distance by rail, diff to distance by road etc. (J)*

This student explained distance to a friend as real life distance which could be measured between two points in different ways (as the crow flies, rail, road, etc). He/she seemed to understand that different ways of measuring could be used and so he/she used examples from real life.

Some students (H, L and M) also used examples to illustrate the meaning of distance to friends. One of them explained that:

*I would give several simple examples of how different metrics can be used, and also mention the discrete metric. Standard distance is an obvious one to explain; as everyone knows the difference between 2 and 5 is 3. (L)*

These students preferred to use the simple examples to explain the distance to friends and to show them how different distances could be used. The examples that students used in their explanations might help them to understand the notion of a metric.

The student N answered that:

*The distance between two points.*

This student explained to a friend that the distance is between two points; he/she also defined the distance as a non negative number between two points in the answer to the first question and he/she did not explain more.

We conclude that, many students understand distance as a measure of similarity between two elements or as a relation between two points; others incorrectly think that it is a measure of similarity between two functions/metric spaces; others think of a distance as a number represents the distance between two points; and for other students the examples that are given to them in the course to explain the distance in metric spaces help them to understand the idea of the distance.

### 3.3.2.3 Summary

We see from the answers to these questions, both in the questionnaire and the interviews, that the majority of students think of distance as a method of comparing elements in a set. They seem to be heavily influenced in this by the lecturer's explanations. Many of them, when asked for a definition, point out that distance in a metric space is not the same as physical distance. Only one student in the interviews, and two in the questionnaire, gave a mathematical definition of distance. When explaining the notion to friends, students were more likely to use examples.

### 3.3.3 Students' Responses to the Statements

In the interviews we asked the students their opinions about some statements related to the open set concept; they had to answer whether if they agreed or disagreed with the following statements and to explain their answers:

*Please indicate whether you agree or disagree with the following statements and justify your choice:*

- (a) *A set is open if all its points are the centre of open balls.*
- (b) *A set is open if it lacks its ends.*
- (c) *If a set is not open then it is closed.*
- (d) *A set is open if all its points are near to each other.*
- (e) *A set is open if all its points are similar.*
- (f) *A set is open if its complement is closed.*

Students were asked this question to explore further their conceptions of the concept of open set. All the students answered this question except the 10<sup>th</sup> student and we will show the students' responses to each of those statements:

**(a) A set is open if all its points are centre of open balls.**

All the students agreed with this statement for a variety of different reasons based on their ideas of open sets. For some of the students (4<sup>th</sup>, 5<sup>th</sup> and 11<sup>th</sup>), since if every point in a set is a centre of an open ball, then it means that, the set is a collection of open balls, and thus is open. Some of those students argued that:

*Because there's an open ball within that open set, that all the points have open balls within that set, so it's true because um, the open set will be union of open balls. (4<sup>th</sup>)*

And,

*Um, all its points the centre of open balls, so it's just uh collection of open balls, so that's the open set. (5<sup>th</sup>)*

So those students agreed with this statement because that was how they understand the open set, that is as a union of open balls.

For other students (2<sup>nd</sup>, 7<sup>th</sup> and 9<sup>th</sup>), because all the points in a set are the centre of open balls then for them it means that none of the boundary points are included in that set. Some of those students' answers were:

*I agree, because none of boundary points are included. It means you can have open balls getting to every point which just the smallest radius the further you go out in the set. (7<sup>th</sup>)*

And,

*My way of thinking of open set is that, as close as you get to the boundary of the set which isn't contained in the set, you can still keep taking an open ball around it, so you know if you were to take an open ball around a point and then take another point that's just inside that, then you keep it'll just have to keep getting smaller. (9<sup>th</sup>)*

We can see that those students related the given statement to their conceptions of an open set, that is to the set's boundary. For them since every point is a centre of an open ball then the radius of the open balls at the points near to the boundary gets smaller as the points get closer to the boundary points without including them. Both of these students mentioned that the boundary is not included. The 2<sup>nd</sup> student also indicated the boundary. He said:

*Uh, agree, because that set doesn't really have boundary if all its points are interior points!*

Other students (1<sup>st</sup> and 8<sup>th</sup>) agreed with the statement because it is the definition of an open set for them. One of the students commented that:

*I agree because that's the definition, yes! (8<sup>th</sup>)*

For this student the statement represents the definition of open set and this matches her conception of open set which is based on the formal definition.

One student agreed with this statement for a different reason. She said that:

*I agree because it's the points are an open ball and convergent to one point, I think that. (6<sup>th</sup>)*

This student seemed to have the idea that the points of open ball are close to the centre so they converge to one point which is the centre. When this student tried to explain an open set to a friend, she explained it as a union of open balls but she also added that '*I know you could state it by sequence of convergence as well, where it converges to a number, but I can't remember exactly*' so she seemed to associate the idea of an open set with convergence. The 6<sup>th</sup> student has difficulty with the English language, so it might be that the word 'convergence' has a different meaning for her other from its use with sequences. This student refers to 'convergence' many times in her arguments as we will see later.

The remaining student is the 3<sup>rd</sup> student, this student agreed with the statement but his reasoning was not obvious. He said that:

*I think so, yeah, so you it is just a one point, zero point, so is open, I just I guess, I think so.*

This student was not clear in his comment but he guessed that the statement is true and so he agreed by using his intuition. Recall that I am using Fischbein's definition of intuition '*Intuitive knowledge is immediate knowledge; that is, a form of cognition which seem to present itself to a person as being self-evident (p. 6)*', which for him covers notions such as guesswork, common sense, belief, and insight, and from now on this is the sense of the word that I will employ.

From the results above we found that all the students explained the given statement in terms of their dominant conceptions of open set.

***(b) A set is open if it lacks its ends.***

This statement was given to students to explore how students would react about the end points of a set. The students gave different answers to this statement. Some of

them (1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup>, 4<sup>th</sup>, 5<sup>th</sup>, 7<sup>th</sup> and 9<sup>th</sup>) agreed, the 6<sup>th</sup> student was not sure, another student (11<sup>th</sup>) disagreed, and the 8<sup>th</sup> student gave two answers to the statement.

Those students who agreed with the statement gave different justifications for their answers. For the 5<sup>th</sup> and 7<sup>th</sup> students the word ‘ends’ in the statement sounded like the ends of the real line  $\mathbf{R}$ . Some of them mentioned that:

*Um... yeah that is kind of end line of  $\mathbf{R}$  not including its boundary points, I think so. (5<sup>th</sup>)*

Or,

*By ends then I automatically think of like a one dimensional space, I automatically think of the real line, is a line with, you know, you’re curved brackets being at the ends. So, if the set is not the closure of that set, kind of thing, yeah I agree. (7<sup>th</sup>)*

The 7<sup>th</sup> student seems to be referring to open intervals on the real line. So, it seems that for those students, the real line  $\mathbf{R}$  does not contain its end points and open intervals in  $\mathbf{R}$  do not contain their end points, and based on these examples the statement is true.

The other students (1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup> and 9<sup>th</sup>) interpreted ‘ends’ as boundary points and therefore they agreed with the statement. One student said that:

*Yeah, say if there is no boundary, so that is open ball, open set, I think. (3<sup>rd</sup>)*

For those students the statement is true as it is one of the ways of thinking of an open set.

The 4<sup>th</sup> student agreed with the statement for another reason. She commented that:

*Oh! I never think about it like that. I suppose, an open set like  $(0, 1)$  is an open set, and it doesn’t contain its ends, but I don’t know if that true for every thing so! It’s definitely true that’s an open set, um so I’ll say for that, true.*

From the information given in the definition question this student’s conception of open set is as a union of open balls and she said that she never thinks of open sets as the statement says. However the example she used seemed to agree with the statement and thus she agreed also, however she was not sure if it is true for all sets.

The 6<sup>th</sup> student did not give her opinion and she argued that:

*I’m not sure, I can’t remember that the ends are included or not.*

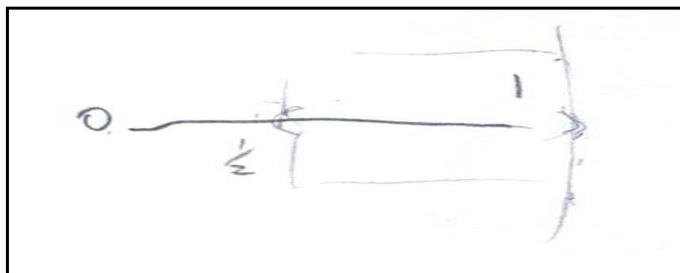
She understood the open set as a union of open balls as she said in the question about the definition, but she was not sure of the truth of this statement because she could not remember.

The 11<sup>th</sup> student, at the beginning agreed with the statement when he said that:

*I would say yes, because if it has its ends, then its end is boundary point, so if it has boundary point then it can't be an open set.*

So in the case of the end points being included in a set then it will not be an open set. But then he changed his mind when he used an example and explained that:

*If your set is  $[0, 1] \cup [2, 3]$  and then you take the point 1 and a half say, using the standard distance, then the open ball will be, around 1 will be sort of, go there, but it will be open because that is not part of the set, I think so, I'd change my mind; I think I'll going to disagree now on that, I'm not sure. (11<sup>th</sup>)*



**Figure 3. 1: The 11<sup>th</sup> student's response to Statement (b)**

This student apparently thought of the metric space  $[0, 1] \cup [2, 3]$  with the usual metric on it and also seemed to take the open ball  $B(1, \frac{1}{2})$  about 1 which is the set  $(\frac{1}{2}, 1]$ . This student was confused when he found that the point 1 is an end point of  $(\frac{1}{2}, 1]$  and the set is still open. The example seemed to be a counterexample for him and thus he disagreed.

The 8<sup>th</sup> student tried to use the definition of open set to think about the statement so she gave two answers to it. At first she claimed that:

*Um, no it's uh, false, I disagree with that because it doesn't, sound like a proper definition.*

It seems that it is the word 'ends' that she objected to because when we replaced the statement by 'A set is open if it lacks its boundaries' she agreed and argued that:

*I'd say agree then, maybe I suppose I just have a picture of the ball and then the boundary around the outside but I know that the definitions might contradict*

*that. So I'm not actually sure. Um..., but it kinda makes logical sounds so my idea is that, I would agree, but by definition I'm not sure. (8<sup>th</sup>)*

This student understands an open set as a set that has an open ball about any point in it as she explained in her definition of an open set. But when she tried to picture a ball and the boundary of it then the statement sounded logical for her and seemed to agree with it and in the meantime she was not sure if her definition of open set would agree with this statement. So this student is apparently indecisive because from her intuition she would like to agree but she was uncertain when using the definition.

From the findings related to this statement we found that some students related the statement to their conceptions of open set, some students were unsure about it because it is different from their conceptions, others used examples to decide their agreement or disagreement.

***(c) If a set is not open then it is closed.***

All the students disagreed with this statement except for the 11<sup>th</sup> student who agreed with it. Most of them (2<sup>nd</sup>, 3<sup>rd</sup>, 4<sup>th</sup>, 5<sup>th</sup> and 8<sup>th</sup>) disagreed for the same reason and the common argument given by them was:

*Uh... no, a set can be open and closed. (2<sup>nd</sup>)*

These students know that some sets can be open and closed at the same time and for that reason they disagreed. They seem to realise that closed is not the opposite of open in this situation (unlike in the use of these words in the English language). However, they do not address the statement correctly, in that they do not consider the case when a set is not open.

Other students (6<sup>th</sup> and 7<sup>th</sup>) disagreed for the same reason and also added another one. One of them said that:

*No, I just remembered that not all sets, they can be open and closed at same time, and uh if it's not open, doesn't necessary mean it's closed. It might be something else. (6<sup>th</sup>)*

For these students, in addition to the fact that some sets can be open and closed, they explained that if a set is not open then it might be closed or could be something else (i.e. not closed). So they gave us more insight into their ideas about the statement. The 7<sup>th</sup> student also remarked that it is possible to have sets that are neither open nor closed as well as ones that are both.

The 1<sup>st</sup> student also disagreed with the statement and he commented:

*No, it is a definition, um a set can be not open, but not open does not mean the same thing as closed.*

This student thought that what he said in his argument is a definition and it seems for him the statement is the negative of that definition so he disagreed. Also both students (1<sup>st</sup> and 6<sup>th</sup>) realise that if a set is not open then it is not necessarily closed, they did not offer explanations of why this is true though.

The 9<sup>th</sup> student disagreed and she gave an example for her argument. She explained that:

*No, well you could have um, you know  $(0,1]$  which is half open, but that is not, it's not fully open, but it's not fully closed because if it's fully closed then it will have to contain all its limit points, but it doesn't contain 0, and if it was fully open then it'll be able to take an open ball about 1 that contains everything, but it doesn't, like if this is subset of  $\mathbf{R}$ .*

The student here gave an example of a set that is not open but also it is not closed and she explained in detail why it is not open and why it is not closed, and by her example showed that if a set is not open does not mean it is closed and that was the reason for her disagreement.

The last student is the 11<sup>th</sup> who agreed with the statement as it sounded true to him. His claim about the statement was:

*Ok, so that would be true. So before if there's a set that can be neither open nor closed and the set is defined to be closed if its complement is open. But the set is not open, so it is not a union of open balls. I wish I knew the correct definition of open set now. So the set is not a union of open balls, so pictorially, the set would contain its boundary, so the complement is open, so the set would be closed. So I would agree.*

This student tried to make a proof of the statement based on his conception of open sets and closed sets and he wished to know the exact definition of an open set to be more certain. He started with a good sentence which is 'before if there's a set that can be neither open nor closed' which means he knows that there is a set that is not open and at the same time not closed, but he did not make use of it to give a final answer to the statement, he might have been unsure of something related to this statement and wanted to figure it out. He thought correctly that if a set is not open then it is not a

union of open balls, then incorrectly thought that if it is not a union of open balls then all the boundary points would be included in it, and thus its complement is open, and if a complement of a set is open then that set is closed by the definition of a closed set. So he started with a set that is not open and ended up with a set which is closed and thus he agreed with the statement. This student's intuition and his picture of the statement led him astray. That is, the incorrect connections between this student's ideas (e.g. 'the set is not a union of open balls, so pictorially, the set would contain its boundary, so the complement is open'), led him to give an incorrect answer to this statement.

From the answers to this question, we see that the students in this group do not seem to have the misconception that closed is the opposite of open. All but one of the students disagreed with the statement and the one who agreed did so after constructing an argument and not as the result of a spontaneous conception concerning the English meaning of the words open and closed. However the reasons that the other students gave were not very strong. Only one student gave a counterexample and the majority of the others disagreed with the statement because they knew of sets that were both open and closed and so they did not really address the question.

***(d) A set is open if all its points are near to each other.***

This statement was given to see if the nearness of points in a set has an effect on the students' conception of open sets. We asked this statement because the lecturer spoke of metrics as measuring nearness or similarity, and we wanted therefore to see if students would accept this statement as a definition. All the students disagreed with the statement except the 6<sup>th</sup> and 9<sup>th</sup> students who agreed with it.

The 1<sup>st</sup> student disagreed for the following reason as he commented:

*No, um, because that is not a definition.*

For this student, the statement is not true because it is not a definition of an open set.

The 4<sup>th</sup> student also disagreed with the statement. For her that is not necessarily true and argued that:

*No that's not necessarily true, because um a set is open if each of its point has as open ball within that set, and just because the points are near doesn't mean that has to be, no, it doesn't have to be open.*

She probably disagreed because the statement does not match her idea of the formal definition of an open set, and according to her the nearness between the points of a set does not mean the set is open. So she seems to judge the statement by comparing it with the definition of an open set.

The 2<sup>nd</sup> and 11<sup>th</sup> students gave similar reasons for their disagreement. One of them said that:

*No, um... you can still have, you can still have a boundary even if the points are near to each other like. (2<sup>nd</sup>)*

The others said, you can have a closed set, which is a very small closed set, where all the points are near to each other. So for these students even if points of a set could be considered as near to each other they realise that boundary points might be included in that set, or that there are closed sets where their points are near to each other and so they disagreed. The 2<sup>nd</sup> student was the only one to query the word 'near' and asked for a definition of it.

The 3<sup>rd</sup>, 5<sup>th</sup>, 7<sup>th</sup> and 8<sup>th</sup> students gave counterexamples to the statement. One of them argued that:

*No, because you can have like a union of two open sets and that would be open, but they might not be beside each other. Just on the real line you can have an open set like  $(-\infty, -100)$  and  $(100, \infty)$  you take the union of them I think it's still an open set but the -100 and 100 are not necessarily near to each other. (5<sup>th</sup>)*

All those students disagreed and to explain their opinions gave examples of open sets where some points are not near to each other.

The 6<sup>th</sup> and 9<sup>th</sup> students agreed with the statement for different reasons. The 6<sup>th</sup> student once again referred to convergence and commented that:

*Is that like saying that they're converging! Yeah.*

It seems for this student if the points are near to each other then it sounds to her like the points converge together and we saw in her answer to the first statement above that this student has the idea that points of open balls converge to the centre, so here she related her conception with this statement and she agreed.

The 9<sup>th</sup> student claimed that:

*Yeah, I suppose so because if every point is contained in some open ball, then that open ball is kinda full of points because you know, it's difficult to explain,*

*but you know, if they weren't near to each other then there would be a finite amount of them, I think.*

This student seems to hold a continuum idea of an open ball and that was clear when she thought of open balls as being 'full' of points. This idea seems to be related to conceptions based on balls in  $\mathbf{R}$  or  $\mathbf{R}^2$ .

We found that the majority of the students disagreed with this statement. Some of them related it to their conceptions of open set (2<sup>nd</sup> and 5<sup>th</sup>), others gave evidence from examples, and some students disagreed because the statement is not the definition of an open set. Also two students agreed with it, one of them related it to her idea of an open set and the other one showed her misconception of an open ball as being full of points.

***(e) A set is open if all its points are similar.***

We asked some students this statement to see the effect of the word 'similar' on students' conceptions of an open set as they have learned that it is helpful to think of metrics as measures of similarity between elements.

Only six students (1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup>, 4<sup>th</sup>, 7<sup>th</sup> and 8<sup>th</sup>) were asked to give an answer to this statement. All of these students gave answers to this statement after first giving their answers to the statement '*A set is open if all its points are near to each other*'. This was in order to see whether they would give the same or different answers to both statements.

From analysing those students' answers regarding Statement (e) we found that some students' answers and reasons for this statement are the same as their answers and reasons for the Statement (d), others gave different answers for different reasons, and some gave the same answers for different reasons.

The 1<sup>st</sup> student disagreed with the statement and answered by:

*No, because there is no definition for that.*

This student's answer to this statement is the same as his answer to the statement (d). He is the only student who gave short and definite reasons to the most of the given statements, and most of his answers are 'there is no definition for that' or 'is it a definition' and he did not give more explanations.

The 2<sup>nd</sup> student also disagreed with the statement and explained that:

*Um... similar to each other, so a metric on that, they would all be the same roughly the same points very close together. Um, it doesn't mean open though, it can still be closed.*

This student found that if the points are similar then there is a metric on the space in which the points would be close together but they could be close in a closed set too. This student appeared to relate the word 'similar' to metrics, and he realised that openness cannot be defined in terms of similarity or nearness.

The 3<sup>rd</sup> and 4<sup>th</sup> students' answers to the statement are different from their answer to statement (d). The 3<sup>rd</sup> student agreed with this statement and argued that:

*They have the same to like a ball or thing, yeah so that's true for similar I think.*

For this student, it seems that the word 'similar' made a difference for him in this statement; he seemed to think that the points are similar means they are in the same ball and thus the set is open. This student has the idea that, if a set is open then there is an open ball in it as he mentioned in the definition question, so he might relate this statement to his conception of open set and so he agreed with it while he disagreed with the statement (d).

The 4<sup>th</sup> student also agreed with this statement and her reason was not far from the 3<sup>rd</sup> student's reason. She said that:

*Yeah I think that's true because um that means that, there's open balls within that open set. Yeah no it is because um, there's an open ball for every point; yeah I think that's true. (4<sup>th</sup>)*

This student seems to relate 'similar' with open balls and think that, points are similar means they are all in open balls and so the set is open. She might link similarity with metrics and link that with open balls and thus with open sets and recall that her definition of an open set was based on the formal definition which she refers to again here.

The 7<sup>th</sup> student disagreed with the statement. He commented that:

*I haven't ever really thought of openness as being anything to do with the closeness of elements. I thought that was all more to do with compactness, and to things like that. I always thought just the open and closed, was just kinda the boundary part and how neat it was, how define the edges work.*

This student never thinks of open and closed sets in terms of closeness and always thinks of them in terms of the boundary points and fuzziness/continuum between

points as he mentioned in the question about the definition of open set. So he disagreed as this is different from his conception of open and closed sets.

The 8<sup>th</sup> student disagreed with this statement as she did with the statement (d) but her reason was different. She claimed that:

*So the distance between them would be the same, no! The distance is similar, but if the distance isn't similar, if like one is close, and one is far apart. Um..., no, I'd say even if the word dissimilar you can probably find an open ball around them. So I would disagree.*

This student also related the word 'similar' with the notion of distance, and she seemed to think that if the distance is similar then points are close and if the distance is dissimilar then the points would be far apart. And she thought that even if the points are far apart they might be included in an open ball and thus she disagreed.

From the analysis of the responses to Statement (e), some students seem to think incorrectly that if the points of a set could be considered as similar then there is an open ball which contains them. While other students realised that the notion of similarity is not precise enough to use to define openness, and on its own it is not sufficient to decide if a set is open or not. Also, it seems that some of these students were confused about how to measure similarity by metrics. It might be that all students know that they can think of a distance as a measure of similarity but they seem to interpret similarity differently and that affected their answers to this statement.

***(f) A set is open if its complement is closed.***

This statement explores if the students have thought of an open set in terms of the definitions of a closed set and to see how they use their ideas to judge the statement. Students were given the definition of a closed set as 'A set is closed if its complement is open'. All students except the 7<sup>th</sup> student answered this question.

The 2<sup>nd</sup> student agreed with the statement and did not give a reason for that. Other students (1<sup>st</sup>, 5<sup>th</sup> and 6<sup>th</sup>) agreed with the statement because for them that is the definition. One of them said that:

*Yeah, I think that is true, um...that is by definition. (1<sup>st</sup>)*

So for this student the statement represents a definition, and so it is true. These students might or might not have noticed the subtle difference between the definition of a closed set and this statement.

The 3<sup>rd</sup> student could not give his opinion about the statement and said that:

*I don't know, I know if set is closed then its complement should open. For open, it could be open or not. I'm not sure, so you can't to say this if it's open, it's closed. It could be closed that what I know.*

This student knew the definition of a closed set and saw that this statement was slightly different but he was not sure about this statement. He did not seem able to use the definition of closed sets to reason this out.

Three students (4<sup>th</sup>, 8<sup>th</sup> and 9<sup>th</sup>) disagreed with the statement and their answers were based on examples which they provided, but their arguments were different from each other. The 4<sup>th</sup> student claimed that:

*No never thought of it in that way before! No, that can't be true, um because um an open set like  $(0, 1)$  its complement is real line minus, or the  $\mathbf{R}^2$  minus  $(0, 1)$  and um that's open as far as I think.*

Recall that this student, as we showed earlier, understands an open set as a union of open balls. She tried to think of the complement of the set  $(0, 1)$  within the metric space  $\mathbf{R}$  or  $\mathbf{R}^2$  but she claimed incorrectly that this complement is an open set. So her incorrect thinking about this specific example led her to give an incorrect answer.

The 8<sup>th</sup> student commented on her disagreement by:

*Ok, the set is closed if its complement is open. This is, a set is open if its complement is closed. I'm just try to think of different metric spaces that I know are open but the ones that I'm thinking of their complement is closed. Discrete metric usually contradicts things. (A, discrete) is a metric space, so this is a subset of the metric space, um  $(0, 1)$  is an open interval in discrete metric is open, um I don't think its complement is closed. I'm going to disagree ok.*

This student was sure of the definition of a closed set, but to make sure of the given statement she tried to think of an example to examine this statement. She used the discrete metric space to see if the statement is false as she says that 'Discrete metric usually contradicts things', so it is a good source of counterexamples. She knows that the set  $(0, 1)$  is an open set in her metric space but incorrectly assumed that the complement of  $(0, 1)$  is not closed, while the truth is that any subset of a discrete

space is both open and closed so the complement of  $(0, 1)$  is both open and closed, and therefore by her incorrect thinking concerning the complement she disagreed with the statement. Even though this student did not give a correct answer, she showed some mathematical maturity by searching for a counterexample and by realising that the discrete metric space might be a good place to find one.

The 9<sup>th</sup> student seemed to get confused with some infinite sets when she disagreed with the given statement. She argued that:

*A set is closed if its complement is open, right! I'm never thought about it. So if you have an open set  $(0, 1)$  yeah, the complement of that would be closed I think. Oh no! It's not closed because this would be  $(-\infty, 0] \cup [1, +\infty)$  and I don't know what happens at infinity. I don't think it is, because if it closed in there and closed here, and that was a subset of  $(\mathbf{R}, d_{st})$ , like the whole metric space is open, so that, hmm, no I don't think that the complement of an open set is closed. I don't think it's either, because you could have sequence that goes to  $-\infty$ .*

At the beginning this student apparently agreed with the statement when she commented that the complement of  $(0, 1)$  would be closed. But then she seemed to get confused when she looked at  $-\infty$  and  $+\infty$  in the complement. She argued that the complement is not open which is true and added that it is not closed because for her there are some sequences that tend to  $-\infty$ , where  $-\infty$  is not part of that set. She seems to conclude that  $-\infty$  would be a boundary point of this set and thus the set does not contain all of its boundary. So this student started well but her conception of infinity dominated her thinking on the statement and thus she gave an incorrect answer.

The 11<sup>th</sup> student began by disagreeing with the statement. He said that:

*No, because you can have an open set but it's also closed set so its complement is open, so no. But that just means is not every open set has closed complement. But if the complement is closed, does that mean the set is open? Um the reason being that I suppose pictorially again, if the complement is closed, the complement contains the boundary so, oh no!*

This student at the beginning thought of a set that is both open and closed and found that since it is closed then its complement is open and so for him the complement is open and not closed and thus he disagreed. He appeared to consider the complement of the set as being closed and forgot to consider also the complement of the same set as being open as well and therefore he gave an incorrect answer. Then he wondered if

the complement of a set is closed means that the set must be open. He then realised that he had thought of the statement incorrectly and commented that:

*Hang on; the set is the complement of its complement. , I'm going to draw out this thing again sorry. Ok, so the definition of closed set is a set its complement is open. The complement of a complement of a set A is the set A, so therefore by definition I agree. (11<sup>th</sup>)*

The student thought of the statement in an easier way when he used the property 'a complement of a complement of a set is the set itself'. So he chose a set A whose complement was closed and then by the definition of closed set the complement of the complement of A must be open, and so by the property the set A is open and therefore he agreed with the statement. This student was the only one to appreciate what the statement was saying and to prove that it was correct.

The findings of the analysis of the responses concerning this statement showed that the students have different ideas related to this statement. Some students see no difference between this statement and the definition of a closed set. So for them the statement is a definition and they agreed. Others had never thought of open sets like that before and tried to use examples to test the statement but they were incorrect in their reasoning. With the statement one student could not give an answer because he was not sure of it. Another one made use of mathematical properties and definitions to give the correct answer.

### **3.3.3.1 Summary**

From the results of students' responses to all statements, we conclude the following points:

- All the students linked the Statement (a) to their conceptions and definitions of open sets.
- Regarding the Statement (b) some students linked it to their conceptions of open sets; for some students who have many conceptions of open sets this is another idea of an open set; others were not sure about the statement because it differs from their conceptions; and other students used examples to examine the statement.
- Concerning the Statement (c), the students did not show use of definitions and of their conceptions of open sets that they gave when asked of definitions except

the 11<sup>th</sup> student who used his conception and intuition with this statement. Most of them know that there are sets that are both open and closed. Another student used example and intuitions for this statement.

- The Statement (d) does not match the definition of open sets for some students; others used counterexamples to disagree; other students related it to their ideas of open balls and open sets.
- Statement (e) seemed to raise different thinking related to the word ‘similar’ when students were considering an open set. For many students the word is used when thinking of metrics. In the responses to Statements (e) and (d), it was evident that some students seem to have the continuum idea of open sets when reasoning about these two statements.
- For some students the Statement (f) seems to be a definition; others used examples to test the statement; one student could not judge the statement; and only one student used a property and a definition correctly.
- Only one student showed a correct use of a formal definition and he showed it only in one statement. Many students tried to use examples to examine the statements and some of those students who used examples seem to get confused when they found that the examples they used conflict with their ideas of open sets and they usually followed their examples to answer the statements.
- Some statements seemed to be considered as definitions by the students and some others were rejected because they did not sound like definitions.
- Some students had problems when thinking of complements of sets.

### **3.4 Conclusion**

In this chapter I addressed students’ thinking about definitions in both general mathematics and in the specific topics of open sets and distance in metric spaces.

The questions which were designed to discover students’ views about mathematical definitions showed that students seemed to value the significance of definitions in mathematics in general, however most of them were not comfortable with them and they seemingly have some trouble when trying to understand them especially the ones in abstract context. Students also appeared to adopt many strategies in order to grasp a new mathematical definition, and these strategies include: visualisations, focusing on

the main elements of the definition, translating to different words, using examples, making intuitive ideas about it, and memorisation.

In this chapter I looked also beneath students' conceptions about an open set. I have found that students have several conceptions concerning this concept and these conceptions are based on the formal definition; the boundary idea; and the union of open balls. Concerning the notion of distance in a metric space, the results pointed out some of the conceptions related to this notion, namely: conception based on the formal definition; conception based on the measure of similarity between points; conception based on the difference between points; conception based on the comparison between points; conception of distance as a real number; and conception about distance as different from physical distance. Most of the students seemed to be affected by the lecturer's explanations which motivated them to think of distance in metric spaces as a means of comparing elements in a set instead of thinking of it as similar to physical distance.

In accordance with the statements about open sets, the findings revealed that many students seem to rely on their conceptions of open sets that they gave in the definition question when reasoning about the statements. When students accepted statements, it was often because they deemed them to be definitions or because of the use of examples. Some other statements were rejected because they did not sound like definitions or because of counterexamples. I found some instances of cognitive conflict; for example one of the students who used examples to argue about statements seemed to get confused when he found that the examples he used conflicted with his ideas of open sets and he eventually followed the example to answer the given statement.

## **Chapter 4**

# **Analysis of Students' Responses to the Task Questions**

### **4.1 Introduction**

The main mathematical concept in my study is that of the open set. Students learn about different types of open sets during the metric spaces course and had some experience of open sets in Euclidean space from other previous courses. So it was possible for us to ask students different questions involving this concept. There are six problems in our study which are related to the concept of open sets. Students in the questionnaire were asked to solve four problems (1, 2, 3 and 6) and students in the interviews were asked to solve two or three of the four problems 2, 3, 4 and 5. In this chapter I will analyse each problem and the analysis which I carry out here is based on the types of arguments given by students in the questionnaire and in the interviews to justify their answers to the problems. Using the responses to these problems I will try to reveal how students regard open sets and also to discover the dominant concept images that students used most frequently.

### **4.2 Analysis of the Responses to Each Problem**

In this section I will consider the responses of the students to the tasks that appeared on the questionnaire and in the interviews. I will consider each problem separately below.

#### **4.2.1 Analysis of the Responses to Problem 1**

Problem 1 was given to the students in the questionnaire. The text of the problem is given below. This was the first and the easiest problem that students had to work on. I chose this as the first task on the questionnaire because I wanted the students to be

relaxed and comfortable. They are very familiar with these kind of sets and with this metric space so I thought that they will not have any difficulty working with this question. However, the results showed that even this question, which all students had seen many examples of before, caused problems and many students provided incorrect answers.

**Problem 1:**

Consider the metric space  $(\mathbf{R}, d)$  where  $d$  is the standard metric, and let  $A = [0, 2)$ .

Is the set  $A$ :

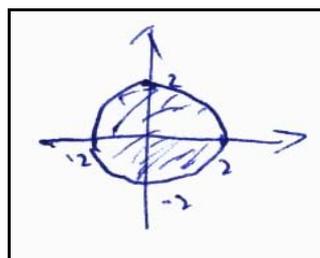
Open                      Closed                      Both                      Neither

Please explain your answer!

(This problem is number 3 on the questionnaire in Appendix 1)

In this problem we consider the set  $A = [0, 2)$  as a subset of  $\mathbf{R}$  (the real line) with the standard metric. To decide if  $A$  is open, the students have to see if every point in  $A$  is the centre of an open ball in  $\mathbf{R}$  that is contained in  $A$ . Clearly the set  $A$  has no open ball in  $\mathbf{R}$  about the point 0 that is still included in  $A$ , therefore the set  $A$  is not an open set in  $\mathbf{R}$ . It is also not a closed set because the set  $A$  does not contain all of its limit points (i.e. the point 2 is a limit point for  $A$  but it is not included in it). Another way of seeing this might be to consider the complement of  $A$  which is  $(-\infty, 0) \cup [2, \infty)$ . This set is not an open set in  $(\mathbf{R}, d)$  as there is no open ball in  $\mathbf{R}$  centered at 2 which belongs to  $(-\infty, 0) \cup [2, \infty)$ . Thus the set  $A$  is neither open nor closed in this metric space.

The students gave different answers to this problem. Three of the students (C, N and P) answered that the set  $A$  is an open set. Students C and N did not comment further on their answers and just drew a picture of the set. The drawing of the interval  $[0, 2)$  on the real line by Student C was correct however the other student (N) drew it incorrectly. He/she drew:



**Figure 4.1: Student N's illustration of Problem 1**

He/she drew a circle on  $\mathbf{R}^2$  and shaded the area inside that circle. So these two students seemed to use their intuitions to imagine and draw the set, but their pictures led them to provide incorrect answers. The answer given by the other student of this group of three was different; he/she seems to have the conception that if a set is not closed then it is open, and this conception was evident in his/her argument:

*Open, it does not contain all its limit points so it cannot be closed. (P)*

So he/she correctly used a theorem on closed sets to conclude that the set is not closed and then, chose the answer *Open* seemingly because the set is not closed. It might be that the meanings of the words *open* and *closed* in everyday life (where if a thing is not closed means it is open and vice versa) play a big role in building the concept image of the mathematical terms *open* and *closed* for this student. Thus it seems that a spontaneous conception (Cornu 1991) is being held by this student.

Only one student out of all the students who did the questionnaire answered that the set A is a closed set. This student wrote that:

*Closed,  $[0, 2)$  is not open.  $[2, \infty) \cup (-\infty, 2)$  is closed. (K)*

This student wrote correctly that the set A is not an open set but the reason was not explained. The student also found the complement of A correctly, but commented incorrectly that the complement is a closed set and once again the reason was not explained. But for the whole response for this question he/she chose the answer 'Closed'. From this student we can see that there are two possible reasons for his/her misconception: one is that the student answered that the set is closed because it is not open which might be due to a spontaneous conception from everyday life as described above; the second one might be the difficulty of thinking of sets that extend to  $\infty$  on one side (i.e. are they open or closed?). In this student's argument, the evidence for the second reason was not clear, but I gave this possibility because we have seen a student before (in Chapter 3) who struggled with sets that extend to  $\infty$  on one side. Recall that the 9<sup>th</sup> student in the interview became confused when considering the complement of the set (0, 1) and she did not accept that its complement  $(-\infty, 0] \cup [1, \infty)$  is a closed set and in that case, as we have seen in last chapter, the reason was related to the fact that the intervals extended to  $\infty$  and  $-\infty$ .

I found that three of the students who answered this question said that the set  $A$  is *both* open and closed, and all of them used their intuitions which seemed to be based on their previous knowledge to answer the problem. Their answers are given below:

*Both, the subset is closed at the point 0 and open at the point 2. (E)*

And:

*Both, it is bounded below and not bounded above. (F)*

And:

*Both, it is both because it includes 0 but does not include 2. (M)*

These three students seem to have the same idea but they each expressed it in their own words. The reason for these arguments might be their previous experience with this kind of set. The first encounter for the students with this kind of set was probably when they learned about it as a half open or half closed interval. This might lead them to think of the interval as open on one side and closed on the other and might be the reason that they said it was both open and closed. Note also, the word “bounded” which is used by Student F does not seem to refer to the mathematical meaning of this word. The student may be referring to the notion of *boundary* rather than *bounded* here. It is likely then, that all three of these students have concept images based on previous knowledge.

In the analysis of students’ answers, I also observed that many students responded correctly that the set  $A$  is *neither* open nor closed. Five students used their idea of the formal definition of open set to answer that the set  $A$  was not open and either used the definition of a closed set or a theorem on closed sets to answer that  $A$  was not closed, and thus chose the answer ‘*neither*’. A common response was:

*Neither, it isn't closed as it doesn't contain all its limit points (2 is a limit point).*

*It isn't open as you can't draw an open ball around 0 which will be in  $A$ . (G)*

One student gave the answer *neither* but did not use any definition or formal statement to explain his/her response and only commented that:

*Neither, it is half open. (I)*

This student used his/her intuition which seems to be based on previous knowledge. It might be that for him/her the set is half open and so it is not fully open and also it is half closed so it is not fully closed and this might mean for the student that it is neither open nor closed. The other two students, out of the 16, did not attempt this problem.



(This problem is number 5 on the questionnaire in Appendix 1, and it is number 1 on the interview task problems in Appendix 2)

The first part of this question asks whether the finite set  $B$  is an open ball in the metric space  $(\mathbf{Z}, d_{\mathbf{Z}})$  or not. According to the definition of the open ball above, the set  $B$  is an open ball if there is some element  $a \in B$  and  $r \in \mathbf{R}$  such that  $B = B(a, r)$  (that is, if all the elements of  $B$  are at distance from the centre  $a$  that is strictly less than some radius  $r$ ). From the set  $B$  above it is clear that the distance between any element and the centre (which is  $m$ ) is 1 or 0, and the only elements of  $\mathbf{Z}$  which are closer than a distance of 2 from  $m$  are  $m-1$  and  $m+1$ . Thus, in the metric space  $(\mathbf{Z}, d_{\mathbf{Z}})$  the set  $B$  is an open ball with centre  $m$  and radius  $r \in (1, 2]$ . It might have been easier for students if I had asked this question directly, that is, I could have asked them, ‘what are the elements of the open ball  $B(m, 2)$  in  $\mathbf{Z}$ ?’. But the question I asked tests the students’ understanding of what an open ball is; their understanding of the space  $(\mathbf{Z}, d_{\mathbf{Z}})$ , and it also gives insight into the methods (definition or conceptions) that students use in their answers. In the second part, this question asks if there is a subset  $C$  of the given set  $B$  which is an open ball, and we can see that the sets  $\{m-1\}$ ,  $\{m\}$ ,  $\{m+1\}$  and  $B$  itself are open balls in  $(\mathbf{Z}, d_{\mathbf{Z}})$  which are also subsets of the set  $B$ . I will analyse the students’ responses to this question below; I will deal first with the responses of the students in the interviews and then I will consider the responses on the questionnaire.

#### **4.2.2.1 The Interviews**

All students except for the 10<sup>th</sup> student answered this question. The students’ responses differed. When they read this question, some of them understood it correctly and tried to see if the set  $B$  is an open ball, and some of them misunderstood it and tried to see if  $B$  is an open set. In either case, there was not much difference between students’ answers because they all have to think about open balls, however we divided their answers to the first part of the question into two groups.

##### **4.2.2.1.1 Students who tried to see if $B$ is an open ball**

There were seven students in this group (2<sup>nd</sup>, 4<sup>th</sup>, 5<sup>th</sup>, 6<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>). Five of them agreed that  $B$  was an open ball (2<sup>nd</sup>, 5<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) and one of them disagreed (4<sup>th</sup>) and the 6<sup>th</sup> student could not give an answer.

The students who agreed that the set B is an open ball gave different reasons for their answers. Some seemed to use the definition of the open ball and got the correct answers for the centre and radius of the open ball. For example:

*So, if you had a ball around  $m$  and radius was  $3/2$ , [...] that would include  $m$ ,  $m+1$  and  $m-1$ , but it wouldn't include anything else, but it would still be an open ball. (9<sup>th</sup>)*

And,

*If you take, ok, so  $m$  will be the centre point of this ball, if it is an open ball, so, if a radius is exactly 1 then, neither  $m-1$  nor  $m+1$  would be in it. But if one was taking the open ball of radius  $3/2$ , then they would be contained in it, and nothing else would be, so yes. (11<sup>th</sup>)*

The following student agreed also that the set B is an open ball and she was the only student who argued that this set could be also a closed ball ( $\overline{B}(m, 1)$ ) when you take the radius to be 1. She commented that:

*I think I said yes and no, because it depends on what you take the distance (radius). If the radius of the ball is one then it was closed and if the radius was 2 then it was open, uh... centre be  $m$ . (5<sup>th</sup>)*

It is interesting to look at the answers that are very close to the correct answers and also at the incorrect answers. One student was very close to the students who got the right answer. This student argued that:

*Yeah it's open, um, because it's contained, all the elements in it are contained in the big set  $\mathbf{Z}$ , yeah  $m$  is the centre of the ball, it's just distance one way from this element, one way from that element, and radius of the ball is between  $1 \leq r < 2$ . (2<sup>nd</sup>)*

For this student the set B is an open ball with centre  $m$  and the radius is  $1 \leq r < 2$ . It is as if he mixed up distance with the radius in the left hand side of the inequality, because based on his choice of  $1 \leq r < 2$ , the radius  $r$  could be 1 but the ball  $B(m, 1) = \{m\} \neq B = \{m-1, m, m+1\}$ .

Another student said that she had not been asked if this kind of set is an open ball before so she could not picture this set, thus she could not tell. But by her intuition B could be an open ball about  $m$  of radius 1:

*I don't think I have seen a set like that and been asked if it's an open ball; so I can't really picture it. [...] but if I was guessing, I'd go with yes. It's like*

*restricted kind of the values are kinda close together. So it would have to seen kind of radius, but something to do with 1, and I would say  $m$  be the centre. It be able to be all held in some kind of shape, I would say is an open ball (8<sup>th</sup>)*

So, this student is not familiar with open balls in this kind of situation, but she seems to be influenced by the form of the set (e.g. restricted, values close together and held in shape). She guessed that the centre is  $m$  and the radius is 1, so she seemed to confuse the radius with distance to the ‘end-points’ and her intuition led her to give an incorrect answer for the radius of the ball. The student might have gotten used to the radius of a ball being specified from the centre and the end points. For instance, the interval  $(1, 3)$  on the real line is an open ball in  $\mathbf{R}$ , students can simply deduce that the centre is 2 which is the middle point and the radius is 1 which is the distance between this centre 2 and either of the end points. This kind of routine method of specifying the radius could be a reason why some students answered that the radius was 1 in this open ball and therefore it is likely that previous knowledge plays a role in students’ conceptions.

One student thought that this set  $B$  could be an open ball and the centre would be  $m$  and radius is 1, like the previous student, but she quickly changed her mind because by her intuitive conception, these kinds of limited (finite) sets cannot be open balls and the set  $B$  is finite and it has only three points, therefore it is not an open ball. This student claimed that:

*I think I’m going that,  $r$  is 1 and  $x$  is centre  $m$ . But, no, that is not open because it doesn’t contain all the points. It’s only contains these three points, it’s limited meets these three points. (4<sup>th</sup>)*

We noticed that, this student firstly tried to make a reasonable deduction that there is a centre and a radius for the set  $B$ , but her continuum idea about open balls stopped her from continuing with her first thoughts. She has the idea that open balls consist of unlimited (or an uncountable number of) points, and the set  $B$  contains only three points so it is limited so it cannot be an open ball. Thus she eliminated her first thoughts and went with her continuum idea which dominated her conception.

Another student seemed to forget how to use the definition of an open ball. She argued that:

*I try to figure out the open ball, that’s the open ball; it is open ball of radius  $r$  about  $x$   $B(x, r)$ . I’ve forgotten how to do that, I’m sorry. (6<sup>th</sup>)*

This student knew how to represent an open ball using its centre and radius but she did not know how to define it and so she could not proceed with the problem.

From the results we found that, some students (5<sup>th</sup>, 9<sup>th</sup>, and 11<sup>th</sup>) used the definition of an open ball and so gave correct answers to the question. They, along with the 2<sup>nd</sup> student, seem to have no problems working in this unfamiliar metric space. For the other students, previous knowledge seemed to play a role in their conceptions of open balls, and we noticed that clearly with the 2<sup>nd</sup>, 4<sup>th</sup> and 8<sup>th</sup> students when they thought that the radius of the set B was 1. Moreover, for some students the intuitive conceptions of open balls such as the continuum idea of an open ball have a significant effect on to the formulation of their concept image, we saw this here for instance with the 4<sup>th</sup> student.

#### 4.2.2.1.2 Students who tried to see if B is an open set

I decided not to exclude the answers of this group from the analysis of this question because in order to see if a set is open or not, formally, you need to think of open balls. There were three students in this group (1<sup>st</sup>, 3<sup>rd</sup> and 7<sup>th</sup>), two of them (1<sup>st</sup> and 3<sup>rd</sup>) agreed that the set B is an open set and the other one (the 7<sup>th</sup>) said that the set B cannot be an open set. The reasons given by the two students, who decided that B is an open set, were not too different from each other.

The 1<sup>st</sup> student tried to use his own definition of open set. Recall that when asked for a definition of an open set, this student gave an answer which was based on the formal definition; however, when working on this problem he seems to use a different (incorrect) definition. He seems to think that the set B is open if he can find one open ball inside it. This student's idea seemed to be that, if one open ball could be drawn about the middle element  $m$  of set B and that open ball is included inside B then the set B is open. The student commented that:

*B is open if we can draw an open ball around 1 which is inside the set. If its centre at 1 then the open ball would include 0 and 2. Um..., if the radius is greater than 1 then it would be outside the set, so, then it wouldn't be an open set. No actually no! I think it might be open, cause if um..., you take a radius less than 1, then it just has the point 1 inside it, so then it is open. (1<sup>st</sup>)*

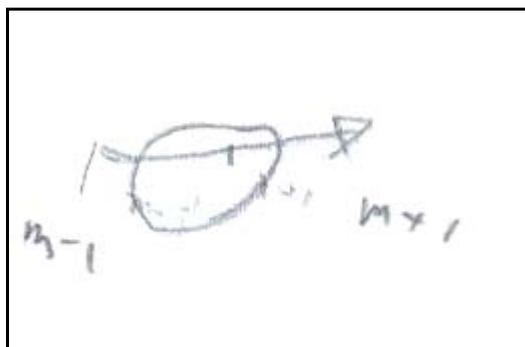
This student replaced  $m$  by 1 so that he worked with the set  $B = \{0, 1, 2\}$ . For him, the set B itself cannot be an open ball because this would require the radius to be

greater than 1 about the centre  $m$ , and he says that such a ball would be outside the set  $B$ . But when he made the ball smaller with radius less than 1 about  $m$ , this open ball is included in the set  $B$ , and therefore, according to him,  $B$  is an open set. This student's conception of open set is based on visualising a ball inside the given set which cannot exceed the end points of that set. This student seems to understand the metric space  $(\mathbf{Z}, d_{\mathbf{Z}})$  as he did not think of other points like those on the real line between 1 and 2 which are not in  $\mathbf{Z}$ , but may have a problem with his visualisation of open balls especially at the end points, and he also seems to think that a set is open if it contains an open ball.

The other student's idea is also that if you can find an open ball in  $B$  then the set  $B$  is open, and he commented that:

*Yeah I think so, it's open because you can..., B of a radius  $m$  so will be inside B. If the set has open ball so will be open. Oh! It can be,  $m$  be centre, so this is  $m + \frac{1}{2}$  and this says  $m - \frac{1}{2}$  will be inside  $\mathbf{Z}$ ,  $m$  so is here as  $m-1, m+1$  so this for the centre, [...] yeah, I find another ball inside it, so it should be open. (3<sup>rd</sup>)*

He was not clear in his work. At first he confused the centre with the radius and said that the radius is  $m$ , but then realised that the centre can be  $m$ . He imagined the set  $B$  on a line centred at  $m$  with ends at  $m-1$  and  $m+1$ , and he also mentioned that  $m - \frac{1}{2}$  lies between  $m$  and  $m-1$ , and  $m + \frac{1}{2}$  lies between  $m$  and  $m+1$ .



**Figure 4.2: The 3<sup>rd</sup> student's response to Problem 2**

Using his picture he found that the points  $m - \frac{1}{2}$ ,  $m$  and  $m + \frac{1}{2}$  are between the end points on the line, so he drew a ball (circle) about these three points and this ball is still inside the end points. Therefore, he said the set  $B$  is open because it contains an open ball inside it. However the open ball seems to be the interval  $(m - \frac{1}{2}, m + \frac{1}{2})$  which is open in  $(\mathbf{R}, d)$  but is not a subset of  $(\mathbf{Z}, d_{\mathbf{Z}})$ . So the latter student had a similar idea to the former one, the difference is that the latter did not understand the metric space

$(\mathbf{Z}, d_{\mathbf{Z}})$  as he used  $m+1/2$  and  $m-1/2$  which are not elements of the set  $B$  (which is in  $\mathbf{Z}$ ). Note that this student drew a circle (in  $\mathbf{R}^2$  when attempting to picture a ball in  $B$ ). It may be that the word ‘ball’ carries significance from its meaning in the English language.

The third student in this group, who tried to see if the set  $B$  is an open set, decided that it was not open and tried to use the definition of an open set to support his claim. He also thought of the boundary idea at the same time:

*Um... first glance I'd say no. I'm gonna have a bit more of look at it. So, let's see, we have a ball of radius  $r$  about  $m-1$   $B(m-1, r)$ , so that means there's some  $r > 0$  which will be contained in the set, but you know, this is cut off exactly at  $m-1$ , so there's no  $r$  that'll cover this without going outside your  $m-1$ . So, no, it's not going to be open. (7<sup>th</sup>)*

For him there is no open ball about  $m-1$  because it is an end point and for him that means that distance to boundary at this point is 0. That contradicts the definition of open ball where any open ball requires  $r > 0$ , thus he says that the definition is not applicable and then the set is not open.

We can see that the 1<sup>st</sup> and 7<sup>th</sup> students seem to have misconceptions concerning the radius of the ball, especially the radius of balls in metric spaces with isolated points, and that led them to use the definition incorrectly. They did not think of the definition of the ball in this metric space but rather they just thought of drawing the radius of the ball, because for both of them the radius cannot exceed the end points of a ball. I think these students have not met examples like this metric space previously and their concept image is based on the picture of the ball and it is built through working a lot with balls in  $\mathbf{R}^n$ . Also the first student has the misconception that if there is one open ball inside a set then that set is open.

#### **4.2.2.1.3 The Second Part of Problem 2**

This question which we analyse here had another part; this part was not given to all interviewed students and only four students (7<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) were asked to do this part. The second part of the question was:

*Can you find an open ball  $C$  which is a subset of  $B$ ?*

*Yes*

*No*

*Explain your answer!*

Two of the students gave  $B(m, \frac{1}{2})$  as their answer, which is correct as  $B(m, \frac{1}{2}) = \{m\}$ . They were able to use the definition of  $d_{\mathbf{Z}}$  and of an open ball to reach this answer. One of them argued that:

*Um..., if you have a ball about  $m$  of radius  $\frac{1}{2}$  then, that's just  $m$ . Yes I think is open because this is only one point and you can take a ball around it then. (9<sup>th</sup>)*

The other one said:

*Ok, um..., an open ball of centred at  $m$ , but with radius  $\frac{1}{2}$ , I'm so would say yes, and that would be it. (11<sup>th</sup>)*

Another student used her intuition to guess the answer which is the ball of radius  $\frac{1}{2}$  and centre  $m$  and she commented that:

*Um..., if you would,  $m$  and  $m + \frac{1}{2}$  and  $m - \frac{1}{2}$ . If my way of thinking is the right way of thinking, then that would work. (8<sup>th</sup>)*

This student incorrectly thought that, the set  $C$  would be  $\{m - \frac{1}{2}, m, m + \frac{1}{2}\}$ . We can see that she represented the elements of  $C$  incorrectly as a subset of the set  $B$ , and rather than thinking of the metric space  $(\mathbf{Z}, d_{\mathbf{Z}})$  which the set  $B$  belongs to, she seemed to be influenced by the metric space  $(\mathbf{R}, d)$ , as the points  $m - \frac{1}{2}$  and  $m + \frac{1}{2}$  do not belong to the set  $B$ . That is she seemed to think of the ball  $B(m, \frac{1}{2})$  incorrectly within  $(\mathbf{Z}, d_{\mathbf{Z}})$ . So this is evidence of the power that the set  $\mathbf{R}$  has to formulate students' conception of open balls.

The other student considers the ball of radius  $\frac{1}{2}$  around  $m$  inside the set  $B$ , but he discounts this set to be an open subset of  $B$  and he claimed that:

*I could find an open ball around  $m$ , um..., yes. If you make  $r$  smaller than 1, um..., sorry hold on! Um..., so we have  $r$  is say a  $\frac{1}{2}$ . [...] um..., a subset of  $B$  which is open, no, oh! We'd only got three elements, but these elements all have space of the exactly one. So you either have a gap of 0 or 1 between them. There is no kind of fuzziness in between, so you can't make it open. Like, it'll either contain them or not. (7<sup>th</sup>)*

We have seen previously that the notion of 'fuzziness' is very important in this student's concept image of open sets, and the absence of fuzziness here in this set  $B$  means that for him  $B$  is not open and does not contain any open balls. This student does not seem to consider the definition of  $d_{\mathbf{Z}}$  here but seems to be working instead in  $\mathbf{R}$  with the standard metric. Thus, for him, any subset of  $B$  cannot be an open set because any subset will have a gap of exactly 0 or 1 between the points and his

conception of open sets requires fuzziness between the points where gaps cannot be determined. Therefore for him, any subset of  $\mathbf{Z}$  cannot be an open set because there are no open balls about the end points and there is no fuzziness between them (the fuzziness that he seems to conceptualise is likely to be related to  $\mathbf{R}^n$  with the usual metric where the radii of open balls about points of a set gets smaller as you keep going closer to the boundary of a set without actually reaching them).

From the answers to the second part of the question, we found that different conceptions influence students' concept images. Some students succeeded in their answers, so their concept image might be based on definitions and intuitions together. However the students whose intuitions dominated their concept images and who did not refer to the metric  $d_{\mathbf{Z}}$  did not succeed with the correct answers. Some of them thought that there is a subset of  $\mathbf{B}$  which is an open ball but they represented the points incorrectly, for example the 8<sup>th</sup> student (arising from a confusion of  $\mathbf{Z}$  with  $\mathbf{R}$ ). What we also notice is that one student (the 7<sup>th</sup>) found an open ball inside  $\mathbf{B}$  but he did not consider that open ball as an open subset of  $\mathbf{B}$ , so it seems that his conceptions of open ball and open set are different (it seems to be important for him that open sets must have fuzziness between their points). For those students who used their intuition, their concept image apparently was based on the Euclidean  $\mathbf{R}^n$ .

#### 4.2.2.2 The Questionnaire

All the students who did the questionnaire (except for Student B) tried to answer Problem 2 stated above, in particular the first part. Concerning the first part of the problem, most students tried to address whether the set  $\mathbf{B}$  could be an open ball in  $(\mathbf{Z}, d_{\mathbf{Z}})$ , except for two students (K and O) who interpreted the question as asking if the set  $\mathbf{B}$  is an open set.

From those who tried to see if  $\mathbf{B}$  is an open ball, seven (out of 14) agreed that the set  $\mathbf{B}$  would be an open ball and gave various different reasons. It is possible that one of these students (Student A) had a conception based on the definition of the open ball and so he/she ended up with the correct answers for the centre and the radius of the ball, this student wrote that:

*Yes,  $m = \text{centre and radius} \in (1, 2)$  e.g.  $3/2$ . (A)*

Another one commented that:

*Yes, centre  $m$  and radius,  $m+2$ . (H)*

This student could have a conception based on the definition of the open ball, he/she seems to understand the metric  $d_Z$  and how the radius could be specified by this metric but he/she represented the radius incorrectly which is  $m+2$  rather than 2 and this hindered his/her work. However, on the whole, Student H seemed to understand the problem.

Others used their intuition to solve the problem and did not succeed, mainly because they confused the radius with the distance to the endpoints or boundary of the set. For instance, a student thought that the set B could be represented by the open ball

$B(m, 1)$ , and claimed that:

*Yes,  $m$  is the centre of the ball. 1 is the radius of the ball. (E)*

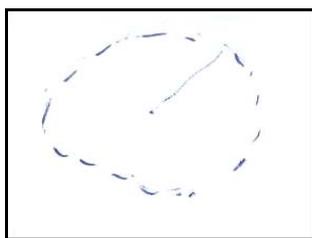
So it is clear that this student confused the distance to the endpoints with the radius of the ball so he/she gave an incorrect radius. It is possible that this might be due to the student's routine method of specifying the radius as we mentioned before.

Another one also claimed that:

*Yes, centre =  $m$ . radius =  $m+1$ . (J)*

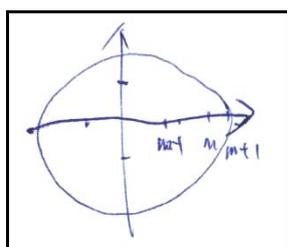
This student gave an incorrect answer for the radius similar to the previous one, but also wrote it in an incorrect way which is  $m+1$  rather than 1. These students seem to have concept images based on their experience of specifying the radius from the centre to end points when finding the radius of open balls in  $\mathbf{R}^n$ .

Others (Students F and N) drew pictures of balls (circles) about a point in  $\mathbf{R}^2$  and so they agreed that B is an open ball.



**Figure 4.3: Student F's response to Problem 2**

And,



**Figure 4.4: Student N's response to Problem 2**

Their concept image seems to be based on the picture of the open ball but we do not have any further information about their thinking on this question. The last student (Student C) in this group did not argue for his/her answer but just stated that B was an open ball.

From those who looked to see if B is an open ball, seven (out of 14) disagreed that B is an open ball, and also gave a variety of different arguments. Student G used his/her ideas concerning the boundary of open balls; he/she found that the end points are included in the set B so it cannot be an open ball and so it is closed. This student commented that:

*No, B is a closed set. The points,  $m-1$  &  $m+1$  are on the boundary.  $\therefore$  A boundary exists.  $\therefore$  B is not an open ball. (G)*

Another student seemed to understand the metric space  $(\mathbf{Z}, d_{\mathbf{Z}})$ , and was correctly able to write B as a closed ball, however he/she were not able to see that B could be expressed as an open ball also:

*No,  $B = \{x \in \mathbf{Z} \mid |x - m| \leq 1\}$ . If this was an open ball, this inequality would be strictly  $<$ . (I)*

Another student tried to define the open ball  $B(a, r)$  but his definition was incorrect . He/she wrote that:

*No,  $B(a, r) = \{a \in \mathbf{Z} \mid a \leq r\}$ . (D)*

Others used their intuition and for some of them the fact that the set is finite was important and they argued that:

*No, an open ball is an open set, and finite set cannot be open. (L)*

So, this student's concept image might be based on the continuum of points in an open ball, like the open balls in  $\mathbf{R}^n$ . Other students found that the layout (look) of the set B is not the same as what they were used to. They were used to the ball  $B(x, r)$ , and claimed that:

*No, because an open ball has only two parameters  $x, y$ .  $B(x, y)$ , where here B has 3;  $m-1, m$  and  $m+1$ . (M)*

And,

*No, as open balls are written  $B(x, r)$ . Where  $x$  is the centre and  $r$  is the radius of the ball. (P)*

The concept images of those students might be based on the look (layout) of the set or the mathematical symbols used to describe sets.

Two students tried to see if the set  $B$  is open set and made use of the definition, one of them argued that:

*No, if you take  $B(m-1, \varepsilon)$ , to be contained in  $B$ ,  $\varepsilon = 0$ , but by definition of open,  $\varepsilon > 0$ . (K)*

So for this student, there is no open ball about  $m-1$  and so  $B$  is not an open set. The student confused the radius in  $(\mathbf{Z}, d_{\mathbf{Z}})$  with the radius in  $(\mathbf{R}, d)$ , and therefore the familiar metric space  $(\mathbf{R}, d)$  influenced his/her conception of open balls in this case. Another student commented that there is some open ball around the point  $m$  which is not in the set  $B$  and so it is not open:

*No, there  $\exists \varepsilon > 0$  such that  $B(m, \varepsilon) \not\subset B$ . (O).*

It may be that this student is thinking of  $(\mathbf{R}, d)$ , since in that metric space such a ball would not be contained in the set  $B$ . He/she also seems to think it is enough to show there is one ball centered at  $m$  which is not contained in  $B$ .

In general, using the definition seemingly led the students to provide correct answers, but maybe a lack of familiarity with the metric space  $(\mathbf{Z}, d_{\mathbf{Z}})$  caused these two students (K and O) to give an incorrect answer. We have seen that many of our students were confused about this metric space and used their intuition gained from the real line to answer the question.

Most of the students attempted the second part of this question, except for Students B and D. The students' answers were not very different from students' answers in the previous part and in the interviews. Some of them (A, H and K) found open balls such as  $B(m, \frac{1}{2})$  or  $B(m, 1)$ . Some others (E, J, M and P) represented the ball or the points of the ball incorrectly. One answered 'No' because all subsets of  $B$  possess boundary points (G). Another one also answered 'No' because the  $B$ , for him/her, is a finite set (L). One draw a picture and answered yes (N). And others answered 'No' and did not give a reason (C, F, I and O).

#### **4.2.2.3 Summary**

We have seen that some students (both in the interviews and in the questionnaires) tackled this question by thinking about the definition of an open set or an open ball as given in lectures. Most of these students were successful in answering this question,

however a lack of awareness of the metric space  $(\mathbf{Z}, d_{\mathbf{Z}})$  caused problems for some of them, especially when they seemed to rely on concept images generated by the familiar metric space  $(\mathbf{R}, d)$ . The influence of Euclidean space was evident in the majority of answers from the whole group. Some of the misconceptions that seem to stem from an over-reliance on images based on the real line include the idea that finite sets could not be open, and that if a set contains endpoints that it cannot be open. Some students also seem to suffer from a confusion concerning the radius of a ball and the distance from the centre of a ball to an extreme point. Once again, this seems to arise from concept images based on the situation in  $(\mathbf{R}, d)$  and a misunderstanding of the metric space  $(\mathbf{Z}, d_{\mathbf{Z}})$ .

Yet another complication in this question came from the fact that the set B can be expressed as both an open and a closed ball. Students had previously seen sets that were both open and closed but it is possible that this was their first encounter with a ball with this property.

### 4.2.3 Analysis of the Responses to Problem 3

The question was given to all 16 students in the questionnaire and also given to three students during the interviews. The question was aimed at finding out if the students pay attention to the main set X in the metric space  $(X, d)$  when they think of the openness of a subset of the space, and also to explore which approach to deciding if a set is open is familiar to the students and which aspects they focus on in their arguments. Thus the question examines if the students think of the set Y as the main metric space which they have to focus on when they try to see if the set A is open, or if they use the real line  $\mathbf{R}$  as the main metric space to decide on the openness of the set A. As was the case with Problem 2, this question probes students' understanding of the importance of the set X in the metric space  $(X, d)$ , (which is the set Y in our problem). It tests also the students' thoughts on the end points of a set, or whether the statement "an open set does not contain its boundary points" has an influence on students' conception of open sets. The problem is given below:

#### **Problem 3:**

*Let  $Y = [0, 2]$  and consider the metric space  $(Y, d_Y)$  where  $d_Y$  is the standard metric on Y inherited from  $\mathbf{R}$ . Let  $A = [0, 2)$ . Is the subset A open? Explain!*

(This problem is number 6 on the questionnaire in Appendix 1, and it is number 2 on the interviews task problems in Appendix 2)

To answer this problem the students have to keep in mind that the metric space is  $(Y, d_Y)$  where  $Y = [0, 2]$  and the subset is  $A = [0, 2)$ , so they have to check if the set  $A$  is an open set within the metric space  $Y$ . Note that the basic open sets in this metric space are sets of the form  $(a, b) \cap Y$  and so could be  $(a, b)$  (where  $0 \leq a < b \leq 2$ ),  $[0, b)$  (where  $0 < b \leq 2$ ),  $(a, 2]$  (where  $0 \leq a < 2$ ) and  $[0, 2]$ . The open sets in this metric space  $(Y, d_Y)$  are formed from the unions of these basic open sets. Using the definition of an open set, the set  $A$  is an open set because if we take any element  $x$  in  $A$  we can find an open ball in  $Y$  around it which stays in  $A$ . If  $x \neq 0$ , we can do this by letting  $r$  be half the minimum of the distances from  $x$  to 0 and from  $x$  to 2, clearly  $B(x, r) = (x-r, x+r)$  is an open ball in  $(Y, d_Y)$ , centered at  $x$  which lies in  $A$ . If  $x = 0$ , consider  $B(0, 1) = \{z \text{ in } [0,2] \mid d_Y(z, 0) < 1\} = [0, 1)$ , this is an open ball in  $(Y, d_Y)$ , centered at  $x = 0$ . Another possible answer is that, the students could find the complement of the set  $A$  in the metric space  $Y$  (the complement is the set  $\{2\}$ ), and show that it is a closed set in  $(Y, d_Y)$  and therefore by the definition of a closed set, since the set  $\{2\}$  is closed then its complement must be an open set, hence the set  $A = [0, 2)$  is an open set in this metric space.

When attempting this problem, some of the students tried to use the definition of an open set and some of them used their conceptions and I will try to analyse each student's answer.

#### 4.2.3.1 The Questionnaire

In my findings I noticed that, seven of the students in the questionnaire agreed that the set  $A$  is an open set in  $Y$  and they gave different reasons for this. For four students,  $A$  is open because there is an open ball for any element in  $A$ , and they argued that:

*Open,  $\forall x \in [0, 2), \exists \varepsilon(x)$  s. t.  $B(x, \varepsilon(x)) \subset A$ . (J)*

Another student tried to explain his/her answer and argued that:

*-Not closed as doesn't contain all its limit points.*

*-Open as any point in it can have an open ball drawn around it which is still in  $A$ , (even, 0, because the main set is  $[0, 2]$ ). (G)*

And,

*Open, we can find an open ball about any point in A. This is because  $d_Y$  is restricted and so  $B(0, \varepsilon) = [0, \varepsilon)$ , where  $\varepsilon < 2$ . (L)*

And,

*Open, you can find  $B(x, \varepsilon)$ ,  $x \in A$ , s.t.  $B(x, \varepsilon) \subset Y$ . (O)*

Student J seems to answer correctly but does not elaborate on what form  $B(x, \varepsilon(x))$  would take, especially at  $x = 0$ . Student G found that A is not a closed set but is open and used the definition to explain the answer. Again, this student seems to understand the concept but does not give further information concerning the open balls in question. The other two students answered the problem correctly by using the definition of open sets. From the comments they gave we can see that each of them missed an important aspect of the full definition. Student L says that it is possible to find open balls centered at any point in A but only demonstrates this for the point  $x = 0$ . The other student (O) wrote that ' $B(x, \varepsilon) \subset Y$ ' but he/she neglected to mention that  $B(x, \varepsilon)$  must be contained in A. We see that the main part of the definition that most students seem to focus on is that "any point in the subset must have an open ball about it", but one student did not find an open ball for every point but just for one, and one student did not show that the open ball was contained in A.

A different student answered that A is an open set and gave a short explanation;

*Open,  $A = B(0, 2)$ . (I)*

For this student the set A is just the open ball  $B(0, 2)$  and therefore A is open, as since this student could express A as a single open ball it is therefore an open set. There are lots of students who have the conception that if one open ball contains all the elements of the given set and it is completely contained in the set then that set is open. This is of course true, but some students seem to think that every open set needs to have this property and this may cause confusion.

The other two students (F and N) answered that A is an open set but did not explain their choice.

The results also showed that two students answered that the set A is closed for various reasons. One student used intuition to answer the problem, and thought that since the set A is contained inside a closed set then it is a closed set; he/she commented that:

*Closed, A is closed since it is entirely contained in the closed set Y. (E)*

This student stated that the set  $Y$  is a closed set which is true, as is the case with the underlying set in any metric space. It may be, in this case though, that this student is influenced by the fact that the set  $Y$  is a closed interval in  $\mathbf{R}$ . Furthermore the student has the misconception that a subset of a closed set is a closed set too and this is might be due to previous mathematical experience.

The other student who answered that  $A$  is a closed set commented that:

*Closed, as its complement is open. (P)*

I do not know why this student answered like that; he/she did not say what is the complement of  $A$  or why that is an open set as he/she did not provide any other comments.

One student answered that the set  $A$  is neither open nor closed, and commented that:

*Neither, Only  $\varepsilon$  satisfying  $B(0, \varepsilon)$  for  $0 \in A$  is  $\varepsilon = 0$ , but  $\varepsilon > 0$ , so  $A$  is not open.*

*$\{2\} \subset [0, 2]$  is not open. (K)*

For this student the point 0 is an end point of the set  $A$  and it is included in it, he/she explained that there are no open balls around 0 because at that included end point  $\varepsilon$  must be 0, and an open ball requires  $\varepsilon > 0$  and therefore the set  $A$  is not an open set. This student is probably confused in his/her conception of an open set by the situation in the whole real line  $\mathbf{R}$  and does not seem to consider the form of open sets in the metric space  $(Y, d_Y)$ . Also this student correctly considered the complement of the set  $A$  within the metric space  $Y$  which is the single point  $\{2\}$  and this set  $\{2\}$  is not an open set.

From the findings, I also observed that three students (A, C and M) answered that the set  $A$  is 'Both' open and closed. One of these three (Student C) did not give a reason for his/her answer. Another student said that the set  $A$  is both open and closed for the reason:

*Both, all points in  $A$  have an open ball which is a subset of  $A$ . It is also closed, as its complement is  $\{2\}$  which is open. (A)*

This student used the definition for the set to be open but did not give details of the forms of the open balls. He/she used the closed set definition incorrectly in order to assert that  $A$  was closed.

Another student claimed that:

*It is both because it includes 0 but does not include 2. (M)*

The student used his/her concept image to answer this problem and that may stem from the type of brackets used to define the set A; there was no use of the definition in this answer.

Of the other students in the questionnaire, two of them (B and D) did not attempt the problem, and Student H answered 'Open' and 'Both' and did not give reasons for the answer.

#### 4.2.3.2 The Interviews

The question was also given to only three of the interviewed students (1<sup>st</sup>, 3<sup>rd</sup> and 6<sup>th</sup>) as we mentioned before. The results showed that those three students used different ideas in their responses. One student did not use any definition for this problem and instead remembered examples from his notes which are similar to this set. He had mentioned previously when asked how he deals with a new definition that he always looks at examples in order to understand definitions and so it is likely that his concept image is influenced by the examples. In his argument he stated that:

*I saw an example like this in the notes. Um well, the subset A is inside Y. and Y is closed, so I think A is closed, because it is inside a closed interval. I can't remember definition, but I remember example like this, where there was a subset inside a bigger set. (1<sup>st</sup>)*

The student has met many sets similar to Y so for him there is no need to use the definition. From his experience, he knew that the set Y is a closed set, this could be due to the brackets of the set Y in the same way as Student E (in the questionnaire), and since A is completely contained in Y so it must be closed as well. So this student has the misconception that a subset of a closed set is closed too.

A different student answered that the set A is an open set by using the closed set definition and commented that:

*Because its complement, uh its complement is  $Y - A = \{2\}$ , uh, because  $\{2\}$  is closed so A is open, A is open I think... and there's open ball in there. (3<sup>rd</sup>)*

This student found the complement of the set A which is the point  $\{2\}$  and since this set is a closed set as he knew then the set A must be an open set by the definition of closed set. This conclusion did not prevent the student from thinking of his definition

of open set because for him if the set  $A$  is open set that means there is an open ball inside it. However, no justification is given as to why  $\{2\}$  is closed in this metric space.

The last student found also that  $A$  is an open set. She wrote what open and closed sets mean for her but did not use any one of her definitions and used her conception of open sets and explained that:

*Is open, you can get  $\alpha$  that in such that  $\alpha$  is another point in the set, if there another point in the set it's open. (6<sup>th</sup>)*

This student could not remember the exact definition of open set, but she understands it as, there is always a point between the end points of the set  $A$  which is included in the set  $A$ . So it seems that her conception of an open set is the continuum of its points.

#### **4.2.3.3 Summary**

In the questionnaire seven students (out of 16) got this question correct. Of the five who gave reasons for their answers, four used the definition and one observed that  $A$  is an open ball in  $(Y, d_Y)$ . The students who answered incorrectly had a variety of approaches to the question. Only one of them used the definition, the others relied on their concept images.

From the findings above, we found that the students possess different concept images related to the conception of open set, especially related to the openness of the set in the given problem. We have seen that the most important part of the definition of an open set that is considered by many students is 'any point in a set must have an open ball about it which is completely contained in the set', and these students did not seem to consider the metric space itself when thinking of open balls. This lack of consideration of the metric space, and certainly the metric subspace, could cause trouble for students. I have also observed here that some students think of open sets in terms of the boundary points. We also have observed some intuitive conceptions about open and closed sets. These intuitive conceptions might be due to previous experience with examples (e.g., a subset of a closed set is also a closed set), and also might be due to the brackets used to define the set (e.g.  $[0, 2)$  is open and closed, and the set  $Y = [0, 2]$  is closed set).

#### 4.2.4 Analysis of the Responses to Problem 4

This question was used only in the interviews and was asked of seven students. It was given to uncover the role of intuition and concept image in the solution of problems in comparison to formal methods based on definitions. Students have different ways of understanding open sets and this question aimed to examine if the students are aware of the all aspects of the formal definition. In particular, the question investigates whether the students pay attention to the metric space in which they work or whether intuition or previous experience would influence their answers.

The students in this group were very familiar with open sets on the real line  $\mathbf{R}$  and in particular with open intervals such as the one that appears in this question (see the statement of the problem below). The set  $S$  is a subset of  $\mathbf{R}^2$  but it can be thought of as an interval on the  $x$ -axis. The question started with, 'Let  $(a, b)$  be an interval in  $\mathbf{R}$ ' to see if this sentence has an effect on the students, because the visualisation of  $S$  as a subset of  $\mathbf{R}^2$  is very similar to a visualisation of the interval  $(a, b)$  in  $\mathbf{R}$ . Therefore we expected that the students might think of the set  $S$  as a subset of  $\mathbf{R}$  instead of as a subset of  $\mathbf{R}^2$ . If students were overly influenced by this representation of  $S$ , it would lead them to an incorrect answer, otherwise they should have ended up with the right answer.

##### **Problem 4:**

*Let  $(a, b)$  be an interval in  $\mathbf{R}$  and  $S = (a, b) \times \{0\}$ , and let  $d$  be the standard metric on  $\mathbf{R}^2$ . As a subset of  $(\mathbf{R}^2, d)$ , is  $S$ :*

*Open                      Closed                      Both                      Neither*

*Please explain your answer!*

(This problem is number 3 on the interview task problems in Appendix 2)

To answer this question the students should know what kind of open ball they must think of in order to test the openness of the set  $S$ . By the definition of an open set, the subset  $S$  of  $(\mathbf{R}^2, d)$  is open if for any point  $x$  in  $S$  there exists  $\varepsilon(x) > 0$  and an open ball  $B(x, \varepsilon(x))$  in  $\mathbf{R}^2$  such that this ball  $B(x, \varepsilon(x))$  is contained in the set  $S$ . But clearly, if  $(x, 0)$  is an element of  $S$ , then any ball  $B(x, \varepsilon(x))$  will contain points whose  $y$ -coordinates are non-zero (for example the point  $(x, \varepsilon(x)/2)$  is an element of this ball) and therefore these points cannot be elements of  $S$ . Thus any open ball in  $\mathbf{R}^2$  about any

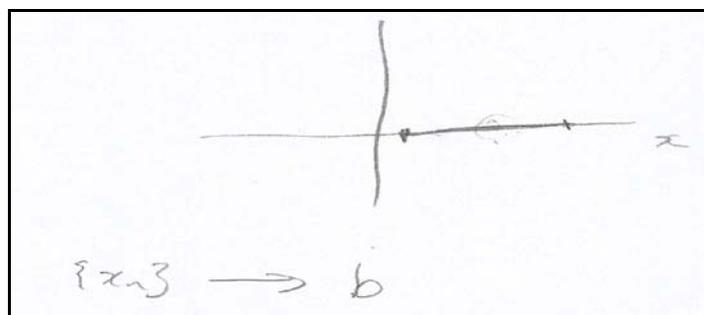
point in  $S$  cannot be completely included in the set  $S$  and therefore  $S$  is not an open set in  $(\mathbf{R}^2, d)$ . It is also easy to see that  $S$  is not closed in this metric space as it does not contain the points  $(a, 0)$  and  $(b, 0)$  which are limit points of the set. Therefore  $S$  is neither open nor closed in  $(\mathbf{R}^2, d)$ .

This problem was given to seven of the interviewed students; these were the 2<sup>nd</sup>, 4<sup>th</sup>, 5<sup>th</sup>, 6<sup>th</sup>, 7<sup>th</sup>, 8<sup>th</sup>, and 9<sup>th</sup> students. Only one student was able to give the correct result, this student started by defining the set  $S$  mathematically, and she defined it by  $\{(x, 0): a < x < b\}$ , and without any difficulty she found that,  $S$  as a subset of  $\mathbf{R}^2$  cannot be open. She drew  $S$  on  $\mathbf{R}^2$  and argued that:

*I don't think this is open, because this (S) will be on the x-axis and you know, it'll be along here maybe somewhere along here maybe. But if you take any ball about it it's gonna cut into the Y axis as well. But there's no elements on the Y that.., there's no elements here that are in it, so no open ball about any point is going to be in it. (9<sup>th</sup>)*

This student seemed to understand each part of the formal definition, and her concept image is guided by the formal definition, and she made a correct drawing of the set  $S$  and considered  $S$  as a subset of  $(\mathbf{R}^2, d)$  not as a subset of  $(\mathbf{R}, d)$ , recall however she did not give a full definition of open set when we asked her to state the definition of open set. Also, this student was also the only student who recognised that the set  $S$  cannot be closed either, and she argued:

*But I don't think it's closed either because um..., if you take a sequence along this way, it doesn't contain its end points, so it won't take it, it won't have all limit points. (9<sup>th</sup>)*



**Figure 4.5: The 9<sup>th</sup> student's illustration for Problem 4**

She also used a formal statement to argue for her answer here, and therefore she chose the answer 'Neither' for the whole problem. So we can say that she refers to the

definition and to results that have been proved. She also does not forget which metric space she is working in and that led her to answer correctly and quickly.

It is interesting to analyse the incorrect answers. The result of this problem showed that the majority of students based their answers on their concept images rather than consulting the formal definition. Their experience with open sets in the real line  $\mathbf{R}$  really seemed to affect their conceptions related to the set  $S$ . I found that their conceptions vary, for some of them the set  $S$  in  $\mathbf{R}^2$  is still an open interval, and one student commented that:

*Umm, well, in  $\mathbf{R}^2$  it won't uh make a difference, because you're going to have your, uh, your interval  $(a, b)$  cross zero. So that is just gonna be the set of order pairs  $(x, 0)$  for any  $x$  in  $(a, b)$ , so you can immediately just bring it back to your  $\mathbf{R}$  and imagine, and it's never going to reach them (the points  $a$  &  $b$ ), and it's still open. (7<sup>th</sup>)*

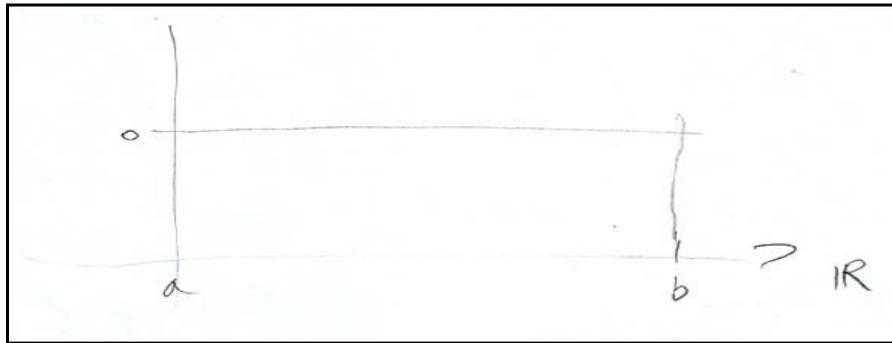
Another student also claimed that:

*So cross with zero is zero, so it's still in  $\mathbf{R}^2$  an open interval. I'm gonna say it's open. I don't think it'll change a lot much If it's open in  $\mathbf{R}$  because it's an open interval in  $\mathbf{R}$  it's still then open in  $\mathbf{R}^2$ . (8<sup>th</sup>)*

Both of these students graphed the set  $S$  correctly on  $\mathbf{R}^2$  and the former also expressed  $S$  mathematically and both realised that it is still an interval  $(a, b)$  on one line of the axes and used their imaginations to picture it as on  $\mathbf{R}$  and so for them the set is open and they were sure of their answers. This concept image is based on the correct picture of the set, but the students did not apply the definition here and this led them to provide incorrect answers. So, visualisation did not seem to help these students in answering the problem.

Another student was in line with the previous ones, this student also tried to draw  $S$  but his drawing was not correct. He argued that:

*Um... a line cross a single element still is going to be a line. So a line as a subset of  $\mathbf{R}^2$  is just a line and is open at both ends (i.e.  $a$  &  $b$ ) and to the sides it has no width is just a line, so it's open, I'd say it's open. (2<sup>nd</sup>)*



**Figure 4. 6: The 2<sup>nd</sup> student's drawing to Problem4**

This student, similarly to the others, found that this set is still a line and that the end points of that line are not part of it as it is an open interval. However he also considered  $S$  as being in  $\mathbf{R}^2$  when he spoke of the sides of  $S$  (width), but the fact that a line has no width may have led him back to thinking about  $S$  as a subset of  $\mathbf{R}$  and he answered incorrectly that  $S$  is an open set. So his experience and knowledge of lines influenced his conception and may have blocked any other ideas concerning  $S$  as a subset of  $\mathbf{R}^2$ .

From the results of this question, we found that some students did not understand the set  $S$  or they did not know how to think of  $S$ . The 4<sup>th</sup> student did not know how to start and preferred to see if the set is closed first and commented that:

*Um..., actually, it really does not come to my head for this one. Right I'll think about it it's closed, is it closed? Because um I think it's easier if sometimes to prove if it's closed first. Um, I'm going to see if the complement is open, and the complement is open, but I also think that, the set is open. It's now I'm thinking that the both open, uh the subset I mean is open. Um..., because the complement is definitely open I think. I'm just going to say (Both). (4<sup>th</sup>)*

We noticed that this student used her intuition to give an answer and did not use any definitions or theorems; the only formal thing is the definition of a closed set, but she used it incorrectly. In an earlier question, when asked if she agreed with the statement 'A set is open if its complement it's closed' she said that:

*No, I never thought of it in that way before! No, that can't be true, because um an open set like  $(0, 1)$  its complement is real line minus or the  $\mathbf{R}^2$  minus  $(0, 1)$  and um that's open as far as I think. (4<sup>th</sup>)*

So she might have difficulties in thinking of the complement of sets which are subsets of Euclidean space  $\mathbf{R}^n$ . Regarding her argument for this question above, she

might understand that the sets  $\mathbf{R}$  and  $\mathbf{R}^2$  are obviously open sets, but she might have the misconception that the complement of the set  $S$  which is  $\mathbf{R}^2 - S$  is an open set. She does not explain what the complement is or why she thinks that it is open. Then she deduced that since the complement of the set  $S$  is an open set, then  $S$  must be a closed set. Also, she did not give an argument for saying that  $S$  is open too, but the reason might be the line does not include its end points, and therefore she chose 'Both' for her answer.

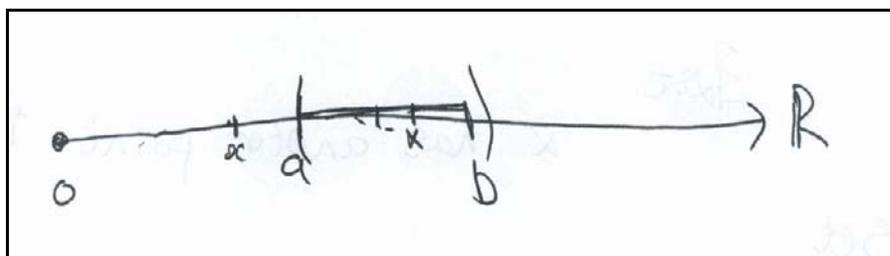
Also, another student answered intuitively that the set is open without any reason, but then she took her answer back and thought again of the cross-product symbol ( $\times$ ) in the set  $S$  and she got confused with it, and commented that:

*Is that ( $\times$ ) suppose, is that a union, is it! So it's  $(a, b)$  union  $0$ , is it or is it just... cross product! I don't know. I think it's open, but it might be closed as well. I don't know. I'm not sure what  $S$  in itself, I think it just might be points  $(a, 0)$  and then  $(0, b)$  I dunno that could be completely wrong. I haven't seen a set like that before. (5<sup>th</sup>)*

The student could not get a picture of the set  $S$  into her head due to the symbol ( $\times$ ), so because she had forgotten the meaning of this symbol she was not able to understand the set  $S$ . She said it might be open and closed as well, but she could not give an argument for that and did not continue with this problem.

The last student used another idea; she seemed to base her answer on the idea of continuum (or a connected set), and tried to draw the set on the real line  $\mathbf{R}$ . She recalled the definition of a closed set but did not use it and claimed that:

*It's closed I think because um... that there is a point, you can always get a point in the interval (between  $0$  and  $(a, b)$ ) and no matter where in interval, the point will land in the interval between them, where there is no a point in the set  $S$ . (6<sup>th</sup>)*



**Figure 4.7: The 6<sup>th</sup> student's illustration for Problem 4**

The student graphed  $S$  on the real line  $\mathbf{R}$  and thought of  $S$  as if it consisted of the point  $\{0\}$  and the interval  $(a, b)$ , she drew  $(a, b)$  to the right of  $0$  and assumed that there was a gap between them. No matter how the big or small a gap there is, for this student there is a point in between that is not a part of  $S$  and therefore she concluded that  $S$  is a closed set. The answer she provided seems to be based on her continuum idea of points in an open set or more precisely on the idea that an open set should be connected, and since  $S$  does not satisfy these conditions she deduced that it is not open and then that the set  $S$  it is a closed set.

#### 4.2.4.1 Summary

From our results above, we found that most students did not pay attention to the metric space and followed their conceptions in this problem; they ignore the fact that the metric space is  $(\mathbf{R}^2, d)$  and work instead in  $(\mathbf{R}, d)$ . Then their knowledge of  $(\mathbf{R}, d)$  takes over and the reason for that could be when students work on the definition of open sets in a metric space they do not focus on an important part of the definition of open sets which is ‘there is exist an open ball of  $(X, d)$ ’. That is they sometimes forget the importance of the underlying set  $X$  and work only in a subspace of  $X$ .

Only two students (2<sup>nd</sup> and 9<sup>th</sup>) of the seven students who answered this question, showed evidence of considering  $S$  as a set in  $\mathbf{R}^2$  but the 2<sup>nd</sup> student discounted that fact and followed his conception and gave an incorrect answer, while the 9<sup>th</sup> student used that fact by referring to the definition and gave a correct answer.

We have noticed many concept images related to this problem. Some concept images are based on the visualisations of the set, but even when the visualisation was correct with some students it led them to give incorrect answers as they think of the metric space  $(\mathbf{R}, d)$  rather than  $(\mathbf{R}^2, d)$ . Some concept images seemed to be based on the open sets in  $\mathbf{R}^n$  (e.g. the 4<sup>th</sup> student). Moreover, some concept images seemed to be based on the continuum or the connectedness idea of the points of a set.

#### 4.2.5 Analysis of the Responses to Problem 5

Problem 5 was the most difficult task in this study. It was asked in order to see what students did when faced with a new definition and to see how they coped with the definition of a metric in an unfamiliar setting. I was interested in whether the unfamiliar setting would affect their work on open balls and open sets. This problem

was given to all students in the interviews and it consists of seven questions all of which should be answered by using the definition given in the problem. All the students were asked to answer the first question ‘Can you describe this metric in words?’ and each of them was asked to answer other two or three questions depending on a student’s ability to understand the given metric.

### **Problem 5:**

*Let  $X$  be the set of all real sequences. Define:*

$$d(\{a_k\}, \{b_k\}) = \begin{cases} 0 & \text{if } a_k = b_k \text{ for any } k \in \mathbb{N} \\ 1/k & \text{if } k = \min_{n \in \mathbb{N}} \{n : a_n \neq b_n\} \end{cases}$$

*-Can you describe this metric in words? Or*

*-What do you think this metric measures?*

*-Let  $\{0\} = \{0, 0, 0, \dots\}$ , if  $d(\{a_n\}, \{0\}) = 1$  what can you say about  $\{a_n\}$ ?*

*-What is  $B(\{0\}, 1)$ ? Or*

*-What is  $B(\{0\}, 1/2)$ ?*

*-Is the set of sequences  $\{\{a_n\} : a_1 = 0 \text{ or } 1\}$  open? Or*

*-Is the set of sequences  $\{\{a_n\} : a_1 = 0\}$  open?*

(This problem is number 4 on the interview task problems in Appendix 2)

#### **4.2.5.1 The First Part of Problem 5**

The first part of the problem was ‘Can you describe this metric in words?’. This question was given to assess students’ understanding of the definition of the given metric because that definition is not easy to grasp; it tests students’ ability to make sense of the given metric. The definition requires the students to make an interpretation of the phrase “for any” as meaning “for all” because students often interpret it as “for one or for some”, thus understanding this phrase correctly or incorrectly would affect students’ conceptions of the metric. In the English language, the word ‘any’ sometimes is used in place of ‘all’ and sometimes is used in place of ‘some’. In mathematical language, the word ‘any’ is genuinely ambiguous as well (Rowland 2002). Usually the meaning of ‘any’ is obvious from the context, but this is not always true. For example, the definition of an open set in a metric space is given in (Simmons, 1963, p. 60) as ‘A subset  $G$  of the metric space  $X$  is called an open set if given any point  $x$  in  $G$ , there exists a positive real number  $r$  such that  $S_r(x) \subseteq G$ ’.

Simmons clarifies this and says ‘i.e. if each point in  $G$  is the centre of some open sphere contained in  $G$ ’. Here ‘any’ is used as ‘all’ or ‘every’ or ‘each’.

In this situation (the definition of the metric  $d$  given above) the meaning ‘for all’ was meant. I did not intentionally use the ‘for any’ to add complications for the students and as we will see later, if students seemed confused about this point they were told that ‘for any’ should be replaced by ‘for all’. So, the first part of the definition of the metric ‘ $d(\{a_k\}, \{b_k\}) = 0$  if  $a_k = b_k$  for any  $k \in N$ ’ means if all the corresponding terms of two sequences are the same then the distance between them is 0, that is if two sequences are equal then the distance between them is zero. The second part of the metric ‘ $d(\{a_k\}, \{b_k\}) = 1/k$  if  $k = \min_{n \in N} \{n : a_n \neq b_n\}$ ’ means if two sequences are not the same then the distance, as given by the metric, between them is  $1/k$  where  $k$  is the first position of the terms of the two sequences where they differ (e.g. if  $\{a_k\} = \{1, 2, 3, 4, 4, 4, 4, \dots\}$  and  $\{b_k\} = \{1, 2, 3, 3, 3, 4, 4, \dots\}$  then  $d(\{a_k\}, \{b_k\}) = 1/4$ ;  $k = 4$  here since the two sequences differ first at the fourth term). This metric is quite complicated to deal with, so constructing a concept image of it in the students’ head is difficult and we expected that students might have difficulty absorbing it.

All the students in the interviews tried this problem except for one student. The 10<sup>th</sup> student refused to do any problem in the interview because he had not recently gone through his notes and so did not remember anything about the concept of open sets. From the ten students who attempted the problem, the 6<sup>th</sup> student had problems with English and had also encountered difficulties when she worked on the other interview tasks, so I asked her only to explain the definition of the metric and she commented that:

*It looks like the discrete metric or some version of, because if they’re equal, it’s 0 but instead of it equal to 1, it’s divided by  $k$ . if the sequence  $\{a_k\}$  and  $\{b_k\}$  are equal, then the metric is 0 and if it’s divided by  $k$ , it’s equal to the min of  $k$  values, where  $a_k$  is not equal to  $b_k$ . (6<sup>th</sup>)*

This student thought that this metric looks like the discrete metric and the difference between the given metric and the discrete metric was that instead of a distance equal to 1 in the discrete metric, in the given metric it is  $1/k$  where  $k$  is the smallest natural number where  $a_k \neq b_k$ . This student did not show good understanding of the defined metric and she only read the metric from the sheet and was not able to describe it

using her own words. Also, when I asked her to give an example to explore her understanding of this metric she wrote the sequences  $\{1/n\}$  and  $\{n\}$  to illustrate the second part of the metric where the sequences are not equal, but she was not able to say what the distance between these sequences would be. She gave the same sequences as examples for the first part of the definition of this metric but she was not able to comment on the sequences that she gave and said ‘I don’t really understand sequences’. So I did not force her to answer the rest of the questions on this topic.

All the other nine students tried their best to answer this problem. The majority of them interpreted the phrase “for any” in the first part of the definition of the metric as meaning “for some” and that led them to understand both parts of the definition incorrectly. To make sure of their understanding, I asked them to give examples of sequences and to apply the metric to those examples. According to the examples they provided, I realised that their misunderstanding of the metric is based on their interpretations of the phrase ‘for any’.

There was only one student (7<sup>th</sup>) who interpreted the phrase ‘for any’ as ‘for all’ and so he understood the metric directly and correctly. This student argued that:

*Um..., if two sequences aren't equal, the point that which they start to variate is the number you take so say if it was  $a_7$ , you take this seven, so the distance between them is going to be a seventh to 1 over that  $k$ , that which they start to diverge, and then if they're equal to each other the whole way, and you just say there's no difference between them which satisfies your axioms of the definition of the metric anyway. (7<sup>th</sup>)*

From the argument above, we noticed that this student understood the full idea of the metric properly and the examples he gave showed his correct understanding of the given metric.

Another student (5<sup>th</sup>) did not show that she understood the metric correctly but she got most of the idea of it. She commented that:

*For all the terms  $a_k$  is equal to  $b_k$ , their metric is 0. But whenever  $\{a_k\}$  is not equal to  $\{b_k\}$ , the smallest value of  $k$  is that's that one. So it's one over that. So if you had  $a_1$  is equal to  $b_2$  then you get 1. (5<sup>th</sup>)*

From what she said in her comment at the beginning, it seemed that she got the idea of the metric but when she said ‘if you had  $a_1$  is equal to  $b_2$  then you get 1’, we

noticed that she might not be fully aware of idea of the metric. For her if  $\{a_k\} \neq \{b_k\}$  then to get the metric  $1/k$ , she seemed to think that  $k$  is not the smallest  $k$  where  $a_k \neq b_k$  but  $k$  is the smallest place where  $a_n = b_m$  and  $n \neq m$ , so by her example, if  $a_1 = b_2$  then the smallest  $k$  is 1 which gives the distance 1. It could be that she implicitly assumed here that if  $a_1 = b_2$  then  $a_1 \neq b_1$  and for that reason she took  $k$  to be 1. So she understood the idea of this metric but she was not very clear and this may have limited her answers to the other parts of the problem.

I classified the other students who did not understand the given definition into two groups where each group understood the metric in a way which is different from the other according to their interpretations of the phrase ‘for any’. The first group consists of the 1<sup>st</sup> and 8<sup>th</sup> students. These students thought that the metric between the sequences is defined term-wise. One student (1<sup>st</sup>) from this group asked before explaining the metric that ‘*I’m not sure of  $k$  is the subscript or the value of...*’ and I answered him ‘ *$k$  is the subscript, yeah*’. Then he said that:

*I’d say that, um there is two sequences  $\{a_k\}$  and  $\{b_k\}$ , and um for each term in each sequence, if the corresponding terms are equal then the distance between each term is zero. If the corresponding terms aren’t equal, then the distance between the two sequences is 1 over  $k$  where  $k$  is the minimum value of  $k$  in the sequence. (1<sup>st</sup>)*

This student thought that the metric is defined between the sequences term by term so that if any two corresponding terms are the same then the distance between them is zero and if they are not the same then the distance is 1 over  $k$  where  $k$  is the position where  $a_k \neq b_k$ . So he seemed to consider that  $k$  is the place of the terms where the sequences differ but he seems to misunderstand the metric by thinking of it as a term-wise metric.

The other student (8<sup>th</sup>) commented that:

*Well, if two elements of the sequence in the same place or the same, is going to be zero, if the two elements in sequence don’t equal to each other, then you take the min of..., that was elements, um..., I put number 1 over the min. (8<sup>th</sup>)*

This student also understood that the distance is defined term by term so that if the corresponding terms of two sequences are equal then the distance between these terms

is zero, but if the corresponding terms are not equal then the distance between them is  $1/k$  where  $k$  is the minimum of these terms.

So we can see that, these students thought of  $k$  differently but both understood the whole metric incorrectly.

The other group comprises of the 2<sup>nd</sup>, 9<sup>th</sup> and 11<sup>th</sup> students, they understood the metric in a different way; and thought for the first part of the definition that if any two terms of the two sequences in the same place are equal then the distance between the sequences is zero, and the second part of the metric definition holds if all the corresponding terms of the two sequences are not equal. They commented that:

*Yeah. Ok, so  $d$  on two sequences. For any  $k$  an element of  $N$  ok. it's equal to zero if any of the elements in the both sequences are equal for any  $k$ . (2<sup>nd</sup>)*

And:

*If you have two sequences and they're the same for any point, the  $k^{\text{th}}$  terms the same for any of them, then the difference between them is 0, but if none of the terms are the same, then  $1/k$  if  $k$  is the min. (9<sup>th</sup>)*

The other student said that:

*So we have two sequences called  $\{a_k\}$  and  $\{b_k\}$ , and if the same term in the two sequences has the same value for no matter which term that is, then the distance is zero. For the second part, because the distance is not equal to 0 we know that  $a_k$  is not equal to  $b_k$  for any  $k$ , so  $1/k$  is just going to be the first term of the sequence then, because would be the min so it would be 1. (11<sup>th</sup>)*

So we can see that, for those students, if there are equal corresponding elements in any (that is some) place in the sequences then the distance is 0, and if there are no corresponding terms that are equal then the distance between the sequences is  $1/k$  where  $k$  is the min place which they are different, and  $k$  is always 1 as the sequences will differ always at the first term so the metric is always 1.

The remaining two students (3<sup>rd</sup> and 4<sup>th</sup>) understood the metric incorrectly and their understandings were somewhat different from the other students. Similarly to the 6<sup>th</sup> student, the 3<sup>rd</sup> student found that, the metric looks like the discrete metric and commented that:

*Uh..., it is similar to discrete metric, yeah, because this is 0 or 1, it says  $1/k$ . So this is, because  $\{a_k\}$  equal  $\{b_k\}$  so same sequence, so it's 0. So if it's different sequences so then would be  $1$  over a min, small one. (3<sup>rd</sup>)*

This student realised that the given definition is similar to the definition of the discrete metric (recall that we define  $d$  to be the discrete metric on a set  $X$  as follows:  $d(x, y) = 0$  if  $x = y$  and  $d(x, y) = 1$  otherwise for  $x$  and  $y$  elements of  $X$ ). And for the metric given in this task, the distance is 0 if two sequences are equal and it is  $1/k$  if two sequences are not equal where  $k$  is the minimum (we will see from his answer to the next part of the problem that what he meant here was the minimum of the values of the sequence). He did not give much explanation for the metric; so it is difficult to know if he understood the metric correctly but his work on the other parts of the problem showed that he misunderstood the metric.

The other student (4<sup>th</sup>) understood the first part of the definition correctly and the example she gave showed good understanding, but she had difficulty in providing an example for the second and for her explanation of the metric, said that:

*Ok, if  $\{a_k\}$  and  $\{b_k\}$  are equal then the metric um is gonna be equal to 0, because there's basically no distance between them, and then, if  $\{a_k\}$  and  $\{b_k\}$  aren't equal when the two  $k$ 's are elements of the natural numbers, we take the smallest one of it and then the metric is gonna be equal to 1 over it.*  
(4<sup>th</sup>)

This student for the first part gave the example, let  $\{a_k\} = \{1/n\}$  and  $\{b_k\} = \{1/n\}$ , so the both sequences are equal and the distance between them then is 0. For the second part she chose  $\{a_k\}$  to be  $\{1/n\}$  but she could not think of an example for  $\{b_k\}$  and commented that:

*Um..., I dunno I can't really picture this at all to be honest. Um..., usually when I'm given stuff like this, I have couple of days to do it.* (4<sup>th</sup>)

So she could not understand the full metric and her work on the rest of the problem showed that she understood the metric incorrectly.

From the responses above, we found that some students seemed to understand the metric correctly and other students understood it incorrectly. For those who misunderstood the metric, the reasons were the interpretation of the phrase 'for any' for some students and the inability to grasp the second part of the definition for other students.

The interviews were individual, so I tried to help those students who were confused by 'for any' when I found that they misunderstood it. Those students were the 2<sup>nd</sup>, 8<sup>th</sup>,

9<sup>th</sup> and 11<sup>th</sup> students. I changed ‘for any’ to the phrase ‘for all’ to them to see if that change would make a difference to their understanding of the given metric. I noticed that the change made a major difference for the 2<sup>nd</sup>, 9<sup>th</sup> and 11<sup>th</sup> students, as they changed their conceptions of the whole metric and seemed to understand the idea of it correctly.

One of those students commented that:

*Oh! Sorry, yeah that’s much better yeah. So there are equal to zero if all, if the two sequences are just the same sequences then and equal to 1 over k, k is equal to the minimum number after which they become not equal, yeah understand them now, ok. (2<sup>nd</sup>)*

So when I changed the phrase for this student, the metric becomes easier for him to understand and he got the right meaning of the metric quickly and without any difficulty. So I see that a small phrase in the definition could pose a huge problem for the students if they could not interpret it in the right way.

I asked another student (9<sup>th</sup>) if the change of ‘for any’ to ‘for all’ would make a difference in her idea of the metric and she replied that:

*Oh! For all k, yeah that does make a difference, cause that would mean the sequences would have to be the same k. Oh! That’s interesting. Normally, I would see for any as the same as for all, but when I read this out loud, it sounded like just for, like if any of them, because I often see any and just think of you know if you say anyone, kinda it means like someone, like just one. Ok so two sequences that they aren’t the same and it’s the smallest k where they aren’t the same. (9<sup>th</sup>)*

So this student argued that, she normally thinks of ‘for any’ as ‘for all’, but for this problem ‘for any’ sounded to her as meaning ‘for some’. When I changed this to ‘for all’ she seemed to understand the defined metric correctly.

I also encouraged the student (11<sup>th</sup>) to think about the phrase, so I asked him what the phrase ‘for any’ means to him and he said that:

*I assumed you were talking about one specific one, is it! Is for all? Oh! No, I’ve completely misunderstood it now. Well that means that, if the distance is 0, the sequences are completely identical, if they’re not term by term identical the distance between them will be 1 over the number of the first term in which they differ. (11<sup>th</sup>)*

This student assumed that the meaning of the phrase is ‘for one’ but then he wondered is it ‘for all’ and realised that he misunderstood the metric and immediately gave a correct interpretation of the definition of the metric.

We can see that the definition of the metric became clear for those students when they replaced ‘for any’ by ‘for all’ in this problem.

For the 8<sup>th</sup> student, the change of the phrase made a difference in her understanding of the first part of the metric, so she thought again about the first part and got it correct but understood the second part incorrectly. She commented that:

*Well then, it would only be zero if the whole sequences the same. But if one is {1, 2, 3, 4} and the other is {1, 5, 7, 9} I'd say then all hold to the second one. So, the second condition would apply to the whole sequence. (8<sup>th</sup>)*

So she thought of the first part of the metric correctly, but as her work later will show, she still considers  $k$  as the value of the smallest term of the sequence which does not match with its corresponding term.

So the change of ‘for any’ into ‘for all’ did not help all the four students to understand the full metric correctly. In general, the students usually use ‘for any’ as the meaning of ‘for all’ because they used this meaning when they tried to state the definition of the open set but in this situation they interpreted it differently and they did not use the same meaning as they used for the open set which led them, in this case, to misunderstand the metric.

Following on from the first part of the problem we asked some students (7<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) the question ‘*what do you think this metric measures?*’ The answers of those students were not too different from each other and their answers were:

*The similarity between sequences, how long it takes the sequence to not become the same, so the longer they stay the same, the closer they're going to be in terms of the metric. (7<sup>th</sup>)*

The other one said that:

*I just looked at it and it looked too confusing to try and think of a picture of sequences, so try to think of how far they're apart. (9<sup>th</sup>)*

The last student said:

*Um..., how soon in sequences their terms differ. (11<sup>th</sup>)*

So we can see that most of these students understand a metric as a measure of how similar two sequences are or how different two sequences are, so they did not find it difficult to answer this question.

From all students' responses to the first part of Problem 5 above we summarise that, five students (2<sup>nd</sup>, 5<sup>th</sup>, 7<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) seemed to understand correctly the full idea of the metric defined on the set of real sequences, and four students (1<sup>st</sup>, 3<sup>rd</sup>, 4<sup>th</sup> and 8<sup>th</sup>) did not show full understanding of the given metric.

From the results we found that, the phrase 'for any' plays a crucial role in students' understanding of the definition of the metric, and that changing 'for any' to 'for all' helped some students to gain a clear understanding of the defined metric. We also noticed that two students compared the given metric with the discrete metric. Moreover, students had difficulty in thinking of a metric in the context of sequences. Students were not able to understand some parts of the definition where some of them considered  $k$  in the definition of the metric given above as a term of a sequence instead of a place of a term of a sequence.

#### 4.2.5.2 The Second Part of Problem 5

The second part of the problem was '*Let  $\{0\} = \{0, 0, 0, \dots\}$ , if  $d(\{a_n\}, \{0\}) = 1$ , what can you say about  $\{a_n\}$ ?*'

This question was given to the all students who attempted this problem except the for the 6<sup>th</sup> student. It also concerned the students' understanding of the given metric. It required students to think about what it means for a sequence  $\{a_n\}$  to be a distance of 1 from the zero sequence in this metric space. Since the distance is 1 then by the definition of the metric  $a_n \neq b_n$  when  $n = 1$  that is,  $a_1 \neq b_1$ , but  $b_1 = 0$  so we must have  $a_1 \neq 0$ . Thus what we can say about  $\{a_n\}$  is that the first element of this sequence is not equal to 0.

Nine students were asked to consider this problem. Four of them (2<sup>nd</sup>, 7<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) gave correct answers to this question based on their correct understanding of the metric, one student (5<sup>th</sup>) gave an answer that we cannot say is incorrect but it was not the most general possible answer to the question, two students' answers (1<sup>st</sup> and 4<sup>th</sup>) were not incorrect but they understood the metric completely incorrectly, and the last two students (3<sup>rd</sup> and 8<sup>th</sup>) experienced difficulties with thinking of the elements of the

sequence as their incorrect understanding of the metric did not help them to think of the sequence.

The four students, who answered this part of the problem correctly, showed good understandings of the definition of the given metric, one of them, the 7<sup>th</sup>, understood the metric directly without my help and the other three 2<sup>nd</sup>, 9<sup>th</sup> and 11<sup>th</sup> got the idea of the metric after I helped them to think of the phrase ‘for any’ as meaning ‘for all’. The common answer for those four students was:

*So, you can say that, um, the first element of  $\{a_n\}$  is not equal to 0. (2<sup>nd</sup>)*

Another student (9<sup>th</sup>) also answered the question correctly and she used the example that she had given in the previous part of this problem to help her to reason here. She said that:

*You know like when I had this example here, the difference between these two is the number where the terms are different. So the distance between this and this is 1. Then it must be the first term is different, so the first term of  $\{a_n\}$  is not 0. (9<sup>th</sup>)*

So this student made use of the example that she gave when she explained the metric. So for her since the given distance in this question is 1 then the first terms of both sequences are different and since the sequence  $\{0\}$  has its first term equal to 0 then  $a_1 \neq 0$  in  $\{a_n\}$ .

When those students got the sense of the metric they gave the correct answer for this part without any difficulty and they were sure of their answers.

The other student (5<sup>th</sup>) did not show full understanding of the metric as we showed above in the last part of the problem. In her answer to this part she said that:

*Um, I'd say that,  $a_1$  was non zero, because if  $a_1$  is non zero, then you're going to get 1. But if all the terms are zero, then you gonna get 0. I think the first term is non zero and all other terms are zero. (5<sup>th</sup>)*

This student answered that the first term of the sequence is not 0 which is correct but also added that the other terms of the sequence are zeros. This answer is not incorrect, but also it is not the general answer, it is just one of an infinite number of possibilities as it is not important for the terms  $a_n$  when  $n > 1$  to be zero and they could be any number, and the most important thing is that  $a_1 \neq 0$ . We can see that her answer is based on her understanding of the metric that we explained it previously, i.e.  $k$  is the

smallest place where  $a_n = b_m$  and  $n \neq m$ . In my opinion, she might have thought that in order to have  $k = 1$  she needs  $b_1 = a_n$  for some  $n \geq 2$  (that is, since  $b_1 = 0$  then  $a_1 \neq 0$  but  $a_n = 0$  for  $n \geq 2$ ).

There were two students (1<sup>st</sup> and 4<sup>th</sup>) who provided good examples of  $\{a_n\}$  but they gave incorrect reasons for their answers based on their incorrect understanding of the metric. One of them commented that:

*All right, if the distance between them is 1 then  $k$  has to be 1. Therefore, the minimum  $k$  is 1, so then uh  $a_1$  equals 1 um, not sure about the rest of the sequence. If  $a_1$  is 1 then every term after that would be zero, if like from  $a_2$  onwards if all the elements are zero then the difference between these points and the points in  $a_n$  will be zero, so the rest of the terms will be zero. (1<sup>st</sup>)*

This student, as I showed before in the first part of the problem, thought incorrectly that the distance is defined between each pair of corresponding terms and considered  $k$  as the place where the corresponding terms are not equal. In his answer to this part of the problem he also thought that when  $k$  is 1 then  $a_1$  must be 1 so he probably considered  $k$  to be the position and the value at the same time. So he thought of a good sequence to answer the question but he gave an incorrect reason for his answer as the sequence  $\{1,0,0,0,\dots\}$  is just one of infinite number of possibilities of  $\{a_n\}$ , but he considered the metric term by term which may have hindered his work. He also confused the term  $a_k$  with the number  $k$ .

The other student (4<sup>th</sup>) answered that:

*Well, in that case I think  $\{a_n\}$  is the sequence 1 then. Oh! No, no sorry! I dunno what to think of the sequence maybe we can think of the sequence  $\{a_n\}$  as sequence with -1 and because we dealing with um the metric then the, you can't have a negative emm distance.*

This student, at first answered that  $\{a_n\}$  is the sequence  $\{1, 1, 1, \dots\}$  but then she excluded that sequence and replaced it by the sequence  $\{-1, -1, -1, \dots\}$  which probably matched her understanding of the metric. As we mentioned previously in the last part of the problem, she thought of the minimum  $k$  where  $a_k \neq b_k$  as the smallest number of the corresponding terms of the sequences (that is  $k = \min\{a_k, b_k\}$ ) and the distance is 1 over  $\min\{a_k, b_k\}$ . So when she chose  $\{a_n\}$  to be the sequence of ones then the

minimum of  $a_k$  and  $b_k$  (i.e. the minimum of 0 and 1) is 0 and so the distance will be  $1/0$  which is impossible as she commented:

*Because I just, I don't want to divide by 0. (4<sup>th</sup>)*

Thus to get rid of dividing by 0 she changed  $\{a_n\}$  to be  $\{-1, -1, -1, \dots\}$ . Note that using her interpretation,  $\min\{a_k, b_k\} = \min\{-1, 0\} = -1$ , and so the distance is  $1/-1 = -1$ . She also thought that, the negative sign does not matter as the distance is always positive.

The last two students (3<sup>rd</sup> and 8<sup>th</sup>) could not answer this part of the problem because they misunderstood the metric and their conceptions confused them and did not help them to get the right answer. Their understanding of the given metric was not far from the last student's conception, the 3<sup>rd</sup> student commented that:

*Ok, it can be minus so k is goes min, so 1 over k equal 1. Oh! k is going to be in  $\mathbf{N}$ , so that's, if  $\{a_n\}$  is bigger than 0, it can't be smallest so it going to be smallest of 0, No k can't because 0, ..., so if all are 1's then min  $a_k$  and  $b_k$  would be 0 not 1, hmm, I don't know what  $\{a_n\}$  what to be. (3<sup>rd</sup>)*

This student, as we said before in the first part of the problem, thought that  $k$  is value of the smallest term of both sequences, and since one sequence is the zero sequence which consists entirely of zeros, the other one must, for him, have values less than zero, (otherwise he thinks that he will have to divide by  $k = 0$ ). Therefore he chose  $\{a_n\}$  to be the sequence  $\{-1, -1, \dots\}$  so that the minimum value  $k$ , for him, between the corresponding terms is -1 so that he can divide by  $k$  to get the distance 1, and maybe, like the last student, he thought that he can get the rid of the minus because it is a distance. Then he recognised that  $k$  is in  $\mathbf{N}$  (the set of natural numbers) which means  $k$  cannot be -1 and he reasons that at the same time  $\{a_n\}$  cannot be  $\{1, 1, 1, \dots\}$  as the other sequence is  $\{0\}$  and then the minimum  $k$  is going to be 0 and so  $1/k$  means  $1/0$  which is undefined. Thus he got confused and could not figure out what  $\{a_n\}$  could be. We can see that this student also confuses the value of the term  $a_k$  with the number  $k$ .

The 8<sup>th</sup> student is similar and claimed that:

*Well if I had um, sequence of the all of ones, and it was gonna be go between 0, 0, 0 and 0 and the ones, then I'm gonna have 1 over the min of these two, which*

*is zero, So we're gonna have 1 over 0. So the distance..., so I don't know how you would get a distance of 1, in the way that I'm thinking of it maybe I think of it wrong. (8<sup>th</sup>)*

So this student also thought of the same idea that the 4<sup>th</sup> and 3<sup>rd</sup> students had, but she realised that her understanding could not be correct as it led her to an answer which cannot be true as it is impossible to divide by zero. I noticed that the last two students (3<sup>rd</sup> and 8<sup>th</sup>) found themselves in a critical situation (dividing by 0) and that prevented them from continuing to work on this part of the problem, but they did not seriously question their understanding of the definition of the metric.

From above regarding the second part of the problem, the results give us more insight about students' understandings of the defined metric and showed that about half of the students had an incorrect understanding of the metric and were not able to get the sense of the given definition.

#### **4.2.5.3 The Third Part of Problem 5**

The third part of the problem was: 'What is  $B(\{0\}, 1)$ ?' And 'what is  $B(\{0\}, \frac{1}{2})$ ?' This part concerns the two open balls  $B(\{0\}, 1)$  and  $B(\{0\}, \frac{1}{2})$ , and the students have to find the elements of the open ball  $B(\{0\}, 1)$  and the elements of the open ball  $B(\{0\}, \frac{1}{2})$ . This question explores students' ability to think of open balls in metric spaces that are not familiar to them, and also investigates the approaches that they use to think about open balls. Nine students attempted this part of the problem, and I asked most of them to think of only one of the balls above; I asked some of them to consider both balls if they found the correct answer for one ball without any difficulty, or if they found it difficult to think of one ball I thought that asking them to try the other ball could help them.

These two questions require the students to think of what kind of elements (in this case sequences) which the open balls can have (i.e. what are the common properties that the sequences should have in each open ball). To answer this part of the problem the students have to use the definition of an open ball within the given metric. For the first open ball  $B(\{0\}, 1)$ , students have to think of the definition of this ball which is:  $B(0, 1) = \{x \in X : d(x, 0) < 1\}$ . In our problem,  $X$  is the set of all real sequences so that,  $x$  is a sequence  $\{a_k\}$ , and 0 is the zero sequence  $\{0\}$  and 1 is the radius of the ball, so we can rewrite the definition of the open ball  $B(0, 1)$  using the given metric in

the problem and the ball is:  $B(\{0\}, 1) = \{\{a_k\} \in X \mid d(\{a_k\}, \{0\}) < 1\}$ . Also, from the definition of the metric which is given in the problem we can see that the possible distances between any two sequences in  $X$  are  $0, 1, 1/2, 1/3, 1/4, 1/5$ , etc. So for this open ball since the distance between the zero sequence and any other sequence is strictly less than 1 and then it cannot be 1 and so it could be  $0, 1/2, 1/3, 1/4$ , etc. If the distance between  $\{a_n\}$  and  $\{0\}$  is 1, this means that  $k = 1$  and so their first elements are not equal i.e.  $a_1 \neq 0$ , but if the distance is less than 1 then  $k \neq 1$  and so  $a_1 = 0$ . Thus all the sequences in the open ball  $B(\{0\}, 1)$  must have their first element equal to 0.

The same thinking can be used for the other open ball  $B(\{0\}, 1/2)$ . We can define this ball as  $B(\{0\}, 1/2) = \{\{a_n\} \in X \mid d(\{a_n\}, \{0\}) < 1/2\}$ . So since the distance  $d$  here is strictly less than  $1/2$  so it cannot be 1 or  $1/2$  and it could be  $0, 1/3, 1/4, 1/5$ , etc. which means the first two elements of all the sequences in this ball cannot differ from the first two elements of  $\{0\}$  and so the first and second elements of all the sequences in that open ball must be zeros and the other elements could be anything.

Not all of the students gave the right answers to this part of the problem. Even though some students had a correct understanding of the metric, they were not able to answer this part of the problem correctly.

Two of our students (7<sup>th</sup> and 11<sup>th</sup>) answered this question correctly. As we have seen previously these two students showed a correct understanding of the metric.

The 11<sup>th</sup> student was asked to consider only the open ball  $B(\{0\}, 1/2)$ , and he argued that:

*Ok, so it will be everything, everything with the distance strictly less than a  $1/2$  between distance, so they cannot differ, distance cannot be equal to 1 or  $1/2$ , so they must not differ in their first two terms. So the ball will be all sequences starting with 0, 0 (the first two terms). (11<sup>th</sup>)*

We asked the other student (the 7<sup>th</sup>), to consider the ball  $B(\{0\}, 1)$ , and in his answer he said that:

*So less than 1; um..., it contains any sequence  $um$ , for which the first term is zero, because you want anything which is less than 1, so say a  $1/2$  where  $a_2$  is different, will be within that. So anything that starts off with  $a_1$  being 0, then goes up to anything will be in that ball. (7<sup>th</sup>)*

So we can see that those students succeeded in their answers and it was easy for them to get the right answers and to comment on the open balls which they described correctly.

The 5<sup>th</sup> student was asked to consider both of the open balls and she said in relation to the open ball  $B(\{0\}, 1)$  that:

*It's going to be all the points such that they're less than 1. So you can have like a half, a third, a fourth, a fifth, a sixth. So all points basically where  $k$  is in  $N$ , and  $a_k$  is not equal to  $b_k$ . So I think it would probably be all of them except for  $a_1$  and  $b_1$ . (5<sup>th</sup>)*

Her comment was not too clear, and it was difficult to say that she answered correctly, so I asked her to describe the second  $B(\{0\}, \frac{1}{2})$  and she commented that:

*So probably be all of them except  $a_1, a_2$  and  $b_1, b_2$ . It just 0 so they're to the same is to there and if they're different as long as is not 1 or 2 then it's going to be smaller than a half, because it's going to be 3, 4, 5, 6..., just you're going to get 1 over that which is neither 1 nor half, I think. (5<sup>th</sup>)*

So she explained that since the distance is less than a half then any two points (sequences) in that ball must have zeros in the first two terms and they are might be different anywhere except for those two first terms. Even though this student had good understanding of the points of the ball, however she seemed to explain the ball  $B(\{0\}, \frac{1}{2})$  incorrectly (i.e. rather than considering the distance between the centre of the ball (zero sequence) and any sequence to be less than  $\frac{1}{2}$ , she considered the distance between any two sequences of the ball). She concludes that the first two terms of any two sequences are zero rather than saying that the first two terms of any sequence in that ball are zero. It is possible that in her explanation she means the sequence  $\{b_n\}$  to be the zero sequence, but she does not make this clear. Therefore she might have good idea of the sequences in these balls but she did not give clear explanations.

The other six students in the interviews did not give the correct answer to this part of the problem. Some of those students understood the metric correctly but were confused about the radius of the open balls, and some others did not have a good understanding of the metric and this was the reason they provided incorrect answers.

One of the students who understood the metric correctly gave a correct answer for the previous parts of the problem but did not succeed in this part of the problem. I

asked this student (2<sup>nd</sup>) to answer the question about the open ball  $B(\{0\}, \frac{1}{2})$  and he answered that:

*Ok. Yeah so its sequences in which the first term is equal to the, equal to 0 and the second term is not equal to 0.*

Even though this student understood the metric correctly, he gave an incorrect answer. He confused the radius of the ball with the distance between the centre and the boundary of the ball. This student when he saw the radius of  $\frac{1}{2}$  in the ball, immediately seemed to look for sequences which were a distance of  $\frac{1}{2}$  from  $\{0\}$ . He saw that these sequences would start to differ from  $\{0\}$  at the second term and the first terms would be 0. So although he understood the metric correctly, he did not apply the definition of the open ball correctly, i.e. that the distance is strictly less than  $\frac{1}{2}$  and not exactly equal to  $\frac{1}{2}$  and that led him to give an incorrect answer.

The 9<sup>th</sup> student succeeded in giving the correct answers to the previous parts in the problem, however in this part of the problem she did not succeed and that gave us more insight about her conception of the metric. We asked this student to answer the question on the open ball  $B(\{0\}, 1)$  and she commented that:

*All this would be  $\{x_n\}$  I suppose and the distance between  $\{x_n\}$  and zero is strictly less than 1, what if something like  $\frac{3}{4}$  then you'd have 1 over ..., that won't work. I'm just skipping on to this part now because I think it might help me understand it. (9<sup>th</sup>)*

We can see that she thought of the open ball correctly but the distance that she chose to test some sequences of the ball did not fit with the given metric (as there is no such distance as  $\frac{3}{4}$  in the given metric), thus she realised that it will not work and asked to try the question concerning the other open ball  $B(\{0\}, \frac{1}{2})$  as it might be easier for her to understand. Regarding the second open ball she answered that:

*If the distance between  $\{x_n\}$  and  $\{0\}$  would be strictly less than a half, that means that the  $k$  is less than 2, where  $k$  is the min thing, so this would be the set of the zero sequence and all the sequences where the first term isn't zero. (9<sup>th</sup>)*

And, about the first ball  $B(\{0\}, 1)$  she commented that:

*Um..., that was  $k$  is less than 1, going from this here, so it's that just the zero sequence...., yeah that's just zero. (9<sup>th</sup>)*

So from her arguments above, we noticed that while she said that the distance between any  $\{x_n\}$  and  $\{0\}$  in the ball  $B(\{0\}, \frac{1}{2})$  is less than  $\frac{1}{2}$ , rather than thinking of

$k$  as greater than 2, she incorrectly deduced that  $k$  must be less than 2. So the possible distance for her are 0 and 1 and based on that she produced her answer which is, that all the sequences in that open ball must be either the zero sequence or a sequence whose first term is not zero (i.e.  $k = 1$ ). And she used the same idea for the open ball  $B(\{0\}, 1)$ , so that since the distance between  $\{0\}$  and another point  $\{x_n\}$  is less than 1 then  $k$ , for her, is less than 1, so the only possible distance is 0 and based on that she answered that all the sequences in the ball  $B(\{0\}, 1)$  are equal to the zero sequence. So this student seemed to understand the idea of the defined metric but she was confused about the possible values of  $k$  when she tried to think of a distance less than the radius. She appeared to consider that, to have a distance  $1/k$  which is less than a radius  $1/n$ , then  $k < n$  instead of  $k > n$ . This led her to give incorrect answers for both open balls.

All of the remaining students gave incorrect answers based on their misunderstanding of the given metric or were not able to make sense of it.

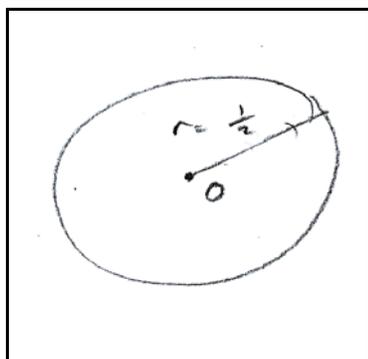
The 4<sup>th</sup> student started to answer this question well, but she was possibly not able to use the definition of the metric and think of the definition of an open ball at the same time. It seems hard for her to manage these two things together, thus she got lost and could not continue to answer the question. She was asked to look at the question about the open ball  $B(\{0\}, 1/2)$  and said that:

*The radius is half, yeah, means that there're  $B(0, 1/2)$  and  $d(0, a) < 1/2 \forall a \in X$ , for um all points in  $X$  it's gonna be less than a half, the points have to be within that distance, So um, we're gonna have our sequence  $b_2$ , I dunno, I'm trying to think of sequence that will work, and um, I don't like sequences. (4<sup>th</sup>)*

This student tried to think of the general definition of the open ball  $B(\{0\}, 1/2)$ , but her definition is not accurate because she said that  $d(0, a) < 1/2 \forall a \in X$  and that means all the real sequences of the set  $X$  would be within that distance which is not true. She might have the correct idea of that open ball in her head but she did not phrase it correctly. She did focus on the second term of the sequence ( $b_2$ ) but was not able to get any further and was not able to think of an example. Also we mentioned earlier that this student did not understand the second part of the definition of the metric and that could be the reason for her difficulty in thinking of sequences that are included in the open ball. When she became confused, she commented that she does not like sequences.

Another student (1<sup>st</sup>) understood the second part of the given metric incorrectly. This student was asked to try to look at the open ball  $B(\{0\}, \frac{1}{2})$ , and he claimed that:

*So to get a half, uh...  $k$  equals 2, so 2 is the minimum  $k$  where  $um... a_k$  is not equal to  $b_k$ . So  $a_k$  must be different from  $b_k$ . Um ... I'm not sure what to do after that. (1<sup>st</sup>)*

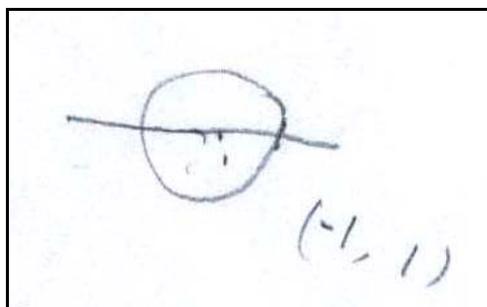


**Figure 4.8: The 1<sup>st</sup> student's picture of  $B(\{0\}, \frac{1}{2})$  in Problem 5**

This student did not use the definition of the open ball that related to the mentioned open ball correctly. He seems to confuse the radius of the ball with the distance between the centre point of the ball and any element of it, which is similar to the work of the 2<sup>nd</sup> student. So rather than thinking of the distance to be less than a  $\frac{1}{2}$ , he seems to consider the distance as equal to  $\frac{1}{2}$  and worked towards that. He also drew a circle to picture the ball with a centre 0 and radius a  $\frac{1}{2}$ , but because he did not understand the metric correctly, the picture did not help him to visualise sequences and so he could not continue to work on this part of the problem.

The (3<sup>rd</sup>) student also understood the metric incorrectly; I asked him to answer the question about the open ball  $B(\{0\}, 1)$  and he claimed that:

*The set! So would be minus 1 and 1. (3<sup>rd</sup>)*



**Figure 4.9: The 3<sup>rd</sup> student's picture of  $B(\{0\}, 1)$  in Problem 5**

This student seemed to ignore the given metric in this part of the problem and possibly answered the question according to his conception of the open balls on the

real line; also he drew a line and also drew a circle on that line to represent the ball on it. So the real line  $\mathbf{R}$  seems to affect this student's conception of open balls.

The 8<sup>th</sup> student also misunderstood the second part of the metric as we showed before where she considered the distance to be defined between corresponding terms and thought that  $k$  was the smallest nonzero number of the corresponding terms. I asked her to answer the question about the open ball  $B(\{0\}, 1)$  and she answered that:

*Well the distance between zero and any point in the set has to be one, so um...,  $k$  has to be 1, cause it's gonna hold to the second condition. Um..., so if I have a sequence may um,  $\{2, 3, 4, 5\}$  and  $\{1, 1, 1, 1\}$  so 2 doesn't equal 1, so the min is going to be 1, so you can get 1 over 1. Um, it is gonna be all sequences, for any thing, that's gonna be greater than 1 in this one, and a string of 1's here. (4<sup>th</sup>)*

This student also seems to confuse the radius of the ball with the distance between the centre point and any point in the ball (similar to the 1<sup>st</sup> and 2<sup>nd</sup> students), and so she gave an incorrect answer for this question. She used her misconception of the metric to obtain the distance 1 rather than obtaining distances less than 1. For her, it seems that since the distance has to be 1 then the smallest nonzero number of all corresponding terms must be 1. The example of the finite sequences she gave confirmed her incorrect thinking about the metric, where the smallest nonzero number between any corresponding terms is 1 so in her understanding each distance between terms is 1 and thus the whole distance is 1. So this student had an incorrect understanding of the metric and also thought of the open ball incorrectly.

The results that related to this part of the problem showed that students' inability to understand the definition of the metric caused them to give incorrect answers, and this was evident especially in this part of the problem. Some students (1<sup>st</sup>, 2<sup>nd</sup> and 8<sup>th</sup>) when asked to describe  $B(\{0\}, r)$ , looked for sequences for which  $d(\{0\}, \{a_n\}) = r$  instead of  $d < r$ , so they confused the radius of open balls with the distance between the centre and the other points and it seemed that their conception of the metric prevented them from thinking of the definition of open balls correctly. Also some students seemed to make an incorrect description of the open ball that is, instead of describing the distance between the centre and a point of a ball they intuitively described the distance between any two points in the ball. Also, some students tried to draw pictures of the open balls that were given to them, but the drawings did not lead them to give the right answers.

#### 4.2.5.4 The Fourth Part of Problem 5

The last part of this problem is: ‘*Is the set of sequences  $\{\{a_n\} : a_1 = 0 \text{ or } 1\}$  open? Explain!*’ or ‘*Is the set of sequences  $\{\{a_n\} : a_1 = 0\}$  open? Explain!*’

This question investigates students’ abilities to deal with open sets in non-familiar metric spaces and how they consider them. It examines whether the students are able to use the given definition to solve the problem or if they follow their own conceptions of open sets to give the answers. This part was given only to five students in the interviews (2<sup>nd</sup>, 5<sup>th</sup>, 7<sup>th</sup>, 8<sup>th</sup> and 9<sup>th</sup>); students were only asked to do this part of the problem if they had performed well on the previous parts. Three of the students (2<sup>nd</sup>, 5<sup>th</sup> and 9<sup>th</sup>) were asked if the set  $\{\{a_n\} : a_1 = 0 \text{ or } 1\}$  is open, and were then asked if the set  $\{\{a_n\} : a_1 = 0\}$  is open when I found that they struggled with the first set; and the other two students were asked only to consider if the set  $\{\{a_n\} : a_1 = 0\}$  is open.

In this part of the problem, to find out if the set  $\{\{a_n\} : a_1 = 0 \text{ or } 1\}$  is open, it is useful to think of this set as made up of two parts, that is a union of two sets:  $(\{\{a_n\} : a_1 = 0\} \cup \{\{a_n\} : a_1 = 1\})$ . If we can prove that each of these sets is open then the union is also open. We know from the last part of the problem that  $B(\{0\}, 1) = \{\{a_n\} : a_1 = 0\}$ . Similarly  $B(\{1\}, 1) = \{\{a_n\} : a_1 = 1\}$ . So our set can be written as  $B(\{0\}, 1) \cup B(\{1\}, 1)$  and since the union of open balls is always open, the set  $\{\{a_n\} : a_1 = 0 \text{ or } 1\}$  is open.

Four students of the five students who tackled this question did not succeed in giving the correct answer and only the 7<sup>th</sup> student gave an answer that was close to the correct answer, but even he was not sure. Some of the students who did not succeed answered the question according to their conceptions of open sets. Some tried to picture the set and the fact they were not able to picture it prevented them from applying their understanding of open sets. I will now show the work of each of them.

The 2<sup>nd</sup> student was asked to answer if the set  $\{\{a_n\} : a_1 = 0 \text{ or } 1\}$  is open. This student was confused by this set and he commented that:

*Um, first thing I think of is, can I take a ball about every element in it. Show it that the all balls are included in the set. It doesn't include all sequences*

*because,  $a_1$  can be equal to other element. So it's not the full set, um... yeah I'm starting to think it's closed, but it all depends on the metric. Um... it could be 0 or 1..., yeah, so all the points in the set here can be broken up into the two types, one of which is the first element is 1, our metric 0 away one from each other, and the other one is the 1 and they are in just a distance zero away from each other So they look like two very similar, I'd say it's open. I'm not sure why though. (2<sup>nd</sup>)*

This student started by thinking of the set in terms of his definition of the open set, so he tried to think if he can find an open ball about each element in the set which is contained in the set. However for him there are other sequences that start with  $a_1$  not 0 or 1 and thus the set does not contain all the sequences (that is it is not all of X), and this seemed to suggest to him that the set is closed. Then he thought of the set again and realised that its elements could be broken into two types: sequences that start with 0; and sequences start with 1. He seemed to deem these two types of sequences to be very similar to each other which led him to think that the set could be open. This student started with a definition of an open set and then used his intuition and then did not return to the definition of an open set. He also did not really think about the definition of the metric in this case and so he gave an intuitive answer. When I realised this student's confusion, we asked him to find out if the other set  $\{ \{a_n\} : a_1 = 0 \}$  is open? I thought that it might be easier for him to think of this set, but again I found that he did not use the metric correctly. He argued that:

*um... no if it just 0 it will be closed because all the elements in this set are at distance 1 away from each other, but there are with the other sequences which would be less than one away. So if I draw a picture, there will always be sequences close to distance 1 over  $n$  an element of  $N$ . so you can get as close as to like to our elements of our set, so there is no ball that will only contain elements of our set are not any of these other sequences. So that'll be closed then. (2<sup>nd</sup>)*

This student drew a line and marked points on it at distances less than 1 from the centre point to represent the sequences that, for him, did not belong to the set. The student thought that all the sequences in the mentioned set are exactly at a distance 1 away from each other and so if he drew an open ball about any point (sequence) of the set this ball would contain other sequences in X that are a distance less than 1 from the

centre and which are not in the given set. He concluded that the set contains no open balls and so the set is closed. We noticed that the student thought of the sequences of the set within the metric incorrectly ‘all the elements in this set are at distance 1 away from each other’ whereas in fact they are all less than a distance of 1 away from each other. Also, since he was not able to find open balls around elements of this, he said that the set is closed rather than saying the set is not open. This student seems to have gotten confused about distance. He seems to say that if  $a_1 = b_1$  then  $d(\{a_n\}, \{b_n\}) = 1$ . This and the fact that he found it difficult to visualise balls in this metric space meant that he could not answer this question.

The 5<sup>th</sup> student tried to answer all the previous parts of this problem based on her idea of the given metric which unfortunately was not quite correct. We asked her in this part to answer if the set  $\{a_n : a_1 = 0 \text{ or } 1\}$  is open. With respect to this question she said that:

*Yeah, I just try to think if it uh open ball of every point in it, and is every open ball... So if  $a_1$  is 0 then, yeah um... and if  $a_1$  1 then is, is there an open ball in that? I don't think there is. I don't think it is open ball really. [...] No, because  $a_1$  at any thing you gonna get either 1 or 0, so is not open because it's not going to contain all the other points. (5<sup>th</sup>)*

At first, this student tried to see if she can find an open ball centred at every point in the set, and seemed to conclude that if  $a_1$  is 0 then there is an open ball, and if  $a_1 = 1$  then there is no open ball, and thus the set is not open. She did not explain her reasons though, but then she used her intuition to answer this question and argued that since there are other sequences in X which start with  $a_1$  not equal 0 or 1 and these are not contained in our particular set, then this set is not going to be an open set. It is possible that this student might think of the distance between the first elements of the sequences in the set instead of thinking of the distance between the sequences themselves and that might be the reason for her incorrect answer. It is also possible that she is thinking of an open set as a continuum of points because she rejects the possibility that the set is open because ‘it is not going to contain all the other points’. Also she commented according to the other question ‘is the set of sequences  $\{a_n : a_1 = 0\}$  open?’ that:

*If you think of all the points around 0 to make the radius small enough you're gonna have points neither aren't in.... (5<sup>th</sup>)*

Here, she seemed to think only of the point 0 (or possibly the sequence  $\{0\}$ ). She seems to use the same idea as in the previous part of the question here that is, rather than thinking of the whole sequence  $\{a_n\}$  she thinks only of the first term  $a_1$ . Also she thinks that if you take a small ball around  $\{0\}$  then it will have to contain elements outside the given set. So she answered that the set is not open which is not correct, while she gave a better answer for the question 'What is  $B(\{0\}, 1)$ ?' . She did not seem to realise that these two questions concern the same set. The representation of the set had a big impact on her answer. It seems that students accept the openness of sets when written in some ways and refuse that with other versions of the same sets.

The 7<sup>th</sup> student was asked to answer the question 'is the set of sequences  $\{\{a_n\} : a_1 = 0\}$  open? And he answered that:

*So, in another way is, can you create a ball about 0, some radius  $r$  contained within that, so you have  $B(\{0\}, r) = \{\{x_n\} : d(\{x_n\}, \{a_n\}) < r\}$ . [...] I think it'll be open, because you will have lots sequences  $u_n$ ,  $\{x_n\}$  an element of your  $X$  for which it takes a long time to diverge of 0, and you know, if you make your  $\varepsilon$  smaller than you just take a sequence which staying with 0 a little bit longer so I'm thinking that you will always be able to find a sequence which is  $u_n$ ..., smaller than  $\varepsilon$ , type of the thing  $u_n$ ..., but I'm not entirely sure. (7<sup>th</sup>)*

This student has a correct conception of the metric and used his conception to make sense of this set. He thought of the definition of the open set but he tried to find out if there is an open ball only around the point 0 (or the sequence  $\{0\}$ ) and did not look at the other elements with  $a_1 = 0$  (maybe for him the point 0 is the main point in that set). He seemed to be saying that, there is always a sequence  $\{x_n\}$  in the set  $\{\{a_n\} : a_1 = 0\}$  such that the first  $n$  terms of this sequence are zeros, and the distance between this type of sequence and the zero sequence gets smaller as the number of zero terms at the beginning of the sequence gets larger. Thus he could find an open ball around  $\{0\}$  of radius  $\varepsilon$  which stays in  $\{\{a_n\} : a_1 = 0\}$  and he concludes that the set is open. However he is not entirely sure of his answer. This student, when we asked him about his idea of an open set, mentioned that '*kind of fades of infinitesimally close to boundary, but it never quite gets out. Fuzzy at the edges*', so

for him open set has fuzziness between its points and that is what he found in this set. Therefore his conception of open set helped him here to be very close to the correct answer to this part of the problem.

The (8<sup>th</sup>) student was also asked if the set of sequences  $\{\{a_n\} : a_1 = 0\}$  is open. She started by using the definition of the given metric and said that:

*Well the distance between any two of these sequences is not gonna be 0. The first two are gonna match but it mightn't necessarily match for all of them. so the distance between the first two is gonna be 0, um..., the distance between these it could be 0, or you know any thing else, so it could be anything.[...] I can't just try to think of the ball do what kind of radius it would have to be an open set. [...] I can't tell, I don't know. (8<sup>th</sup>)*

This student tried to think of the elements of the set within the given metric. At the beginning she found that the distance between any two sequences cannot be zero if the  $a_n$  terms where  $n > 1$  do not match. But then she got confused when she used her misconception of the second part of the definition of this metric (i.e. the distance is defined between each corresponding term) and found that the distance between the first terms of any two sequences is 0, and the distance between the other corresponding terms could be anything. She tried to use the definition of the open set when she thought of a ball in the set, but she could not find a radius to have the set to be open and so got confused and then gave up answering the question. Probably the fact that she misunderstood the metric caused her to have difficulty thinking of open balls in this metric space and so considering the openness of the set was difficult.

The remaining student who attempted this part of the problem was asked if the set of sequences  $\{\{a_n\} : a_1 = 0 \text{ or } 1\}$  is open. Her comment on this question was:

*So, if I call this A, I'll be saying that, A is open if  $\forall \{a_n\} \in A$ , is either 0 or 1,  $\exists \varepsilon > 0$ , s. t.  $B(x, \varepsilon) \subset A$ . A ball around  $\{a_n\}$  of radius  $\varepsilon$  would be...,  $B(\{a_n\}, \varepsilon) = \{\{b_n\} \subset A \mid d(\{a_n\}, \{b_n\}) < \varepsilon\} \subset A$ . So is there  $\varepsilon$  for every sequence; that is either 0 or 1. If I chose  $\varepsilon$  to be a half, then  $a_n$  and  $b_n$  only differ at the first term..., so do I want that or do I want they don't differ at the first term, I want they don't..., I dunno! (9<sup>th</sup>)*

This student started correctly by calling the set  $\{\{a_n\} : a_1 = 0 \text{ or } 1\}$  to be A, and defining how the set A could be an open set, and she also correctly defined an open

ball about any element in that set. When she started to think of an open ball she chose  $\varepsilon$  to be  $\frac{1}{2}$  and then she tried to test this radius on the given set. According to her argument ‘*If I chose  $\varepsilon$  to be a half, then  $a_n$  and  $b_n$  only differ at the first term*’, she seemed to think of  $k$  inversely, so to get a distance less than  $\frac{1}{2}$  she thought that  $k$  is less than 2 which would mean that the first terms differ. She also appeared to get confused when she found that the first terms of sequences in the set A might be the same but might also be different. After her confusion, we asked her to answer if the set of sequences  $\{ \{a_n\} : a_1 = 0 \}$  is open, to make the set easier for her to think of, and she replied that:

*If it's gonna be open, then it needs to be in A, um..., so yeah,  $\varepsilon$  would have to be..... They would have to be the same for the first term so  $k$  is..., I don't think it's open because if the first term is the same then  $k$  equals..., it would, um, the distance between them would be 2 or more so you wouldn't be able to choose an  $\varepsilon$  so that..., can I just say that I don't think it is open! (9<sup>th</sup>)*

She struggled with this set too. I think that her inverse thinking of  $k$  to choose smaller or bigger distances led her to say that the set is not open. It appeared that the reason for her answer is that, since only the first terms are the same then the other terms of all the sequences could be the same or could be different so that  $k \geq 2$  which means for her the distance between the sequences could only be bigger than 2. (She seems to confuse  $k$  and  $1/k$  here again.) She concludes that it is not possible to find a small value of  $\varepsilon$  such that a ball with this radius would be contained in this set, and therefore she thought that the set cannot be open. So her misconception of choosing  $k$  caused her to struggle with the sets and to give incorrect answers to this part of the problem.

From the results of this part of the problem we notice that, only one student tried to think of the set  $\{ \{a_n\} : a_1 = 0 \text{ or } 1 \}$  as  $\{ \{a_n\} : a_1 = 0 \} \cup \{ \{a_n\} : a_1 = 1 \}$  and no one made the connection to  $B(\{0\}, 1)$ . Also all of the students who attempted this part of the problem tried to use the open ball definition of open sets here but found it difficult to apply. The fact that they could not visualise open balls hindered them from progressing and the unfamiliarity of the metric and also spaces of sequences made the task difficult.

I conclude with some points related to the students' answers to this part of the question:

- The different representations of a set could give rise to completely different answers, for instance the sets  $B(\{0\}, 1)$  and  $\{a_n : a_1 = 0\}$  are the same set but students treated them differently. (5<sup>th</sup>)
- The misunderstanding of the metric prevented some students from thinking of the openness of the given sets correctly. (8<sup>th</sup> & 9<sup>th</sup>)
- Rather than saying a set is not open some students instinctively say that the set is closed and this could be evidence of the spontaneous conception ‘if a set is not open then it is closed’. (2<sup>nd</sup>)
- Some students’ idea of open sets (e.g. keep getting smaller distance) helped them to find the answer to this question but did not help them in the other problems. (7<sup>th</sup>)
- Students have difficulty with visualising balls in non-familiar spaces.
- The continuum idea of open balls influenced some students’ answers. (2<sup>nd</sup>, 5<sup>th</sup>)
- Thinking of  $k$  inversely to have distances less than  $1/k$  led some students to give incorrect answers. (9<sup>th</sup>)

#### **4.2.5.4.1 Summary of results about each student’s answers to this problem**

##### **The 1<sup>st</sup> student**

This student seemed to misunderstand the metric given in this problem since he explained that the metric is defined between the terms of two sequences instead of between the two sequences themselves. He gave a correct example for a sequence in answer to the second part of the problem; however his reason was based on an incorrect understanding of the metric. This student sometimes mixed up  $k$  and the value  $a_k$ . In his answer to the question about the open ball in the third part of the problem, he confused the distance between the centre and any point of the ball, with the radius of the ball. He found it very difficult to visualise an open ball in this setting. He did not do the fourth part of the problem. It was difficult for this student to think of the metric correctly, perhaps because he was not used to thinking of spaces of sequences and was not able to visualise elements or subsets of the space, and thus he did not succeed with this problem.

### **The 2<sup>nd</sup> student**

The phrase ‘for all’ made a difference to this student’s understanding of the given metric and so he understood the idea of the metric and he explained it correctly. He provided a correct answer to the second part of the problem based on his good conception of the metric. In the third part of the question, this student confused the radius of the ball with the distance between the centre of the ball and the other points (i.e. instead of describing the distance as being less than the radius, he described the distance as being equal to the radius) and that caused him to answer the question incorrectly. In the fourth part of the problem he was the only student who thought of the set  $\{ \{a_n\} : a_1 = 0 \text{ or } 1 \}$  as the union of two similar types of sets. However he seemed to think of the distance between the first terms of sequences in the set instead of the distance between the sequences themselves which prevented him from giving a correct answer to this question. So he succeeded in some parts of the problem and was confused with other parts where he gave incorrect answers.

### **The 3<sup>rd</sup> student**

This student compared the given metric with the discrete metric so for him they are similar and he did not show a good understanding of the metric. He could not give an answer to the second part of the problem and his answer showed his incorrect idea of the metric. It seemed that he considered  $k$  to be the minimum of the terms  $a_k$  and  $b_k$ . Recall that when finding the distance between  $\{1\}$  and  $\{0\}$  he decided  $k = 0$  (since the minimum of 1 and 0 is 0) and the distance is  $1/k$  which means he will divide by 0 which is impossible. Also he gave an incorrect answer to the third part of the problem, his answer was based on his conception of open balls in  $\mathbf{R}$  where  $B(0, 1) = (-1, 1)$  and he did not use the given metric in his answer. He did not do the last part of the problem. This student did not succeed with any part of the problem and that might be due to his unfamiliarity with this metric space and his incorrect understanding of the metric.

### **The 4<sup>th</sup> student**

This student found it difficult to understand the second part of the definition of the metric as this metric space contains sequences and she could not picture it. In the second part of the problem she gave a correct example of a sequence but her reason was incorrect and was based on her misunderstanding of the metric; where  $k$  for her is

the smallest term of the corresponding terms of two sequences and recall that based on that she chose  $\{a_n\}$  to be  $\{-1\}$ , so that  $k = \min \{-1, 0\}$  and the distance is  $1/k$  or  $-1$ . She added that to get the distance 1 she can get rid of the negative sign as a distance is always positive. In the third part of the problem she tried to define the open ball  $B(\{0\}, \frac{1}{2})$  in general, but she could not visualise the sequences within that ball so she could not make progress with the question. She did not do the last part of the problem. This student found it difficult to visualise this metric space which includes sequences and so it seemed that she could not understand the defined metric and did not succeed with any part of this problem.

### **The 5<sup>th</sup> student**

This student appeared to mostly understand the given metric based on her explanation of it however she may have had some misunderstandings which limited her answers to the other parts of the problem. Her answer to the second part of the problem was not quite perfect as it was not the most general answer to the question but she seemed to have the right idea. In the third part of the problem she showed a good understanding of the balls but she did not describe them formally (i.e. instead of describing the distance in the ball between the centre and any other point she described it as it between any two points in the ball). She gave an incorrect answer to the last part of the problem and the reason for that might be her conception of the distance between the first terms of the sequences in the given set instead of the distance between the sequences of that set. So she did not give full answers to most parts of the problem and did not succeed with the last part.

### **The 6<sup>th</sup> student**

This student was similar to the 3<sup>rd</sup> student in that she compared the defined metric with the discrete metric. She did not explain the metric well and only read it from the given sheet. She was not able to apply the definition to any example and she did not do any of the parts of the problem.

### **The 7<sup>th</sup> student**

This student constructed a good understanding of the given metric without any help, and he gave a correct explanation of it. He succeeded in the second and third parts of the problem and found the answers without any difficulty. He was the only student

who found the answer to the fourth part of the problem but he was not fully sure of his answer. So his correct understanding of the metric directly enabled him to think of the other parts of the problem easily and correctly.

### **The 8<sup>th</sup> student**

This student misunderstood the metric even when we changed ‘for any’ to ‘for all’ as she explained it as being defined term by term between any two sequences and also thought that  $k$  is the smallest term of the corresponding terms of the sequences. In the second part of the problem she got confused when she tried to find the distance between the sequences  $\{0\}$  and  $\{1\}$ . Because of her misunderstanding of the metric she thought  $k = \min \{1, 0\}$  and so the distance would be  $1/0$  which she realised is impossible and so she could not answer the question. Also she gave an incorrect answer to the third part of the problem and she confused the distance between the centre and the other points in the ball with the radius of the ball and also described the ball based on her incorrect idea of the metric. In the last part of the problem, she tried to think of an open ball within the given set but she had difficulty to think of a radius for any ball and she could not answer the question. So this student also had difficulty thinking of this metric space as it seems to be hard for her to think of open balls of sequences and this along with her misunderstanding of the metric meant that she did not succeed with this problem.

### **The 9<sup>th</sup> student**

This student seemed to understand the given metric well when we changed the phrase ‘for any’ to ‘for all’. Also she gave a correct answer to the second part of the problem. But she answered the third and the fourth parts of the problem incorrectly, and that was based on her incorrect thought about  $k$  when she tried to choose a distance less than the radius  $1/k$  of a ball. So she succeeded in some parts of the problem and did not succeed with other parts based on her confusion with distances that are less than  $1/k$ . However this confusion did not stem from her understanding of the metric.

### **The 11<sup>th</sup> student**

This student displayed a correct understanding of the metric when he replaced ‘for any’ with ‘for all’. Also he provided correct answers to the second and third parts of

the problem without any difficulty and that was based on his good understanding of the metric. He did not do the other part of the problem. So he succeeded with all the parts of the problem that he worked on.

#### **4.2.5.5 Summary of Findings on Problem 5**

Problem 5 was the most difficult problem that the student were asked to work on. It was difficult for two main reasons; firstly the students were not very comfortable working with sequence spaces, and secondly the metric was complicated and not easy to understand. I saw that the students who were able to grasp the meaning of the metric were more likely to succeed on this task. This is not surprising. I also saw that mathematical language can cause problems, as it did here with the difference between ‘for all’ and ‘for any’. I also observed that the way that mathematical objects are represented can influence students’ thinking, for example the form of the metric immediately led two students to think of the discrete metric, and even students who had previously described the ball  $B(0, 1)$  correctly, did not recognise it when written as  $\{\{a_n\}: a_1 = 0\}$ .

Half of the students in this group were able to understand the definition of the metric in this problem correctly, while the other half displayed various misunderstandings. I were able to gain a lot of information about these misunderstandings from the students’ explanation of the metric, but even more from their work on the rest of the problem.

#### **4.2.6 Analysis of the Responses to Problem 6**

The discrete metric is very different from other common metrics, the elements of a discrete metric space are isolated from each other in a special sense. The question was given to the students in the questionnaire and it was designed to detect the students’ knowledge of the discrete metric and to investigate how they think of the unit circle within that metric.

##### **Problem 6:**

*If  $d$  is a discrete metric on  $\mathbf{R}^2$ , describe the unit circle, i.e. the set of  $x \in \mathbf{R}^2$  such that  $d(0, x) = 1$ .*

(This problem is number 4 on the questionnaire in Appendix 1)

The definition of the unit circle is given in the question which is the set of points  $\{x \in \mathbf{R}^2: d(0, x) = 1\}$ . The discrete metric in  $\mathbf{R}^2$  is defined by:  $d(x, y) = 0$  if  $x = y$  and  $d(x, y) = 1$  if  $x \neq y$ , where  $x, y \in \mathbf{R}^2$ . Thus any point in  $\mathbf{R}^2$  is in the unit circle except the point  $(0, 0)$ , i.e. the unit circle in this metric is  $\mathbf{R}^2 - \{(0, 0)\}$ .

Five of the 16 students (A, G, J, L and O) were able to give the correct solution to this problem, two students (D and N) gave unclear answers, seven incorrect answers were given by the students (C, E, H, I, K, M and P), and the other two students (B and F) did not attempt this problem.

A common answer for students who gave correct answers was:

$$\text{Unit circle is } \mathbf{R}^2 \setminus \{(0, 0)\} \text{ as, } d((0, 0), (0, 0)) = 0 \text{ but } d((0, 0), (x, y)) = 1, \\ \forall (x, y) \in \mathbf{R}^2. (O)$$

All of the five students who answered correctly made sense of the discrete metric in  $\mathbf{R}^2$  and were able to find the unit circle in this metric space.

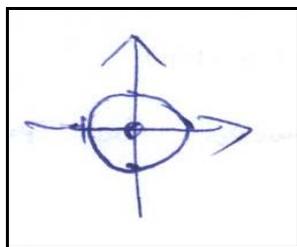
The two students who gave insufficient responses tried to write the definition of the discrete metric but they did not comment on that definition and did not apply it to this problem successfully. One of them wrote:

$$d(0, x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}. (D)$$

This student seems to understand the metric and is very close to giving an answer to this problem but does not actually say what the unit circle is in this situation.

The other student answered that:

$$\begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}, \text{ Not } 0. (N)$$



**Figure 4.10: The Student N's illustration for Problem 6**

This student seemed to attempt to write the definition of discrete metric but the definition was not completed. This student also gave an incorrect picture of the unit circle. He/she drew the intuitive picture of a circle on the plane  $\mathbf{R}^2$  with the usual metric rather than using the discrete metric. This student commented 'Not 0', and so

he/she seemed to understand that the zero point is not included in the unit circle with the discrete metric, but there was not enough explanation to be sure if he/she understood.

We see that those two students appeared to have the idea of the set by the definitions they tried to write but they were not able to give sufficient descriptions. For example, do they think that the only points on the unit circle are those on  $\{(x, y) \mid x^2 + y^2 = 1\}$ ?

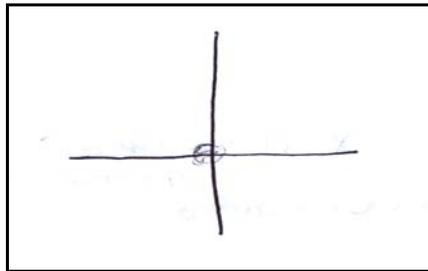
As we mentioned earlier seven students in the questionnaire answered the problem incorrectly, and they gave different responses to the problem. Some of them (E and M) wrote only their general definitions of the discrete metric and unit circle but they did not comment or explain these definitions, therefore we classified them as incorrect answers. One of them wrote that:

$$\begin{aligned} \text{Unit circle: } x^2 + y^2 = 1, \text{ discrete metric } d(x, y) = 0 \text{ if } x = y \\ d(x, y) = 1 \text{ if } x \neq y. \text{ (E)} \end{aligned}$$

These students did not apply their definitions of the discrete metric to the problem and appeared to have a conception of unit circle which comes from the usual metric on  $\mathbf{R}^2$ .

The students (C, G, I and K) thought incorrectly that the unit circle with the discrete metric is only the origin  $\{(0, 0)\}$ . One of them claimed that:

$$(X, d) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases} \Rightarrow \text{the set in } \mathbf{R}^2 = \{0\} \text{ as } x < 1. \text{ (C)}$$



**Figure 4.11: Student C's response to Problem 6**

This student tried to think of the definition of the discrete metric but his/her way of writing of the definition was not correct because he/she incorrectly wrote ' $(X, d) = \{ \dots \}$ ' rather than ' $d(x, y) = \{ \dots \}$ '. Also it seems that, this student, instead of thinking of the distance of points on the unit circle to the origin in  $\mathbf{R}^2$  being equal to 1, he/she thought of the distance being less than 1; so if the distance is going to be less than 1, then by the discrete metric the only possible point is the origin. So this student

described the unit ball rather than the unit circle and this is could be the reason for the other students (G, I and K) to provide the same incorrect answer.

Another student answered incorrectly that:

*The unit circle is the origin, discrete metric requires; if  $d(x, y) = 1, x = y$  so  $d(0, x) = 1, x = 0$ . (H)*

So this student defined the discrete metric incorrectly which led him/her to give an incorrect answer to the problem.

Another student who gave an incorrect answer commented that:

*For every point from  $[0, 1]$  in the circle, every point outside that is not. (P)*

This student confused the real line with the plane and also confused the closed ball with the unit circle. So there are some students who thought incorrectly that the circle can be described in terms of the real line and this student provides evidence for this conception.

From the results of this problem we can conclude that, some students appeared to have a sensible meaning of the discrete metric and thus they gave correct answers to this problem, while other students did not give clear explanations in their answers. We noticed that there are some students who were not able to write definitions correctly and that might be a reason for giving incorrect answers. We found also that some students used their conception of the unit circle in  $\mathbf{R}^2$  with the usual metric to describe the unit circle asked in this question. In addition, there some students who confused the unit circle with the ball.

### **4.3 Conclusion**

In this chapter I discussed an analysis of students' responses to the given problems in the questionnaire and in the interviews. Six problems were designed in relation to the concept of open set. Some of these problems (1, 2, 3 and 6) were given to the students in the questionnaire, and in the interviews each student was given two or three of the problems 2, 3, 4 and 5. Based on students' justifications of their answers I discovered some information about students' conceptions concerning the open set concept and also their conception concerning other notions (distance and open balls) which are basic for the concept of open sets. We have found that many of students' concept image of open sets in a metric space is affected by previous experience with

sets (intervals) in the metric space  $\mathbf{R}$ . We have also noticed that some students' thoughts about sets in metric spaces seemed to be based on everyday life usage of language (e.g. if a set is not open then it is closed). The influence of the Euclidean space on students' concept images was evident in most of students' answers to the problems asked in both the questionnaire and the interviews and students had difficulties in thinking of unfamiliar metric spaces, and visualising sets.

Many students seem to think that they understand the most important parts of the definition of an open set, and they seemingly use their understanding when working on problems in place of the definition. The analysis also showed that ignorance of some parts of the definition (which might be seen as unimportant) could cause problems for many students (that was clear in many students' responses to Problems 3 and 4).

Furthermore, students had difficulty in thinking of metric spaces in unfamiliar settings (especially in the context of sequences). The majority of students in the interviews struggled with the notions of metric and open balls and open sets in this context and they seemed to use all kinds of ways that were based on their understanding when considering the notions mentioned rather than using the formal definitions.

The findings also have pointed to the ambiguity of mathematical language. We have noticed the students' confusion with the mathematical quantifier 'for any', where some of them used it as to mean 'for some'. There was genuine ambiguity concerning the correct meaning in this case. This suggests that lecturers should be very careful when using this phrase.

I will consider students' concept images in more depth in Chapter 5. I will also investigate students' consistency in relation to the use of definitions.

## Chapter 5

# Students' Conceptions and Consistency

### 5.1 Introduction

This chapter represents the categories of students' conceptions concerning the notions of open sets and distance in a metric space. Firstly I will categorise students' concept images of the open set concept, secondly I will consider the categories of students' concept images about the distance in a metric space, I will also consider some of the influences on the formation of students' concept images, and I will then address some of the misconceptions that students have about open sets. Finally I will discuss the consistency between students' definitions and students' conceptions when working on task problems. This chapter combines the information gathered from both the questions on definitions and the task problems about each concept.

### 5.2 Concept Images of Open Sets

Resulting from the data which arose from the analysis of students' answers to the questions asked in the questionnaire and the interviews, we categorised students' concept images of open sets in metric spaces. Our categories do not indicate every possible conception of open sets in metric spaces as there might be other conceptions that did not arise in our study. Five main categories of concept images of open sets in metric spaces were obtained from our findings; these categories specifically are: conception based on the boundary idea, conception based on the formal definition, conception based on openness in  $\mathbf{R}$ , conception based on the union of open balls, and conception based on visualisation. I present and elaborate on each category and show examples of each one in the following sections.

### 5.2.1 Concept Image Based on the Boundary of Sets

The intuitive conception of an open set which is that it does not contain any of its boundary points, dominates many students' conceptions of open sets. Eight students (1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup>, 4<sup>th</sup>, 5<sup>th</sup>, 7<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) in the interviews and eleven students (A, D, E, F, G, H, I, K, L, M and N) in the questionnaire refer to open sets using the boundary idea. These students used words such as boundary, end, edge, border, outline, bounded, region and perimeter when referring to this concept image. We found that many students used this boundary idea to describe how they think of open sets; others showed this idea in their work on some of the task problems.

Of the students who used the boundary idea in their descriptions of an open set in metric spaces, we have these examples:

*Open set something which doesn't have a clear boundary, you can get as close as you like but never get the actual end of the set. (2<sup>nd</sup>)*

And:

*The open set is basically, it isn't like say straight edges, is kinda fuzz out, because it doesn't contain border elements, [...], or it's based where it doesn't contain all its boundary points (7<sup>th</sup>)*

And:

*A space which doesn't have its perimeter as a part of it. (G)*

These students' understanding of an open set is based on the fact that 'an open set does not contain any of its boundary points'. For these students the boundary, end, edge, border or perimeter points of open sets are unreachable points and they cannot be included in the set.

As an example of the other students who use this idea of open set in their answers to the problems, this student answered Problem 1 by:

*Both, it is bounded below and not bounded above. (F)*

For this student the word 'bounded' seems to mean 'has a boundary' rather than 'is contained in a ball with finite radius'. Another student answered Problem 2 by:

*No, B is a closed set. The points,  $m-1$  &  $m+1$  are on the boundary.  $\therefore$  A boundary exists.  $\therefore$  B is not an open ball. (G)*

These students clearly showed their conception of open sets using the boundary idea, so that for them if 'the boundary points' are contained in a set or the set is

bounded then the set for them is closed and if the set is not bounded or ‘does not contain its boundary points’ then the set is open.

The fact that ‘an open set in a metric space does not contain any of its boundary points’ is of course true but it seems that many students have misconceptions about the boundary points of a set. They sometimes think of these points intuitively and not mathematically (by definition). We can see this when students use terms such as border points, outlines, end points, perimeters, etc. which come from real life to indicate the boundary points of a set, and they think that the boundary points of a set are the same as the end (border) points of a set. The difference between the boundary points and the other kind of points (e.g. ends, outlines, etc.) is intuitively subtle but it is very important. The boundary points of a subset  $A$  in a metric space are defined to be ‘*the set of points of the closure of  $A$  less the interior of  $A$* ’ but students often do not use this definition while thinking of boundary points and they use their intuitive conception to look at these points instead.

In the interviews we asked the students to give their opinion about some statements related to open sets and also to explain their thinking. One of these statements was Statement (b) which says that ‘*A set is open if it lacks its end points*’. This statement was given to students particularly to discover if the boundary idea could affect students’ conceptions of an open set. We found that many students are influenced by this conception. For example, one student when he gave his opinion about Statement (b) answered that:

*If it lack its ends yeah! It’s open then, yeah I agree, just because it doesn’t contain either of its boundaries. (2<sup>nd</sup>)*

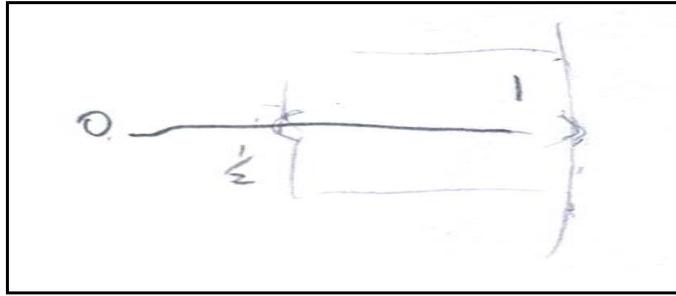
Another student also commented that:

*I would say yes, because if it has its ends, then its end is boundary point, so if it has boundary point then it can’t be an open set. (11<sup>th</sup>)*

These students seem to automatically deem an end point of a set as a boundary point and thus they agreed with the statement using this conception of open set which is based on the boundary idea. Recall that the latter student (11<sup>th</sup>) tried to provide an example to examine his opinion about the statement and argued that:

*So if your set is  $[0,1] \cup [2,3]$  and then you take the point 1 and  $\frac{1}{2}$  say  $[B(1, \frac{1}{2})]$  using the standard distance, then the open ball will be sort of go there,*

*but it will be open because that is not part of the set, I think so, I'd change my mind; I think I'll going to disagree now on that, I'm not sure. (11<sup>th</sup>)*



**Figure 5.1: The 11<sup>th</sup> student's response to Statement (b)**

This student considers the set  $[0,1] \cup [2,3]$  as the whole metric space and thinks of an open ball about the point 1 in his particular set as this point is an end or a boundary point for him. However he found that there is an open ball centered at this point and so he encountered a cognitive conflict when he realised that his set has an end point and it is still open. However, by definition and in this context the set  $[0,1] \cup [2,3]$  has no boundary points and also the point 1 is not a boundary point for the subset  $(\frac{1}{2},1]$  in the metric space  $[0,1] \cup [2,3]$  while the point  $\frac{1}{2}$  is a boundary point for that subset.

The confusion that students have on this point may stem from their familiarity with intervals on the real line. For a bounded interval (in the standard metric on  $\mathbf{R}$ ) the endpoints and boundary points coincide and students are not used to applying the definition of a boundary point in this context.

This misconception concerning boundary points might be the reason for the students who thought that subsets of a metric space that consists of isolated points cannot be open sets. For example some students answer that the set B in Problem 2 is not open because it contains boundary points where, mathematically, these kinds of subsets in  $\mathbf{Z}$  do not have boundary points. Also, this misconception of boundary points might be the reason for students who answer that the set S in Problem 4 is an open set. For instance:

*Um... a line cross a single element still is going to be a line. So a line as a subset of  $\mathbf{R}^2$  is just a line and is open at both ends [i.e. a and b] and to the sides it has no width is just a line, so it's open, I'd say it's open. (2<sup>nd</sup>)*

So this student found that the points  $a$  and  $b$  are not included in the set S which is a line segment of the  $x$ -axis and thus, it seems that based on his conception of an open set, he incorrectly answered that the set is open.

We conclude that many students refer in their conception of an open set to the boundary point idea, and some of them also show this conception while working on the problems. We find also that some students have misconceptions concerning the boundary points of sets in metric spaces which affect their concept images of open sets in metric spaces in general.

### 5.2.2 Concept Image based on the Formal Definition

This category comprises five students (1<sup>st</sup>, 2<sup>nd</sup>, 4<sup>th</sup>, 7<sup>th</sup>, and 9<sup>th</sup>) from the interviews and seven students (A, B, E, J, K, L and O) from the questionnaire. The findings of our study show that many students have a conception of an open set which comes from the formal definition of an open set in a metric space but this conception often does not include all aspects of the formal definition. I realised that some students do not pay attention to every word or phrase in the formal definition, and they think that the incomplete statement (conception) that they have is the formal definition of an open set. Most of students define an open set in a metric space by saying that ‘A subset  $U$  is open in the metric space  $(X, d)$  if every point in the subset  $U$  must be the centre of an open ball which belongs to the subset  $U$ ’. The formal definition of an open set that was given to students stated that ‘A subset  $U \subset (X, d)$  is an open set if  $\forall x \in U \exists \epsilon(x) > 0$  and an open ball  $B(x, \epsilon(x))$  in  $(X, d)$  s. t.  $B(x, \epsilon(x)) \subset U$ ’. I noticed that when some students consider an open set within a metric space they think that a subset is open in a metric space if there exists an open ball about each element of the subset which is completely contained in the subset but they do not mention that the open balls must be also considered within the metric space  $(X, d)$ . I observed that many students refer to this conception of an open set in their definitions of an open set. For example, one student defined an open set by:

*The set is open if for any point in the set you can draw an open ball around it which is contained in the set. (1<sup>st</sup>)*

Another student defined it as:

*$U$  is an open set in a metric space if every element of  $U$ , has a ball, whose radius depends on that element, which is entirely contained in  $U$ . (K)*

These students gave their conception of an open set in words which comes from the definition of open sets in metric spaces and both of them did not mention that the ball should be open in  $(X, d)$ .

Other students used mathematical language and symbols to define an open set. For instance, this student answered that:

*A set  $U \subset (X, d)$  is open if,  $\forall x \in U \exists \varepsilon(x) > 0$  s. t.  $B(x; \varepsilon(x)) \subset U$ . (J)*

This student also seems to have conception arising from the definition of an open set in a metric space.

Other students show this conception of an open set in their work on the problems. One student in his/her answer to Problem 3 wrote that:

*$\forall a \in A$ , we let  $x = |2 - a|$  then  $B(a, \frac{x}{2}) \subseteq A$  and so  $A$  is open. I.e. all points in  $A$  have an open ball which is a subset of  $A$ . (A)*

This student shows evidence of the conception which is based on the formal definition of open sets in metric spaces.

This conception which is based on the formal definition of an open set might not been seen, by some lecturers, as an issue for many students. The fact that students do not emphasise that the open ball needs to be considered within the metric space  $(X, d)$  is usually not a problem for them, especially when there is no ambiguity over which metric space is in question. However, when working with subspaces, confusion can arise.

The incomplete conception of the definition of open sets in metric spaces might be the reason why many students think incorrectly that the set  $S$  is an open set in their answer to Problem 4. As an instance of this, one student commented that:

*Ok, interval from  $a$  to  $b$  and, so cross with zero is zero, so it's still in  $\mathbf{R}^2$  an open interval. I'm gonna say it's open. I don't think it'll change a lot much. If it's open in  $\mathbf{R}$ , because it's an open interval in  $\mathbf{R}$  it's still then open in  $\mathbf{R}^2$ . (8<sup>th</sup>)*

Another student also claimed that:

*Umm, well, in  $\mathbf{R}^2$  it won't uh make a difference, because you're going to have your, uh, your interval  $(a, b)$  cross zero. So that is just gonna be the set of order pairs  $(x, 0)$  for any  $x$  in  $(a, b)$ , so you can immediately just bring it back to your  $\mathbf{R}$  and imagine, and it's never going to reach them  $[a \& b]$ , and it's still open. (7<sup>th</sup>)*

These students referred to the set  $S$  as a subset of  $\mathbf{R}^2$  in their answers but for them the subset  $S$  is still an interval on the  $x$ -axis and they thought of it as an open interval on  $\mathbf{R}$  and thus it is an open for them. These students seem to consider the metric on

the set  $S$  which comes from  $\mathbf{R}$  and do not consider the metric on the metric space  $\mathbf{R}^2$  which differs from the metric on  $\mathbf{R}$ , and so they incorrectly think that  $S$  is open in  $\mathbf{R}^2$  as they already have experience and know that for any point in this type of interval on  $\mathbf{R}$  there is an open ball which is contained in it and do not think of open balls in  $\mathbf{R}^2$  as the definition requires.

This incomplete conception of an open set which based on the formal definition might be also the reason for students who answered that the set  $B$  in Problem 2 is not open. As an example for that, this student answered that:

*No, if you take  $B(m-1, \varepsilon)$ , to be contained in  $B$ ,  $\varepsilon = 0$ , but by definition of open,  $\varepsilon > 0$ . (K)*

This student seems to be thinking of the metric space  $(\mathbf{R}, d)$  instead of  $(\mathbf{Z}, dz)$ . In that space the point  $m-1$  would not have an open neighbourhood in  $B$  and this is what seems to be confusing for them. Furthermore, this conception could be the source of trouble for students who gave incorrect answers to the second part of Problem 2: ‘Can you find an open ball  $C$  which is a subset of  $B$ ?’ To show this, one student claimed that:

*Um..., if you would,  $m$  and  $m + \frac{1}{2}$  and  $m - \frac{1}{2}$ , if my way of thinking is the right way of thinking, then that would work. (8<sup>th</sup>)*

In another example, this student answered that:

*Yes,  $C = \{m - \frac{1}{2}, m, m + \frac{1}{2}\}$ .  $C$  is an open ball centred at  $m$  with radius  $\frac{1}{2}$ . (E)*

So it seems that these students also do not pay attention to the metric space  $\mathbf{Z}$  and they incorrectly include the points  $m - \frac{1}{2}$  and  $m + \frac{1}{2}$  in the set  $C$  which should be a subset of  $\mathbf{Z}$ .

We see therefore that students encounter problems when they are working in subspaces. They seem to get confused between the metric space and its subspace. I suggest that one remedy for this may be an increased awareness of the part of the formal definition of an open set given in this course which emphasises that the open ball should come from the metric space  $(X, d)$ . The lecturer reported to me that this was his intention, but we see that the students did not place the same importance on this part of the definition as the lecturer did. In some textbooks this is dealt with by using the notation  $B_d(x, \varepsilon)$  for an open ball, thus reinforcing the connection between the open ball and the metric in question.

From our examples above, we summarise that, many students have an incomplete conception of an open set in metric spaces which is based on the definition of an open set, that is many students do not consider all the aspects of formal definitions and they ignore some aspects which seem to be unimportant to them. This incomplete idea of an open set is obvious in students' definitions and descriptions of an open set and also some students showed this conception while working on the problems. This conception of open sets might come from students' experience with metric spaces that have no ambiguity between them and their subspaces but it might lead students to have incorrect concept images of open sets in general as we observed above.

### 5.2.3 Concept Image Based on Open Sets in Euclidean Space

This category is assigned to students' conception of an open set which comes from their experience and ideas concerning openness in  $\mathbf{R}^n$ . The results from our data analysis of students' work show that five students in the interviews (3<sup>rd</sup>, 5<sup>th</sup>, 7<sup>th</sup>, 8<sup>th</sup> and 9<sup>th</sup>) and also four students in the questionnaire (E, J, L and N) have a general conception of an open set that is based on their conception of open sets in  $\mathbf{R}$ . We found this type of conception in some students' description of an open set, and in other students' work on the problems.

In the definition and description questions of open sets in a metric space, two students refer to open sets in the set  $\mathbf{R}$ . One of them commented that:

*I always think of it [open set] in just in  $\mathbf{R}$  in the real line, they're really obvious open sets. (5<sup>th</sup>)*

This student explained that she would use open sets on the real line  $\mathbf{R}$  as an example whenever doing problems because they are really obvious and simple sets for her. So when she considers an open set she always seems to base her ideas on open sets in  $\mathbf{R}$ .

Another student when defining an open set wrote that:

*$(-\infty, a)$  doesn't reach the two terms but gets infinitely close to them (N).*

This student also mentioned an example in the set  $\mathbf{R}$  to illustrate his/her idea of an open set in a metric space.

We also noticed the influence of some students' conceptions of open set and open balls in  $\mathbf{R}$  in their work on the problems. Some students specify the radius of open balls in metric spaces such as  $\mathbf{Z}$  incorrectly and we notice this clearly in their answers to Problem 2. For example, one student claimed that:

*Yes,  $m$  is the centre of the ball, 1 is the radius of the ball. (E)*

This student answered correctly that the set  $B = \{m - 1, m, m + 1\}$  is an open ball in the metric space  $(\mathbf{Z}, dz)$  with the centre  $m$  but he/she incorrectly specified the radius of this open ball to be 1. This incorrect specification of the radius of this kind of set might be due to this student's experience of finding the radius of open balls or closed balls in the metric spaces  $\mathbf{R}^n$  where the radius can be calculated simply by finding the distance between the end points and the centre point.

Another student (8<sup>th</sup>) also said that:

*Um, if you would,  $m$  and  $m + \frac{1}{2}$  and  $m - \frac{1}{2}$ ; if my way of thinking is the right way of thinking, then that would work.*

When I asked this student about the centre of her ball she replied ' $m$ ' and about the radius she said '*a half*'. This argument was given by this student when she was asked to give a subset  $C$  of the set  $B = \{m - 1, m, m + 1\}$  such that  $C$  is an open ball. In this student's argument there are two pieces of evidence which point to a conception that is based on  $\mathbf{R}$ : The first one is when the student thought incorrectly of the set  $\{m - \frac{1}{2}, m, m + \frac{1}{2}\}$  as a subset of  $B$ . The student seems to think incorrectly that any point between  $m$  and  $m + 1$  and also between  $m - 1$  and  $m$  is included in  $B$  and thus for her the set  $\{m - \frac{1}{2}, m, m + \frac{1}{2}\}$  could be a subset of  $B$ . So this student seems to think of the set  $B$  as  $[m - 1, m + 1]$  in  $\mathbf{R}$ . The second piece of evidence is when she answered incorrectly that the radius of her open ball  $C$  is a  $\frac{1}{2}$ , which might come from her experience of specifying the radius of open balls in  $\mathbf{R}$ .

Here is another indication of the conception that is based on  $\mathbf{R}$ : in Problem 5 when one student was asked to describe the open ball  $B(0, 1)$  in the metric space  $(X, d)$  where  $X$  is the set of all real sequences with the metric  $d$  that was defined in that problem, this student answered that:

*The set  $B(0, 1)$ ! So would be minus 1 and 1  $(-1, 1)$ . (3<sup>rd</sup>)*

This student did not use the given metric in the problem and he probably answered the question using his conception of open balls on the real line.

From the results above we saw that some students intuitively use their conceptions of open sets and open balls in  $\mathbf{R}$  to think or consider open sets and open balls in any metric space.

As a special case of this conception of an open set which is based on the experience of openness in Euclidean spaces, we recognised another conception of open sets which is based on the continuum idea of points or the idea that open sets must be connected. This conception seems to come from the continuum of points in the metric spaces  $\mathbf{R}^n$ . We noticed that some students consider that any open set should possess this continuum property between its points, that is open sets must contain infinitely many points and be connected. Some of the students who answered Problem 2 by saying that the set  $B = \{m - 1, m, m + 1\}$  is not an open ball (not open set) show indications of this continuum conception. One student argued that:

*I think I'm going that,  $r$  is 1 and  $x$  is centre  $m$ . But, no, that is not open because it doesn't contain all the points. It's only contains these three points, it's limited, meets these three points, I don't think it's open. (4<sup>th</sup>)*

This student seemed to start by thinking that the set B could be an open ball with centre  $m$  and radius 1 (although the radius is incorrect). But then she seemed to think that this set has only three isolated points and that the points in between these three points are not included in the set B, thus for her this set cannot be an open ball.

Another example is the following student's answer to the same problem:

*No, an open ball is an open set, and finite set cannot be open. (L)*

This student knows that any open ball is an open set and the set B is a finite set which is correct, but for him/her a finite set cannot be an open set (open ball). The student seems to have a conception of open set which an open set must be an infinite set so that it has uncountably many (continuum) points.

Another student gave a different explanation of the set B in Problem 2. This student commented that:

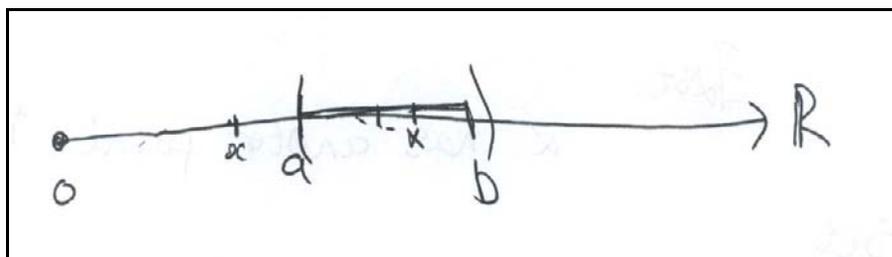
*We'd only got three elements, but these elements all have space of the exactly one. So you either have a gap of 0 or 1 between them. There is no kind of fuzziness in between, so you can't make it open. Like; it'll either contain them or not. (7<sup>th</sup>)*

This student commented on the distance (space) between the points in the set B. He thinks of the points as being separated by gaps of length 1. Recall from the boundary concept image, that this student thinks that an open set does not have straight or well-defined edges so that he can keep taking an open ball getting closer to the edges without including them. This student apparently did not find this 'fuzziness' idea in

the given set  $B$  and came to the conclusion that it cannot be open. This conception of fuzziness between points of an open set might be come from the continuum property of open balls (open sets) in  $\mathbf{R}^n$ .

The 6<sup>th</sup> student's answer to Problem 4 also seems to refer to a conception of open set that is based on the continuum or the connectedness idea. Regarding the set  $S$  this student claimed that:

*It's closed I think because um... that there is a point, you can always get a point in the interval [between 0 and (a, b)] and no matter where in interval, the point will land in the interval between them, where there is no a point in the set S. (6<sup>th</sup>)*



**Figure 5.2: The 6<sup>th</sup> student's response to Problem 5**

This student tried to draw the set  $S = (a, b) \times \{0\}$  on the real line  $\mathbf{R}$  but she seems to think of  $S$  as  $\{0\} \cup (a, b)$ . She explained on her drawing that there will be always a point in the interval (gap) between  $\{0\}$  and  $(a, b)$  which is not in the set  $S$ . So she seems to have a conception that only a connected set could be open.

One of the statements that we asked students in the interviews to give their opinions on was the Statement (d) which said that 'A set is open if all its points are near to each other'. One student, in response to this question said that:

*Yeah, I suppose so because if every point is contained in some open ball, then that open ball is kinda full of points because you know, it's difficult to explain, but you know, if they weren't near to each other then there would be a finite amount of them, I think. (9<sup>th</sup>)*

This argument shows that this student might have the conception that an open ball is full of points which are near to each other. It could be that this student is thinking of balls in  $\mathbf{R}^n$  which have no gaps in them. She also seems to think that if the set contained only finitely many points then it would not be open.

We conclude from this subsection that students' familiarity with the openness concept in  $\mathbf{R}^n$  also has a role in their conception of open ball and open sets in any

metric space. Conceptions which are based on  $\mathbf{R}^n$  could lead some students to think of the radius of balls in metric spaces which are different from  $\mathbf{R}^n$  incorrectly, could also lead some students to get confused with the usual metric  $d_{\mathbf{R}}$  when working with metric spaces such as  $\mathbf{Z}$ , could also lead other students to think that open sets must possess the continuum property between their points and so there are no gaps between them.

#### 5.2.4 Concept Image Based on the Union of Open Balls

The outcomes of our data analysis show that some students describe an open set as a union of open balls. Five students (3<sup>rd</sup>, 4<sup>th</sup>, 5<sup>th</sup>, 6<sup>th</sup> and 11<sup>th</sup>) from the interviews and five students (A, G, H, N and O) from the questionnaire were classified in this category. This kind of conception of an open set is noticed in students' definitions and explanations of open sets or in their opinions about some of the given statements about open sets, and this kind of concept image of open set has not been seen in students' answers to any of the given problems.

Many of the students in this category show this conception in their definitions and explanations of open sets. For example, when asked for a definition of an open set, this student defined it as:

*An open set is a union of open balls. (4<sup>th</sup>)*

Students A, G, H, N and the 3<sup>rd</sup> student also defined an open set using the same statement.

Two other students' definitions of an open set were similar, they said that:

*Open set, it's a collection of open balls. (5<sup>th</sup> and 6<sup>th</sup>)*

So these students seem to use some theorems concerning open sets as definitions.

The 11<sup>th</sup> said in his definition that:

*Oh, the open set one. It's gone when I ended off the exam. Open sets are, I would understand them as union of open balls, um..., in that every point is the centre of an open ball.*

This student admitted that he forgot the formal definition of an open set when the exam was over but he understands an open set as a union of open balls. When this student said '*in that every point is the centre of an open ball*' it appears that he has another idea of open set that has come from the formal definition and he seems to connect the two ideas of an open set together.

Another student gave an explanation of an open set to friends based on this conception of an open set. He/She commented that:

*I would ask them to imagine a ball of some radius  $\varepsilon$  about a point. Now every element of the open set  $U$  is at the centre of an open ball of some radius  $\varepsilon$ . All of these open balls combined create the open set. (O)*

This student preferred to describe an open set to a friend as a combination (union) of open balls, and also he/she seems to connect the idea of open set which comes from the formal definition of an open set to his/her understanding of an open set.

The 4<sup>th</sup>, 5<sup>th</sup> and 11<sup>th</sup> students repeated their conception of an open set when they asked to give their opinions about the statement 'A set is open if all its points are centre of open balls'. One of these three students said that:

*Yeah that's true because there's an open ball within that open set, that all the points have open balls within that set, so it's true because um, the open set will be union of open balls. (4<sup>th</sup>)*

These students seem to interpret the statement in line with their conception of open sets, that is since there is an open ball for each point then the set is a union of open balls and so this statement matches their conceptions.

The 11<sup>th</sup> student once again used his conception of an open set when he elaborated on his opinion about the statement 'A set is open if its complement is closed'.

According to this statement he said that:

*The set is defined to be closed if its complement is open, but the set is not open, so it is not a union of open balls, so pictorially, the set would contain its boundary, so the complement is open, so the set would be closed. So I would agree. (11<sup>th</sup>)*

So this student shows much evidence for his conception of an open set as a union of open balls.

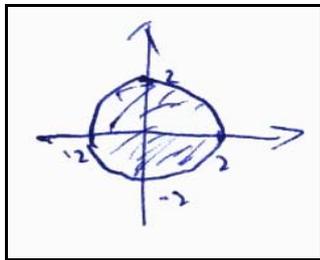
We conclude that the concept image based on the idea of the union/collection of open balls is another concept image of open sets in metric spaces that some students have. This conception might come from a theorem concerning open sets which says that 'A subset  $(U, d_U) \subset (X, d)$  is open  $\Leftrightarrow$  it is a union of open balls of  $(X, d)$ ' and the students in this category seem to build their conception on this theorem. However the students who have this concept image did not seem to use it when attempting the problems in the interviews or questionnaire.

### 5.2.5 Concept Image Based on Visualisation

This category consists of the conception of open sets which are based on generic pictures of open balls and visualisations of open sets. Our study pointed out that there were seven students (1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup>, 4<sup>th</sup>, 7<sup>th</sup>, 8<sup>th</sup> and 9<sup>th</sup>) from the interviews and also two students (F and N) from the questionnaire who gave evidence for this kind of conception of an open set. We noticed this concept image of an open set in those students' answers to some of the given problems. Regarding Problem 1, one student (in the questionnaire) answered:

*Open. (N)*

Also drew:



**Figure 5.3: Student N's answer to Problem 1**

This student did not comment or explain on his/her answer about whether the set  $A = [0,2)$  is an open set in  $\mathbf{R}$  but only drew an incorrect picture of  $A$ . This student might understand that the subset  $A$  is on the real line  $\mathbf{R}$  but he/she seemed to visualise it incorrectly in the plane  $\mathbf{R}^2$  as an open ball (as 2 is not part of the set) with centre 0 and radius 2.

For most students their picture of an open ball is based on pictures from Euclidean space. For example the 1<sup>st</sup> student (in the interview) when considering Problem 2 commented that:

*1<sup>st</sup> B is open if we can draw an open ball around 1 which is inside the set, if its centre at 1 then the open ball would include 0 and 2.*

*I you said here, draw an open ball, so that means you have a picture of open ball in your mind!*

*1<sup>st</sup> yeah!*

*I which kind?*

*1<sup>st</sup> just a ball, a circle, um it has to be inside the set. Um..., if the radius is greater than 1 then it would be outside the set, so, then it wouldn't be an open set.*

This student spoke about drawing an open ball, and when he was asked what was his mental image of a ball he said 'just a ball, a circle'. He seems to imagine an open ball (circle) about the point 1 (the centre of the set B) and when he tried to include the points 0 ( $m-1$ ) and 2 ( $m+1$ ) within the ball he found the ball to be drawn outside the set B and thus with his picture of the ball the set B cannot be open. This seems to be similar for the 8<sup>th</sup> student. When asked to comment on 'A set is open if the boundary is not included' she says:

*I'd say agree then, I just have picture of the ball and then the boundary around the outside!*

It is probable that images or pictures of balls have a role in these students' conception of an open set.

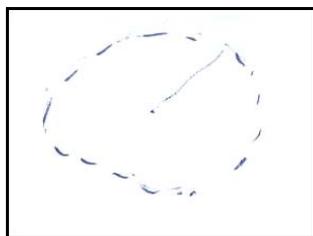
The same student (8<sup>th</sup>) to problem 2 said that:

*I don't think I have seen a set like that and been asked if it's an open ball; so I can't really picture it. (8<sup>th</sup>)*

It seems that this student could not use her imagination to picture the set B as a ball because she had not met a ball like that. So her difficulty of picturing the set prevented her giving a definite answer to this problem.

Two students in the questionnaire answered that the set B (in Problem 2) is open and they only provided pictures with their answers. One of them answered that:

*Yes. (F)*

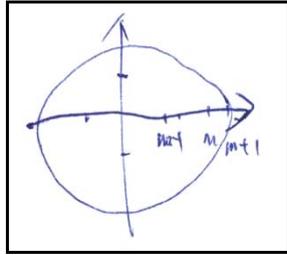


**Figure 5.4: Student F's answer to Problem 2**

This student probably pictured the set B as an open ball (circle) and he/she gave the answer based on the picture.

Similarly, the other one answered Problem 2 by writing:

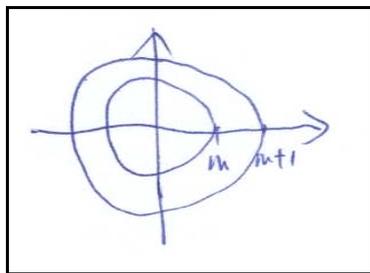
*Yes. (N)*



**Figure 5.5: Student N's answer to the first question of Problem 1**

When addressing the subset C of B in the same problem, the same student (N) answered that:

*Yes. (N)*

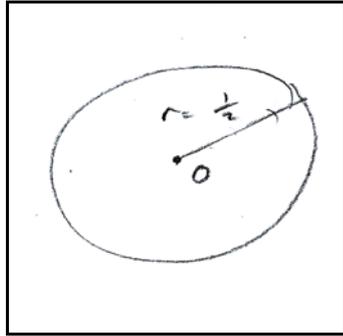


**Figure 5.6: Student N's answer to the second question of Problem 2**

This student seems to conceptualise an open ball to be always a circle or disc in  $\mathbf{R}^2$ , recall that this student used a similar drawing in his/her answer to Problem 1. Here we can see the effect of using one picture of an open ball in  $\mathbf{R}^2$  to be a standard for any open ball in any metric space.

In addition, when some students (1<sup>st</sup> and 3<sup>rd</sup>) were asked to explain the open balls  $B(0, 1)$  and  $B(0, \frac{1}{2})$  within the metric space  $(X, d)$  in Problem 5, these students drew pictures of 'circles' to represent their balls. One of those two students confused the radius of the ball with the distance between the centre and any point in ball in his explanation. He answered that:

*So to get a half, uh... k equals 2, so 2 is the minimum k where um...  $a_k$  is not equal to  $b_k$ . So  $a_k$  must be different from  $b_k$ . Um..., I'm not sure what to do after that. I'm not sure what values are in the sequence for  $b_k$  and uh..., I don't know how to draw an open ball with sequences. (1<sup>st</sup>)*



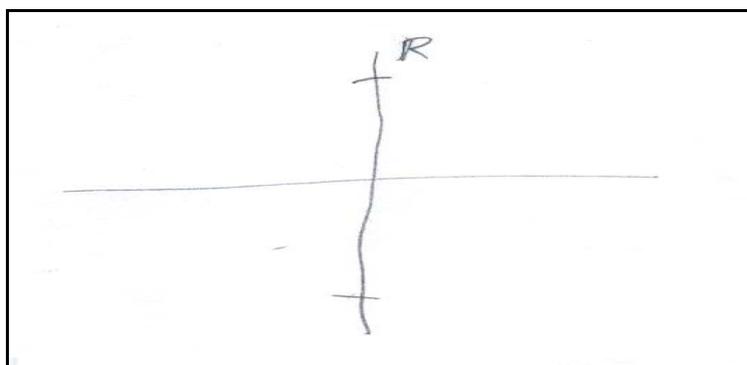
**Figure 5.7: The 1<sup>st</sup> student's response to the fourth part of Problem 5**

This student drew a picture of a circle to represent the open ball  $B(0, \frac{1}{2})$  but his picture did not help him to describe or understand the elements (sequences) within that ball. It may be that this student relies on his imagination in order to understand balls.

We can see that all the former students spoke or used picture of circles to think of balls in a metric space. This picture of a ball might be due to the ‘*spontaneous conception*’ which arises from the real world meaning of ‘ball’.

Moreover, we recognised that drawings or pictures of sets, even if the drawing is correct, can affect students' thinking about open sets. In respect of Problem 4 two students (7<sup>th</sup> and 8<sup>th</sup>) drew the set  $S$  correctly but they gave incorrect answers to the question. For example, one of them answered that:

*Umm, well, in  $\mathbf{R}^2$  it won't uh make a difference, because you're going to have your, uh, your interval  $(a, b)$  cross zero. So that is just gonna be the set of order pairs  $(x, 0)$  for any  $x$  in  $(a, b)$ , so you can immediately just bring it back to your  $\mathbf{R}$  and imagine, and it's never going to reach them, and it's still open. (7<sup>th</sup>)*



**Figure 5.8: the 7<sup>th</sup> student's drawing to Problem 4**

This student gave a correct drawing for the set  $S$  but he seems to be influenced by the picture when the drawing appears to show that the set  $S$  is still an open interval

$(a, b)$  on one axis, and so he thinks incorrectly of  $S$  (the interval) as a subset of  $\mathbf{R}$  instead of  $\mathbf{R}^2$ . So a concept image which is based only on pictures of the set even if the pictures are correct could lead students to confusion.

In working on Problem 5, some students admitted that the problem was not easy because it was hard for them to visualise or picture sequences. When one student was asked to give an example to illustrate the second part of the definition of the metric on the set of sequences  $X$ , she commented that:

*Um..., I dunno I can't really picture this at all to be honest. Um..., usually when I'm giving stuff like this, I have couple of days to do it. (4<sup>th</sup>)*

This student could not understand the metric because she could not picture or imagine this metric space. Also when we asked the 9<sup>th</sup> student to describe the definition of the metric she said that 'I'm going to stop thinking about distance, because it's not distance, so the difference then between them' then when we asked her why she said this is not distance it is difference, she answered that:

*Because I just looked at it to, and it looked too confusing to try and think of a picture of sequences, so try to think of how far they're apart. (9<sup>th</sup>)*

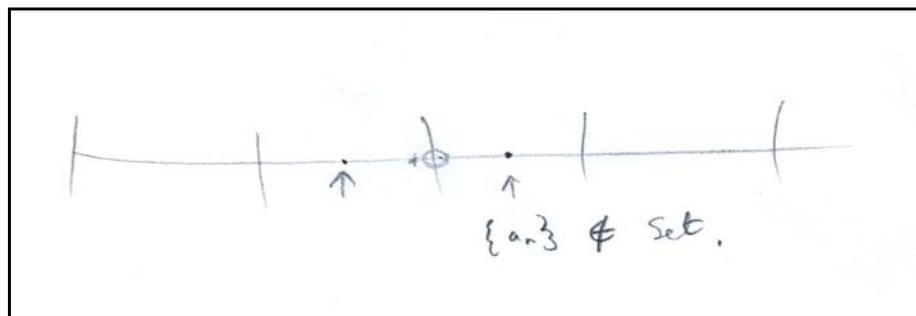
Another student when asked 'Is the set of sequences  $\{\{a_n\}: a_1 = 0 \text{ or } 1\}$  open?' was confused and could not tell if the set open or closed and when we asked him why he could not tell he commented that:

*Um, just not being able to fully see all the elements of the set like. (2<sup>nd</sup>)*

Those students seem to use pictures in thinking of open balls or open sets and so for them it is confusing to picture sequences within open balls or open sets which prevented some of them from answering the given questions related to this problem.

The 2<sup>nd</sup> student drew a picture to represent the sequences of the set  $\{\{a_n\}: a_1 = 0\}$  to explain that this set is not open within the metric space  $X$  in Problem 5. This student commented that:

*um... no if it just 0 it will be closed because all the elements in this set are at distance 1 away from each other, but there are with the other sequences which would be less than one away. So if I draw a picture, there will always be sequences close to distance 1 over  $n$  an element of  $N$ . so you can get as close as to like to our elements of our set, so there is no ball that will only contain elements of our set are not any of these other sequences. So that'll be closed then. (2<sup>nd</sup>)*



**Figure 5.9: The 2<sup>nd</sup> student's illustration for the 4<sup>th</sup> part of Problem 5**

This student drew and thought of the elements of the mentioned set incorrectly to conclude that the set is not open and intuitively said that it is closed. He considered incorrectly that all the sequences in this set are exactly at a distance 1 away from each other instead of being less than 1 from the centre element. So as we have seen the conceptions that based on pictures only could lead students to think incorrectly about open sets.

I conclude that many students have a concept image of an open set and that concept image depends on their pictures and visualisations of the elements of the sets. Also many students picture an open ball as a circle, which might arise from their experiences of open balls in Euclidean space or come from the spontaneous conception of a ball in real life, and they use this picture in all metric spaces to think of an open set. These pictures might be correct or might be incorrect, but it seems that depending on the pictures only without reference to definitions may lead students into confusion about open sets.

### 5.2.6 Summary

We have seen that students possess several and different concept images concerning the concept of open set in a metric space and I have presented evidence for that. I have classified their concept images that arose from our findings into five categories: concept image based on the idea of the boundary points of a set, concept image based on the formal definition, concept image based on the experience of open sets in Euclidean space, concept image based on the collection of open balls, and also concept image based on visualisation.

We have also seen that the first two concept images were apparent in students' definitions as well as in their work on the task problems. The concept image of an open set being a union of open balls was common in students' definitions but they did

not seem to use it on tasks (even in the last part of Problem 5 when it would have been very useful). The other two concept images were most apparent when students were working on problems.

### 5.3 Concept Images of Distance

I classified students' concept images of distance in a metric space using the information gained from the analysis of the students' answers in the questionnaire and interviews. The classification of students' concept images of distance yielded four categories namely: concept image based on the measurement of similarity; concept image based on comparison/difference between points; concept image based on physical distance; and concept image based on definition. I will explain each category in detail.

#### 5.3.1 Concept Image Based on Measurement of Similarity

Most of the students are included in this category. It consists of seven students from the interviews (1<sup>st</sup>, 2<sup>nd</sup>, 4<sup>th</sup>, 6<sup>th</sup>, 7<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) and nine students (A, B, C, D, E, F, I, J and O) from the questionnaire and all of these students mentioned similarity to describe their conception of the distance. Some students commented that:

*It is better to think of it as a measure of similarity between two things. (1<sup>st</sup>)*

And,

*Distance is a way of representing the similarity between two objects with a single number. (I)*

And another student also described that:

*Distance in metric space is a way of comparing how similar two points in a metric space are according to some formula. (O)*

These students seem to be influenced by their lecturer's explanation of the notion of distance in metric spaces when he explained to students that it can be helpful to think of 'distance' as a measure of similarity with smaller distances indicating greater similarity. The lecturer also gave students different examples to describe the meaning of similarity between elements in metric spaces. This way of describing distance in terms of measure of similarity is non-standard and does not appear in many textbooks. However it seems to have caught the students' imagination as almost all of them refer to it

Most of students in this category seemed to understand this idea of distance in a metric space as they used it to explain distance but some of them (E and F) did not show a good understanding of this idea of distance based on their comments. These students seem to describe their idea of distance in certain metric spaces instead of general metric spaces. For example one of them described the distance by:

*It is a measurement of similarity of two functions. (E)*

This student's description of the distance is not true in general but it might be true if the elements of a metric space are functions, but it seems that the student is affected by the examples that were given to them, one of the examples in the lectures was about similarity and dissimilarity in a metric space consisting of functions.

Only one student in the interviews mentioned similarity and that was when answering Problem 5. When I asked some students the question '*what do you think this metric measures?*' This student answered that:

*The similarity between sequences, how long it takes the sequence to not become the same, so the longer they stay the same, the closer they're going to be in terms of the metric. (7<sup>th</sup>)*

From above I conclude that, students are liable to be influenced by the lecturers' explanations. Many of students in this group have a conception of a distance as a measure of similarity. Most of them seem to understand this idea as was explained to them and some of them seem to have subtle misconceptions of this idea which might be due to the effect of the examples on students' understanding. The students do not seem to use this concept image when working on problems.

### **5.3.2 Concept Image Based on Comparison/Difference**

This category arises from students who have a conception of the distance in a metric space as a comparison or difference between two points in a metric space. There were seven students in the interviews (2<sup>nd</sup>, 5<sup>th</sup>, 6<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) and five students in the questionnaire (A, G, K, L and M) in this group. The definitions and the descriptions of distance of these students show this type of conception. For instance, one of these students argued that:

*It's like um comparison rather than actual distance. So I thought that was a good way of explaining it. So it was like comparing two elements rather than looking at the distance between them. (8<sup>th</sup>)*

This concept image seems to be related to the first as the lecturer emphasised that metrics could be thought of as different methods or criteria for comparing objects.

Indeed one student says that:

*Our lecturer actually is really good. He does, it's not as much distance, it is a difference. So you like might choose the difference between them in this way or the difference between them in that way. (5<sup>th</sup>)*

Some students in the questionnaire commented that:

*A measure of how 'close' two things are to being the same based on a particular criterion of measurement. (G)*

And,

*This is a measure of the difference between two elements in a set. (L)*

These students described the notion of distance as a comparison between two points in a set or as a difference between two points.

Also regarding the question 'what do you think this metric measures?' which was given in problem 5 to some students, one student answered that:

*so  $X$  is the set of all real sequences, so if you have two sequences then I'm going to stop thinking about distance, because it's not distance, so the difference then between them [...] I just looked at it and it looked too confusing to try and think of a picture of sequences, so try to think of how far they're apart. (9<sup>th</sup>)*

This student seems to find that it is helpful to think of the given metric in Problem 5 in terms of how different the sequences are.

We can see that, all the students in this category described the notion of distance as a comparison between two points in a set or as a difference between two points. This category might be seen to be similar to the previous one, but it differs in students' use of words to describe their idea. All the students in this category use the words 'comparing' or 'difference' in their answers.

### **5.3.3 Concept Image Based on Physical Distance**

This category includes the students who mentioned the physical/actual distance in their explanations of the notion of distance in a metric space. There were six students in the interviews (1<sup>st</sup>, 3<sup>rd</sup>, 5<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup> and 10<sup>th</sup>) and two students in the questionnaire (G and J) who referred to physical distance in their definitions and explanations of distance in a metric space. These students mentioned it in different ways. The 3<sup>rd</sup>

student explained the notion of distance differently from the other students. He said that:

*Uh... distance just same as physical distance, yeah!*

This student mentioned physical distance in his explanation to friends and for him distance in metric space is the same as physical distance.

The other student (J) in this group described that:

*Distance in a metric space is like distance in real life, it can be defined in many ways. E.g. Distance as the crow flies is diff to distance by rail, diff to distance by road etc. (J)*

This student also indicated the usual distance in his/her conception of distance in a metric space. For this student, distance in a metric space is similar to distance in real life (in the same way as the 3<sup>rd</sup> student), but this student elaborated more on his/her idea when explaining that the distance in real life can be defined in different way (e.g. as crow flies; by rail or by road). This student seems to use examples from real life in order to describe the notion of distance in metric space.

The 1<sup>st</sup>, 5<sup>th</sup>, 8<sup>th</sup>, 9<sup>th</sup>, 10<sup>th</sup> students and Student G indicated in their descriptions that distance in a metric space is different from the usual distance in real life. One of these students said that:

*Distance in metric space not like the distance that we have in everyday's life and the distance in metric space mean something different. (10<sup>th</sup>)*

Another student explained that:

*You measure how different things are based on standard (which might or might not be 'distance' in the way we normally think about it!). (G)*

So these students seem to understand that distance in metric spaces differs from the way we measure distance between two points in real life and it depends on given criteria.

I summarise that, some students have a concept image of distance in a metric space which is based on physical distance. These students use the usual distance in real life differently to describe their conceptions of distance, some of them found that the distance in a metric space is different from physical distance and others found that the distance in metric space is similar to physical distance.

### 5.3.4 Concept Image Based on the Definition

This category comprises of two students (3<sup>rd</sup> and 7<sup>th</sup>) in the interview and three students (D, I and N) in the questionnaire. These students showed in their definitions of distance conceptions based on the formal definition. The 7<sup>th</sup> student gave exactly the full formal definition of a distance in a metric space and when he commented on the definition, said that:

*I know that definition, because the three axioms are kind of important for that definition and the whole definition is kinda based around these axioms, so I'd know this one a lot more formally. (7<sup>th</sup>)*

This student seems to find that the idea of the formal definition is very important to understand the notion of a distance in a metric space as the three axioms of the definition are the basis of the idea of the distance.

The students (D and I) also seemed to have conceptions based on the formal definition of the distance and they tried to give the formal definition but their definitions were not complete. One of them defined a distance as:

$$\begin{aligned}(X, d). \quad & d(x, y) \geq 0 \\ & d(x, y) = 0 \Leftrightarrow x = y \\ & d(x, y) = d(y, x) \\ & d(x, z) \leq d(x, y) + d(y, z); \quad \forall x, y, z \in X. \text{ (D)}\end{aligned}$$

This student tried to list the axioms which must be hold for a distance in metric spaces but he/she did not define the distance ( $d$ ) itself and so the definition is not complete. The other student (I) gave the definition of a distance as:

*(X, d) a metric space,  $d : (X \times X) \rightarrow \mathbf{R}$ , where  $d(a, b) =$  distance between  $a$  and  $b$ . (I)*

This student defined the distance as a function only without stating the axioms of this function and so his/her definition was not complete. So these two students gave evidence for a conception based on the formal definition of a distance.

The 3<sup>rd</sup> and N students mentioned a number in their conception of distance in a metric space. Some of them argued in their definitions of the distance in a metric space that:

*Distance uh bigger or equal than 0. Yeah I think it's a real number or something like. (3<sup>rd</sup>)*

And,

*Distance  $\geq 0$  between two points. (N)*

These two students seemed to formally think about distance in a metric space which is a non negative real number. However they did not explain the axioms that must be satisfied for such a real number.

I deduce that some students have concept image of distance in a metric space which is based on a real number. For some students distance is just non negative number between two points, and only one student in the interview and one in the questionnaire attempted to give a formal definition of distance.

### **5.3.5 Summary**

We have seen that there is evidence for four concept images relating to distance arising from this study: concept image based on similarity, concept image based on comparison, concept image based on the definition, and concept image based on physical distance. The evidence comes mainly from the questions relating to definitions in the interviews and the questionnaire.

## **5.4 Influences on Students' Concept Images**

The results of our study show that students might have many concept images related to the open set concept in metric spaces. I classified these concept images into five categories namely: concept image based on the boundary idea of a set; concept image based on the formal definition of an open set; concept image based on the union of open balls; concept image based on open sets in Euclidean space; and concept image based on visualisation. We also saw that students have many different concept images related to distance in a metric space. In our analysis of the concept images that students have, I discovered that there are many influences on these conceptions, such as: previous knowledge; intuitions; lecturers; and examples. In this section will try to describe the influences on these concept images which arise from our study.

### **5.4.1 Previous Knowledge**

Pre-existing knowledge has a significant role in some students' conceptions of mathematical concepts. In relation to our study, students encounter the real line in their secondary school education and in first year at university, and so they are

familiar with open, closed or half open intervals. The results showed that some students still use their previous experience of intervals in the real line  $\mathbf{R}$  when working with subsets in metric spaces. In one student's answer to Problem 1, he/she claimed that:

*Both, the subset is closed at the point 0 and open at the point 2. (E)*

This student seems to be answering the problem according to his/her experience with the brackets of interval  $[0,2)$  on the real line  $\mathbf{R}$ . It might be that this student thinks that since 0 is included in A then the set A is closed at 0, and that since the point 2 is not included in A then the set A is open at this point and thus the set is 'Both' open and closed.

Another student answered to Problem 3 that:

*It is both because it includes 0 but does not include 2. (M)*

This student also seems to be influenced by his/her experience of the brackets which are involved in the set  $A = [0, 2)$  rather than using the definition to answer the problem, and the reason this student used to justify the answer is probably the same reason that the former student used.

Students' experience with open sets in  $\mathbf{R}$  might affect their thinking of open sets in other Euclidean spaces. Many students gave an answer to Problem 4 based on their experience with open sets in  $\mathbf{R}$ . For example, one student said that:

*Well, in  $\mathbf{R}^2$  it won't uh make a difference, because you're going to have your, uh, your interval  $(a, b)$  cross zero. So that is just gonna be the set of order pairs  $(x, 0)$  for any  $x$  in  $(a, b)$ , so you can immediately just bring it back to your  $\mathbf{R}$  and imagine, and it's never going to reach them and it's still open. (7<sup>th</sup>)*

For this student, it is apparent that his knowledge of an open set in  $\mathbf{R}$  dominates his thinking when giving the answer to this problem, so that he did not think of the metric space  $\mathbf{R}^2$ .

So we can see that mistakes could occur when students have experience with concepts, especially if they did not have to use definitions when they encountered these concepts the first time.

### **5.4.2 Intuitions**

Many of students' conceptions and reasoning about mathematical concepts are based on their intuitions and these intuitions might not be in line with the formal

definitions (Fischbein, 1987). In respect to the open set concept, I found that the intuitive reasoning and conception that students use in answering problems could also have potential to lead students to make incorrect judgements about open sets in metric spaces. To show these intuitive conceptions, recall that one student, related to Problem 1, answered that:

*Open, it does not contain all its limit points so it cannot be closed. (P)*

This student correctly found that the set  $[0, 2)$  in  $\mathbf{R}$  does not contain all its limit points thus the set is not closed but he/she intuitively answered that the set is open and that might be due to the set is not closed.

Another student commented on the statement ‘*If a set is not open then it is closed*’ by saying that:

*Ok, so that would be true. [...] But the set is not open, so it is not a union of open balls. I wish I knew the correct definition of open set now. So the set is not a union of open balls, so pictorially, the set would contain its boundary, so the complement is open, so the set would be closed. So I would agree. (11<sup>th</sup>)*

This student admitted that he did not know the correct definition of an open set, however he commented on the given statement based on his understanding which led him to conclude incorrectly that a set which is not open is closed. So some students still use their intuitive thinking to say that if a set is not closed then it is open and vice versa, and this use of intuition might be caused by spontaneous conceptions which come from the real life use of the words open and closed, where not open means closed and not closed means open.

In answering Problem 2, many students seemed to answer intuitively rather than use definitions and also the intuitive answers were different. For instance, a student commented that:

*I don't think I have seen a set like that and been asked if it's an open ball; so I can't really picture it. [...] but if I was guessing, I'd go with yes. It's like restricted kind of the values are kinda close together. So it would have to seen kind of radius, but something to do with 1 and I would say  $m$  be the centre. (8<sup>th</sup>)*

This student admitted that she has not been asked if a set like this is an open ball, but by guessing (intuition) it could be an open ball. She also seems intuitively to think that

the radius is 1 (which is incorrect) and centre is  $m$ . So it seems that for all students who answered that the radius of the set B is 1, their answers were based on their intuitive thinking instead of formal thinking.

A different intuitive conception related to Problem 2 appears in this student's answer:

*No, an open ball is an open set, and finite set cannot be open. (L)*

This student apparently has intuitive conception of an open set which that an open set cannot be a finite set. This student might not have encountered metric spaces such as  $(\mathbf{Z}, d_{\mathbf{Z}})$  and so did not meet open subsets which might also be finite sets.

With respect to Problem 4, I also found that some students seemed to give answers to this problem based on their intuitions. One student in the interview commented on this problem saying that:

*Actually, it really does not come to my head for this one. Right I'll think about it it's closed, is it closed? Um, I'm going to see if the complement is open, and the complement is open, but I also think that, the set is open. It's now I'm thinking that the both open, uh the subset I mean is open. Um..., because the complement is definitely open I think. I'm just going to say (Both). (4<sup>th</sup>)*

This student thought intuitively that S is closed and this might be the reason for saying that the complement of S is open, but she also seemed to use her intuition to say that the set S is also open and thus her answer was 'Both'. When this student could not understand the set S she seems to use her intuition to give an answer.

Another student answered that:

*It's closed I think because um... that there is a point, you can always get a point in the interval and no matter where in interval, the point will land in the interval between them, where there is no a point in the set S. (6<sup>th</sup>)*

This student also seemed to use her intuition to answer the problem. When she found that there might be a gap between  $\{0\}$  and  $(a, b)$  so that the points in this gap are not part of the set S and for her that means the set S is closed. So might have an incorrect intuition about closed sets which is closed sets are not connected.

From above I conclude that many students use their intuitions to answer problems rather than use formal definitions and these intuitions lead them to think incorrectly about open and closed sets.

### 5.4.3 Lecturer

The lecturers (teachers) could also have an influence on students' understanding of concepts. My study showed that many students in the interviews referred to their lecturer when we asked them to give their conceptions about the distance concept in a metric space. One student said that:

*[The lecture] spent a whole class on explain that through similarities, and I've never had a problem to understand the metrics because of it. (4<sup>th</sup>)*

Another student also commented that:

*Our lecturer actually is really good. He does it's not as much distance it's a difference,[...] and he gave us a great example, he drew a circle and he said you keep going straight down from the top to the bottom straight line, or you can go around that way and you can just go between those two points in two different ways. (5<sup>th</sup>)*

And one student in the questionnaire when I asked how to explain the idea of distance, he/she answered that:

*The way our lecturer did, to show there are different ways to measure things or different ways to travel from point to point. (H)*

These are some examples to show how students referred to their lecturer in their comments on some of the given problems. In relation to the questions about the definition and the explanation to friends, we have seen that most students explained distance using the similarity idea and this is an evidence of how students might be affected by their teachers' explanations of concepts and also how the ways that lecturers use to present concepts could play a role in building students' conceptions of such concepts.

### 5.4.4 Examples

The examples given in students' notes during courses also contribute in the formation of students' conception of mathematical concepts. In some of the problems that I asked, some students mentioned examples in their notes when answering the problems. In one student's answer to Problem 3, he claimed that:

*I saw an example like this in the notes. Um well, the subset  $A$  is inside  $Y$  and  $Y$  is closed, so I think  $A$  is closed, because it is inside a closed interval. I can't*

*remember definition, but I remember example like this, where there was a subset inside a bigger set. (1<sup>st</sup>)*

This student seems to answer the problem based on an example he saw before in the notes. He incorrectly thought that since the set  $A = [0,2)$  is inside the set  $Y = [0,2]$  and also since the set  $Y$  is closed then the set  $A$  is closed as well. So this student might be influenced by an example in the notes when answering this problem.

Also in some students' explanations of the notion of distance in a metric space, these students mentioned examples given to them about the distance. For example, in one student's explanation of a distance to a friend he/she said that:

*I would give them an example, like in the picture, The values that  $g(x)$  and  $f(x)$  are quite similar, but the values of their derivatives are very different. (M)*

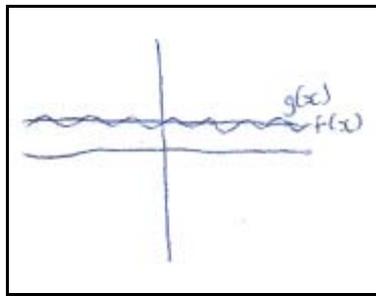


Figure 5.9: Student M's explanation of distance

Another student commented that:

*The example I liked was different ways of travelling to points on a circle, showed obviously that distance depends on how you go about it; ie what metric you use. (H)*

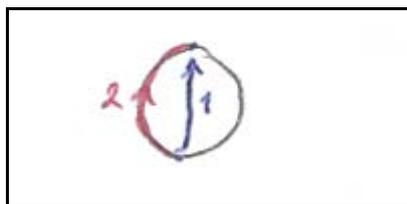


Figure 5.10: Student H's explanation of distance

For these two students it seems that the examples that were given to them during their class helped them to acquire a good understanding of the concept of distance in a metric space.

From this section we can see how students might be affected by the examples that are given to them during courses. We have seen that some of the examples were useful and students could use them to illustrate their understanding and other students seem

to make an incorrect use of the given examples when illustrating their conceptions about concepts and use them as a basis for their reasoning when solving problems.

### **5.4.5 Summary**

In summary, the results of this study have exposed some influences on the formation of students' concept images. We have seen that previous mathematical knowledge could affect students' understanding of the concept of open set. We have also noticed that many students provided intuitive answers, and also spontaneous conceptions based on the real life meaning of words. The role of lecturer in explaining the concept of distance to student was obvious in students' explanations of the concept. Also, we have seen that the examples which have been given to students in classes could influence students' ideas about the concept.

## **5.5 Misconceptions Relating to Open Sets**

While I were analysing students' work in the questionnaire and the interviews I realised that many students have misconceptions about the notions of some concepts that are related to the open set concept. These misconceptions are about the notion of an open ball, the notion of the complement, the interpretation of some quantifiers and misconceptions about the notion of open set itself. I will elaborate in detail on each misconception as follows.

### **5.5.1 Misconception of the Notion of Open Ball Concept**

The notion of the open ball concept is essential to understand the open set concept in a metric space. So, students' conceptions of open balls play a significant role in their understanding of an open set. The findings of our research showed that three students in the interviews (4<sup>th</sup>, 8<sup>th</sup> and 9<sup>th</sup>) and five students in the questionnaire (E, J, L, M and P) have misconceptions about the notion of an open balls. I have spoken about some of these misconceptions when considering students' concept images of open sets.

The students 4<sup>th</sup>, 9<sup>th</sup> and L have a misconception of open balls which is that an open ball must be full of points (i.e. it contains infinitely many points). Some cases of this misconception are:

*No, an open ball is an open set, and finite set cannot be open. (L)*

And,

*No, that is not open because it doesn't contain all the points. It's only contains these three points (4<sup>th</sup>)*

These students maybe thought that an open ball cannot be finite set and it must possess a continuum of points.

Students usually know the difference between notion of the radius of a ball and the notion of distance between the centre of the ball and any point in it, however most of them confused the radius with the distance while thinking or describing open balls in unfamiliar metric spaces. This confusion was observed clearly in many students' answers to Problems 2 and 5. In answering Problem 2, some students (8<sup>th</sup>, E and J) commented that:

*If I was guessing, I'd go with yes. It's like restricted kind of the values are kinda close together. So it would have to seen kind of radius, but something to do with 1, and I would say  $m$  be the centre, I would say is an open ball. (8<sup>th</sup>)*

And,

*Yes,  $m$  is the centre of the ball. 1 is the radius of the ball. (E)*

These students seem to measure the distance between the centre point and an end point to find the radius instead of thinking that this distance must be less than the radius for a ball to be open.

In Problem 5 some students (1<sup>st</sup>, 2<sup>nd</sup> and 8<sup>th</sup>) were asked to answer the questions 'What is  $B(\{0\}, 1)$ ?' and 'What is  $B(\{0\}, \frac{1}{2})$ ?'. One student when answering the question about the ball  $B(\{0\}, 1)$  claimed that:

*Well the distance between zero and any point in the sequence, or in the set has to be one, so um...,  $k$  has to be 1, cause it's gonna hold to the second condition. (8<sup>th</sup>)*

With respect to the ball  $B(\{0\}, \frac{1}{2})$  another student said that:

*Ok. Yeah so its sequences in which the first term is equal to the, equal to 0 and the second term is not equal to 0. (2<sup>nd</sup>)*

These students rather than thinking that 1 and  $\frac{1}{2}$  are the radii of the given balls they were confused and according to the given metric in Problem 5, thought incorrectly that 1 and  $\frac{1}{2}$  are the distances between the centre points and any point in the balls  $B(\{0\}, 1)$  and  $B(\{0\}, \frac{1}{2})$  respectively.

Another misconception of open balls was based intuitively on the layout or notation of the ball. A student answered the question on the set B in Problem 2 by:

*No, because an open ball has only two parameters  $x, y$ .  $B(x, y)$ , where here B has 3;  $m-1, m$  and  $m+1$ . (M)*

And also he/she answered the question on the set C in Problem 2 by:

*Yes,  $C = \{m-1, m\}$  or  $C = \{m, m+1\}$  or  $C = \{m-1, m+1\}$  are all subsets of B and are open balls. (M)*

Another student spoke about the set B in Problem 2 and answered that:

*No, as open balls are written  $B(x, r)$ . Where  $x$  is the centre and  $r$  is the radius of the ball. (P)*

This student also answered in relation to the second part of Problem 2 that:

*Yes,  $B(m, m+1)$  is a subset of this B. (P)*

This student (P) might think that the set  $B = \{m-1, m, m+1\}$  is not an open ball as it is not presented as  $B(x, r)$  while he/she accepted that the set  $B(m, m+1)$  is an open ball as it is presented in the form  $B(x, r)$ .

These students seem to be influenced by the notation of an open ball which is  $B(x, r)$  where two characters (the centre and the radius) are used to represent the ball.

I conclude that, in relation to the open set concept many students could have misconceptions about open ball which are the basis of an open set. We have seen that students confused the radius with the distance between the centre and a point of open balls which caused them to think of open balls in unfamiliar metric spaces incorrectly. Some students think incorrectly that an open ball has to be an infinite set. Also, some students could be influenced by the notation used to describe an open ball and think that an open ball must be presented by the expression  $B(x, r)$ .

### **5.5.2 Misconception about the Notion of Open Set**

The outcomes showed that some students have incorrect ideas about open sets. Some students (1<sup>st</sup> and 3<sup>rd</sup>) incorrectly think that if they can find one open ball inside a set then that set is open. For example, in answering Problem 2 the 1<sup>st</sup> student said that:

*B is open if we can draw an open ball around 1 which is inside the set, if its centre at 1 then the open ball would include 0 and 2. Um..., if the radius is greater than 1 then it would be outside the set, so, then it wouldn't be an open*

*set. No actually no! I think it might be open, cause if um..., you take a radius less than 1, then it just has the point 1 inside it, so then it is open. (1<sup>st</sup>)*

When this student found incorrectly that the first ball he thought of is not included in the set he answered that the set cannot be open, but then he found another ball which is, for him, inside the set and he concludes that the set would be open.

In the same problem, the 3<sup>rd</sup> student argued that:

*Yeah I think so, it's open because you can..., B of a radius m so will be inside B. If the set has open ball so will be open. Oh! It can be, m be centre, so this is  $m + \frac{1}{2}$  and this says  $m - \frac{1}{2}$  will be inside Z, m so is here as  $m-1$ ,  $m+1$  so this for the centre, [...] yeah, I find another ball inside it, so it should be open. (3<sup>rd</sup>)*

This student said clearly that ‘If the set has open ball so will be open’, so he seems to think incorrectly of the set  $\{m-\frac{1}{2}, m, m+\frac{1}{2}\}$  as an open ball which for him is included inside the set **Z** and so since B contains an open ball then it is an open set.

So those two students might have the misconception that if there is an open ball inside a set then that set is open.

Another student tried to describe an open set and she said that:

*The official definition is you can take any open ball around any point and it's still completely contained in the set. (9<sup>th</sup>)*

This student tried to give the formal definition of an open set, but by her definition, it seems that she has the misconception of an open set which is that all the open balls about any point in a set should be considered and must be contained in a set, for that set to be open, and that is not correct. The formal definition of an open set requires that at least one open ball in  $(X, d)$  about each point in a set is considered and that ball is included inside the set.

I deduce that students might have misconceptions related to the notion of open set. Some students could think that one open ball inside a set is enough for the set to be open; others could think that all open balls about any point in a set must be considered for the set to be open.

### **5.5.3 Misconception or Confusion about Quantifiers**

In English, the term ‘any’ sometimes is used to mean ‘for some’ (e.g. any absence of players means the game will be cancelled), and sometimes could be used to indicate ‘for all’ (e.g. any person can see that).

The word ‘any’ is genuinely ambiguous in mathematical definitions (Rowland 2002). It is sometimes used to mean ‘all’ and sometimes it is used to mean ‘some’. For example, recall the use of ‘any’ in the definition of open set in (Simmons, 1963, p. 60) that was given in Chapter 4, Section 4.2.5.1, where in that definition the word ‘any’ means ‘all’. Another example for using ‘any’ in place of ‘all’ is in the statement of the Triangle Inequality given in a well-known analysis textbook:

‘For any  $a, b \in \mathbf{R}$  we have  $|a + b| \leq |a| + |b|$ ’ (Bartle and Sherbert, 1982, p. 42).

In the interviews the expression ‘for any’ was used in the definition of the metric on the set  $X$  (which is the set of all real sequences) in Problem 5. It was intended that in this context it would mean ‘for all’.

In the definition given to the students in this study ‘any’ was used as follows:

$$d(\{a_k\}, \{b_k\}) = \begin{cases} 0 & \text{if } a_k = b_k \text{ for any } k \in N \\ 1/k & \text{if } k = \min_{n \in N} \{n : a_n \neq b_n\} \end{cases}$$

It was meant in the sense ‘for each’ or ‘for all’. However it read to most students (1<sup>st</sup>, 2<sup>nd</sup>, 8<sup>th</sup>, 9<sup>th</sup> and 11<sup>th</sup>) as ‘for some’ or ‘if any’. To show some examples of the students’ interpretations, one of these students commented that:

*Ok, so d on two sequences. For any k an element of N ok. it's equal to zero if any of the elements in the both sequences are equal for any k. (2<sup>nd</sup>)*

This student seemed to understand that the whole metric is equal to zero *if any* (i.e. some) two terms in the same place in both sequences are the same, otherwise the second part of the definition of the metric will be hold.

Another student explained that:

*If you have two sequences and they're the same for any point, the k<sup>th</sup> terms the same for any of them, then the difference between them is 0, but if none of the terms are the same, then 1 over k if k is the minimum. (8<sup>th</sup>)*

This student seemed to understand that the metric is defined termwise between any two sequences, so that if two corresponding terms in both sequences are equal then the distance between these terms is zero, and if two corresponding terms are not equal then the metric between them is  $1/k$  where  $k$  is the minimum term.

So most of the interviewed students interpreted the phase ‘for any’ incorrectly as ‘if any’ or ‘for some’ and this influenced their understanding of the whole definition of the given metric.

When we tried to stimulate some students to change their interpretations of ‘for any’ to be ‘for all’, one student commented that:

*Oh! For all k, yeah that does make a difference, cause that would mean the sequences would have to be the same k. Oh! That’s interesting. Normally, I would see for any as the same as for all, but when I read this out loud, it sounded like just for, like if any of them, because I often see any and just think of you know if you say anyone, kinda it means like someone, like just one. (9<sup>th</sup>)*

This student admitted that she would see ‘for any’ as ‘for all’ but in our definition it sounded for her like ‘if any’.

We conclude that, many students could have difficulty with the interpretation of some mathematical quantifiers. Even though, students meet these quantifiers in many mathematical definitions or theorems and they are familiar with their meaning, but still some students could get confused with them. It seems that our unfamiliar metric space which contained sequences equipped with an unusual metric caused many students to think of ‘for any’ as to mean ‘if any’, although most of them are used to interpret ‘for any’ as ‘for all’ in their mathematical experience. However, it is difficult to make a strong conclusion in this case as the term ‘for any’ is ambiguous.

#### **5.5.4 Misconception or Confusion about Complements**

During our analysis of students’ conceptions concerning open sets we noticed that some students have misconceptions or confusion in their thinking concerning complements of some sets. The results of the study showed that even if some students could argue correctly how sets are defined to be open or closed, they could have difficulty arguing whether the complements of sets are open or closed. I will break students’ misconceptions of complements into two sections: misconceptions of complements which extend to infinity and intuitive misconceptions about complements. I will elaborate and present examples about each set of misconceptions.

##### **5.5.4.1 Misconceptions about Complements Which Extend to Infinity**

We have noticed that two students (4<sup>th</sup> and 9<sup>th</sup>) have misconceptions about the complements of subsets of  $\mathbf{R}^n$ . In relation to Statement (c) which says ‘A set is open if its complement is closed’ the 4<sup>th</sup> student commented that:

*No, never thought of it in that way before! No, that can't be true, um because um an open set like  $(0, 1)$  its complement is real line minus, or the  $\mathbf{R}^2$  minus  $(0, 1)$  and um that's open as far as I think. (4<sup>th</sup>)*

This student argued incorrectly that the complement  $\mathbf{R} - (0, 1)$  or  $\mathbf{R}^2 - (0, 1)$  is open set. It might be that she used her intuition here and thought that since the sets  $\mathbf{R}$  and  $\mathbf{R}^2$  are open sets (they have no boundary because they extend to infinity) and the complement  $\mathbf{R} - (0, 1)$  extends to infinity as well and so she concludes that it is open.

The following student commented on the statement 'A set is open if its complement is closed' by:

*A set is closed if its complement is open, right! I'm never thought about it. So if you have an open set  $(0, 1)$  yeah, the complement of that would be closed I think. Oh no! It's not closed because this would be  $(-\infty, 0] \cup [1, +\infty)$  and I don't know what happens at infinity. I don't think it is, because if it closed in there and closed here, and that was a subset of  $(\mathbf{R}, d_{st})$ , like the whole metric space is open, so that, hmm, no I don't think that the complement of an open set is closed. I don't think it's either, because you could have sequence that goes to  $-\infty$ . (9<sup>th</sup>)*

At the start this student answered that the complement of the set  $(0, 1)$  is closed, but then she became confused with the expression of the complement  $(-\infty, 0] \cup [1, +\infty)$ .

Her confusion seemed to stem from the fact that the complement extends to infinity. This is similar to the reaction of the 4<sup>th</sup> student. This student seems to be influenced by the look of the set, however she used some mathematical thought to argue for her claim which is there might be some sequence that goes to  $-\infty$  (she thought that  $-\infty$  is a limit point which is not contained in the complement and thus it is not a closed set). So we can see that students could have difficulty with thinking about complements especially if the complements extend to infinity.

#### **5.5.4.2 Intuitive Misconceptions about Complements**

We have noticed another kind of incorrect conception about complements which was not caused by infinity but it seemed instead to be caused by intuition. For example, in answering Problem 1, this student claimed that:

*Closed,  $[0, 2)$  is not open.*

*$[2, \infty) \cup (-\infty, 0)$  is closed. (K)*

This student seemed to know that the set  $[0, 2)$  is not open but seemed to use his/her intuition to answer incorrectly that the subset  $[0, 2)$  of  $\mathbf{R}$  is closed based on the fact that it is not open. Also he/she argued incorrectly that the set  $[2, \infty) \cup (-\infty, 0)$  (which is the complement of  $[0, 2)$  in  $\mathbf{R}$ ) is closed. It might be that this student used the same reason that he/she used for the subset  $[0, 2)$  (i.e. since the complement  $[2, \infty) \cup (-\infty, 0)$  is not open then for him/her is it closed), and this might come from the intuitive conception of not open means closed and vice versa.

Two other students (A and P) in the questionnaire considered the complement of the set  $A = [0, 2)$  in Problem 3 incorrectly. The student A answered that:

*Both, all points in A have an open ball which is a subset of A. It is also closed, as its complement is  $\{2\}$  which is open. (A)*

This student incorrectly argued that the set A is a closed set because its complement in the metric space  $[0, 2]$  (which is  $\{2\}$ ) is open. It is not clear why this student felt that the set  $\{2\}$  is open. It seems that this student considered the set A formally (however it was incorrect) but it might be that he/she considered the complement intuitively.

Another student was similar to the previous one. In the same problem he/she claimed also that:

*Closed, as its complement is open. (P)*

This student also answered incorrectly that the complement of the set A is open and thus the set A itself is closed. Even though, these two students tried to use some formal statements in the arguments, their incorrect thinking about the complements hindered their work.

We have also found that some students (8<sup>th</sup> and 11<sup>th</sup>) got confused with the complements of sets that are both open and closed. One of these students in answer to Statement (c) firstly claimed that:

*No, because you can have an open set but it's also closed set so its complement is open, so no. But that just means is not every open set has closed complement. But if the complement is closed, does that mean the set is open? Um the reason being that I suppose pictorially again, if the complement is closed, the complement contains the boundary so, oh no! (11<sup>th</sup>)*

This student at first disagreed with the statement because he thought of a set that is both open and closed so that its complement is also open and closed. But he got

confused when he concluded that his set is open and the complement is also open and he incorrectly deduced that the complement of every open set is not always closed. This shows that some students find it difficult to make deductions about complements.

We conclude that even though many students could consider open sets and closed sets correctly; these students could get confused when considering whether the complements of some sets are open or closed sets even if they are familiar with the types of the sets in question.

### **5.5.5 Summary**

We have seen that students have a variety of misconceptions which can cause them to have difficulties when working with concepts. Some of the misconceptions are due to language, some are due to students misremembering a definition which leads to problems with their concept image, and some are due to problems with reasoning.

## **5.6 Students' Consistency While Working on Problems**

Knowing information about students' concept images and concept definitions is important to find out about the difficulties that students meet in their understanding of mathematical concepts. In relation to the concept of open set in a metric space we asked questions to investigate students' concept definition and concept images and we tried to discover which definitions and concept images students use when they work on problems related to open sets. I also want to find out if there are other different conceptions that might be used by students. So I would like to see if there is consistency between students' concept images and concept definitions.

According to students' concept definitions we found that many students adapt the definition of open set in terms suitable to their concept images so that students define open sets differently based on their conceptions. Also many students define open sets by conceptions different from the conceptions that they use when they work on problems related to open sets.

### **5.6.1 The Interviews**

In the interviews there were three problems related to open sets (Problem 3, Problem 4 and the fourth part of Problem 5) and students had to work on one or two of them as

shown in Table 5.1. Using this table we can follow each student's definitions and explanation of an open set and also we can follow the conceptions that each student used when working on the problems.

In Problem 3 which concerned ' $A = [0, 2) \subset [0, 2]$ ', it is clear that none of the three students who worked on this problem used his/her definitions or explanation of open sets and so there is no consistency between their definitions and their work on the problems. Only one student (3<sup>rd</sup>) succeeded to get the correct answer and formally deduced that since the complement of the set  $A$  is a closed set then the set  $A$  must be an open set; the other two students used their intuitions and did not succeed.

In Problem 4 which concerned ' $S = (a, b) \times \{0\} \subset \mathbf{R}^2$ ', from Table 5.1, we can see that most of the students who work on this problem did not use their definitions or explanations while solving the problem except for two students (2<sup>nd</sup> and 9<sup>th</sup>). Also all students did not succeed in finding the correct answer except for the 9<sup>th</sup> student who was the only student who used the exact formal definition of an open set in this problem, although this student did not give the exact formal definition when asked for the definition of an open set.

In the fourth part of Problem 5 which concerned the set ' $\{a_n : a_1 = 0 \text{ or } 1\}$ ' we can see from Table 5.1 that all students who worked on this unfamiliar problem used the formal definition idea while working on it but none of them succeeded in getting the right answer except for the 7<sup>th</sup> student who also used the formal definition idea and his continuum idea of open set (i.e. there is fuzziness between the elements of the set where he can keep finding open balls with smaller radius).

In the interview there were also two problems (Problem 2 and the third part of Problem 5) related to the concept of an open ball which is the basis of the open set concept, and most of students worked on both problems as is shown in Table 5.2. From this table we notice that, in the first part of the Problem 2 regarded the set ' $B = \{m-1, m, m+1\} \subset \mathbf{Z}$ ' it seems that students who used the definition of an open ball and thought of the centre and the radius within the set  $\mathbf{Z}$  correctly succeeded to answer the problem correctly, and it seems that those who were confused with the radius of open balls in the set  $\mathbf{Z}$  and who used other ideas (e.g. intuitions or boundary points) did not succeed in getting the correct answer for this problem. Also in the second part of the Problem 2 which was related to the set ' $C \subset B = \{m-1, m, m+1\} \subset \mathbf{Z}$ ', Table 5.2 shows that, students who worked on this part and used the definition correctly (i.e.

they think of the centre and the radius of open balls in  $\mathbf{Z}$  correctly) succeeded in getting the right answer, and it seems that those who used other ideas (such as intuitions or continuum) instead of the formal definition did not get the correct answer.

### 5.6.2 The Questionnaire

In the questionnaire, the students were asked to answer two problems related to the open set concept (Problem 1 and Problem 3), and these problems are shown in Table 5.3. According to Problem 1 in this table ' $A = [0, 2) \subset \mathbf{R}$ ', we can see that only four students (F, J, L and O) used their definitions of open sets while working on the problem and the others used other ideas different from what they had given as their definitions and explanations. Also the table shows that the students who succeeded in finding the right answer were the only students who used the definition of an open set along with a theorem on closed sets, and those who used other approaches (e.g. graphs, intuition, boundary points and just a theorem on closed sets) did not get the right answer for this problem.

In the Problem 3 ' $A = [0, 2) \subset Y = [0, 2]$ ', from Table 5.3 we conclude that only four students (I, J, L and O) used their definitions of open sets on this problem and the other students used different ideas from what they provide in their definitions and explanations of open sets. Also only five students (G, I, J, L and O) succeeded in getting the correct answer to this problem and all these five students used the formal definition in their reasoning, and all the other students who used other ideas failed to give the right answer including Student A who used the formal definition idea and thought of the complement of the set A incorrectly.

Table 5.4 shows students' work in the questionnaire on Problem 2 which is related to the concept of an open ball. From this table and regarding the first part of this problem ' $B = \{m-1, m, m+1\} \subset \mathbf{Z}$ ' we can see that, it seems that the students who used the definition of an open ball within the integers correctly (i.e. they thought of the centre and the radius of B correctly in  $\mathbf{Z}$ ) answered the problem correctly, and those who used other ideas or got confused with the radius of open balls in  $\mathbf{Z}$  seemed to fail to answer the problem correctly. Also from the second part ' $C \subset B = \{m-1, m, m+1\} \subset \mathbf{Z}$ ' of the same problem, it seems that here too the students who think of the centre and the radius of an open ball within  $\mathbf{Z}$  achieved the correct answer to this part,

and those who used other ideas (i.e. conceptions of  $\mathbf{R}$ , intuitions, boundary ideas or graphs) seemed to fail to answer this part of the problem correctly.

### 5.6.3 Summary

We have seen that there was very little consistency between students' concept definitions and their work on the task problems.

We see that six students in the interviews and three in the questionnaire spoke about open sets as union of open balls when asked for their definitions of open sets but we did not see this idea being used in the solutions to the problems.

Similarly the boundary idea of open sets was mentioned by two students in the interviews and two students in the questionnaire when asked for a definition of an open set, and by three others in the interviews and two others in the questionnaire when asked how to explain the concept to a friend. However, once again, students did not make good use of this conception in the task solutions.

We saw that students often used their intuition in these solutions but many also used the formal definition of an open set, even if they had not given this definition previously. We also saw that students who used this definition were more successful than those who did not.

Student	Definition	Explanation	$A = [0, 2) \subset [0, 2]$		$S = (a, b) \times \{0\} \subset \mathbf{R}^2$		$\{ \{ a_n \} : a_1 = 0 \text{ or } 1 \}$	
			Success	Use	Success	Use	Success	Use
1 <sup>st</sup>	Formal definition	Formal def	No	Intuition	-	-	-	-
2 <sup>nd</sup>	Boundary idea	Boundary idea	-	-	No	Boundary idea	No	Incomplete def
3 <sup>rd</sup>	Union of open balls	Union of open balls	Yes	Complement	-	-	-	-
4 <sup>th</sup>	Union of open balls	Formal	-	-	No	Intuition	-	-
5 <sup>th</sup>	Union of open balls	Boundary points	-	-	No	Intuition	No	Incomplete def
6 <sup>th</sup>	Union of open balls	Union of open balls	No	Continuum	No	Continuum	-	-
7 <sup>th</sup>	Boundary idea	Boundary idea	-	-	No	Intuition	Yes	Formal & Continuum idea
8 <sup>th</sup>	Formal def	Formal def	-	-	No	Intuition	No	Incomplete def
9 <sup>th</sup>	Formal def	Boundary idea	-	-	Yes	Formal Definition	No	Incomplete def
11 <sup>th</sup>	Union of open balls	Boundary idea	-	-	-	-	-	-

**Table 5.1: Students' definitions and explanations of open sets and the ideas they use when working on Problems 3 & 4 and the 4<sup>th</sup> part of Problem 5 in the Interviews**

Student	$B = \{m-1, m, m+1\} \subset \mathbf{Z}$		$C \subset B = \{m-1, m, m+1\} \subset \mathbf{Z}$		$B(\{0\}, 1)$ or $B(\{0\}, \frac{1}{2})$ in $X = \{\{a_k\}\}$	
	Success	Use	Success	Use	Success	Use
1 <sup>st</sup>	No	Incorrect radius	-	-	No	Visualisation
2 <sup>nd</sup>	No	Incorrect radius	-	-	No	Confused $d$ with $r$
3 <sup>rd</sup>	No	Graph & Intuition	-	-	No	Conception of $\mathbf{R}$
4 <sup>th</sup>	No	Intuition	-	-	No	Visualisation
5 <sup>th</sup>	Yes	Definitions	-	-	No	Incorrect distance
6 <sup>th</sup>	No	None	-	-	-	-
7 <sup>th</sup>	No	Boundary idea	No	Continuum	Yes	Definitions
8 <sup>th</sup>	No	Intuition	No	Intuition	No	Incorrect use of definitions
9 <sup>th</sup>	Yes	Definition	Yes	Definition	No	Incorrect use of $d < 1/k$
10 <sup>th</sup>	-	-	-	-	-	-
11 <sup>th</sup>	Yes	Definition	Yes	Definition	Yes	Definitions

**Table 5.2: Students' conceptions concerning open balls when working on Problem 2 and the 3<sup>rd</sup> part of Problem 5 in the Interviews**

Student	Definition	Explanation	$A = [0, 2) \subset \mathbf{R}$		$A = [0, 2) \subset [0, 2]$	
			Success	Use	Success	Use
A	Union of open balls	Boundary idea	Yes	Def & Complement	No	Def & Complement
B	-	Formal def	-	-	-	-
C	-	-	No	Graph	No	No reason
D	-	-	-	-	-	-
E	Formal def	Formal def	No	Intuition	No	Intuition
F	Boundary idea	Boundary idea	No	Boundary idea	No	No reason
G	Union of open balls	Boundary idea	Yes	Def & Thm of closed	Yes	Def & Thm of closed
H	Union of open balls	Union of open balls	No	No reason	No	No reason
I	Formal def	-	No	Intuition	Yes	$A = B(0, 2)$
J	Formal def	Formal def	Yes	Def & Thm of closed	Yes	Definition
K	Formal def	Formal def	No	Intuition	No	Boundary & Complement
L	Formal def	Boundary idea	Yes	Def & Thm of closed	Yes	Definition
M	-	-	No	Intuition	No	Intuition
N	Boundary idea	Union of open balls	No	Graph	No	No reason
O	Formal def	Union of balls	Yes	Def & Thm of closed	Yes	Definition
P	Formal def	Unclear	No	Thm of closed	No	Intuition & Complement

**Table 5.3: Students' definitions and explanations of an open set and the ideas they use when working on Problems 1 & 3 in the Questionnaire**

Student	$B = \{m-1, m, m+1\} \subset \mathbf{Z}$		$C \subset B = \{m-1, m, m+1\} \subset \mathbf{Z}$	
	Success	Use	Success	Use
A	Yes	Definition	Yes	Definition
B	-	-	-	-
C	No	-	No	-
D	No	Incorrect thinking of $r$	No	-
E	No	Incorrect thinking of $r$	No	Distance in $\mathbf{R}$
F	No	Graph	No	-
G	No	Boundary idea	No	Boundary idea
H	Yes	Definitions	No	Incorrect thinking of $r$
I	No	Boundary idea	No	-
J	No	Graph & incorrect thinking of $r$	No	Conception of $\mathbf{R}$
K	No	Incomplete def & Boundary idea	Yes	Definition
L	No	Intuition	No	Intuition
M	No	Intuition	No	Intuition
N	No	Graph	No	Graph
O	No	$\exists \varepsilon > 0$ s.t. $B(m, \varepsilon) \not\subset B$	No	-
P	No	Intuition	No	Intuition

**Table 5.4: Students' conceptions concerning open balls when working on Problem 2 in the Questionnaire**

## 5.7 Conclusion

Using the outcomes found in Chapters 3 and 4 of this thesis, I tried in this chapter to look at the main conceptions and misconceptions that students might possess about the concepts of open sets and distance in a metric space and I also mentioned some possible effects on the formation of students' concept images that emerged from our study.

The findings showed that different concept images appeared in relation to the open set concept. These concept images were based on: the boundary idea, the formal definition, openness in Euclidean space, unions of open balls, and visualisation. We have seen that the students' confusion about boundary points and endpoints of a set could cause them to think of open sets incorrectly. The lack of focusing on every part of formal definitions (especially the ones which might be seen as trivial) could also lead students to have incorrect thoughts about the concept of an open set in metric subspaces. Moreover we noticed that the previous experience of open sets in  $\mathbf{R}^n$  has its effect on students' understanding of openness in general metric spaces. We also observed that some students used their visualisation of an open ball as a 'circle' and they appeared to base pictures on this when thinking of open balls instead of using definitions. The results also indicated the big role that lecturers play in forming students' concept images.

This chapter also explored some concept images related to the concept of distance in metric spaces. These concept images are: concept image based on definition, concept image based on the measurement of similarity, concept image based on physical distance; and concept image based on comparison/difference between points.

I looked also in this chapter at some effects on students' conceptions and I concluded as well that students still use their intuition when reasoning about mathematical concepts, however they all know the basic role that definitions play in mathematics, and some of students use the examples given in courses as a basis when working on task problems.

In this chapter I also considered some students' misconceptions relating to open sets. We realised that many students have misconceptions about the notion of open balls and also misconceptions concerning the notation related to open balls. Some others hold misconceptions about open sets themselves. Furthermore, we realised that

students have difficulty with interpreting some mathematical quantifiers such as ‘for any’. In addition, we noticed that while some students seemed to be able to think about the openness of subsets of metric spaces, they appeared to have trouble in thinking about the openness of the complements, especially if these complements extend to infinity or are both open and closed.

Furthermore in this chapter I addressed students’ consistency when working on task problems and the results revealed that the consistency is rarely observed on students’ work. Most of the students did not use their personal definitions when working on task problems and they used other methods instead (e.g. intuitions, graphs, previous experience, etc.) as well as the formal definition.

## Chapter 6

# Discussion and Conclusion

The goal of my thesis was to examine students' understanding of mathematical concepts at higher levels of university studies. The main concept which I considered is that of open sets in a metric space, I also, investigated the concept of distance which is closely connected to that of open sets. This thesis is meant to contribute to the shedding of light on students' conceptual thinking in an area in which only a little work has been done previously.

Students from different levels, who were enrolled in the metric spaces course, were asked to fill in a questionnaire and to be interviewed. Using the theory of concept image and concept definition by Tall and Vinner (1981) I deeply analysed students' writing and verbal responses to the given questions, and I gained much information about the ways that students learn mathematical concepts in the advanced levels.

Using the analysis discussed in the previous chapters, in this chapter I will discuss the results that I found in my study in relation to my research questions stated in Chapter 1. I restate them here:

- 1- Do students understand the role and the use of definitions in mathematics?
- 2- What definitions and images do students have about open sets and distance in metric spaces?
- 3- Which definition and images do students use while working on problems?
- 4- Are students consistent in the use of their concept definition and concept image? And also, is there consistency between students' conceptions and the formal definition?

In the following, Section 6.1 discusses the results obtained regarding the first research question, Section 6.2 discusses the findings gained in relation to the second and third research questions, Section 6.3 discusses the results which emerged regarding the fourth research question, and Section 6.4 outlines the conclusion of this thesis.

## 6.1 Understanding Definitions

Definitions have a fundamental role in advanced mathematics and in order to get a good understanding of a topic an individual needs to understand fully its definition. In the following subsections I will consider the main conclusions concerning students' understanding of mathematical definitions.

### 6.1.1 The Role of Definitions in Mathematics

In this study, I tried to examine students' understanding of the role and nature of definitions in mathematics and to discover their experience with definitions while learning mathematics, using the questions that I discussed and analysed in Chapter 3. Our results illustrated that students seemed to know the important role that definitions play in mathematics. In section 3.2.2 they generally pointed to the main characteristics assigned, by them, to mathematical definitions such as: solving problems; and giving the exact meaning of concepts. They noted that mathematical definitions are specific and they cover all possibilities of a concept; they are methodical/formal, rigorous and there is no ambiguity about them; they are the basis of concepts and the building blocks of the subject. Students also appeared to be aware of the difference in the role that definitions play in mathematics in comparison to other subjects. As we saw in section 3.2.12, most of them indicated that mathematical definitions in mathematics are precise and constant, while in other subjects, definitions are vague and there is more room for interpretation.

From the results I noticed that the characteristics of mathematical definitions that students mentioned in the study are very similar to the roles of definitions in mathematics which are used in the mathematics community and listed in Zaslavsky and Shir (2010):

- Introducing the objects of theory and capturing the essence of a concept by conveying its characterized properties.
- Constituting fundamental components for concept formation.
- Establishing the foundation for proofs and problem solving.
- Creating uniformity in the meaning of concepts. (p. 317)

Zaslavsky and Shir (2010) also noticed indirectly (through students' discussions about accepting or excluding statements as a definition of a certain concepts) that their

students alluded to most of the roles of mathematical definitions listed above. The characteristics of definitions in mathematics which were pointed out by students in my study, also seemed to be roughly similar to the characteristics quoted from Leikin and Zazkis (2010), especially numbers 1, 4, and 5, and 7. That is, definitions contain a list of properties of the concept; they are the basis of the concept, so that any statement about the concept follows logically from the definition; they are crisp and not fuzzy, theorems may lead to equivalent definitions of a concept. In my study, it was evidence that some students' definitions of an open set came from theorems on the concept, for example, students defined open sets as being unions of open balls. The students did not mention the other attributes listed by Leikin & Zazkis (2010). For example, they did not speak about the fact that an example must fit all and not just some of the criteria in the definition. They also did not speak of the desirability of definitions being minimal. But this may be because of the types of questions asked in this study that is, students were asked about the point of definitions and not what features they should have.

### **6.1.2 Use of Definitions**

In section 3.2.8, the direct question about students' use of definitions showed us that the majority of students reported that they would use definitions when working on problems, in particular when doing their assignments. Some students found that, using definitions repeatedly when doing homework helped to get familiarity with them.

While working on the task problems in this study, from Table 5.1 in Chapter 5 we can see that one student from those who answered Problem 4, and all the five students who tried the 4<sup>th</sup> part of Problem 5 seem to use the formal definition of an open set, and that only one student succeeded on each problem. For the questions on open balls, Table 5.2 shows that three students (out of ten) appeared to use the definitions involved in Problem 2, and that all got the correct answers. Moreover, these were the only students to succeed on this question. On Problem 5, two students invoked the definition and were the only successful students.

Tables 5.3 shows that five students in the questionnaire seem to use the formal definition of an open set in both Problems 1 and 3, and one student in Problem 3 only, and that only one of them did not succeed with the correct answer. And Table 5.4 illustrates that three students seem to use definitions in Problem 2 and that they were

successful on the whole. So we can notice that the use of definitions did appear in some of students' work, and students who seemed to make a good use of the definition were able to achieve the correct answers, and those who did not seem to make a good use of it were not successful. We can see that many students' answers were based on their ideas and their understanding of the definition of the concept under consideration. That is, the students often used their concept images or their intuition when working on task problems instead of referring only to the definition.

This result appears to be similar to the result of Vinner (1991). Vinner used the theory of concept definitions and concept image developed by Tall and Vinner (1981), which tries to explain how the learning of mathematics knowledge occurs, and how definitions are used, to describe the interplay between students' concept images of a certain concept and its formal definition involved within the formation of the concept. He explained that when a problem is given to students in a technical context, they are formally required to consult the concept definition before answering (as illustrated in Figure 1.2, Chapter 1). However, he asserted that students often base on their concept image instead (as shown in Figure 1.3, Chapter 1). I have found similar behavior in this study.

The results of my project also explored the idea that students seem to think that their conceptions about a concept cover the full idea of its definition. Many of them, when giving a definition or working on task problems said that they forgot the exact definition and what they have offered instead is how they understand the concept. It was also apparent that when trying to solve the task problems, some of them recognised their need of the exact definitions.

Other students considered what they use as a definition because they used phrases such as 'the definition of ...is that...'. Similar to this, Przenioslo (2004) noticed that students often treat their own associations as definitions of concepts. Wawro et al. (2011) found that students interpret formal definitions in ways consistent with their conceptions. It might be that, students' ideas are applicable and useful when they use them with familiar metric spaces and thus they rely on them in place of the formal definition. I found that many students seem to encounter trouble when they try to apply their ideas in unfamiliar metric spaces, and this is maybe why they resorted to consulting the formal definition instead (this was evident especially in Problem5).

In addition, when students tried to judge the statements given to them, some of them treated some statements as definitions of the concept when accepting them, and some of them rejected some of the statements as they did not look like proper definitions or because the students had not seen something like that before and were then able to proceed no further.

I found also that, some of students seem to be confused between definitions and theorems on concepts as most of them use some theorems as definitions (especially in the definition of an open set). In addition, in section 3.2.9 when students were asked if they focus on all parts of a definition in order to understand it, the results revealed that many of them said that they only focus on what they think are the most important parts of a definition, and when working on task problems the results confirm what the students said. When I asked for a definition of an open set all students who tried to give the formal definition appeared to ignore some parts which might be considered as trivial by them and these omitted parts seemed to cause students to have trouble with open sets in metric subspaces. They might have felt that there is no need to state them and so they just stated their understanding of the definition. Furthermore, in some problems, some students tried to use their incomplete idea of the definition of an open set which led them to provide correct answers to problems that involved sets which were familiar to them; however this incomplete idea seemed to cause them to give incorrect answers to more tricky problems that involved subspaces. Edwards and Ward (2004) found that students would use definitions when solving problems but they do not use them the way mathematicians would and in particular they rely more on their conceptions and intuitions.

### **6.1.3 Understanding New Definitions**

How students initially understand or deal with new definitions is an important issue that I have addressed in this thesis. The initial reactions to new definitions might contribute to the formation of the students' first images about a concept and some students may continue using these first images even if they are given further information about the concept. The findings in section 3.2 showed that students used many strategies in order to deal with new definitions which included: visualisation; using examples; focusing on the key parts of a definition; using intuition to grasp the meaning; using their own words; and sometimes memorising if a definition is

complicated or long. I found that a student might use more than one strategy to handle new definitions. The findings of my study were similar to those of Dahlberg and Housman (1997) who observed some initial strategies used by students when presented with new definitions, and some of these strategies are: example generation (in which a student generates an example to illustrate the concept); reformulation (i.e. a students reformulate the definition in a way different from the given statement); and memorisation. I found also that, most students said that they would connect new definitions with previous learned material when possible. However, whether they did this or not seemed to depend on the type of material under consideration and none of the students mentioned this spontaneously when asked what they did with new definitions. In addition, it emerged that students did not routinely question why a certain definition was needed but did this if they could not understand the definition or they had multiple definitions for the same concept.

Regarding visualisation, the results in section 3.2.7 of this study showed that visualisation seemed to be appreciated by students as they consider it to be a very helpful in order to understand a concept, and they appeared to know that it is easier to use in some mathematics topics rather than other more abstract ones, and some of them were aware of the possible pitfalls of using images.

The results of the analysis of the task problems also showed that many students seemed to count on visualisation when thinking about some problems. As I discussed in Chapter 4, many of them were not able to understand the new definition of the metric given in Problem 5, and so they could not think clearly about the sets and the balls involved, and that was at least partly due to the difficulty of visualisation in the given setting. Therefore, the lack of visualisation caused difficulty in grasping the meaning of the new definition for some students. However, the use of visualisation when it is possible does not help if is not completely and accurately linked with the exact definition. As an example for that, many of the students seemed to be affected by the picture of the set  $S$  in Problem 4 when they imagined it as an interval on the  $x$ -axis and did not think of it as part of the metric space  $\mathbf{R}^2$  which in turn caused them to give incorrect answers, and in this case visualisation did not help them. Alcock and Simpson (2004) similarly found that visualisation which is linked strongly with formal representations is very helpful to grasp the meaning of concepts, but that students who used visualisation without a link to the formal definition could face problems.

Memorisation did not appear to be used by students when dealing with new definitions and it seems that students did not like the use of memorisation in the normal learning of mathematics. The results in section 3.2.4 showed that memorisation is appreciated by students in the case of the examinations and especially if the definitions are long or complicated, and some students think that it is important to be able to do this to keep the definitions in mind. When I asked for definitions during students' work on questions, none of them showed a repetition of the exact words of the definitions given in the course and all responded with answers based on their conceptions.

## **6.2 Students' Concept Images**

This study confirmed that students possess many concept images about any mathematical concept and at any level of education. In the following I will address the results which I found concerning students' concept images related to the open set and distance concepts in a metric space, the misconceptions that students' use when thinking about them, and the role of intuition in students understanding of such concepts.

### **6.2.1 Students' Concept Images of Open Sets and Distance in Metric Spaces**

Regarding the concept of an open set, in section 5.2 I found five dominant concept images which are used by students when reasoning about the concept. They are: concept image based on the boundary points idea of sets; concept image based on the definition; concept image based on the union of open balls; concept image based on familiarity of openness in  $\mathbf{R}$ ; and concept image based on visualisation.

Other concept images emerged from the results of my study in section 5.3 when students considered the notion of distance in a metric space, these consisted of: concept image based on definition; concept image based on the measurement of similarity; concept image based on difference between elements of sets; and concept image based on physical distance.

Regarding the concept images which are based on the formal statements, the results revealed that some of the students tried to use the definitions and some theorems or lemmas on the concepts when thinking about such concepts or working on problems

but the majority of them seemed to have misconceptions when using such formal statements, and I will consider this further in the following section.

In relation to concept images that I detailed above, the study also discovered some influences which appeared to contribute to the formation of the emerged concept images. As I stated in section 5.4 these influences include: previous mathematical knowledge; intuition; lectures; and examples. Concerning previous mathematical experience, the results revealed that many students appeared to revert to their old ideas about sets and it seemed to be easier for them to think about the concepts in a familiar context. However, many of students provided incorrect answers and some had trouble when their old thoughts did not work in a different context. This could be similar to what has been found in Vinner & Dreyfus (1989); Vinner (1991); and McGowen & Tall (2010). Recall that McGowen & Tall (2010) spoke about ‘met-befores’ and asserted that students’ previous experience could have both positive and negative effects on learning. This study has also seen that students may not be aware of the influence of their previous knowledge on their conceptions. This concurs with Maracci’s (2008) findings on tacit models and their implicit influence on knowledge construction.

The role of the lectures and the examples used to describe concepts, was clearly seen in students’ definitions and explanations related to the concept of distance in a metric space, as most of them explained the concept in ways that their lecturer explained it to them, and they also used the same examples given to them in class. So there is no doubt that teaching methods can have a major effect on the formation of students’ concept images. Bingolbali & Monaghan (2008) and Maull and Berry (2000), when considering the mathematical education of engineers, observed that the development of students’ concept images is affected by teaching practice. Chapter 5 contains a broader discussion of students’ concept images.

### **6.2.2 Students’ Misunderstanding**

As I mentioned in the previous section, some students have concept images that are based on formal statements on the examined topics. However, based on students’ arguments, I found also that many of these students’ concept images include incorrect conceptions related to the concept. In section 5.5 we saw that there were misconceptions concerning the notion of open set itself, for example, some students

thought if one open ball is found inside a set then that set is open; while for others all open balls about all elements of the set should be included inside the set in order for it to be open. Another misconception is related to the mathematical quantifier ‘for any’. Most of students misinterpreted this as ‘for some’ rather than ‘for all’ even though all of them have worked with this quantifier ‘for any’ in terms of ‘for all’ in their previous mathematical studies. Pimm (1984) and Rowland (2001) observed the ambiguity of the word ‘any’ in the learning of mathematics.

Another misconception was related to the notion of an open ball in a metric space, where some students confused the radius of a ball with the distance from the centre to any point in a ball. An extra misconception or confusion was found in relation to the complements of sets in metric spaces, especially ones which extend to infinity, that is some students do not consider the sets  $\mathbf{R}-(a, b)$  to be closed sets.

This study also confirmed what Tsamir (2001) found, that is that different representations for the same task problem can lead students to give different answers. This phenomenon was obvious in some students’ responses to the third and fourth parts of Problem 5 where students did not realise that they were working with the same set represented in two different ways.

### **6.2.3 The Role of Intuition in Students’ Understanding**

The results of this study showed that intuitions play an undeniable role in students’ understanding of mathematical concepts. In this study I used the definition of the term ‘intuition’ suggested by Fischbein (1987) stated in Chapter1, to mean ‘an intuitive conception’, so that all students’ reasoning about a concept which is appeared to be based on immediate knowledge is considered as intuition.

Concerning the concept of an open set, the findings showed that many students used their intuition when thinking about this concept. For example, based on students’ work on task problems I noticed that some students have an intuitive conception of ‘if a set is not open then it is closed or vice versa’, however, in the interviews, the majority of students rejected this when asked directly. As another example, some students show their intuitive thoughts of a ball (or circle) in general metric spaces as a normal circle. All these intuitive conceptions seem to be due to spontaneous conceptions due to the use of words in everyday life (Cornu, 1991)

Moreover, some students appeared to think that open sets cannot be finite sets and similarly for open balls, and also that an open set should have ‘fuzziness’ between its points. These students seemed to use their intuition as a general standard to think about the concept. This might support Fischbein’s (1987) statement stated in Chapter 1, in which some particular instances of a concept may become, for some students, universal representations of such a concept

I realised that these kinds of intuition caused students to think and reason incorrectly about open sets. Interestingly, I noticed that the student who held the intuitive conception that an open set has fuzziness between its points understood the definition given in Problem 5 clearly and was the only student who answered all questions related to it correctly and without any trouble.

Many of our students had concept images of open sets which included ideas relating to a continuum. We have seen that according to Kreyszig (1997), this notion of the ‘mathematical continuum’ was the one originally generalised from Euclidean space when mathematicians began to think about topological spaces. It was only later that the concept of open neighbourhood was seen as the basis of Topology. The students in this study may be progressing along similar lines to early topologists.

### **6.3 Students’ Consistency**

Tall and Vinner (1981) defined students’ concept images about a certain concept to include all their associations related to that concept. These associations might or might not be in line with the formal definition. The main concept that I considered in this thesis is that of the open set.

In relation to the consistency concerning the concept images of an open set that arose from students definitions and those that emerged from their work on the task problems, the findings revealed that most the concept images which were obtained from students’ definitions are similar to those that emerged when students approached the problems. The concept image that is based on the union of open balls was only noticed in students’ definitions, while the concept image that is based on visualisation was noticed in students’ work only.

In this thesis I also examined how consistent students’ own definitions are with their conceptions while working on problems. In section 3.3.1, I have discussed students’ definitions in relation to the concept of open set that emerged from the results of my

study. These definitions are: formal definitions, definitions based on the boundary idea, and definitions based on the union of open balls. We can notice that all these definitions are apparently based on formal statements (boundary idea, i.e. a set is open if it does not contain its boundary points; a set is open if and only if it is a union of open balls; and the formal definition). However, many students who gave these definitions did not seem to make good use of them when solving the given problems, and they often seemed to rely on other conceptions instead.

Table 5.1 in Chapter 5 illustrated that only three students (out of ten) in the interviews tried to use their own definitions (as opposed to the formal definition) about an open set on Problems 3 and 4 and the 4<sup>th</sup> part of Problem 5, and only one of these three succeeded to provide the right answer to one problem. Table 5.3 also indicated that, only five students (out of sixteen) appeared to use their own definitions on Problems 1 and 3, however not all of those five achieved the right answers. To explain further, the students who gave a definition based on the boundary idea did not make good use of it when working on problems and that led them to provide incorrect answers, and the students who provided definitions based on the union of open balls did not use them at all on task problems. A few students who gave definitions based on the formal definition did not show a correct use of it when working on some of the given problems. From this I can say that it is apparent that there is little consistency between students' concept definitions and their work on task problems. This is similar to the findings of Edwards and Ward (2004).

## **6.4 Summary**

To sum up, from this study I can conclude that, students appeared to understand the role that definitions play in advanced mathematics. In general, they seem to know that definitions in mathematics are different from definitions in other subjects, and that mathematical definitions are the basis of concepts and the building blocks of the subject. However, most of them did not rely on definitions when working on task problems and instead based their responses on their concept images, and many of them seemingly think that their versions of definitions cover all aspects of the concept in question. When understanding new definitions, students appear to use a number of strategies which include: focusing on the main points of definitions; using examples; visualisation; grasping the meaning intuitively; using different words; and lastly memorisation.

Concerning my study, I found that a variety of concept images are held by students when thinking of the concept of open set, these images are: concept image based on the boundary points of sets; concept image based on the definitions; concept image based on the union of open balls; concept image based on visualisation; and concept image based on familiarity of openness in  $\mathbf{R}$ .

Students also have other concept images that are related to the concept of distance in a metric space, namely: concept image based on definition; concept image based on difference between points; of sets concept image based on the measurement of similarity; and concept image based on physical distance.

The result of the study also explored some students' misconceptions when thinking of the particular concepts. These misconceptions were about the notion of an open ball; the notion of the complement; the notion of open set itself; and misinterpretation of some mathematical quantifiers.

To consider students' consistency in relation to the definition of an open set, it seems that most of the concept images that appeared in students' definitions were similar to the concept images used by them when working on task problems. The only difference is that the concept image that was based on the union of open balls was only noticed in students' definitions, while the concept image that was based on visualisation was only observed in students' work on task problems. However, I found that most students used their concept image rather than the concept definition when answering problems.

In relation to the concept of distance in metric spaces, there was no obvious consistency between students' ideas of the notion of distance and their work on the metric given in Problem 5. The concept images which were based on the measurement of similarity and the difference between points appeared with only once in the tasks, in particular, in explaining the metric given in Problem 5.

## **6.5 Limitations of the Study**

Most of the students were in the penultimate year of a mathematics degree and some of them were doing a Higher Diploma in mathematics course, and so the number of students who were enrolled in the metric spaces course was small and also all participants came from the same class with the same lecturer. Therefore, the size of

the sample of students who volunteered to investigate the conceptions related to the concept under consideration of my thesis was small. However, the number of students is not too small compared with other studies which use individual interviews, because as known this type of interview is time-consuming.

In the interviews, students were not all given exactly the same questions either because I followed up what they said or because they did not succeed with earlier task problems. Also as English is not my mother language, their answers might be affected by their perceptions of my presentation of the questions.

Regarding the use of the phrase ‘for any’ in Problem 5 which was given to students in the interviews, it might have been better if I had used the phrase ‘for all’ instead. However, the phrase ‘for any’ gave me a chance to explore students’ confusion with mathematical language and quantifiers.

## **6.5 The Implications of This Research**

We know that the open set concept is a central concept in the area of Topology, and so it was an interesting topic to study. To the best of my knowledge, no other study of concept images in this area has been carried out. In this section, I will point out some implications for teaching the metric spaces course which inspired my thesis.

- 1- Students might be aware of the role of definitions in mathematics, however when I asked for definitions the majority of them were not able to give the full formal definition. Also, concepts such as open sets and distance have been encountered by students in their previous mathematical studies and some of the words used carry meaning in everyday life. Thus, it is important from time to time to examine students’ knowledge of the formal definitions involved in a course. It might be that, if students would be able to recall formal definitions, then they would be able and more likely to use them in any task question.
- 2- Some parts of definitions, especially in the open set definition, seemed to be neglected by students and sometimes by the teacher as well when considering the definition. Therefore, a lecturer should place emphasis on using each part of the given definitions in his/her course.
- 3- Many formal statements are given to students about a certain concept including the definition and theorems on the concept. The study found that some students

seem to be confused concerning the distinction between definitions and theorems. Consideration concerning this point should be paid to help students overcome this confusion.

- 4- Lecturers should be aware of the common concept images students have in relation to these involved concepts and know the ways that students interpret their definitions. Students should know the distinction between the statement of the formal definitions and the description of the definition which might limit reflection on all of the definitions' aspects (Giraldo, 2003).
- 5- The findings of this study might enable teachers to get insight into how students might understand the basic ideas involved in the notion of metric space and how such understanding is linked to the more general idea. It is important to consider these findings as it might advocate some improvements for future teaching in the area.

## **6.6 Implications for Further Research**

As I indicated before, one of the limitations of this study was the small number of students in the sample used. Therefore, similar studies could be tried with more students and in particular, with students from different classes, perhaps in a different university or even in a different country.

Mathematicians could be asked the same questions about definition and concept images associated with the concept of open set. In this way, it might be possible to compare experts' attitudes to definitions to those of novice learners. It would also be interesting to compare the concept images of both groups.

In addition, different topics in advanced mathematics could be examined in order to find out more about the effects on students' understanding of concepts in the high levels at universities. To date, much of the research in this area has dealt with topics in introductory analysis and linear algebra but very little is known about subjects that are encountered later in university degree programmes.

## **6.7 Final Remarks**

This thesis, in particular, is meant to give insight into students' conceptions about the open sets in metric spaces and the related notion of distance which is essential to

it. The study reported here addresses how students conceptualise the notions of open sets and distance, and discusses the methods that students use when approaching problems associated to them. The work of this thesis is significant to the area of mathematics education in order to improve the information available on the learning of mathematics at higher levels of education.

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## Appendix 1: Questionnaire

This is a study into the way you think about topics in Topology.

Group:.....

(for example, 2<sup>nd</sup> Arts, Higher Diploma, etc.)

Thank you for your participation!

---

1.(i) Define the term: Open set in a metric space.

(ii) How would you explain the idea of an open set in a metric space to a friend of yours?

2.(i) Define the term: distance in a metric space.

(ii) How would you explain the idea of distance in a metric space to a friend of yours?

3. Consider the metric space  $(\mathbf{R}, d)$  where  $d$  is the standard metric, and let

$A = [0, 2)$ . Is the set A:

Open

Closed

Both

Neither

Please explain your answer!

4. If  $d$  is a discrete metric on  $\mathbf{R}^2$ , describe the unit circle, i.e. the set of  $x \in \mathbf{R}^2$  such that  $d(0, x) = 1$ .

5. Consider the metric space  $(\mathbf{Z}, d_{\mathbf{Z}})$  where  $d_{\mathbf{Z}}$  the standard metric inherited from  $\mathbf{R}$ , and let  $B = \{m-1, m, m+1\}$ . Is B an open ball?

Yes

No

- If Yes, please specify the centre and the radius of the ball.

- If No, please explain the reason.

- Can you find an open ball C which is a subset of B?

Yes

No

Explain your answer!

6. Let  $Y = [0, 2]$  and consider the metric space  $(Y, d_Y)$  where  $d_Y$  is the standard metric on Y inherited from  $\mathbf{R}$ , Let  $A = [0, 2)$ . Is the subset A:

Open

Closed

Both

Neither

Please explain your answer!

## Appendix 2: The Interview Questions

### General Questions:

1. Do you like mathematical definitions?
2. What do you think is the point of definitions in mathematics?
3. When you are presented with a new definition in mathematics, what do you do?
4. Do you memorise definitions?
5. Do you try to relate the definition to material you know already?
6. Do you try to understand the reason the definition was made?
7. Do you use pictures in your mind to understand definitions?
8. Do you refer to definitions when you read or are working on problems?
9. When you try to understand a definition, do you focus on every single word? Or do you think some parts are more important than others?
10. If you are asked to state a definition, do you state it as in your notes or as you understand it?
11. Is it easy for you to understand mathematical definitions?
12. Do you find definitions in maths are different from definitions in other subjects?

### Questions about the notions of open set and distance:

1. (i) Define the term *open set* in metric space.  
(ii) How would you explain the idea of an open set in a metric space to a friend of yours?
2. (i) Define the term *distance* in a metric space.  
(ii) How would you explain the idea of distance in a metric space to a friend of yours?
3. Please indicate whether you agree or disagree with the following statements and justify your choice:
  - (g) A set is open if all its points are centre of open balls.
  - (h) A set is open if it lacks its ends.
  - (i) If a set is not open then it is closed.
  - (j) A set is open if all its points are near to each other.
  - (k) A set is open if all its points are similar.
  - (l) A set is open if its complement is closed.

