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Clebsch–Gordan and 6j-coefficients for rank 2 quantum groups

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Abstract
We calculate (q-deformed) Clebsch–Gordan and $6j$-coefficients for rank 2 quantum groups. We explain in detail how such calculations are done, which should allow the reader to perform similar calculations in other cases. Moreover, we tabulate the q-Clebsch–Gordan and $6j$-coefficients explicitly, as well as some other topological data associated with theories corresponding to rank 2 quantum groups. Finally, we collect some useful properties of the fusion rules of particular conformal field theories.

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1. Introduction

Clebsch–Gordan coefficients and $6j$-symbols have long played an important role in representation theory and also in quantum mechanics, where they traditionally appear in the addition of angular momenta and of internal quantum numbers of particles, such as isospin and color. Recently, models with the so-called *topological phases* have come under intense investigation in condensed matter physics and these provide another arena where explicit knowledge of generalized Clebsch–Gordan and $6j$-coefficients is of great importance. Part of the current excitement is generated by the prospect that the unusual properties of the excitations of these phases can be harnessed and put to use in fault tolerant 'topological quantum computers' [1, 2]. In view of this, we believe it is useful to provide the explicit data needed to compute time evolutions in a number of topological phases, with a description of the methods employed, so as to assist the reader in calculating similar data for other systems.

In the low energy limit, all physical information in a topological phase of matter can be expressed in the framework of topological field theory. Practical expressions for any physical amplitude in a topological field theory can be obtained in terms of generalized (often q-deformed) $6j$-symbols, usually called F-symbols, and a further set of numbers called the

R-symbols. These data can be obtained from the representation theory of quantum groups. For an important class of topological models, namely those associated with $su(2)_k$ Chern–Simons theory, there exists an explicit expression for the F-symbols, which was derived from the associated quantum group $U_q(sl(2))$. However, for most other topological models, no general formulas are available. Nevertheless, whenever the quantum group that describes a topological model is known, the F-symbols can be calculated explicitly. We explain in detail how such calculations can be done, focusing mostly on the quantum groups arising from q-deformations of simple Lie algebras, which describe the Chern–Simons gauge theories based on the corresponding simple Lie groups. We provide explicit tables of q-Clebsch–Gordan coefficients and F-symbols, as well as R-symbols for rank 2 quantum groups.

Although we mainly focus on multiplicity-free cases (that is, cases in which there are no fusion multiplicities greater than 1), we include one example of a theory which does contain a nontrivial fusion multiplicity, namely the $su(3)_3/Z_3$ orbifold theory. We are aware of only one other instance in which the F-symbols were calculated for a theory with a fusion multiplicity, namely in [3], for a theory with three types of particles. In contrast to the example treated here, that theory does not allow for a consistent braiding, and hence does not have an associated R-matrix.

This paper is organized as follows. In section 2, we give an overview of the structure of topological models in general. The 'topological data' specifying topological models are introduced, and a quick introduction on how these can be used to calculate physical observables is given. Section 3 introduces the basic representation theory of quantum groups which is used throughout the paper to calculate topological data. In particular, the definition of the q-Clebsch–Gordan coefficients and F-symbols is given. In section 4, we collect useful data on anyon theories. In particular, we list those theories (based on affine Lie algebras), which do not have fusion multiplicities. It turns out that many of the multiplicity-free theories at high rank have the same fusion rules as other, 'simpler' or better known theories, such as theories at lower rank or orbifolds of the chiral boson. We give such identifications with simpler theories for all affine Lie algebras of types C, D, E, F and G.

A rather detailed description of the calculation of q-Clebsch–Gordan coefficients is given in section 5, which should enable the readers to carry out such calculations themselves. Using these methods, we give a simple expression for the R-symbols in section 6. A summary of the topological data considered in this paper is provided in section 7. Finally, in the last part of the main text (section 8), we give some applications of these topological data.

About half of this paper consists of appendices, in which we tabulate, amongst other things, the q-Clebsch–Gordan coefficients and the F- and R-symbols for various theories. These include $su(3)_2$ (appendix A), $su(3)_3/Z_3$ (appendix B) and the non-simply laced cases $so(5)_1$ (appendix C) and $g_{2,1}$ (appendix D). In particular, this covers all the rank 2 algebras at the lowest non-trivial (i.e. non-Abelian) level. We also cover one of the simplest theories containing a fusion multiplicity, namely $su(3)_3/Z_3$. For completeness, we give explicit expressions for the case $su(2)_k$ as well, in appendix E. The q-Clebsch–Gordan coefficients and F-symbols have been collected in four mathematica notebooks, which are available via http://arxiv.org/src/1004.5456/anc.

2. Topological models

Topological phases are phases of matter characterized by the property that their low energy sectors may be described in the language of topological field theory [4–7]. In particular, these phases have a finite number of types of gapped low energy excitations which may exhibit nontrivial behavior under fusion and exchanges. In planar systems, the exchange behavior is governed by a nontrivial representation of the braid group and the excitations are called *anyons*. The different types of anyons are said to have different *topological charges*. The amplitude for low-energy processes involving anyons are calculated using diagrams that may be interpreted as spacetime diagrams for the processes. The type, or topological charge, of each anyonic particle is indicated as a label on its worldline in such a diagram. As particles fuse or split, this gives rise to vertices where three worldlines meet. As the particles move around and exchange positions, this induces braiding of their world lines. The amplitude associated with a diagram is invariant under continuous deformations and may be calculated by the application of certain moves, most notably the so-called F-moves. Using the F-moves one may express a diagram as a linear combination of diagrams in which one four-particle process in the history depicted by the diagram has been recoupled. The coefficients that appear in these linear combinations are called the F-symbols and they essentially determine the value of any observable quantity in the theory. Most of this paper is devoted to an explicit calculation and tabulation of the F-symbols for a number of theories.

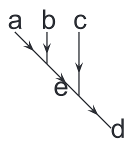
Before we get started with the definition of the F-symbols, we first quickly introduce the concept of fusion rules. The fusion rules of a topological theory state how many times the particle type c appears in the decomposition of the (fusion) product of two particles of types a and b:

$$a \times b = \sum_c n_{a,b}{}^c c. \qquad (1)$$

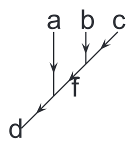
Here, the fusion coefficients $n_{a,b}{}^c$ are non-negative integers. When the TQFT is described using a quantum group (see also section 3), the fusion coefficient $n_{a,b}{}^c$ is just the multiplicity of the quantum group representation labeled c in the decomposition of a suitably defined tensor product of the representations labeled a and b.

The fusion rules are required to be associative. One also requires that $n_{a,b}{}^c = n_{b,a}{}^c$, and that $n_{1,a}{}^{a'} = \delta_{a,a'}$, where 1 denotes the trivial particle. Each particle a has a unique anti-particle $\bar{a}$ for which $n_{a,\bar{a}}{}^1 = 1$ (note that a can be its own anti-particle, as is the case for all representations of $su(2)_k$).

We now define our standard set of F-symbols as the coefficients appearing in the diagrammatic equation



$$= \sum_f (F_d^{a,b,c})_{e,f}$$



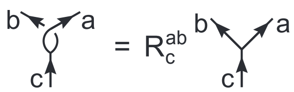
(2)

There are a number of remarks to be made before we continue. First of all, when the TQFT is described using a quantum group, the three lines labeled a, b and c represent three irreducible quantum group representations that are tensored together in two different orders, leading to different intermediate representations with irreducible components e and f. The line labeled d indicates that an irreducible component of type d is selected from the threefold tensor product as the 'overall topological charge'. The F-symbols then give a map between these components of the threefold tensor product (again see section 3 for more detail).

Second, the F-symbols as defined here are strictly speaking not numbers, but matrices which map between two topological vector spaces characterized by the fusion trees in the diagrams. We mainly restrict our attention to *multiplicity-free* TQFTs, for which all such fusion spaces are either one dimensional or zero dimensional. In the second case, the diagram has amplitude zero (we say that it is not allowed by the fusion rules). In the first case, all the vertices correspond to one-dimensional 'fusion spaces', and the F-symbols can be viewed as numbers (they are 1×1 matrices). We can then view $\left[F^{abc}_{d}\right]$ as a matrix, whose rank is given by the number of consistent choices of e (or f, which is equivalent), such that the fusion rules are satisfied.

Since the F-symbols are the coefficients of a transformation between two topological vector spaces determined by fusion trees, they are only fully determined after the bases for these spaces are chosen. A change of basis transforms the F-symbols and since every choice of basis is equally good this gives us a gauge freedom in the F-symbols. Since we (mostly) deal with theories without fusion multiplicities and we use only orthonormal bases, this gauge freedom reduces to a choice of a phase for every vertex in the two diagrams that appear in the definition of the F-symbols.

It is not too difficult to see that repeated application of recoupling moves such as that shown in equation (2) allows us to reduce any diagram that involves only fusions and splittings to a standard form, in effect fixing the time evolution of the state of the system (see for instance [5] for more details). More generally, diagrams may have crossings of the particle lines, corresponding to exchanges of the particles. These may by removed by R-moves, such as the one shown below.



(3)

The R-symbol R^{ab}_{c} that appears in this equation is in principle a unitary matrix, but in a fusion multiplicity-free theory, this reduces to a number (in fact, a phase, since R^{ab}_{c} is a unitary 1×1 matrix in this case). The R-symbols, like the F-symbols, depend on the choice of bases in the topological vector spaces (an exception are the R-symbols R^{aa}_{b} which are gauge invariant), but often this gauge freedom is exhausted once the F-symbols are fixed—this is the case for the theories examined in this paper.

Using both R-symbols and F-symbols, all diagrams that can occur may be reduced to a standard form and hence a knowledge of these symbols completely fixes the gauge invariant physical observables of the theory. Some important gauge invariant quantities characterizing topological models are quantum dimensions, twist factors, Frobenius–Schur indicators and the central charge. Formulas for these quantities in terms of the F- and R-symbols are given in section 7.

3. Quantum groups, CG coefficients and 6j-symbols

Quantum groups are algebras which have a number of structures that make sure that it is possible to define a tensor product on their representations and associated Clebsch–Gordan coefficients and $6j$-symbols. This makes them useful in physics especially in the theory of integrable models and models of anyonic systems. In the quantum group description of anyonic systems the irreducible representations of the quantum group correspond to the topological superselection sectors of the anyon model. The decomposition of tensor products of representations corresponds to fusion and $6j$-symbols correspond to the F-symbols. There is also a structure called the universal R-matrix which provides for braiding and in particular gives the values of the R-symbols. The structures of the quantum group are defined in such a way that the F-symbols and R-symbols obtained from the quantum group's representation theory have all the required properties to fulfill their role in the corresponding anyon model. In particular, fusion is associative and the F-symbols and R-symbols satisfy the pentagon and hexagon equations (see appendix F for the most general form). For a detailed introduction to quantum groups the reader may for instance consult [8]. Here we do not even review all of the structure of a quantum group. Instead, we discuss just enough of the structure to be able to define Clebsch–Gordan coefficients, $6j$-symbols and R-symbols. We explicitly give the relevant structures for the quantum groups we are interested in, the q-deformed universal enveloping algebras $U_q(\mathfrak{g})$ based on the semisimple Lie algebras.

Let $\mathfrak{g}$ be a semisimple Lie algebra and denote its simple roots by α_i. To each simple root, we associate three generators H_i, L_i^+ and L_i^-. These generate $U_q(\mathfrak{g})$ as an algebra, subject to the relations

$$[H_i, H_j] = 0 \qquad \left[H_i, L_j^\pm\right] = \pm A_{ij} L_j^\pm \qquad \left[L_i^+, L_j^-\right] = \delta_{ij} \lfloor H_i \rfloor_{q_i} \tag{4}$$

and

$$\sum_{s=0}^{1-A_{ij}} (-1)^s \begin{bmatrix} 1-A_{ij} \\ s \end{bmatrix}_{q_i} \left(L_i^\pm\right)^{1-A_{ij}-s} L_j^\pm \left(L_i^\pm\right)^s = 0 \qquad (\text{for } i \neq j). \tag{5}$$

Here, $A_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ are the elements of the Cartan matrix of g and we have defined $q_i = q^{1/t_i}$, where $t_i = \frac{2}{(\alpha_i,\alpha_i)}$ are integers in the set $\{1, 2, 3\}$ such that the matrix with the (i, j)-element $t_i A_{ij}$ is symmetric. In addition, these are the elements of the inverse of the quadratic form matrix. When $q = 1$, these relations reduce to the relations for the Chevalley–Serre basis of the universal enveloping algebra $U(\mathfrak{g})$. The q_i-number $\lfloor n \rfloor_{q_i}$ is given by

$$\lfloor n \rfloor_{q_i} = \frac{q_i^{n/2} - q_i^{-n/2}}{q_i^{1/2} - q_i^{-1/2}} = \sum_{m=1}^{n} q_i^{\frac{n+1}{2}-m}$$

and the q_i-binomials that appear are defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_{q_i} = \frac{\lfloor n \rfloor_{q_i}!}{\lfloor m \rfloor_{q_i}! \lfloor n-m \rfloor_{q_i}!},$$

where for $n > 1$ we introduced

$$\lfloor n \rfloor_{q_i}! = \prod_{m=1}^{n} \lfloor m \rfloor_{q_i}$$

and for $n = 0$ we take $\lfloor 0 \rfloor_{q_i}! = 1$. When $t_i = 1$, we often drop the subscript i from q_i, and the subscript q_i from the q-numbers and factorials altogether, i.e. $\lfloor n \rfloor_{q_i} = \lfloor n \rfloor_q = \lfloor n \rfloor$.

Any quantum group A has a *coproduct*, usually denoted as Δ, which is a homomorphism of algebras from A into $\mathsf{A} \otimes \mathsf{A}$, which is coassociative, that is,

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta. \tag{6}$$

Given two representations π^1, π^2 of A, one can define a tensor product representation $\pi^1 \otimes \pi^2$ by the formula

$$\pi^1 \otimes \pi^2 : x \to (\pi^1 \otimes \pi^2)(\Delta(x)). \tag{7}$$

Since Δ is a homomorphism, this will indeed be a representation and since Δ is coassociative, this tensor product will be associative, that is, different orders of tensoring multiple representations lead to the same overall representation of A.

For $U_q(\mathfrak{g})$, the coproduct is given on the generators by the formulas

$$\begin{aligned} \Delta(H_i) &= 1 \otimes H_i + H_i \otimes 1 \\ \Delta\left(L_i^\pm\right) &= L_i^\pm \otimes q_i^{H_i/4} + q_i^{-H_i/4} \otimes L_i^\pm. \end{aligned} \tag{8}$$

If q is not a root of unity, the representation theory of $U_q(\mathfrak{g})$ is very similar to the representation theory of $\mathfrak{g}$. The irreducible representations of $U_q(\mathfrak{g})$ are labeled by the dominant integral weights of $\mathfrak{g}$. The module of the representation labeled by the weight λ has a basis of eigenstates of the H_i, so that each such state is itself labeled by a weight of $\mathfrak{g}$. The action of the generators $L_i^{\pm}$ on a vector of weight μ sends this state either to zero or to a state with weight $\mu \pm \alpha_i$. The weight of a state is in general not enough to fix the state up to a constant, as there are cases where the common eigenspaces of the H_i are higher dimensional. When we describe representations in detail in subsequent sections, we make some arbitrary choices to fix a basis for these higher dimensional eigenspaces (see section 5.2 for this). However, when no such choices are required, the matrix elements of the generators $L_i^{\pm}$ can be written in a form which directly mirrors the undeformed case. For example, the action of $L^{\pm}$ for the states in the irreducible representations of $U_q(sl(2))$ is given by

$$L^{\pm}|j,m\rangle = \sqrt{\lfloor j \mp m\rfloor_q \lfloor j \pm m + 1\rfloor_q}\,|j, m \pm 1\rangle. \tag{9}$$

Here $j \in \frac{1}{2}\mathbb{Z}$ and $m \in \{-j, -j+1, \ldots j-1, j\}$ label the allowed z-components of the 'q-spin' or, equivalently, the eigenvalues of $H/2$ in the representation. This formula easily translates into the general multiplicity-free case. When q is a root of unity, one finds that some of the q-numbers in the formula above become equal to zero and this has some interesting consequences. It turns out that the representations given above remain well defined, but many are no longer irreducible and in particular there are indecomposable representations. In the following, we focus on the irreducible representations.

Let us now introduce the q-Clebsch–Gordan coefficients for the tensor product of irreducible representations. As usual, these Clebsch–Gordan coefficients relate the product basis of the tensor product to a basis which is compatible with the decomposition of the tensor product as a direct sum of irreducible representations. To write explicit CG coefficients, we need to introduce the notation for the canonical basis states of an irreducible $U_q(\mathfrak{g})$-module. As noted before, each irreducible module is labeled by a weight of $\mathfrak{g}$ and has a basis of eigenstates of the H_i, which are themselves each labeled by a weight of $\mathfrak{g}$. If all weight spaces which occur in the module have dimension equal to 1, we can denote the basis states for the module simply as $|j,m\rangle$, where j is the weight labeling the module and m is the weight labeling the weight space in the module. One may now define the Clebsch–Gordan coefficients by

$$|j,m\rangle = \sum_{m_1,m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q |j_1,m_1\rangle|j_2,m_2\rangle. \tag{10}$$

In other words, the CG coefficients $\left[\begin{smallmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{smallmatrix}\right]_q$ are the coefficients of the decomposition of the state $|j,m\rangle$ in the irreducible j-component in the tensor product representation $j_1 \otimes j_2$ into the standard product basis of this tensor product representation.

Of course, the basis states that appear above are only fixed up to a phase by the labels j and m, so their phases may still be chosen to give 'nicer' CG coefficients. For tensor products that are not multiplicity free, more labels are needed to distinguish the various irreducible representations of the same weight that may occur in the tensor product. This is dealt with in some detail in section 5.2, but it does not change the structure of what follows here (we can imagine the extra labels to be implicit in the m-labels).

The q-Clebsch–Gordan coefficients are orthogonal for $q \in R$ and we can use analytic continuation to obtain an orthogonality relation for arbitrary q, which we checked to hold for the q-CG coefficients we calculated:

$$\sum_{m_1,m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{bmatrix}_q = \delta_{j,j'}\delta_{m,m'}. \tag{11}$$

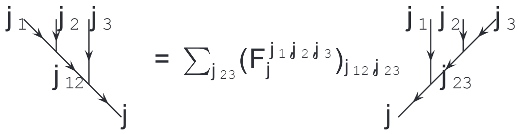
We use the 'inner product' which is defined, like the usual inner product on $\mathbb{R}^n$, as the sum of the products of the coefficients of the two vectors, *without complex conjugation*. This inner

Table 1. Theories with fusion multiplicities

$A_{r,k}$	$\equiv$	$su(r+1)_k$	$k > 3 \wedge r > 2$	$E_{6,k}$	$k > 3$
$B_{r>2,k}$	$\equiv$	$so(2r+1)_k$	$k > 3$	$E_{7,k}$	$k > 3$
$C_{r>3,k}$	$\equiv$	$sp(2r)_k$	$k > 2$	$E_{8,k}$	$k > 4$
$D_{r>4,k}$	$\equiv$	$so(2r)_k$	$k > 3$	$F_{4,k}$	$k > 3$
				$G_{2,k}$	$k > 3$

product is positive definite for all real $q > 0$, but for other values of q this is not necessarily true. With this inner product the formula above guarantees that states in different irreducible subrepresentations of a tensor product are orthogonal *for all q*.

We calculate the F-symbols, or $q-6j$ coefficients, by making use of the q-Clebsch–Gordan coefficients. The graphical representation of these symbols in our current notation is



Thus, at real positive q, each F-symbol can be obtained as the inner product between two states obtained from the two different ways of fusing the three particles (or representations) j_1, j_2 and j_3. These states themselves can be written in terms of the q-Clebsch–Gordan coefficients. Hence, we obtain the formula

$$\left[F_j^{j_1,j_2,j_3}\right]_{j_{12},j_{23}} = \sum_{m_1,m_2,m_3,m_{12},m_{23}} \begin{bmatrix} j_1 & j_2 & j_{12} \\ m_1 & m_2 & m_{12} \end{bmatrix}_q \begin{bmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & j \end{bmatrix}_q \times \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{bmatrix}_q \begin{bmatrix} j_1 & j_{23} & j \\ m_1 & m_{23} & j \end{bmatrix}_q . \qquad (12)$$

The inner product (without complex conjugation) used to derive this formula is really only well defined (positive definite) for $q > 0$ real, but by analytic continuation, the formula nevertheless remains true for arbitrary q.

It would be natural to continue with a description of the R-symbols in terms of quantum group data. However, these are most easily expressed in terms of particular q-Clebsch–Gordan coefficients. Therefore, we will wait with the R-symbols until section 6, which immediately follows the section in which we give a detailed explanation of the calculation of the q-Clebsch–Gordan coefficients.

4. On fusion multiplicities

While working on this manuscript, we often wondered about properties of the fusion rules of particular CFTs. Which theories have fusion multiplicities, and what is the structure of those that do not? Although this is an issue which is slightly off topic, we do take this paper as an opportunity to gather this undoubtedly known information here. We should note that in gathering this information, we benefited much from the program *Kac* by Schellekens [9].

In table 1, we list the theories, based on affine Lie algebra's, or WZW CFTs, which do have fusion multiplicities.

In table 2, we give the list of WZW theories, without multiplicities, for which we can identify the fusion rules as a 'known' fusion ring. Two of the entries are a tautology, namely

Table 2. Identification of the fusion rules without fusion multiplicities, in terms of perhaps 'better'-known fusion rings. The $k = 2$ entries for $A_{r>2}$ and B_r are a tautology.

	$k = 1$		$k = 2$	$k = 3$
$A_{r>2}$	$\mathbf{Z}_{r+1}$		$su(r+1)_2$	
B_r	$su(2)_2$		$so(2r+1)_2$	
C_r	$su(2)_r$			
D_r	$\mathbf{Z}_2 \times \mathbf{Z}_2$ $\mathbf{Z}_4$	(r even) (r odd)	Z_2 orbifold $R = 2r$	
E_6	$\mathbf{Z}_3$		$so(3)_5 \times \mathbf{Z}_3$	
E_7	$\mathbf{Z}_2$		$so(3)_3 \times su(2)_2$	
E_8	$\mathbf{Z}_1$		$su(2)_2$	$so(3)_9$
F_4	$so(3)_3$		$so(3)_9$	
G_2	$so(3)_3$		$so(3)_7$	

$A_{r>2,2}$ and $B_{r,2}$. For completeness, we note that of course the theories $A_{1,k} \equiv su(2)_k$ do not have fusion multiplicities for arbitrary k. Again, we used *Kac* to gather this information. For example, the fusion rules of $D_{r,k=2}$ are the same as the fusion rules of the Z_2 orbifold of the chiral boson at radius $R = 2r$, which we checked explicitly up to $r = 20$. We note that with $so(3)_k$, we denote the integer spin sector of the $su(2)_k$ theory. In particular, $so(3)_3$ corresponds to the Fibonacci theory (sometimes denoted as Fib), which consists of two anyon types $\mathbf{1}$ and τ, with the non-trivial fusion rule $\tau \times \tau = \mathbf{1} + \tau$.

5. Calculating the q-Clebsch–Gordan coefficients

In this section, we show how to calculate the q-Clebsch–Gordan coefficients in full detail. While the calculational techniques presented in the rest of this paper are valid for all of the quantum groups $U_q(\mathfrak{g})$, we use the case $\mathfrak{g} = su(3)$ in our explicit examples, because it is probably the simplest case to display all of the relevant features. Hence, we will often be dealing with the quantum group $U_q(su(3))$. For details on the representation theory of the corresponding finite-dimensional Lie algebra $su(3)$, we refer to appendix A and, for instance, to the books [11–13].

We start with some straightforward examples which explain the structure of the calculation, without having to worry about additional complications. The complications consist of two types of multiplicities, namely weight-space multiplicities and fusion multiplicities, which will be dealt with after we have completed explaining the structure of the calculations.

We first introduce some notation needed in this discussion. The representations of the rank 2 algebra $su(3)$ are labeled by the Dynkin labels of the highest weight, (Λ_1, Λ_2). In addition, we also denote the representations by their dimensions in boldface (and an overline to denote the conjugate representations). The weights of the representations are labeled by (λ_1, λ_2). For example, the eight-dimensional adjoint representation $\mathbf{8}$ has the highest weight $(\Lambda_1, \Lambda_2) = (1, 1)$. The other weights in this representation are $(-1, 2)$, $(2, -1)$, $(0, 0)_+$, $(0, 0)_-$, $(-2, 1)$, $(1, -2)$ and $(-1, -1)$, which we include here, in order to show the notation of the two-dimensional weight space $(0, 0)$. We note that for $q = e^{2\pi i/5}$ (or $k = 2$), $\mathbf{8}$ is the only representation with a weight-space multiplicity. In addition, the following representations $\mathbf{1} = (0, 0)$, $\mathbf{3} = (1, 0)$, $\bar{\mathbf{3}} = (0, 1)$, $\mathbf{6} = (2, 0)$, $\bar{\mathbf{6}} = (0, 2)$ and $\mathbf{8} = (1, 1)$ are present at $k = 2$.

As stated in section 3, the q-Clebsch–Gordan coefficients express the tensor product representations in terms of the direct sum of irreducible representations

$$|j, m\rangle = \sum_{m_1,m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q |j_1, m_1\rangle |j_2, m_2\rangle. \tag{13}$$

5.1. *The structure of calculating q-Clebsch–Gordan coefficients*

The calculation of q-Clebsch–Gordan coefficients closely follows the calculation of Clebsch–Gordan coefficients in the 'classical' case, see for instance [10]; however, there are some important differences.

The most obvious difference is that the explicit representation matrices for the elements of $U_q(\mathfrak{g})$ contain q-numbers instead of integers, as illustrated in formula (9), and these q-numbers propagate into the Clebsch–Gordan and $6j$-coefficients for $U_q(\mathfrak{g})$. We follow a policy of expressing all explicitly given coefficients in terms of q-numbers, as much as possible, because this allows one to immediately obtain the corresponding classical coefficients by replacing q-numbers by the corresponding integers[4].

When q is a root of unity, it turns out that a tensor product of two irreducible representations often contains indecomposable representations. To deal with this problem, one has to introduce a 'truncated' tensor product, in which the unwanted, indecomposable representations no longer appear. This 'truncated' tensor product precisely corresponds to the fusion product (as introduced in section 2) for physical systems described by the Chern–Simons theory based on $\mathfrak{g}$ at level k, where k is related to q through

$$q = \mathrm{e}^{\frac{2\pi \mathrm{i}}{k+g}}.$$

Here g is the dual Coxeter number of $\mathfrak{g}$. This makes the case where q is a primitive root of unity the most interesting from the perspective of the physics of anyons. For the actual calculation of Clebsch–Gordan coefficients which involve only irreducible representations, the truncation of the tensor product makes little difference—the main change in our treatment compared to the classical case is that we will not consider the other components of the tensor product when q is a root of unity.

The third and perhaps the most important difference between the deformed and classical calculation of CG coefficients occurs due to the deformation of the coproduct. As can be seen from equation (8), the action of the raising and lowering operators of $U_q(\mathfrak{g})$ in a tensor product representation involves factors of different powers of q, depending on wether the raising/lowering operator acts on the left- or right-hand factor of the tensor product. These factors of powers of q also end up in the CG coefficients. As a result of the deformation of Δ, the various components of a tensor product representation are no longer simply symmetric or antisymmetric under exchange of the tensor factors, which gives an indication that there is no longer a representation of the permutation group S_N on an N-fold tensor product, but instead a representation of the braid group B_N. The R-symbols which describe the action of the braid group are calculated in section 6.

Let us now explain the calculational algorithm from scratch (that is, without presuming a knowledge of the classical algorithm), by taking a simple concrete example, based on the

[4] It should be noted that the opposite process, obtaining the q-deformed coefficients from the classical ones by replacing integers by q-integers, is not easily accomplished, because the various different ways in which integers may be represented would lead to different q-deformed results. For example, if the integer 9 appears in a classical coefficient, it is not clear whether this should be replaced by $\lfloor 9\rfloor_q$ or $\lfloor 3\rfloor_q^2$ or $1 + \lfloor 8\rfloor_q$, to name just a few of the possibilities.

$su(3)$ tensor product $\mathbf{3} \otimes \mathbf{3} = \bar{\mathbf{3}} \oplus \mathbf{6}$, which for $k > 2$ corresponds to the fusion product as well: $\mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6}$. We start with the Clebsch–Gordan coefficients for the representation $\mathbf{6}$ in the tensor product decomposition of $\mathbf{3} \otimes \mathbf{3}$.

The highest weight of the representation $\mathbf{6}$ is $(2, 0)$, which is the sum of the highest weights of the two representations $\mathbf{3}$, namely $(1, 0)$. Thus, the highest weight of the representation $\mathbf{6}$ uniquely decomposes, and we find the first (trivial) Clebsch–Gordan coefficient:

$$|(2,0)(2,0)\rangle = |(1,0)(1,0)\rangle \otimes |(1,0)(1,0)\rangle, \tag{14}$$

namely

$$\begin{bmatrix} (1,0) & (1,0) & (2,0) \\ (1,0) & (1,0) & (2,0) \end{bmatrix}_q = 1. \tag{15}$$

This coefficient has norm 1, and we choose to set the phase to 0 by convention. By acting with the lowering operator L_1^- on both sides of equation (14), we find other Clebsch–Gordan coefficients. On the right-hand side, we actually need to use $\Delta(L_1^-)$, which is given by $\Delta(L_1^-) = L_1^- \otimes q^{H_1/4} + q^{-H_1/4} \otimes L_1^-$. This leads to the following:

$$\begin{aligned} |(2,0)(0,1)\rangle &= \frac{L_1^-}{\sqrt{\lfloor 2 \rfloor}} |(2,0)(2,0)\rangle = \frac{\Delta(L_1^-)}{\sqrt{\lfloor 2 \rfloor}} |(1,0)(1,0)\rangle \otimes |(1,0)(1,0)\rangle \\ &= \frac{q^{1/4}}{\sqrt{\lfloor 2 \rfloor}} |(1,0)(-1,1)\rangle \otimes |(1,0)(1,0)\rangle + \frac{q^{-1/4}}{\sqrt{\lfloor 2 \rfloor}} |(1,0)(1,0)\rangle \otimes |(1,0)(-1,1)\rangle. \end{aligned} \tag{16}$$

We put in the factor $1/\sqrt{\lfloor 2 \rfloor}$ 'by hand' to make sure that the states are normalized throughout the calculation. Of course, one could also ignore these factors at first and normalize the states later on. To be able to discuss the normalization, we recall that we defined the inner product in the usual way for $q \in \mathrm{R}$, that is, by simple multiplication of the coefficients, and for arbitrary q by analytic continuation of this definition. We note that this inner product does not involve complex conjugation of the coefficients. Recalling in addition that we used the following convention for the q-numbers:

$$\lfloor n \rfloor = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \tag{17}$$

it easily follows that the right-hand side of equation (16) is indeed normalized.

So, we found the additional q-Clebsch–Gordan coefficients

$$\begin{bmatrix} (1,0) & (1,0) & (2,0) \\ (1,0) & (-1,1) & (0,1) \end{bmatrix}_q = \frac{q^{-1/4}}{\sqrt{\lfloor 2 \rfloor}} \tag{18}$$

$$\begin{bmatrix} (1,0) & (1,0) & (2,0) \\ (-1,1) & (1,0) & (0,1) \end{bmatrix}_q = \frac{q^{1/4}}{\sqrt{\lfloor 2 \rfloor}}. \tag{19}$$

We can now continue this procedure by acting with additional lowering operators. In fact, in this case, we can either act with L_1^- to obtain the state $|(2,0)(-2,2)\rangle$, or with L_2^- to find the coefficients for $|(2,0)(1,-1)\rangle$. To see with which lowering operators one can act, we refer to appendix A containing the structure of the representations we consider in this paper. All the q-Clebsch–Gordan coefficients for the $\mathbf{6}$ in $\mathbf{3} \times \mathbf{3}$ are collected in appendix A.3.1.

We now explain how to obtain the q-Clebsch–Gordan coefficients for the representation $\bar{\mathbf{3}}$ in the tensor product decomposition of $\mathbf{3} \otimes \mathbf{3}$. To start with, we need to express the (highest weight) state $|(0,1)(0,1)\rangle$ in terms of the states $|(1,0)(1,0)\rangle \otimes |(1,0)(-1,1)\rangle$ and

$|(1,0)(-1,1)\rangle \otimes |(1,0)(1,0)\rangle$. We do this by using that $|(0,1)(0,1)\rangle$ is a highest weight state, i.e $L_1^+|(0,1)(0,1)\rangle = L_2^+|(0,1)(0,1)\rangle = 0$. In this case, we only need to consider

$$\begin{aligned} L_1^+|(0,1)(0,1)\rangle &= \Delta\left(L_1^+\right)(a|(1,0)(1,0)\rangle \otimes |(1,0)(-1,1)\rangle \\ &\quad + b|(1,0)(-1,1)\rangle \otimes |(1,0)(1,0)\rangle) \\ &= (aq^{-1/4} + bq^{1/4})|(1,0)(1,0)\rangle \otimes |(1,0)(1,0)\rangle = 0, \end{aligned} \tag{20}$$

from which it follows that

$$|(0,1)(0,1)\rangle = \frac{q^{1/4}}{\sqrt{\lfloor 2\rfloor}}|(1,0)(1,0)\rangle \otimes |(1,0)(-1,1)\rangle - \frac{q^{-1/4}}{\sqrt{\lfloor 2\rfloor}}|(1,0)(-1,1)\rangle \otimes |(1,0)(1,0)\rangle, \tag{21}$$

where the overall sign (or better, phase) is our convention. In general, we choose the overall phase to be such that for $q = 1$, the Clebsch–Gordan coefficient with the highest possible weight on the left-hand side of the tensor product is real and positive (we spell out all our conventions in detail below). We should note that this state $|(0,1)(0,1)\rangle$ is orthogonal to the state $|(2,0)(0,1)\rangle$ (equation (16)), as it should. Thus, we find the following q-Clebsch–Gordan coefficients:

$$\begin{bmatrix} (1,0) & (1,0) & (0,1) \\ (1,0) & (-1,1) & (0,1) \end{bmatrix}_q = \frac{q^{1/4}}{\sqrt{\lfloor 2\rfloor}} \tag{22}$$

$$\begin{bmatrix} (1,0) & (1,0) & (0,1) \\ (-1,1) & (1,0) & (0,1) \end{bmatrix}_q = -\frac{q^{-1/4}}{\sqrt{\lfloor 2\rfloor}}. \tag{23}$$

We can find the other coefficients related to the states $|(0,1)(1,-1)\rangle$ and $|(0,1)(-1,0)\rangle$ by first acting with the lowering operator L_2^- and subsequently with L_1^- on the state $|(0,1)(0,1)\rangle$. Without spelling out all the details, this gives the following q-Clebsch–Gordan coefficients:

$$\begin{bmatrix} (1,0) & (1,0) & (0,1) \\ (1,0) & (0,-1) & (1,-1) \end{bmatrix}_q = \frac{q^{1/4}}{\sqrt{\lfloor 2\rfloor}} \qquad \begin{bmatrix} (1,0) & (1,0) & (0,1) \\ (0,-1) & (1,0) & (1,-1) \end{bmatrix}_q = -\frac{q^{-1/4}}{\sqrt{\lfloor 2\rfloor}} \tag{24}$$

$$\begin{bmatrix} (1,0) & (1,0) & (0,1) \\ (-1,1) & (0,-1) & (-1,0) \end{bmatrix}_q = \frac{q^{1/4}}{\sqrt{\lfloor 2\rfloor}} \qquad \begin{bmatrix} (1,0) & (1,0) & (0,1) \\ (0,-1) & (-1,1) & (-1,0) \end{bmatrix}_q = -\frac{q^{-1/4}}{\sqrt{\lfloor 2\rfloor}}. \tag{25}$$

With the information we have given so far, it is possible to obtain all the q-Clebsch–Gordan coefficients which do not involve the representation **8**. In the following subsection, we explain the subtleties which arise when dealing with this representation.

5.2. Dealing with multiplicities

In this section, we explain how to deal with the two kinds of multiplicities, namely weight space multiplicity and fusion multiplicity.

The eight-dimensional adjoint representation of $su(3)$ has the property that the 'weight' $(0,0)$ corresponds to a two-dimensional weight space. In other words, the two states $L_2^- L_1^-|(1,1)(1,1)\rangle$ and $L_1^- L_2^-|(1,1)(1,1)\rangle$ are linearly independent. Thus, we choose a basis for this weight space. We pick the following basis, which is orthonormal:

$$|(1,1)(0,0)_\pm\rangle = \frac{\left(L_1^- L_2^- \pm L_2^- L_1^-\right)}{\sqrt{2(\lfloor 2\rfloor \pm 1)}}|(1,1)(1,1)\rangle. \tag{26}$$

We now have the following relations for the action of the lowering operators on the states 'above' the weight (0, 0) space

$$L_1^-|(1,1)(2,-1)\rangle = \sqrt{\frac{\lfloor 2\rfloor+1}{2}}|(1,1)(0,0)_+\rangle + \sqrt{\frac{\lfloor 2\rfloor-1}{2}}|(1,1)(0,0)_-\rangle \tag{27}$$

$$L_2^-|(1,1)(-1,2)\rangle = \sqrt{\frac{\lfloor 2\rfloor+1}{2}}|(1,1)(0,0)_+\rangle - \sqrt{\frac{\lfloor 2\rfloor-1}{2}}|(1,1)(0,0)_-\rangle, \tag{28}$$

while the action of raising and lowering operators on $|(1,1)(0,0)_\pm\rangle$ is given by

$$\begin{aligned} L_1^+|(1,1)(0,0)_\pm\rangle &= \sqrt{\frac{\lfloor 2\rfloor\pm 1}{2}}|(1,1)(2,-1)\rangle \\ L_2^+|(1,1)(0,0)_\pm\rangle &= \pm\sqrt{\frac{\lfloor 2\rfloor\pm 1}{2}}|(1,1)(-1,2)\rangle \end{aligned} \tag{29}$$

$$\begin{aligned} L_1^-|(1,1)(0,0)_\pm\rangle &= \sqrt{\frac{\lfloor 2\rfloor\pm 1}{2}}|(1,1)(-2,1)\rangle \\ L_2^-|(1,1)(0,0)_\pm\rangle &= \pm\sqrt{\frac{\lfloor 2\rfloor\pm 1}{2}}|(1,1)(1,-2)\rangle. \end{aligned} \tag{30}$$

The fact that one has to choose a basis for the two-dimensional weight space $|(1,1)(0,0)_\pm\rangle$ is not the only subtlety which arises in conjunction with the representation **8**. Consider the $su(3)$ tensor product $\mathbf{8}\otimes\mathbf{8} = \mathbf{1}\oplus\mathbf{8}\oplus\mathbf{8}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{27}$. One finds that in the decomposition, the eight-dimensional representation appears twice. At level $k = 2$, or in other words, for $q = \mathrm{e}^{2\pi \mathrm{i}/5}$, only one of these two eight-dimensional representations is present in the fusion product, which reads $\mathbf{8}\times\mathbf{8} = \mathbf{1}+\mathbf{8}$. One is thus led to the question, how can one decide which eight-dimensional representation to pick?

In solving the highest weight conditions $L_1^+|(1,1)(1,1)\rangle = L_2^+|(1,1)(1,1)\rangle = 0$, one finds a two-dimensional space of solutions, as one should. A convenient way of writing the two solutions is as follows:

$$\begin{aligned} |(1,1)(1,1)\rangle_1 = {} & \frac{q^{3/4}}{\sqrt{\lfloor 4\rfloor+1}}|(1,1)(1,1)\rangle\otimes|(1,1)(0,0)_+\rangle \\ & + \frac{q^{-3/4}}{\sqrt{\lfloor 4\rfloor+1}}|(1,1)(0,0)_+\rangle\otimes|(1,1)(1,1)\rangle \\ & - \sqrt{\frac{\lfloor 2\rfloor+1}{2(\lfloor 4\rfloor+1)}}(|(1,1)(-1,2)\rangle\otimes|(1,1)(2,-1)\rangle \\ & + |(1,1)(2,-1)\rangle\otimes|(1,1)(-1,2)\rangle) \end{aligned} \tag{31}$$

$$\begin{aligned} |(1,1)(1,1)\rangle_2 = {} & \frac{q^{3/4}}{\sqrt{\lfloor 4\rfloor-1}}|(1,1)(1,1)\rangle\otimes|(1,1)(0,0)_-\rangle \\ & - \frac{q^{-3/4}}{\sqrt{\lfloor 4\rfloor-1}}|(1,1)(0,0)_-\rangle\otimes|(1,1)(1,1)\rangle \\ & + \sqrt{\frac{\lfloor 2\rfloor-1}{2(\lfloor 4\rfloor-1)}}(|(1,1)(2,-1)\rangle\otimes|(1,1)(-1,2)\rangle \\ & - |(1,1)(-1,2)\rangle\otimes|(1,1)(2,1)\rangle). \end{aligned} \tag{32}$$

We observe that for generic q, the states are orthonormal. However, for $q = e^{2\pi i/5}$, we have $\lfloor 4 \rfloor = 1$, which means that the state $|(1,1)(1,1)\rangle_2$ is not normalizable[5]. Thus, we conclude that at level $k = 2$, the state $|(1,1)(1,1)\rangle_1$ corresponds to the eight-dimensional representation which is present in the fusion product $\mathbf{8} \times \mathbf{8} = \mathbf{1} + \mathbf{8}$, while the state $|(1,1)(1,1)\rangle_2$ corresponds to the representation which is present in the tensor product, but not in the fusion product.

At level $k = 3$, i.e. $q = e^{2\pi i/6}$, both eight-dimensional representations are present in the fusion product: $su(3)_3$ is one of the simplest theories which exhibits a fusion multiplicity. Therefore, we also give the q-Clebsch–Gordan coefficients for the $su(3)_3/Z_3$ theory, which contains four representations, **1**, **8**, **10** and $\overline{\mathbf{10}}$, see appendix B for more details. In particular, we have the following fusion product:

$$\mathbf{8} \times \mathbf{8} = \mathbf{1} + \mathbf{8} + \mathbf{8}' + \mathbf{10} + \overline{\mathbf{10}}, \tag{33}$$

where the first **8** in the fusion product corresponds to the one present at level $k = 2$, while the second $\mathbf{8}'$ corresponds to the one appearing for the first time at $k = 3$. So, in the case at hand, there is a natural choice of a basis for the two-dimensional space corresponding to this fusion multiplicity, namely in 'order of appearance' when the level k increases. In fact, for $su(3)$, it is always possible to make such a choice, because the so-called threshold levels for the fusion coefficients of $su(3)$ differ by 1. We note, however, that this property is special for $su(3)$. For more on this, we refer to section 16.4 of [11].

With this information, we dealt with all the subtleties arising from the adjoint representation of $su(3)$.

5.3. Gauge convention for the q-Clebsch–Gordan coefficients

In the calculation of the q-Clebsch–Gordan coefficients, one must choose conventions for the overall phases of the coefficients. In this section, we explicitly describe our choice. To fix the gauge for the symbols $\begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q$ for a particular choice of j_1, j_2 and j, it suffices to fix the phase of one of the symbols. It is convenient to fix the phase of the symbol with $m = j$, and the highest possible weight $m_1 = m_{\max}$. We have chosen this phase in such a way that in the limit $q \to 1$, this symbol $\begin{bmatrix} j_1 & j_2 & j \\ m_{\max} & j - m_{\max} & j \end{bmatrix}_q$ is real and positive.

This gauge choice for the q-Clebsch–Gordan coefficients also completely specifies the gauge degrees of freedom for the $q - 6j$ symbols, because they are fixed by the q-Clebsch–Gordan coefficients. Our gauge choice is such that the $q - 6j$ symbols with a trivial particle on any of the incoming lines is equal to 1, when no vertices with a fusion multiplicity are present. When q is the primitive root of unity corresponding to the associated Chern–Simons theory, the $q - 6j$ symbols with a trivial particle on the outgoing line is also equal to 1 (again assuming there are no vertex multiplicities) .

5.4. Symmetries of the q-Clebsch–Gordan coefficients

In this section, we give the symmetries of the q-Clebsch–Gordan coefficients. Due to these symmetries, we do not have to calculate all the q-Clebsch–Gordan coefficients. We first state these symmetry relations and explain them afterward:

$$\begin{bmatrix} j_2 & j_1 & j \\ m_2 & m_1 & m \end{bmatrix}_q = (-1)^{s_1} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_{\frac{1}{q}} \tag{34}$$

[5] One might be tempted to multiply the given expression for this state by $\sqrt{\lfloor 4 \rfloor - 1}$ to obtain a state with a good limit as $q \to e^{2\pi i/5}$. This indeed yields a state with finite coefficients, but it will have norm zero with respect to our complex conjugation-free inner product.

$$\begin{bmatrix} j_1 & j_2 & j \\ -\overline{m_1} & -\overline{m_2} & -\overline{m} \end{bmatrix}_q = (-1)^{s_2+\#(0,0)_-} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_{\frac{1}{q}} \qquad (35)$$

$$\begin{bmatrix} \overline{j_1} & \overline{j_2} & \overline{j} \\ \overline{m_1} & \overline{m_2} & \overline{m} \end{bmatrix}_q = (-1)^{s_3+\#(0,0)_-} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q , \qquad (36)$$

where, as before, we used the j's and m's to denote the highest weights and the weights of the states in the representations, respectively. By the conjugate weights (denoted by the overline), we mean the following. In the case of $su(3)$ the weight conjugate to $m = (\lambda_1, \lambda_2)$ is $\overline{m} = \overline{(\lambda_1, \lambda_2)} = (\lambda_2, \lambda_1)$. However, for $so(5)$ and G_2, weights are self-conjugate $\overline{m} = \overline{(\lambda_1, \lambda_2)} = (\lambda_1, \lambda_2)$. In the equations above s_1, s_2 and s_3 are either 0 or 1, depending on the case at hand (see the tables in appendix A.3.10 and appendix B.2). In addition, with $\#(0,0)_-$, we mean the number of m_1, m_2, m which are equal to $(0,0)_-$ in the representation **8** of $su(3)$, see below for an explanation.

The first relation (equation (34)) is related to the symmetry of the tensor products. When $j_1 = j_2$, the value of s_1 is gauge invariant and cannot be chosen. However, for $j_1 \neq j_2$, the value of s_1 is a gauge choice. As noted in section 5.3, we always choose the overall phase of the states in such a way that the q-Clebsch–Gordan coefficient with $m = j$ and with the highest possible value of m_1 is positive in the limit $q \to 1$. Note that the highest possible value of m_1 is not always equal to j_1. This choice completely fixes the gauge and hence also s_1. In the tables in appendix A.3.10 and appendix B.2, we give the values of s_1 in our gauge.

The relation (equation (35)) relates the q-Clebsch–Gordan coefficients 'within' a certain representation. This relation stems from the fact that one can equally well obtain the whole representation by acting with raising operators on the lowest weights, instead of acting with lowering operators on the highest weights, as we do here. The value of s_2 depends on the representation at hand; this value is specified in the tables in appendix A.3.10 and appendix B.2. Finally, the last symmetry relation (36) corresponds to the symmetry under complete conjugation.

There is one important issue, which occurs for both equations (35) and (36), which is related to the conjugation of the weight $(0,0)_-$ in the eight-dimensional representation of $su(3)$. Loosely speaking, we have the relation $\overline{(0,0)_-} = -(0,0)_-$. This means that for every time the (conjugated) weight $\overline{(0,0)_-}$ appears in equations (35) and (36), we get an additional sign.

Finally, we also note that the only case in this paper for which $s_3 = 1$ is the $\mathbf{8}'$ representation which appears in the fusion product $\mathbf{8} \times \mathbf{8} = \mathbf{1} + \mathbf{8} + \mathbf{8}' + \mathbf{10} + \overline{\mathbf{10}}$ of $su(3)_3/Z_3$.

6. Obtaining the R-symbols

After having obtained the q-Clebsch–Gordan coefficients, it is rather straightforward to obtain the R-symbols as well. We make use of the explicit expression for the R-matrix. This expression is rather cumbersome for arbitrary quantum groups, but we only need a very limited amount of information contained in this expression to extract the R-symbols. In particular, the expression contains a product over all positive roots. The ordering of these positive roots is important, but we will not specify this ordering, because it is immaterial for our purposes. In addition to this product, there is a pre-factor, which we need to obtain the R-matrix elements.

With these caveats in place, we give the *structure* of the R matrix:

$$R = q^{(F_{\rm qf})_{i,j}\frac{h_i\otimes h_j}{2}} \prod_{\alpha>0} E_{q^{-1}}^{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})(e_\alpha q^{\frac{h_\alpha}{4}}\otimes q^{-\frac{h_\alpha}{4}} f_\alpha)}, \tag{37}$$

where $E_q^x = \sum_{n=0}^{\infty} \frac{q^{\frac{n}{4}(n-1)}}{\lfloor n\rfloor!} x^n$.

The overall factor, $q^{(F_{\rm qf})_{i,j}\frac{h_i\otimes h_j}{2}}$, contains the quadratic form matrix $F_{\rm qf}$, whose components in terms of the inverse Cartan matrix elements and the integers t_j read $(F_{\rm qf})_{ij} = (A^{-1})_{ij}/t_j$, where there is no summation over j. Recall that $t_i = \frac{2}{(\alpha_i,\alpha_i)}$, with α_i being the simple roots of the algebra. For simply laced Lie algebras, such as $su(r+1)$, all the $t_i = 1$. The short roots of B_r, C_r and F_4 have $t = 2$, while for the short root of G_2, one has $t = 3$.

The product is over all positive roots of the algebra and the factors consist of exponentials of raising operators e_α acting on the left tensor factor, and lowering operators f_α acting on the right tensor factors. It is this structure which allows us to obtain the R-symbols from the pre-factor alone, as we explain below, and give an explicit example for clarification.

In the calculation of the R-symbols one first acts with the R-matrix on a particular tensor decomposition, followed by swapping the tensor factors. This combined operation is denoted by σR. This gives rise to the tensor decomposition in which the representations are swapped, up to an overall phase factor. The R-symbol is precisely this phase factor. We now explain the calculation of the R-symbols $R_j^{j_1,j_2}$ in detail.

First of all, the R-symbols $R_j^{j_1,j_2}$ depend only on the representations j_1, j_2 and j, and not on the particular weights within these representations, as is the case for the the F-symbols. We can thus pick a particular weight in the representation j to obtain $R_j^{j_1,j_2}$, which we take to be $m = j$. We restrict ourselves further by only considering a suitably chosen term in the decomposition.

Namely, we pick that component of the tensor decomposition which has a lowest possible weight in the left factor of the tensor product. By this we mean that one cannot subtract a root from this weight and obtain another term in the tensor decomposition.

Because all the raising operators in the R-matrix act on the left factor of the tensor product, the only contribution to the component in the tensor product with this lowest possible weight on the right (after performing the swap σ!) comes from the component with the lowest possible weight on the left. In addition, it is only the identity term in all exponents which contributes, because all the other terms contain raising operators on the left tensor factor. Thus, the knowledge of the two corresponding q-Clebsch–Gordan coefficients and the factor $q^{(F_{\rm qf})_{i,j}\frac{h_i\otimes h_j}{2}}$ suffices to obtain the R-symbols.

To show how this works, we give an explicit example. Let us consider the $\bar{\mathbf{3}}$ in $\mathbf{3}\times\mathbf{3}$, and take the highest weight of $\bar{\mathbf{3}}$:

$$|(0,1)(0,1)\rangle = \frac{q^{1/4}}{\sqrt{\lfloor 2\rfloor}}|(1,0)(1,0)\rangle\otimes|(1,0)(-1,1)\rangle - \frac{q^{-1/4}}{\sqrt{\lfloor 2\rfloor}}|(1,0)(-1,1)\rangle\otimes|(1,0)(1,0)\rangle. \tag{38}$$

We let σR act on the term $-\frac{q^{-1/4}}{\sqrt{\lfloor 2\rfloor}}|(1,0)(-1,1)\rangle\otimes|(1,0)(1,0)\rangle$ and find

$$\begin{aligned}\sigma R\left(-\frac{q^{-1/4}}{\sqrt{\lfloor 2\rfloor}}|(1,0)(-1,1)\rangle\otimes|(1,0)(1,0)\rangle\right) &= \frac{-q^{-1/6}q^{-1/4}}{\sqrt{\lfloor 2\rfloor}}|(1,0)(1,0)\rangle\otimes|(1,0)(-1,1)\rangle \\ &= (-q^{-2/3})\frac{q^{1/4}}{\sqrt{\lfloor 2\rfloor}}|(1,0)(1,0)\rangle\otimes|(1,0)(-1,1)\rangle.\end{aligned} \tag{39}$$

Comparing this result to the first term in equation (38), we find that $R^{\mathbf{3},\mathbf{3}}_{\bar{\mathbf{3}}} = -q^{-2/3}$. In general (including the non-simply laced case), we obtain the following expression for the R-matrix elements

$$R^{j_1,j_2}_{j} = q^{\frac{1}{2}m_{\min}\cdot F_{\mathrm{qf}}\cdot(j-m_{\min})} \frac{\begin{bmatrix} j_1 & j_2 & j \\ m_{\min} & j-m_{\min} & j \end{bmatrix}_q}{\begin{bmatrix} j_2 & j_1 & j \\ j-m_{\min} & m_{\min} & j \end{bmatrix}_q}, \tag{40}$$

where $m_{\min}$ is the lowest possible weight mentioned earlier, i.e. $m_{\min}$ is chosen so that $\begin{bmatrix} j_1 & j_2 & j \\ m_{\min}-\alpha & j-m_{\min}+\alpha & j \end{bmatrix}_q = 0$ for all simple roots α, and nevertheless the CG-coefficients in the formula above are nonzero. Note that we assume that the weights are given in terms of their Dynkin labels (hence the appearance of the quadratic form matrix F_{qf}). Finally, let us note two symmetries of the R-symbols which exist in our gauge:

$$R^{a,b}_{c} = R^{b,a}_{c} \qquad R^{a,b}_{c} = R^{\bar{a},\bar{b}}_{\bar{c}}. \tag{41}$$

7. TQFT data

In this section, we briefly describe some of the topological data associated with a topological quantum field theory. We will not go into a lot of detail to explain the theory behind this data. For this, we refer the reader to, for instance, [1] or [14]. We frequently use the data of $su(3)_2$ as an example and refer to the appendices for the corresponding data of the other models. We note that the formulas for the various quantities hold in the case of unitary anyon models, which corresponds here to the cases where q is a primitive root of unity.

7.1. Quantum dimensions

The quantum dimension d_a associated with a particle of type a is defined in the following way:

$$d_a = \frac{1}{\left|\left[F^{a,\bar{a},a}_{a}\right]_{\mathbf{1},\mathbf{1}}\right|}. \tag{42}$$

Though this is not obvious from this definition, the Hilbert space of n particles of type a grows as d_a^n. The quantum dimensions of the six representations of $su(3)_2$ are given by

$$d_{\mathbf{1}} = 1 \qquad d_{\mathbf{3}} = d_{\bar{\mathbf{3}}} = \lfloor 3 \rfloor \to \phi \qquad d_{\mathbf{6}} = d_{\bar{\mathbf{6}}} = 1 + \lfloor 5 \rfloor \to 1 \qquad d_{\mathbf{8}} = \lfloor 3 \rfloor + \lfloor 5 \rfloor \to \phi, \tag{43}$$

where the numerical values are obtained for $q = \mathrm{e}^{\frac{2\pi \mathrm{i}}{5}}$ and $\phi = \frac{1+\sqrt{5}}{2}$. In particular, the total quantum dimension $D^2 = \sum_a d_a^2 = 3(2+\phi)$.

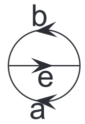
7.2. Frobenius–Schur indicator

The phase or, in our gauge choice, the sign of the F-symbol $\left[F^{a,\bar{a},a}_{a}\right]_{\mathbf{1},\mathbf{1}}$, is called the Frobenius–Schur indicator $\mathrm{fb}_a = d_a \left[F^{a,\bar{a},a}_{a}\right]_{\mathbf{1},\mathbf{1}}$. In the case of a self-dual particle, $\bar{a} = a$, this is a gauge invariant quantity, which takes the values ± 1. In the case of non-self dual particles, the Frobenius–Schur indicator is often defined to be zero, but we will not do this here and we simply keep the definition above. Concretely, this means we have $\mathrm{fb}_{\mathbf{1}} = \mathrm{fb}_{\mathbf{3}} = \mathrm{fb}_{\bar{\mathbf{3}}} = \mathrm{fb}_{\mathbf{6}} = \mathrm{fb}_{\bar{\mathbf{6}}} = \mathrm{fb}_{\mathbf{8}} = 1$.

In section 7.4, we give an expression for the Frobenius–Schur indicator of the self-dual particles (of modular theories) in terms of the so-called twist factors.

7.3. Theta symbols

In this section, we give the value of the 'theta' symbols. These depend on our choice of gauge. By the 'theta' symbols, we mean the following diagrams:



We denote the value of the theta symbols by $\vartheta(a,b,e)$. These symbols can easily be evaluated, by applying the appropriate F-symbol, and by noting that tadpoles give zero contributions:

$$\vartheta(a,b,e) = \left[F_a^{a,b,\bar{b}}\right]_{e,\mathbf{1}} d_a d_b. \tag{44}$$

Our choice of gauge implies that $\left[F_a^{a,b,\bar{b}}\right]_{e,\mathbf{1}} = \pm\sqrt{\frac{d_e}{d_a d_b}}$, so we find that

$$\vartheta(a,b,e) = \pm\sqrt{d_a d_b d_e}, \tag{45}$$

where the sign is just the sign of the F-symbol $\left[F_a^{a,b,\bar{b}}\right]_{e,\mathbf{1}}$. The theta symbols have the following symmetries:

$$\vartheta(a,b,e) = \vartheta(\bar{a},\bar{b},\bar{e}) \qquad \vartheta(a,b,e) = \vartheta(b,a,e) \qquad \vartheta(a,b,e) = \vartheta(\bar{e},b,\bar{a}). \tag{46}$$

Thus, by specifying the following values, all the theta symbols for $su(3)_2$ are determined:

$$\begin{aligned} &\vartheta(\mathbf{3},\bar{\mathbf{3}},\mathbf{3}) = -d_{\mathbf{3}}^{\frac{3}{2}} \qquad \vartheta(\bar{\mathbf{3}},\mathbf{3},\mathbf{6}) = d_{\mathbf{3}}\sqrt{d_{\mathbf{6}}} \\ &\vartheta(\mathbf{3},\mathbf{3},\mathbf{8}) = d_{\mathbf{3}}\sqrt{d_{\mathbf{8}}} \qquad \vartheta(\mathbf{3},\bar{\mathbf{6}},\mathbf{8}) = -\sqrt{d_{\mathbf{3}}d_{\mathbf{6}}d_{\mathbf{8}}} \end{aligned} \tag{47}$$

$$\vartheta(\mathbf{6},\bar{\mathbf{6}},\mathbf{6}) = d_{\mathbf{6}}^{\frac{3}{2}} \qquad \vartheta(\mathbf{8},\mathbf{8},\mathbf{8}) = d_{\mathbf{8}}^{\frac{3}{2}}. \tag{48}$$

Note that if any of the labels is the vacuum representation, the theta symbol is just the quantum dimension of the remaining representation.

7.4. Twist factors

The twist factors or topological spin of a particle a is denoted by θ_a. In general, the twist factors are given by

$$\theta_a = \theta_{\bar{a}} = \mathrm{fb}_a \left(R_{\mathbf{1}}^{\bar{a},a}\right)^*. \tag{49}$$

In those cases where the TQFTs are associated with conformal field theories, θ_a is also given by $\theta_a = \mathrm{e}^{2\pi \mathrm{i} h_a}$, where h_a is the scaling dimension of the corresponding primary field in the CFT. Again, taking $su(3)_k$ as an example,

$$\theta_{\mathbf{1}} = 1 \qquad \theta_{\mathbf{3}} = \theta_{\bar{\mathbf{3}}} = \mathrm{e}^{2\pi\mathrm{i}h_3} = \mathrm{e}^{\frac{8\pi\mathrm{i}}{3(k+3)}} = q^{4/3} \tag{50}$$

$$\theta_{\mathbf{6}} = \theta_{\bar{\mathbf{6}}} = \mathrm{e}^{2\pi\mathrm{i}h_6} = \mathrm{e}^{\frac{20\pi\mathrm{i}}{3(k+3)}} = q^{10/3} \qquad \theta_{\mathbf{8}} = \mathrm{e}^{2\pi\mathrm{i}h_8} = \mathrm{e}^{\frac{6\pi\mathrm{i}}{(k+3)}} = q^3. \tag{51}$$

In addition, one can express the Frobenius–Schur indicator for self-dual particles in terms of the fusion coefficients and twist factors as follows:

$$\mathrm{fb}_c = \frac{1}{D^2}\sum_{a,b} n_{a,b}{}^c \left(\frac{\theta_a}{\theta_b}\right) d_a d_b, \tag{52}$$

where the summation variables a and b run over all particle types, and $n_{a,b}^c$ denote the fusion coefficient. In particular, we find that both $\mathrm{fb}_{\mathbf{1}} = \mathrm{fb}_8 = 1$ for $su(3)_2$.

7.5. Central charge

The central charge of the theory can be determined (modulo 8) by using the result

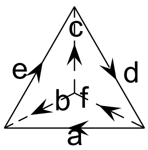
$$e^{\frac{2\pi i c}{8}} = \frac{1}{D} \sum_a d_a^2 \theta_a, \tag{53}$$

which leads to the expected result $c = \frac{16}{5}$ for $su(3)_2$.

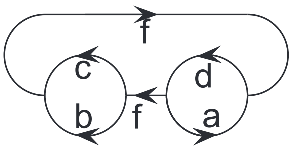
7.6. The tetrahedral symbol

Finally, we mention a relation between the F-symbols and the so-called tetrahedral symbols. The tetrahedral symbols are related to the F-symbols one-to-one, and are proportional the the F-symbols. The advantage of the tetrahedral symbols is that they satisfy more symmetry properties. In actual calculations, however, one always needs the F-symbols, so will tabulate those instead in the appendices.

The tetrahedral symbol is represented as



To calculate the value of the tetrahedral symbols, one applies the F-symbol $\left[F_d^{a,b,c}\right]_{e,f'}$. The only non-vanishing term in the sum (2) is the one with $f' = f$, and the resulting graph is



This graph is easily found to give $\left[F_f^{f,\bar{f},f}\right]_{\mathbf{1},\mathbf{1}} \vartheta(a,d,f)\vartheta(b,c,f)$. Using $\left|\left[F_f^{f,\bar{f},f}\right]_{\mathbf{1},\mathbf{1}}\right| = \frac{1}{d_f}$ and equation (45) for the theta symbols, we find that

$$G(a,b,c,d,e,f) = \operatorname{sgn}(\vartheta(a,d,f))\operatorname{sgn}(\vartheta(b,c,f))\varkappa_f \left[F_d^{a,b,c}\right]_{e,f} \sqrt{d_a d_b d_c d_d}, \tag{54}$$

where $\operatorname{sgn}(\vartheta(a,b,c)) = \frac{\vartheta(a,b,c)}{|\vartheta(a,b,c)|} = \operatorname{sgn}\left(\left[F_a^{a,b,\bar{b}}\right]_{e,\mathbf{1}}\right)$.

As we stated before, the tetrahedral symbols have more symmetry than the F-symbols. This symmetry arises from the symmetry of the tetrahedron. Thus, the following relations generate the full symmetry. Note that we have to take the conjugate representations in those cases where the arrows are reversed before and after 'rotation' of the tetrahedron:

$$\begin{aligned} G(a,b,c,d,e,f) &= G(d,\bar{f},b,e,a,\bar{c}) = G(e,c,\bar{f},a,d,\bar{b}) \\ G(a,b,c,d,e,f) &= G(c,\bar{d},a,\bar{b},\bar{f},\bar{e}) \\ G(a,b,c,d,e,f) &= G(\bar{c},\bar{b},\bar{a},\bar{d},\bar{f},\bar{e}). \end{aligned} \tag{55}$$

8. Applications, discussion and outlook

The topological data calculated in this paper can be applied to calculate various properties of physical systems. Particularly interesting examples of such systems occur in the context of the fractional quantum Hall effect. It has been proposed that (some of) the topological properties of fractional quantum Hall states can be measured by means of interference experiments, see [15] for the Moore–Read state [16] case, [17] for the Read–Rezayi states [18] and [19] for an account of the general case. Recent and promising experiments in this area are described in [20–22].

Of course, apart from what may be experimentally measurable in the immediate future, it is also of interest to determine the full representation of the braid group which governs the exchange properties of the non-Abelian anyons in these Hall states, not in the least because a knowledge of these braid group representations or indeed of the full TQFTs describing these states is needed in the design of gates and algorithms for future topological quantum computers based on these states. In [23], the braiding of quasiparticles in the the Read–Rezayi states was investigated using quantum groups and we hope the present paper makes it obvious how similar calculations can be done for more complicated systems. Braiding properties can also be deduced by considering the full CFT correlation functions describing the non-Abelian excitations, as was done in [24] for the Moore–Read state. The $U_q(su(3))$ coefficients calculated in this paper match up with CFT-correlator calculations in [25], where the $su(3)_k$-based spin-singlet state of Ardonne and Schoutens [26] was considered, as well as with the Read–Rezayi states.

An interesting recent development is the study of anyonic quantum spin chains [27, 28]. Non-Abelian anyons are the basic constituents of these chains, and the F-symbols are essential in the construction of the Hamiltonian. For more details, we refer to [29].

On a more formal level, the coefficients calculated can be used to study relations between different anyon models, such as those based on coset-theories. For instance, from the explicit calculation of the F-symbols of the Z_3-parafermion theory in [30], where the pentagon equations were solved directly, we know that these symbols are equivalent to those calculated here for $su(3)_2$.

A potentially interesting mathematical problem would be to calculate q-deformed Clebsch–Gordan and $6j$-coefficients for the full set of irreducible representations of $U_q(sl(3))$ at all levels. The multiplicities could be dealt with by the method of threshold levels discussed in section 5.2. This should also yield a particularly interesting set of bases for the CG-coefficients and $6j$-symbols of $SU(3)$ itself in the limit where q approaches 1.

Acknowledgments

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Appendix A. The case $su(3)$

In this appendix, we give the topological data, q-Clebsch–Gordan coefficients and F-symbols in the case of $su(3)_2$. However, we first summarize the data for the finite-dimensional Lie-algebra $su(3)$. For much more detail on the theory of finite (and infinite)-dimensional Lie algebras, see for instance [11–13].

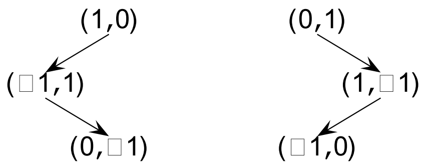

Figure A1. The weights of the $su(3)$ representations **3** and $\overline{\mathbf{3}}$.

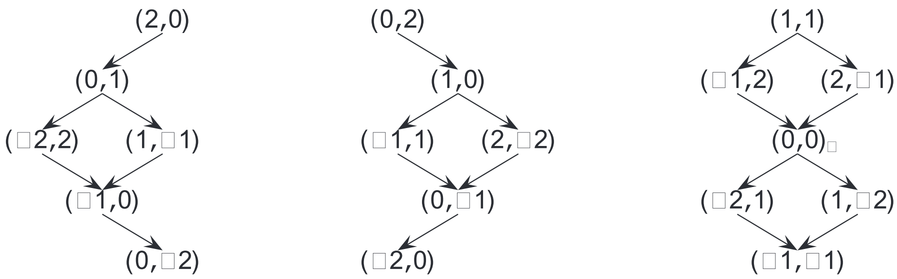

Figure A2. The weights of the $su(3)$ representations **6**, $\overline{\mathbf{6}}$ and **8**.

The q-Clebsch–Gordan coefficients, i.e.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q,$$

are tabulated in the following way. First, the various sections are labeled by the explicit fusion rule $j_1 \times j_2$. In this section, one finds the coefficients of all the possible fusion outcomes j_3. The coefficients for a particular j_3 are given in a table, whose rows are labeled by m_1, and the columns by m_2. In the case that there are no weight space multiplicities, this uniquely specifies $m_3 = m_1 + m_2$. In the case of the eight-dimensional representation of $su(3)$, **8**, the weight $(0, 0)$ corresponds to a two-dimensional space, whose basis states we denote by $(0, 0)_+$ and $(0, 0)_-$. Often we can compactly combine the symbols for $m_3 = (0, 0)_\pm$. In cases where this is not easily possible, the q-CG coefficients with $m_3 = (0, 0)_+$ and $m_3 = (0, 0)_-$ are given in separate tables, while the locations in the original table for which $m_1 + m_2 = (0, 0)$ are marked by X.

A.1. Generalities on su(3)

The Cartan matrix of the simply laced Lie algebra $su(3)$ and its inverse read

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad A^{-1} = F_{\mathrm{qf}} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{A.1}$$

The representations relevant for the $su(3)_2$ theory are $\mathbf{1} = (0, 0)$, $\mathbf{3} = (1, 0)$, $\overline{\mathbf{3}} = (0, 1)$, $\mathbf{6} = (2, 0)$, $\overline{\mathbf{6}} = (0, 2)$ and $\mathbf{8} = (1, 1)$. In figures A1 and A2, we give the structure of these representations for completeness. Here and below, arrows pointing 'to the left' correspond to the subtraction of α_1, while arrows pointing 'to the right' correspond to the subtraction of α_2.

Table A1. The fusion rules of $su(3)_2$.

$\times$	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\mathbf{6}$	$\bar{\mathbf{6}}$	$\mathbf{8}$
$\mathbf{3}$	$\bar{\mathbf{3}}+\mathbf{6}$				
$\bar{\mathbf{3}}$	$\mathbf{1}+\mathbf{8}$	$\mathbf{3}+\bar{\mathbf{6}}$			
$\mathbf{6}$	$\mathbf{8}$	$\mathbf{3}$	$\bar{\mathbf{6}}$		
$\bar{\mathbf{6}}$	$\bar{\mathbf{3}}$	$\mathbf{8}$	$\mathbf{1}$	$\mathbf{6}$	
$\mathbf{8}$	$\mathbf{3}+\bar{\mathbf{6}}$	$\bar{\mathbf{3}}+\mathbf{6}$	$\bar{\mathbf{3}}$	$\mathbf{3}$	$\mathbf{1}+\mathbf{8}$

A.2. The topological data of $su(3)_2$

In this section, we give the relevant topological data for $su(3)_2$. The fusion rules of the theory are given in table A1. The numerical values of the data are obtained from the general ones in terms of q by specifying $q = e^{2\pi i/5}$.

The quantum dimensions are given by

$$d_{\mathbf{1}} = 1 \qquad d_{\mathbf{3}} = d_{\bar{\mathbf{3}}} = \lfloor 3\rfloor \to \phi \qquad d_{\mathbf{6}} = d_{\bar{\mathbf{6}}} = 1 + \lfloor 5\rfloor \to 1 \qquad d_{\mathbf{8}} = \lfloor 3\rfloor + \lfloor 5\rfloor \to \phi, \tag{A.2}$$

which gives $D^2 = \sum_a d_a^2 = 3\phi + 6$. The twist factors are given by

$$\theta_{\mathbf{1}} = 1 \qquad \theta_{\mathbf{3}} = \theta_{\bar{\mathbf{3}}} = e^{2\pi i h_3} = e^{\frac{8\pi i}{3(k+3)}} = q^{4/3} \tag{A.3}$$

$$\theta_{\mathbf{6}} = \theta_{\bar{\mathbf{6}}} = e^{2\pi i h_6} = e^{\frac{20\pi i}{3(k+3)}} = q^{10/3} \qquad \theta_{\mathbf{8}} = e^{2\pi i h_8} = e^{\frac{6\pi i}{(k+3)}} = q^3. \tag{A.4}$$

The Frobenius–Schur indicators of the self-dual particles are $\mathrm{fb}_{\mathbf{1}} = \mathrm{fb}_{\mathbf{8}} = 1$ and finally, the central charge is $c = \frac{16}{5}$.

A.3. The q-CG coefficients relevant for $su(3)_2$

In this section, we give the q-CG coefficients in the case of $su(3)_2$.

A.3.1. $\mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6}$.

	(1, 0)	(−1, 1)	(0, −1)
(1, 0)		$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$
(−1, 1)	$-\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$		$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$
(0, −1)	$-\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	$-\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	

The q-CG coefficients for the $\bar{\mathbf{3}}$ in $\mathbf{3} \times \mathbf{3}$.

	(1, 0)	(−1, 1)	(0, −1)
(1, 0)	1	$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$
(−1, 1)	$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	1	$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$
(0, −1)	$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	1

The q-CG coefficients for the $\mathbf{6}$ in $\mathbf{3} \times \mathbf{3}$.

A.3.2. $\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8}$.

	(0, 1)	(1, −1)	(−1, 0)
(1, 0)			$\frac{q^{\frac{1}{2}}}{\sqrt{\lfloor 3\rfloor}}$
(−1, 1)		$-\frac{1}{\sqrt{\lfloor 3\rfloor}}$	
(0, −1)	$\frac{q^{-\frac{1}{2}}}{\sqrt{\lfloor 3\rfloor}}$		

The q-CG coefficients for the $\mathbf{1}$ in $\mathbf{3} \times \bar{\mathbf{3}}$.

	(0, 1)	(1, −1)	(−1, 0)
(1, 0)	1	1	$\frac{q^{-\frac{1}{4}}}{\sqrt{2(\lfloor 2\rfloor \pm 1)}}$
(−1, 1)	1	$\frac{q^{\frac{1}{4}} \pm q^{-\frac{1}{4}}}{\sqrt{2(\lfloor 2\rfloor \pm 1)}}$	1
(0, −1)	$\pm\frac{q^{\frac{1}{4}}}{\sqrt{2(\lfloor 2\rfloor \pm 1)}}$	1	1

The q-CG coefficients for the $\mathbf{8}$ in $\mathbf{3} \times \bar{\mathbf{3}}$.

A.3.3. $\mathbf{3} \times \mathbf{6} = \mathbf{8}$.

	(2, 0)	(0, 1)	(−2, 2)	(1, −1)	(−1, 0)	(0, −2)
(1, 0)		$\frac{q^{\frac{1}{2}}}{\sqrt{\lfloor 3\rfloor}}$	$q^{\frac{1}{4}}\sqrt{\frac{\lfloor 2\rfloor}{\lfloor 3\rfloor}}$	$\frac{q^{\frac{1}{2}}}{\sqrt{\lfloor 3\rfloor}}$	$\pm q^{\frac{1}{4}}\sqrt{\frac{\lfloor 2\rfloor\pm 1}{2\lfloor 3\rfloor}}$	$q^{\frac{1}{4}}\sqrt{\frac{\lfloor 2\rfloor}{\lfloor 3\rfloor}}$
(−1, 1)	$-q^{-\frac{1}{4}}\sqrt{\frac{\lfloor 2\rfloor}{\lfloor 3\rfloor}}$	$-\frac{q^{-\frac{1}{2}}}{\sqrt{\lfloor 3\rfloor}}$		$\frac{q^{\frac{3}{4}}\mp q^{-\frac{3}{4}}}{\sqrt{2(\lfloor 2\rfloor\pm 1)\lfloor 3\rfloor}}$	$\frac{q^{\frac{1}{2}}}{\sqrt{\lfloor 3\rfloor}}$	$q^{\frac{1}{4}}\sqrt{\frac{\lfloor 2\rfloor}{\lfloor 3\rfloor}}$
(0, −1)	$-q^{-\frac{1}{4}}\sqrt{\frac{\lfloor 2\rfloor}{\lfloor 3\rfloor}}$	$-q^{-\frac{1}{4}}\sqrt{\frac{\lfloor 2\rfloor\pm 1}{2\lfloor 3\rfloor}}$	$-q^{-\frac{1}{4}}\sqrt{\frac{\lfloor 2\rfloor}{\lfloor 3\rfloor}}$	$-\frac{q^{-\frac{1}{2}}}{\sqrt{\lfloor 3\rfloor}}$	$-\frac{q^{-\frac{1}{2}}}{\sqrt{\lfloor 3\rfloor}}$	

The q-CG coefficients for the **8** in $\mathbf{3} \times \mathbf{6}$.

A.3.4. $\mathbf{3} \times \bar{\mathbf{6}} = \bar{\mathbf{3}}$.

	(0, 2)	(1, 0)	(−1, 1)	(2, −2)	(0, −1)	(−2, 0)
(1, 0)			$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$		$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$	$q^{\frac{1}{2}}\sqrt{\frac{\lfloor 2\rfloor}{\lfloor 4\rfloor}}$
(−1, 1)		$-\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor}}$		$-\sqrt{\frac{\lfloor 2\rfloor}{\lfloor 4\rfloor}}$	$-\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor}}$	
(0, −1)	$q^{-\frac{1}{2}}\sqrt{\frac{\lfloor 2\rfloor}{\lfloor 4\rfloor}}$	$\frac{q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$	$\frac{q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$			

The q-CG coefficients for the $\bar{\mathbf{3}}$ in $\mathbf{3} \times \bar{\mathbf{6}}$.

A.3.5. $\mathbf{3} \times \mathbf{8} = \mathbf{3} + \bar{\mathbf{6}}$.

	(1, 1)	(−1, 2)	(2, −1)	$(0, 0)_\pm$	(−2, 1)	(1, −2)	(−1, −1)
(1, 0)				$\frac{q^{\frac{3}{4}}\sqrt{\lfloor 2\rfloor\mp 1}}{\sqrt{2\lfloor 2\rfloor\lfloor 4\rfloor}}$	$\frac{q^{\frac{1}{2}}\sqrt{\lfloor 3\rfloor}}{\sqrt{\lfloor 2\rfloor\lfloor 4\rfloor}}$		$\frac{q^{\frac{1}{2}}\sqrt{\lfloor 3\rfloor}}{\sqrt{\lfloor 2\rfloor\lfloor 4\rfloor}}$
(−1, 1)			$-\frac{\sqrt{\lfloor 3\rfloor}}{\sqrt{\lfloor 2\rfloor\lfloor 4\rfloor}}$	$\frac{(\mp q^{\frac{1}{4}}-q^{-\frac{1}{4}})\sqrt{\lfloor 2\rfloor\mp 1}}{\sqrt{2\lfloor 2\rfloor\lfloor 4\rfloor}}$		$-\frac{\sqrt{\lfloor 3\rfloor}}{\sqrt{\lfloor 2\rfloor\lfloor 4\rfloor}}$	
(0, −1)	$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 3\rfloor}}{\sqrt{\lfloor 2\rfloor\lfloor 4\rfloor}}$	$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 3\rfloor}}{\sqrt{\lfloor 2\rfloor\lfloor 4\rfloor}}$		$\pm\frac{q^{-\frac{3}{4}}\sqrt{\lfloor 2\rfloor\mp 1}}{\sqrt{2\lfloor 2\rfloor\lfloor 4\rfloor}}$			

The q-CG coefficients for the **3** in $\mathbf{3} \times \mathbf{8}$.

	(1, 1)	(−1, 2)	(2, −1)	$(0, 0)_\pm$	(−2, 1)	(1, −2)	(−1, −1)
(1, 0)		$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$		$\pm\frac{q^{\frac{1}{4}}\sqrt{\lfloor 2\rfloor\pm 1}}{\sqrt{2}\lfloor 2\rfloor}$	$\frac{1}{\lfloor 2\rfloor}$	$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	$\frac{1}{\lfloor 2\rfloor}$
(−1, 1)	$-\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$		$-\frac{q^{-\frac{1}{2}}}{\lfloor 2\rfloor}$	$\frac{(\pm q^{\frac{1}{4}}-q^{-\frac{1}{4}})\sqrt{\lfloor 2\rfloor\pm 1}}{\sqrt{2}\lfloor 2\rfloor}$		$\frac{q^{\frac{1}{2}}}{\lfloor 2\rfloor}$	$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$
(0, −1)	$-\frac{1}{\lfloor 2\rfloor}$	$-\frac{1}{\lfloor 2\rfloor}$	$-\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	$-\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 2\rfloor\pm 1}}{\sqrt{2}\lfloor 2\rfloor}$	$-\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$		

The q-CG coefficients for the $\bar{\mathbf{6}}$ in $\mathbf{3} \times \mathbf{8}$.

A.3.6. $\mathbf{6} \times \mathbf{6} = \bar{\mathbf{6}}$.

	(2, 0)	(0, 1)	(−2, 2)	(1, −1)	(−1, 0)	(0, −2)
(2, 0)			$\frac{q^{\frac{1}{2}}}{\sqrt{[3]}}$		$\frac{q^{\frac{1}{2}}}{\sqrt{[3]}}$	$\frac{q^{\frac{1}{2}}}{\sqrt{[3]}}$
(0, 1)		$-\frac{1}{\sqrt{[3]}}$		$-\frac{q^{-\frac{1}{4}}}{\sqrt{[2][3]}}$	$\frac{q^{\frac{3}{4}}}{\sqrt{[2][3]}}$	$\frac{q^{\frac{1}{2}}}{\sqrt{[3]}}$
(−2, 2)	$\frac{q^{-\frac{1}{2}}}{\sqrt{[3]}}$			$-\frac{1}{\sqrt{[3]}}$		$\frac{q^{\frac{1}{2}}}{\sqrt{[3]}}$
(1, −1)		$-\frac{q^{\frac{1}{4}}}{\sqrt{[2][3]}}$	$-\frac{1}{\sqrt{[3]}}$	$-\frac{1}{\sqrt{[3]}}$	$-\frac{q^{-\frac{1}{4}}}{\sqrt{[2][3]}}$	
(−1, 0)	$\frac{q^{-\frac{1}{2}}}{\sqrt{[3]}}$	$\frac{q^{-\frac{3}{4}}}{\sqrt{[2][3]}}$		$-\frac{q^{\frac{1}{4}}}{\sqrt{[2][3]}}$	$-\frac{1}{\sqrt{[3]}}$	
(0, −2)	$\frac{q^{-\frac{1}{2}}}{\sqrt{[3]}}$	$\frac{q^{-\frac{1}{2}}}{\sqrt{[3]}}$	$\frac{q^{-\frac{1}{2}}}{\sqrt{[3]}}$			

The q-CG coefficients for the $\bar{\mathbf{6}}$ in $\mathbf{6} \times \mathbf{6}$.

A.3.7. $\mathbf{6} \times \bar{\mathbf{6}} = \mathbf{1}$.

	(0, 2)	(1, 0)	(−1, 1)	(2, −2)	(0, −1)	(−2, 0)
(2, 0)						$\frac{q}{\sqrt{[5]+1}}$
(0, 1)					$-\frac{q^{\frac{1}{2}}}{\sqrt{[5]+1}}$	
(−2, 2)				$\frac{1}{\sqrt{[5]+1}}$		
(1, −1)			$\frac{1}{\sqrt{[5]+1}}$			
(−1, 0)		$-\frac{q^{-\frac{1}{2}}}{\sqrt{[5]+1}}$				
(0, −2)	$\frac{q^{-1}}{\sqrt{[5]+1}}$					

The q-CG coefficients for the $\mathbf{1}$ in $\mathbf{6} \times \bar{\mathbf{6}}$.

A.3.8. $\mathbf{6} \times \mathbf{8} = \bar{\mathbf{3}}$.

	(1, 1)	(−1, 2)	(2, −1)	$(0, 0)_{\pm}$	(−2, 1)	(1, −2)	(−1, −1)
(2, 0)					$\frac{q^{\frac{3}{4}}}{\sqrt{[4]}}$		$\frac{q^{\frac{3}{4}}}{\sqrt{[4]}}$
(0, 1)				$-\frac{q^{\frac{1}{4}}\sqrt{[2]\pm 1}}{\sqrt{2[2][4]}}$		$-\frac{1}{\sqrt{[2][4]}}$	$\frac{q}{\sqrt{[2][4]}}$
(−2, 2)			$\frac{q^{-\frac{1}{4}}}{\sqrt{[4]}}$			$-\frac{q^{\frac{1}{4}}}{\sqrt{[4]}}$	
(1, −1)		$\frac{q^{-\frac{1}{2}}}{\sqrt{[2][4]}}$		$\frac{(-q^{\frac{1}{4}} \pm q^{-\frac{1}{4}})\sqrt{[2]\pm 1}}{\sqrt{2[2][4]}}$	$-\frac{q^{\frac{1}{2}}}{\sqrt{[2][4]}}$		
(−1, 0)	$-\frac{q^{-1}}{\sqrt{[2][4]}}$		$\frac{1}{\sqrt{[2][4]}}$	$\pm\frac{q^{-\frac{1}{4}}\sqrt{[2]\pm 1}}{\sqrt{2[2][4]}}$			
(0, −2)	$-\frac{q^{-\frac{3}{4}}}{\sqrt{[4]}}$	$-\frac{q^{-\frac{3}{4}}}{\sqrt{[4]}}$					

The q-CG coefficients for the $\bar{\mathbf{3}}$ in $\mathbf{6} \times \mathbf{8}$.

A.3.9. $\mathbf{8} \times \mathbf{8} = \mathbf{1} + \mathbf{8}$.

	(1, 1)	(−1, 2)	(2, −1)	$(0,0)_+$	$(0,0)_-$	(−2, 1)	(1, −2)	(−1, −1)
(1, 1)				$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor+1}}$	0	$\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2(\lfloor 4\rfloor+1)}}$	$\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2(\lfloor 4\rfloor+1)}}$	X
(−1, 2)			$-\frac{\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2(\lfloor 4\rfloor+1)}}$	$\frac{-q^{-\frac{1}{4}}-q^{\frac{1}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$	$-\frac{q^{\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor+1}}$		X	$\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2(\lfloor 4\rfloor+1)}}$
(2, −1)		$-\frac{\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2(\lfloor 4\rfloor+1)}}$		$\frac{-q^{-\frac{1}{4}}-q^{\frac{1}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$	$\frac{q^{\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor+1}}$	X		$\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2(\lfloor 4\rfloor+1)}}$
$(0,0)_+$	$\frac{q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor+1}}$	$\frac{-q^{-\frac{1}{4}}-q^{\frac{1}{4}}+q^{-\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$	$\frac{-q^{-\frac{1}{4}}-q^{\frac{1}{4}}+q^{-\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$	X	X	$\frac{-q^{-\frac{1}{4}}-q^{\frac{1}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$	$\frac{-q^{-\frac{1}{4}}-q^{\frac{1}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$	$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor+1}}$
$(0,0)_-$	0	$-\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor+1}}$	$\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor+1}}$	X	X	$\frac{q^{\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor+1}}$	$-\frac{q^{\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor+1}}$	0
(−2, 1)	$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2(\lfloor 4\rfloor+1)}}$		X	$\frac{-q^{-\frac{1}{4}}-q^{\frac{1}{4}}+q^{-\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$	$\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor+1}}$		$-\frac{\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2(\lfloor 4\rfloor+1)}}$	
(1, −2)	$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2(\lfloor 4\rfloor+1)}}$	X		$\frac{-q^{-\frac{1}{4}}-q^{\frac{1}{4}}+q^{-\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$	$-\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor+1}}$	$-\frac{\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2(\lfloor 4\rfloor+1)}}$		
(−1, −1)	X	$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2(\lfloor 4\rfloor+1)}}$	$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2(\lfloor 4\rfloor+1)}}$	$\frac{q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor+1}}$	0			

The q-CG coefficients for the **8** (when $m \neq (0,0)_\pm$) in $\mathbf{8} \times \mathbf{8}$.

	(1, 1)	(−1, 2)	(2, −1)	$(0,0)_+$	$(0,0)_-$	(−2, 1)	(1, −2)	(−1, −1)
(1, 1)								$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor+1}}$
(−1, 2)							$\frac{-q^{-\frac{1}{4}}+q^{\frac{1}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$	
(2, −1)						$\frac{-q^{-\frac{1}{4}}+q^{\frac{1}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$		
$(0,0)_+$				$\frac{-2(q^{-\frac{1}{4}}+q^{\frac{1}{4}})+q^{\frac{3}{4}}+q^{-\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$	0			
$(0,0)_-$				0	$\frac{q^{\frac{3}{4}}+q^{-\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$			
(−2, 1)			$\frac{q^{-\frac{1}{4}}-q^{\frac{1}{4}}+q^{-\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$					
(1, −2)		$\frac{q^{-\frac{1}{4}}-q^{\frac{1}{4}}+q^{-\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$						
(−1, −1)	$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor+1}}$							

The q-CG coefficients for the **8** (for $m = (0,0)_+$) in $\mathbf{8} \times \mathbf{8}$.

	(1, 1)	(−1, 2)	(2, −1)	$(0,0)_+$	$(0,0)_-$	(−2, 1)	(1, −2)	(−1, −1)
(1, 1)								0
(−1, 2)							$\frac{q^{\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor+1}}$	
(2, −1)						$-\frac{q^{\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor+1}}$		
$(0,0)_+$				0	$\frac{q^{\frac{3}{4}}+q^{-\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$			
$(0,0)_-$				$\frac{q^{\frac{3}{4}}+q^{-\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor+1}}$	0			
(−2, 1)			$-\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor+1}}$					
(1, −2)		$\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor+1}}$						
(−1, −1)	0							

The q-CG coefficients for the **8** (for $m = (0,0)_-$) in $\mathbf{8} \times \mathbf{8}$.

Table A2. The parameters s_1, s_2 and s_3 in the symmetry relations between the q-CG coefficients for $su(3)_2$, as explained in section 5.4.

j_1	j_2	j	s_1	s_2	s_3	j_1	j_2	j	s_1	s_2	s_3
3	**3**	$\bar{\mathbf{3}}$	1	1	0	**6**	**6**	$\bar{\mathbf{6}}$	0	0	0
3	**3**	**6**	0	0	0	**6**	$\bar{\mathbf{6}}$	**1**	0	0	0
3	$\bar{\mathbf{3}}$	**1**	0	0	0	**6**	**8**	$\bar{\mathbf{3}}$	1	1	0
3	$\bar{\mathbf{3}}$	**8**	0	0	0	**8**	**8**	**1**	0	0	0
3	**6**	**8**	1	1	0	**8**	**8**	**8**	0	0	0
3	$\bar{\mathbf{6}}$	$\bar{\mathbf{3}}$	0	0	0						
3	**8**	**3**	0	0	0						
3	**8**	$\bar{\mathbf{6}}$	1	1	0						

	(1, 1)	(−1, 2)	(2, −1)	(0, 0)$_+$	(0, 0)$_-$	(−2, 1)	(1, −2)	(−1, −1)
(1, 1)								$\frac{q}{\sqrt{[2][4]}}$
(−1, 2)							$-\frac{q^{\frac{1}{2}}}{\sqrt{[2][4]}}$	
(2, −1)						$-\frac{q^{\frac{1}{2}}}{\sqrt{[2][4]}}$		
(0, 0)$_+$				$\frac{1}{\sqrt{[2][4]}}$	0			
(0, 0)$_-$				0	$\frac{1}{\sqrt{[2][4]}}$			
(−2, 1)			$-\frac{q^{-\frac{1}{2}}}{\sqrt{[2][4]}}$					
(1, −2)		$-\frac{q^{-\frac{1}{2}}}{\sqrt{[2][4]}}$						
(−1, −1)	$\frac{q^{-1}}{\sqrt{[2][4]}}$							

The q-CG coefficients for the **1** in **8** × **8**.

A.3.10. The relation between the q-CG coefficients. In table A2, we specify the coefficients s_1, s_2 and s_3, which appear in the symmetry relations between the various q-CG coefficients as explained in section 5.4.

A.4. The F-symbols for $su(3)_2$

In total, there are 405 F-symbols. Out of these, there are 147 symbols which have a **1** on at least one of the outer lines. In the gauge we chose, all these symbols are equal to 1. Moreover, because the F-symbols are invariant under the operation of taking the conjugate representation of all six indices, namely $\left[F_d^{a,b,c}\right]_{e,f} = \left[F_{\bar{d}}^{\bar{a},\bar{b},\bar{c}}\right]_{\bar{e},\bar{f}}$, we only list those symbols which either have $a = \mathbf{3}$, $a = \mathbf{6}$ or $a = \mathbf{8}$.

All those symbols which have at least one **6** or $\bar{\mathbf{6}}$ on the outer lines correspond to a one-dimensional transformation and reduce to ±1 for $q = \mathrm{e}^{2\pi \mathrm{i}/5}$, but we list these symbols for general q. It turns out that there are only ten independent values:

$$\begin{aligned} F_{\bar{\mathbf{6}}}^{\mathbf{3},\mathbf{3},\bar{\mathbf{3}}} &= F_{\mathbf{3}}^{\mathbf{3},\mathbf{3},\mathbf{6}} = F_{\bar{\mathbf{6}}}^{\mathbf{3},\mathbf{3},\mathbf{6}} = F_{\bar{\mathbf{6}}}^{\mathbf{3},\bar{\mathbf{3}},\mathbf{3}} = F_{\mathbf{6}}^{\mathbf{3},\bar{\mathbf{3}},\bar{\mathbf{3}}} = F_{\mathbf{3}}^{\mathbf{3},\bar{\mathbf{3}},\mathbf{6}} = F_{\mathbf{3}}^{\mathbf{3},\bar{\mathbf{3}},\bar{\mathbf{6}}} = F_{\mathbf{3}}^{\mathbf{3},\mathbf{6},\mathbf{3}} = F_{\bar{\mathbf{6}}}^{\mathbf{3},\mathbf{6},\mathbf{3}} = F_{\bar{\mathbf{3}}}^{\mathbf{3},\mathbf{6},\bar{\mathbf{3}}} \\ &= F_{\mathbf{3}}^{\mathbf{3},\mathbf{6},\mathbf{6}} = F_{\mathbf{3}}^{\mathbf{3},\bar{\mathbf{6}},\bar{\mathbf{3}}} = F_{\mathbf{3}}^{\mathbf{6},\mathbf{3},\mathbf{3}} = F_{\mathbf{6}}^{\mathbf{6},\mathbf{3},\mathbf{3}} = F_{\mathbf{3}}^{\mathbf{6},\mathbf{3},\bar{\mathbf{3}}} = F_{\bar{\mathbf{3}}}^{\mathbf{6},\mathbf{3},\mathbf{6}} = F_{\bar{\mathbf{3}}}^{\mathbf{6},\bar{\mathbf{3}},\mathbf{3}} = F_{\bar{\mathbf{3}}}^{\mathbf{6},\mathbf{6},\mathbf{3}} = 1 \end{aligned} \qquad \text{(A.5)}$$

$$F_{8}^{3,3,\bar{6}} = F_{6}^{3,3,8} = F_{8}^{3,\bar{6},\bar{6}} = F_{\bar{3}}^{3,\bar{6},8} = F_{\bar{3}}^{3,8,\bar{6}} = F_{6}^{3,8,\bar{6}} = F_{8}^{6,\bar{3},\bar{3}} = F_{\bar{6}}^{6,\bar{3},8} = F_{8}^{6,6,\bar{3}}$$
$$= F_{3}^{6,6,8} = F_{3}^{6,8,\bar{3}} = F_{\bar{6}}^{6,8,\bar{3}} = F_{6}^{8,3,3} = F_{6}^{8,3,\bar{6}} = F_{\bar{6}}^{8,\bar{3},\bar{3}} = F_{\bar{6}}^{8,\bar{3},6} = F_{3}^{8,6,\bar{3}} = F_{3}^{8,6,6}$$
$$= F_{\bar{3}}^{8,\bar{6},3} = F_{\bar{3}}^{8,\bar{6},\bar{6}} = -\frac{\lfloor 3\rfloor}{\lfloor 2\rfloor\sqrt{\lfloor 5\rfloor + 1}} \to -1 \tag{A.6}$$

$$F_{6}^{3,\bar{3},6} = F_{3}^{3,\bar{6},6} = F_{6}^{6,\bar{3},3} = F_{6}^{6,6,\bar{6}} = F_{3}^{6,\bar{6},3} = F_{\bar{6}}^{6,\bar{6},\bar{6}} = \frac{1}{\sqrt{\lfloor 5\rfloor + 1}} \to 1 \tag{A.7}$$

$$F_{6}^{3,6,\bar{3}} = F_{\bar{6}}^{3,\bar{6},\bar{3}} = F_{3}^{6,3,\bar{6}} = F_{\bar{3}}^{6,\bar{3},\bar{6}} = -\frac{1}{\sqrt{\lfloor 5\rfloor + 1}} \to -1 \tag{A.8}$$

$$F_{\bar{6}}^{3,\bar{3},\bar{6}} = F_{3}^{3,6,\bar{6}} = F_{6}^{6,3,\bar{3}} = F_{\bar{3}}^{6,\bar{6},\bar{3}} = -\frac{\lfloor 2\rfloor}{\lfloor 3\rfloor} \to -1 \tag{A.9}$$

$$F_{8}^{3,6,8} = F_{\bar{6}}^{3,8,8} = F_{8}^{6,3,8} = F_{\bar{3}}^{6,8,8} = F_{8}^{8,3,6} = F_{8}^{8,\bar{3},\bar{6}} = F_{8}^{8,6,3} = F_{8}^{8,\bar{6},\bar{3}}$$
$$= F_{\bar{6}}^{8,8,3} = F_{6}^{8,8,\bar{3}} = F_{\bar{3}}^{8,8,6} = F_{3}^{8,8,\bar{6}} = -\frac{1}{\sqrt{2}\lfloor 2\rfloor}\sqrt{\frac{\lfloor 2\rfloor + \lfloor 3\rfloor}{\lfloor 3\rfloor - 1}} \to -1 \tag{A.10}$$

$$F_{8}^{3,\bar{6},3} = F_{6}^{3,\bar{6},8} = F_{6}^{3,8,3} = F_{8}^{6,\bar{3},6} = F_{3}^{6,\bar{3},8} = F_{3}^{6,8,6} = F_{\bar{3}}^{8,3,\bar{6}} = F_{3}^{8,\bar{3},6} = F_{\bar{6}}^{8,6,\bar{3}}$$
$$= F_{6}^{8,\bar{6},3} = \frac{1}{\lfloor 4\rfloor} \to 1 \tag{A.11}$$

$$F_{8}^{6,\bar{6},8} = F_{6}^{6,8,8} = F_{8}^{8,6,\bar{6}} = F_{8}^{8,\bar{6},6} = F_{6}^{8,8,6} = F_{\bar{6}}^{8,8,\bar{6}} = -\frac{1}{\lfloor 4\rfloor} \to -1 \tag{A.12}$$

$$F_{8}^{3,8,6} = F_{8}^{6,8,3} = F_{8}^{6,8,\bar{6}} = F_{\bar{6}}^{8,3,8} = F_{6}^{8,\bar{3},8}$$
$$= F_{\bar{3}}^{8,6,8} = F_{6}^{8,6,8} = F_{3}^{8,\bar{6},8} = F_{\bar{6}}^{8,\bar{6},8} = \frac{\lfloor 3\rfloor}{\lfloor 3\rfloor + \lfloor 5\rfloor} \to 1 \tag{A.13}$$

$$F_{6}^{6,\bar{6},6} = \frac{1}{\lfloor 5\rfloor + 1} \to 1. \tag{A.14}$$

The remaining 72 symbols correspond to 18 different labelings of the external lines. We give these 18 matrices explicitly below. Again, the internal labels e and f have to be inferred from the others, and the ordering in the matrices is always according to $\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}, \mathbf{6}, \bar{\mathbf{6}}, \mathbf{8}$:

$$F_{\bar{3}}^{3,\bar{3},\bar{3}} = F_{3}^{3,3,\bar{3}} = \begin{pmatrix} \frac{-1}{\sqrt{\lfloor 3\rfloor}} & \frac{\lfloor 3\rfloor-1}{\sqrt{\lfloor 5\rfloor+1}} \\ \frac{\lfloor 3\rfloor-1}{\sqrt{\lfloor 5\rfloor+1}} & \frac{1}{\sqrt{\lfloor 3\rfloor}} \end{pmatrix} \to \begin{pmatrix} -1/\sqrt{\phi} & 1/\phi \\ 1/\phi & 1/\sqrt{\phi} \end{pmatrix} \tag{A.15}$$

$$F_{3}^{3,\bar{3},3} = \begin{pmatrix} \frac{1}{\lfloor 3\rfloor} & \frac{\lfloor 2\rfloor\sqrt{\lfloor 3\rfloor-1}}{\lfloor 3\rfloor} \\ \frac{\lfloor 2\rfloor\sqrt{\lfloor 3\rfloor-1}}{\lfloor 3\rfloor} & \frac{-1}{\lfloor 3\rfloor} \end{pmatrix} \to \begin{pmatrix} 1/\phi & 1/\sqrt{\phi} \\ 1/\sqrt{\phi} & -1/\phi \end{pmatrix} \tag{A.16}$$

$$F_{8}^{3,\bar{3},8} = F_{\bar{3}}^{8,8,\bar{3}} = F_{3}^{8,8,3} = \begin{pmatrix} \frac{1}{\sqrt{\lfloor 3\rfloor+\lfloor 5\rfloor}} & \frac{-1}{\lfloor 2\rfloor} \\ \frac{\lfloor 4\rfloor+1}{\sqrt{2}\lfloor 2\rfloor\sqrt{\lfloor 5\rfloor+\lfloor 4\rfloor+1}} & \frac{\sqrt{\lfloor 2\rfloor+\lfloor 3\rfloor}}{\sqrt{2}\lfloor 2\rfloor} \end{pmatrix} \to \begin{pmatrix} 1/\sqrt{\phi} & -1/\phi \\ 1/\phi & 1/\sqrt{\phi} \end{pmatrix} \tag{A.17}$$

$$F_{3}^{3,8,8} = F_{8}^{8,\bar{3},3} = F_{8}^{8,3,\bar{3}} = \begin{pmatrix} \frac{1}{\sqrt{\lfloor 3\rfloor+\lfloor 5\rfloor}} & \frac{\lfloor 4\rfloor+1}{\sqrt{2}\lfloor 2\rfloor\sqrt{\lfloor 5\rfloor+\lfloor 4\rfloor+1}} \\ \frac{-1}{\lfloor 2\rfloor} & \frac{\sqrt{\lfloor 2\rfloor+\lfloor 3\rfloor}}{\sqrt{2}\lfloor 2\rfloor} \end{pmatrix} \to \begin{pmatrix} 1/\sqrt{\phi} & 1/\phi \\ -1/\phi & 1/\sqrt{\phi} \end{pmatrix} \tag{A.18}$$

$$F_{8}^{3,3,3} = F_{\bar{3}}^{3,3,8} = F_{\bar{3}}^{3,8,3} = F_{3}^{8,\bar{3},\bar{3}} = F_{\bar{3}}^{8,3,3} = \begin{pmatrix} \frac{-1}{\lfloor 2\rfloor} & \frac{\sqrt{\lfloor 3\rfloor}}{\lfloor 2\rfloor} \\ \frac{\sqrt{\lfloor 3\rfloor}}{\lfloor 2\rfloor} & \frac{1}{\lfloor 2\rfloor} \end{pmatrix} \to \begin{pmatrix} -1/\phi & 1/\sqrt{\phi} \\ 1/\sqrt{\phi} & 1/\phi \end{pmatrix} \tag{A.19}$$

$$F_{\mathbf{8}}^{\mathbf{3},\mathbf{8},\bar{\mathbf{3}}} = F_{\bar{\mathbf{3}}}^{\mathbf{8},\bar{\mathbf{3}},\mathbf{8}} = F_{\mathbf{3}}^{\mathbf{8},\mathbf{3},\mathbf{8}} = \begin{pmatrix} \frac{-1}{\lfloor 3\rfloor+\lfloor 5\rfloor} & -\sqrt{\frac{\lfloor 3\rfloor}{\lfloor 2\rfloor^2}\frac{\lfloor 3\rfloor}{\lfloor 5\rfloor+1}} \\ -\sqrt{\frac{\lfloor 3\rfloor}{\lfloor 2\rfloor^2}\frac{\lfloor 3\rfloor}{\lfloor 5\rfloor+1}} & \frac{\lfloor 3\rfloor}{\lfloor 2\rfloor^2} \end{pmatrix} \to \begin{pmatrix} -1/\phi & -1/\sqrt{\phi} \\ -1/\sqrt{\phi} & 1/\phi \end{pmatrix} \tag{A.20}$$

$$F_{\mathbf{8}}^{\mathbf{8},\mathbf{8},\mathbf{8}} = \begin{pmatrix} \frac{1}{\lfloor 3\rfloor+\lfloor 5\rfloor} & \frac{1}{\sqrt{\lfloor 3\rfloor+\lfloor 5\rfloor}} \\ \frac{1}{\sqrt{\lfloor 3\rfloor+\lfloor 5\rfloor}} & \frac{\lfloor 3\rfloor-\lfloor 2\rfloor-4}{2(\lfloor 2\rfloor+\lfloor 3\rfloor)} \end{pmatrix} \to \begin{pmatrix} 1/\phi & 1/\sqrt{\phi} \\ 1/\sqrt{\phi} & -1/\phi \end{pmatrix}. \tag{A.21}$$

A.5. The R-symbols for $su(3)_2$

We give the R-symbols below. We use the following symmetries to shorten the list

$$R_c^{a,b} = R_c^{b,a} \qquad R_c^{a,b} = R_{\bar{c}}^{\bar{a},\bar{b}}. \tag{A.22}$$

With our conventions, we have

$$R_a^{\mathbf{1},a} = 1 \tag{A.23}$$

$$R_{\mathbf{1}}^{\mathbf{3},\bar{\mathbf{3}}} = q^{-\frac{4}{3}} \qquad R_{\mathbf{1}}^{\mathbf{6},\bar{\mathbf{6}}} = q^{-\frac{10}{3}} \qquad R_{\mathbf{1}}^{\mathbf{8},\mathbf{8}} = q^{-3} \tag{A.24}$$

$$R_{\mathbf{3}}^{\bar{\mathbf{3}},\bar{\mathbf{3}}} = -q^{-\frac{2}{3}} \qquad R_{\mathbf{3}}^{\bar{\mathbf{3}},\mathbf{6}} = q^{-\frac{5}{3}} \qquad R_{\mathbf{3}}^{\mathbf{3},\mathbf{8}} = q^{-\frac{3}{2}} \qquad R_{\mathbf{3}}^{\bar{\mathbf{6}},\mathbf{8}} = -q^{-\frac{5}{2}} \tag{A.25}$$

$$R_{\mathbf{6}}^{\mathbf{3},\mathbf{3}} = q^{\frac{1}{3}} \qquad R_{\mathbf{6}}^{\bar{\mathbf{3}},\mathbf{8}} = -q^{-\frac{1}{2}} \qquad R_{\mathbf{6}}^{\bar{\mathbf{6}},\bar{\mathbf{6}}} = q^{-\frac{5}{3}} \tag{A.26}$$

$$R_{\mathbf{8}}^{\mathbf{3},\bar{\mathbf{3}}} = q^{\frac{1}{6}} \qquad R_{\mathbf{8}}^{\mathbf{3},\mathbf{6}} = -q^{\frac{5}{6}} \qquad R_{\mathbf{8}}^{\mathbf{8},\mathbf{8}} = q^{\frac{3}{2}}. \tag{A.27}$$

Appendix B. The case $su(3)_3/Z_3$

In this appendix, we give the topological data for the theory $su(3)_3/Z_3$, namely the q-CG coefficients, as well as the F and R symbols. This theory contains four particles, which we denote by **1**, **8**, **10** and $\overline{\mathbf{10}}$, where the last two correspond to the $su(3)$ representations (3, 0) and (0, 3), respectively; the weights of these representations are given in figure B1. We note that this theory is a 'sub-theory' of the full, modular $su(3)_3$ theory. However, the theory under consideration is not modular.

The fusion rules are given in table B1.

The quantum dimensions of the particles are given by $d_{\mathbf{1}} = 1$, $d_{\mathbf{8}} = \lfloor 3\rfloor + \lfloor 5\rfloor \to 3$, $d_{\mathbf{10}} = d_{\overline{\mathbf{10}}} = (\lfloor 3\rfloor - 1)\lfloor 5\rfloor \to 1$, where the numerical value is for $q = e^{2\pi i/6}$. The twist factors are given by $\theta_{\mathbf{1}} = 1$, $\theta_{\mathbf{8}} = q^3$ and $\theta_{\mathbf{10}} = \theta_{\overline{\mathbf{10}}} = q^6$. The Frobenius–Schur indicator of the self-dual particles **1** and **8** is the same as for $su(3)_2$, namely $\mathrm{fb}_{\mathbf{1}} = \mathrm{fb}_{\mathbf{8}} = 1$, while also $\mathrm{fb}_{\mathbf{10}} = \mathrm{fb}_{\overline{\mathbf{10}}} = 1$.

B.1. q-CG coefficients for $su(3)_3/Z_3$

In this section, we give the q-CG coefficients in the case of $su(3)_3/Z_3$. The Clebsch–Gordan coefficients for the **1** and **8** in the $\mathbf{8}\times\mathbf{8}$ are given in the section containing the Clebsch–Gordan coefficients for $su(3)_2$.

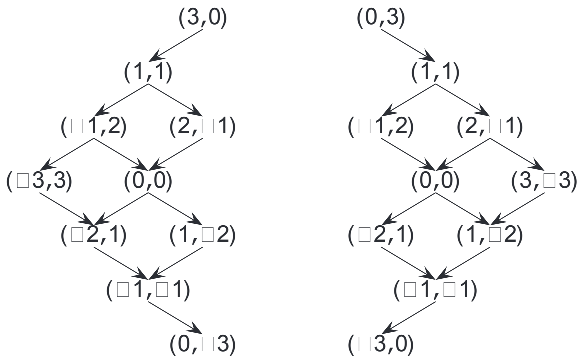

Figure B1. The weights of the $su(3)$ representations **10** and $\overline{\mathbf{10}}$.

Table B1. The fusion rules of $su(3)_3/Z_3$.

$\times$	**8**	**10**	$\overline{\mathbf{10}}$
8	$\mathbf{1}+\mathbf{8}+\mathbf{8}+\mathbf{10}+\overline{\mathbf{10}}$		
10	**8**	$\overline{\mathbf{10}}$	
$\overline{\mathbf{10}}$	**8**	**1**	**10**

B.1.1. $\mathbf{8}\times\mathbf{8}=\mathbf{1}+\mathbf{8}+\mathbf{8}'+\mathbf{10}+\overline{\mathbf{10}}$.

	$(1,1)$	$(-1,2)$	$(2,-1)$	$(0,0)_+$	$(0,0)_-$	$(-2,1)$	$(1,-2)$	$(-1,-1)$
$(1,1)$				0	$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor-1}}$	$\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2(\lfloor 4\rfloor-1)}}$	$-\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2(\lfloor 4\rfloor-1)}}$	X
$(-1,2)$			$-\frac{\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2(\lfloor 4\rfloor-1)}}$	$-\frac{q^{\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor-1}}$	$\frac{-q^{-\frac{1}{4}}+q^{\frac{1}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$		X	$-\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2(\lfloor 4\rfloor-1)}}$
$(2,-1)$		$\frac{\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2(\lfloor 4\rfloor-1)}}$		$\frac{q^{\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor-1}}$	$\frac{-q^{-\frac{1}{4}}+q^{\frac{1}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$	X		$\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2(\lfloor 4\rfloor-1)}}$
$(0,0)_+$	0	$\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor-1}}$	$-\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor-1}}$	X	X	$\frac{q^{\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor-1}}$	$-\frac{q^{\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor-1}}$	0
$(0,0)_-$	$-\frac{q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor-1}}$	$\frac{-q^{-\frac{3}{4}}-q^{-\frac{1}{4}}+q^{\frac{1}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$	$\frac{-q^{-\frac{3}{4}}-q^{-\frac{1}{4}}+q^{\frac{1}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$	X	X	$\frac{-q^{-\frac{1}{4}}+q^{\frac{1}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$	$\frac{-q^{-\frac{1}{4}}+q^{\frac{1}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$	$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor-1}}$
$(-2,1)$	$-\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2(\lfloor 4\rfloor-1)}}$		X	$-\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor-1}}$	$\frac{-q^{-\frac{3}{4}}-q^{-\frac{1}{4}}+q^{\frac{1}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$		$-\frac{\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2(\lfloor 4\rfloor-1)}}$	
$(1,-2)$	$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2(\lfloor 4\rfloor-1)}}$	X		$\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor-1}}$	$\frac{-q^{-\frac{3}{4}}-q^{-\frac{1}{4}}+q^{\frac{1}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$	$\frac{\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2(\lfloor 4\rfloor-1)}}$		
$(-1,-1)$	X	$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2(\lfloor 4\rfloor-1)}}$	$-\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2(\lfloor 4\rfloor-1)}}$	0	$-\frac{q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor-1}}$			

The q-CG coefficients for the $\mathbf{8}'$ (when $m\neq(0,0)_\pm$) in $\mathbf{8}\times\mathbf{8}$.

	$(1,1)$	$(-1,2)$	$(2,-1)$	$(0,0)_+$	$(0,0)_-$	$(-2,1)$	$(1,-2)$	$(-1,-1)$
$(1,1)$								$-\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor-1}}$
$(-1,2)$							$\frac{q^{-\frac{1}{4}}+q^{\frac{1}{4}}-q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$	
$(2,-1)$						$\frac{q^{-\frac{1}{4}}+q^{\frac{1}{4}}-q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$		
$(0,0)_+$				$\frac{-q^{-\frac{3}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$	0			
$(0,0)_-$				0	$\frac{2(-q^{-\frac{1}{4}}+q^{\frac{1}{4}})-q^{-\frac{3}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$			
$(-2,1)$			$\frac{-q^{-\frac{1}{4}}-q^{\frac{1}{4}}+q^{-\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$					
$(1,-2)$		$\frac{-q^{-\frac{1}{4}}-q^{\frac{1}{4}}+q^{-\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$						
$(-1,-1)$	$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor-1}}$							

The q-CG coefficients for the $\mathbf{8}'$ (for $m=(0,0)_-$) in $\mathbf{8}\times\mathbf{8}$.

	$(1,1)$	$(-1,2)$	$(2,-1)$	$(0,0)_+$	$(0,0)_-$	$(-2,1)$	$(1,-2)$	$(-1,-1)$
$(1,1)$								0
$(-1,2)$							$-\frac{q^{\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor-1}}$	
$(2,-1)$						$\frac{q^{\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor-1}}$		
$(0,0)_+$				0	$\frac{-q^{-\frac{3}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$			
$(0,0)_-$				$\frac{-q^{-\frac{3}{4}}+q^{\frac{3}{4}}}{2\sqrt{\lfloor 4\rfloor-1}}$	0			
$(-2,1)$			$-\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor-1}}$					
$(1,-2)$		$\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 3\rfloor}}{2\sqrt{\lfloor 4\rfloor-1}}$						
$(-1,-1)$	0							

The q-CG coefficients for the $\mathbf{8}'$ (for $m = (0,0)_+$) in $\mathbf{8}\times\mathbf{8}$.

	$(1,1)$	$(-1,2)$	$(2,-1)$	$(0,0)_+$	$(0,0)_-$	$(-2,1)$	$(1,-2)$	$(-1,-1)$
$(1,1)$			$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	$\frac{\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{q^{-\frac{1}{2}}}{\lfloor 2\rfloor\sqrt{\lfloor 3\rfloor}}$
$(-1,2)$			$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	$\frac{1}{\lfloor 2\rfloor\sqrt{\lfloor 3\rfloor}}$	$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor\lfloor 3\rfloor}}$
$(2,-1)$	$-\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	$-\frac{q^{-\frac{3}{4}}}{\sqrt{\lfloor 2\rfloor\lfloor 3\rfloor}}$		$\frac{(1-q^{-\frac{1}{2}})\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{(1+q^{-\frac{1}{2}})\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{1}{\lfloor 2\rfloor\sqrt{\lfloor 3\rfloor}}$		$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor\lfloor 3\rfloor}}$
$(0,0)_+$	$-\frac{\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$-\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{(-1+q^{\frac{1}{2}})\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})(\lfloor 2\rfloor+1)}{2\lfloor 2\rfloor\sqrt{\lfloor 3\rfloor}}$	$\frac{1}{2}$	$\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{(1-q^{-\frac{1}{2}})\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$
$(0,0)_-$	$-\frac{\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$-\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{(-1-q^{\frac{1}{2}})\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$-\frac{1}{2}$	$\frac{(q^{-\frac{1}{2}}-q^{\frac{1}{2}})(\lfloor 2\rfloor-1)}{2\lfloor 2\rfloor\sqrt{\lfloor 3\rfloor}}$	$-\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{(-1-q^{-\frac{1}{2}})\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$-\frac{\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$
$(-2,1)$	$-\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor\lfloor 3\rfloor}}$	$-\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	$-\frac{1}{\lfloor 2\rfloor\sqrt{\lfloor 3\rfloor}}$	$-\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$		$-\frac{q^{-\frac{3}{4}}}{\sqrt{\lfloor 2\rfloor\lfloor 3\rfloor}}$	
$(1,-2)$	$-\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor\lfloor 3\rfloor}}$	$-\frac{1}{\lfloor 2\rfloor\sqrt{\lfloor 3\rfloor}}$		$\frac{(-1+q^{\frac{1}{2}})\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{(1+q^{\frac{1}{2}})\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 2\rfloor\lfloor 3\rfloor}}$		$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$
$(-1,-1)$	$-\frac{q^{\frac{1}{2}}}{\lfloor 2\rfloor\sqrt{\lfloor 3\rfloor}}$	$-\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor\lfloor 3\rfloor}}$	$-\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor\lfloor 3\rfloor}}$	$-\frac{\sqrt{\lfloor 2\rfloor+1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$	$\frac{\sqrt{\lfloor 2\rfloor-1}}{\sqrt{2\lfloor 2\rfloor\lfloor 3\rfloor}}$		$-\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 2\rfloor}}$	

The q-CG coefficients for the **10** in $\mathbf{8}\times\mathbf{8}$.

B.1.2. $\mathbf{10}\times\mathbf{8}=\mathbf{8}$.

	$(1,1)$	$(-1,2)$	$(2,-1)$	$(0,0)_+$	$(0,0)_-$	$(-2,1)$	$(1,-2)$	$(-1,-1)$
$(3,0)$						$\frac{q^{\frac{3}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 5\rfloor}}$		$\frac{q^{\frac{3}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 5\rfloor}}$
$(1,1)$				$-\frac{\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor+1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$	$-\frac{\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor-1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$	$\frac{q^{\frac{5}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$	$-\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$	X
$(-1,2)$			$\frac{q^{-\frac{3}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$	$-\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor+1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$	$-\frac{q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor-1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$		X	$\frac{q^{\frac{5}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$
$(2,-1)$		$\frac{q^{-\frac{3}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$		$\frac{(-1+q^{-\frac{1}{2}})\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor+1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$	$\frac{(-1-q^{-\frac{1}{2}})\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor-1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$	X		$\frac{q^{\frac{5}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$
$(-3,3)$			$\frac{q^{-\frac{1}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 5\rfloor}}$				$-\frac{q^{\frac{1}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 5\rfloor}}$	
$(0,0)$	$-\frac{q^{-\frac{3}{2}}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$	$\frac{q^{-\frac{1}{2}}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$	$\frac{q^{-\frac{1}{2}}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$	X	X	$-\frac{q^{\frac{1}{2}}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$	$-\frac{q^{\frac{1}{2}}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$	$\frac{q^{\frac{3}{2}}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$
$(-2,1)$	$-\frac{q^{-\frac{5}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$		X	$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor+1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$	$-\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor-1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$		$-\frac{q^{\frac{3}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$	
$(1,-2)$	$-\frac{q^{-\frac{5}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$	X		$\frac{(1-q^{\frac{1}{2}})\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor+1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$	$\frac{(-1-q^{\frac{1}{2}})\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor-1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$	$-\frac{q^{\frac{3}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$		
$(-1,-1)$	X	$-\frac{q^{-\frac{5}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$	$\frac{q^{\frac{1}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$	$\frac{\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor+1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$	$-\frac{\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor-1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$			
$(-3,0)$	$-\frac{q^{-\frac{3}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 5\rfloor}}$	$-\frac{q^{-\frac{3}{4}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 5\rfloor}}$						

The q-CG coefficients for the **8** (for $m \neq (0,0)_\pm$) in $\mathbf{10}\times\mathbf{8}$.

	(1, 1)	(−1, 2)	(2, −1)	$(0,0)_+$	$(0,0)_-$	(−2, 1)	(1, −2)	(−1, −1)
(3, 0)								
(1, 1)								$\frac{q\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor+1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$
(−1, 2)							$-\frac{\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor+1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$	
(2, −1)						$\frac{(-1+q^{\frac{3}{2}})\sqrt{\lfloor 2\rfloor}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor(\lfloor 2\rfloor+1)}}$		
(−3, 3)								
(0, 0)				$\frac{q^{-1}+q^{-\frac{1}{2}}-q^{\frac{1}{2}}-q}{2\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$	$-\frac{\lfloor 2\rfloor}{2\sqrt{\lfloor 5\rfloor}}$			
(−2, 1)			$\frac{\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor+1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$					
(1, −2)		$\frac{(1-q^{-\frac{3}{2}})\sqrt{\lfloor 2\rfloor}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor(\lfloor 2\rfloor+1)}}$						
(−1, −1)	$-\frac{q^{-1}\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor+1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$							
(−3, 0)								

The q-CG coefficients for the **8** (for $m = (0,0)_+$) in **10** × **8**.

	(1, 1)	(−1, 2)	(2, −1)	$(0,0)_+$	$(0,0)_-$	(−2, 1)	(1, −2)	(−1, −1)
(3, 0)								
(1, 1)								$\frac{q\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor-1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$
(−1, 2)							$-\frac{\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor-1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$	
(2, −1)						$\frac{(-1-q^{\frac{3}{2}})\sqrt{\lfloor 2\rfloor}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor(\lfloor 2\rfloor-1)}}$		
(−3, 3)								
(0, 0)				$\frac{\lfloor 2\rfloor}{2\sqrt{\lfloor 5\rfloor}}$	$\frac{-q^{-1}+q^{-\frac{1}{2}}-q^{\frac{1}{2}}+q}{2\sqrt{\lfloor 3\rfloor\lfloor 5\rfloor}}$			
(−2, 1)			$-\frac{\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor-1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$					
(1, −2)		$\frac{(-1-q^{-\frac{3}{2}})\sqrt{\lfloor 2\rfloor}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor(\lfloor 2\rfloor-1)}}$						
(−1, −1)	$\frac{q^{-1}\sqrt{\lfloor 2\rfloor(\lfloor 2\rfloor-1)}}{\sqrt{2\lfloor 3\rfloor\lfloor 5\rfloor}}$							
(−3, 0)								

The q-CG coefficients for the **8** (for $m = (0,0)_-$) in **10** × **8**.

B.1.3. $\mathbf{10} \times \mathbf{10} = \overline{\mathbf{10}}$.

	(3, 0)	(1, 1)	(−1, 2)	(2, −1)	(−3, 3)	(0, 0)	(−2, 1)	(1, −2)	(−1, −1)	(−3, 0)
(3, 0)					$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$		$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$		$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$	$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$
(1, 1)			$\frac{-q^{\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor}}$			$\frac{-\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{q^{\frac{5}{4}}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{-q^{-1}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{q\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$
(−1, 2)		$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor}}$		$\frac{q^{-\frac{3}{4}}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$		$\frac{-q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$		$\frac{-\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{q^{\frac{5}{4}}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$
(2, −1)			$\frac{-q^{\frac{3}{4}}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$		$\frac{-q^{\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor}}$	$\frac{-q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{-\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{-q^{\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor}}$	$\frac{-q^{-\frac{1}{4}}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	
(−3, 3)	$\frac{-q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$			$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor}}$				$\frac{-q^{\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor}}$		$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$
(0, 0)		$\frac{\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$		$\frac{(q^{-\frac{1}{2}}-q^{\frac{1}{2}})\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{-q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{-q^{\frac{1}{2}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{-\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	
(−2, 1)	$\frac{-q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$	$\frac{-q^{-\frac{5}{4}}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$		$\frac{\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$		$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$		$\frac{-q^{\frac{3}{4}}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{-q^{\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor}}$	
(1, −2)		$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor}}$	$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor}}$	$\frac{q^{-\frac{1}{2}}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{q^{-\frac{3}{4}}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$			
(−1, −1)	$\frac{-q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$	$\frac{-q^{-1}\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{-q^{-\frac{5}{4}}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{q^{\frac{1}{4}}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$		$\frac{\sqrt{\lfloor 2\rfloor}}{\sqrt{\lfloor 3\rfloor\lfloor 4\rfloor}}$	$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor}}$			
(−3, 0)	$\frac{-q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$	$\frac{-q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$	$\frac{-q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$		$\frac{-q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor}}$					

The q-CG coefficients for the $\overline{\mathbf{10}}$ in **10** × **10**.

Table B2. The parameters s_1, s_2 and s_3 in the symmetry relations between the q-CG coefficients for $su(3)_3/Z_3$.

j_1	j_2	j	s_1	s_2	s_3
8	**8**	**1**	0	0	0
8	**8**	**8**	0	0	0
8	**8**	**8**′	1	0	1
8	**8**	**10**	1	1	0
8	**10**	**8**	1	1	0
10	**10**	$\overline{\mathbf{10}}$	1	1	0
10	$\overline{\mathbf{10}}$	**1**	0	0	0

B.1.4. $\mathbf{10} \times \overline{\mathbf{10}} = \mathbf{1}$.

	(0, 3)	(1, 1)	(−1, 2)	(2, −1)	(0, 0)	(3, −3)	(−2, 1)	(1, −2)	(−1, −1)	(0, −3)
(3, 0)										$q^{\frac{3}{2}}$
(1, 1)									$-q$	
(−1, 2)								$q^{\frac{1}{2}}$		
(2, −1)							$q^{\frac{1}{2}}$			
(−3, 3)						-1				
(0, 0)					-1					
(−2, 1)				$q^{-\frac{1}{2}}$						
(1, −2)			$q^{-\frac{1}{2}}$							
(−1, −1)		$-q^{-1}$								
(−3, 0)	$q^{-\frac{3}{2}}$									

The q-CG coefficients for the **1** in $\mathbf{10} \times \overline{\mathbf{10}}$. Each symbol has an additional factor $\frac{1}{\sqrt{[5]([3]-1)}}$.

B.2. The relation between the q-CG coefficients

In table B2, we specify the coefficients s_1, s_2 and s_3, which appear in the symmetry relations between the various q-CG coefficients as explained in section 5.4.

B.3. The F- and R-symbols for $su(3)_3/Z_3$

Because the expressions are rather involved, we will not give the F-symbols for $su(3)_3/Z_3$ as a function of q, but only for the specialization $q = \mathrm{e}^{2\pi \mathrm{i}/6}$. In principle, it is straightforward to obtain them as a function of q from the q-Clebsch–Gordan coefficients.

All the one-dimensional F-symbols with a **1** on an outer line are 1. However, in the case of the presence of a vertex with three **8**'s, one has two-dimensional objects:

$$F_{\mathbf{8}}^{\mathbf{1},\mathbf{8},\mathbf{8}} = F_{\mathbf{8}}^{\mathbf{8},\mathbf{1},\mathbf{8}} = F_{\mathbf{8}}^{\mathbf{8},\mathbf{8},\mathbf{1}} = F_{\mathbf{1}}^{\mathbf{8},\mathbf{8},\mathbf{8}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{B.1}$$

The following non-trivial symbols are 1:

$$F_{\mathbf{10}}^{\mathbf{8},\mathbf{10},\mathbf{8}} = F_{\overline{\mathbf{10}}}^{\mathbf{8},\mathbf{10},\mathbf{8}} = F_{\mathbf{10}}^{\mathbf{8},\overline{\mathbf{10}},\mathbf{8}} = F_{\overline{\mathbf{10}}}^{\mathbf{8},\overline{\mathbf{10}},\mathbf{8}} = F_{\mathbf{8}}^{\mathbf{10},\mathbf{8},\mathbf{10}} = F_{\mathbf{8}}^{\mathbf{10},\mathbf{8},\overline{\mathbf{10}}} = F_{\mathbf{8}}^{\overline{\mathbf{10}},\mathbf{8},\mathbf{10}} = F_{\mathbf{8}}^{\overline{\mathbf{10}},\mathbf{8},\overline{\mathbf{10}}}$$
$$= F_{\mathbf{10}}^{\mathbf{10},\overline{\mathbf{10}},\mathbf{10}} = F_{\overline{\mathbf{10}}}^{\overline{\mathbf{10}},\mathbf{10},\overline{\mathbf{10}}} = 1. \tag{B.2}$$

The following symbols are −1:

$$F^{\mathbf{8,8,10}}_{\mathbf{10}} = F^{\mathbf{8,8,10}}_{\overline{\mathbf{10}}} = F^{\mathbf{8,8,\overline{10}}}_{\mathbf{10}} = F^{\mathbf{8,8,\overline{10}}}_{\overline{\mathbf{10}}} = F^{\mathbf{8,10,10}}_{\mathbf{8}} = F^{\mathbf{8,10,\overline{10}}}_{\mathbf{8}} = F^{\mathbf{8,\overline{10},10}}_{\mathbf{8}} = F^{\mathbf{8,\overline{10},\overline{10}}}_{\mathbf{8}}$$
$$= F^{\mathbf{10,10,8}}_{\mathbf{8}} = F^{\mathbf{10,\overline{10},8}}_{\mathbf{8}} = F^{\mathbf{\overline{10},10,8}}_{\mathbf{8}} = F^{\mathbf{\overline{10},\overline{10},8}}_{\mathbf{8}} = F^{\mathbf{10,8,8}}_{\mathbf{10}} = F^{\mathbf{10,8,8}}_{\overline{\mathbf{10}}} = F^{\mathbf{\overline{10},8,8}}_{\mathbf{10}} = F^{\mathbf{\overline{10},8,8}}_{\overline{\mathbf{10}}}$$
$$= F^{\mathbf{10,10,\overline{10}}}_{\mathbf{10}} = F^{\mathbf{10,\overline{10},\overline{10}}}_{\overline{\mathbf{10}}} = F^{\mathbf{\overline{10},10,10}}_{\mathbf{10}} = F^{\mathbf{\overline{10},\overline{10},10}}_{\overline{\mathbf{10}}} = -1. \tag{B.3}$$

In addition, we have

$$F^{\mathbf{8,8,8}}_{\mathbf{10}} = F^{\mathbf{8,8,10}}_{\mathbf{8}} = F^{\mathbf{8,\overline{10},8}}_{\mathbf{8}} = F^{\mathbf{10,8,8}}_{\mathbf{8}} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \tag{B.4}$$

$$F^{\mathbf{8,8,8}}_{\overline{\mathbf{10}}} = F^{\mathbf{8,8,\overline{10}}}_{\mathbf{8}} = F^{\mathbf{8,10,8}}_{\mathbf{8}} = F^{\mathbf{\overline{10},8,8}}_{\mathbf{8}} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}. \tag{B.5}$$

Finally, the most interesting F-symbol reads

$$F^{\mathbf{8,8,8}}_{\mathbf{8}} = \begin{pmatrix} \frac{1}{3} & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ -\frac{1}{3} & \frac{1}{\sqrt{12}} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{\sqrt{12}} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}. \tag{B.6}$$

Here, we used the following basis. The first row corresponds to the case for which $j_{12} = \mathbf{1}$. The next four rows correspond to $j_{12} = \mathbf{8}$. In this case, there are two vertices with three external **8** lines, which each are two-dimensional. The second and fifth row correspond to the cases in which we took the vertices to be 'the same', while the third and fourth row correspond to the 'off-diagonal' cases. Finally, rows 6 and 7 correspond to the **10** and $\overline{\mathbf{10}}$, respectively. Note that this matrix is not symmetric, but $F^{\mathbf{8,8,8}}_{\mathbf{8}} \cdot F^{\mathbf{8,8,8}^T}_{\mathbf{8}} = \mathbf{1}$, as it should. The R symbols read

$$R^{\mathbf{1},x}_{x} = 1 \tag{B.7}$$

$$R^{\mathbf{8,8}}_{\mathbf{1}} = q^{-3} \to -1 \qquad R^{\mathbf{8,8}}_{\mathbf{8},1} = q^{-\frac{3}{2}} \to -i \qquad R^{\mathbf{8,8}}_{\mathbf{8},2} = -q^{-\frac{3}{2}} \to i \qquad R^{\mathbf{8,8}}_{\mathbf{10}} = -1 \tag{B.8}$$

$$R^{\mathbf{8,10}}_{\mathbf{8}} = -q^{-3} \to 1 \tag{B.9}$$

$$R^{\mathbf{10,10}}_{\overline{\mathbf{10}}} = -q^{-3} \to 1 \tag{B.10}$$

$$R^{\mathbf{10,\overline{10}}}_{\mathbf{1}} = q^{-6} \to 1. \tag{B.11}$$

We verified that the symbols above satisfy the pentagon and both hexagon equations.

Because of the relevance for the quantum dimension and Frobenius–Schur indicator, we give the following F-symbols as a function of q:

$$\left(F^{\mathbf{8,8}}_{\mathbf{8,8}}\right)_{\mathbf{1,1}} = \frac{1}{\lfloor 3\rfloor + \lfloor 5\rfloor} \qquad \left(F^{\mathbf{10,\overline{10}}}_{\mathbf{10,10}}\right)_{\mathbf{1,1}} = \frac{1}{(\lfloor 3\rfloor - 1)\lfloor 5\rfloor}. \tag{B.12}$$

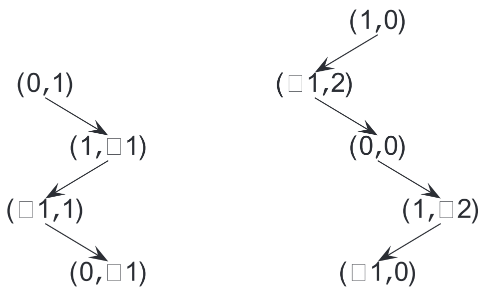

Figure C1. The weights of the $so(5)_1$ representations **4** and **5**.

Appendix C. The case $so(5)_1$

The Cartan matrix, its inverse and the quadratic form matrix for $so(5)$ read

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \qquad A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \qquad F_{\mathrm{qf}} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{C.1}$$

The second root is short, and we have $t_1 = 1$ and $t_2 = 2$, where $t_i = \frac{2}{(\alpha_i,\alpha_i)}$. For $so(5)$, one has that $\overline{\Lambda} = \Lambda$. The weight spaces of the representations $\mathbf{4} = (0, 1)$ and $\mathbf{5} = (1, 0)$, relevant for $so(5)_1$, are given in figure C1.

The non-trivial fusion rules of $so(5)_1$ are

$$\mathbf{4} \times \mathbf{4} = \mathbf{1} + \mathbf{5} \qquad \mathbf{4} \times \mathbf{5} = \mathbf{4} \qquad \mathbf{5} \times \mathbf{5} = \mathbf{1}, \tag{C.2}$$

namely the fusion rules of $su(2)_2$ or the Ising conformal field theory.

Continuing with the topological data, the quantum dimensions are given by $d_{\mathbf{1}} = 1$, $d_{\mathbf{4}} = \lfloor 5\rfloor_2 - 1 \to \sqrt{2}$ and $d_{\mathbf{5}} = \lfloor 4\rfloor + 1 \to 1$, where the numerical values are obtained by setting $q = \mathrm{e}^{2\pi \mathrm{i}/4}$. The twist factors are given by $\theta_{\mathbf{4}} = q^{\frac{5}{4}}$ and $\theta_{\mathbf{5}} = q^2$ and the Frobenius–Schur indicators by $\mathrm{fb}_{\mathbf{1}} = \mathrm{fb}_{\mathbf{5}} = 1$ and $\mathrm{fb}_{\mathbf{4}} = -1$. Finally, the central charge is $\frac{5}{2}$.

C.1. The q-CG coefficients for $so(5)_1$

In this section, we give the q-CG coefficients for $so(5)_1$.

C.1.1. $\mathbf{4} \times 4 = \mathbf{1} + \mathbf{5}$.

	(0, 1)	(1, −1)	(−1, 1)	(0, −1)
(0, 1)		$\frac{q^{\frac{1}{8}}}{\sqrt{\lfloor 2\rfloor_3}}$	$\frac{q^{\frac{1}{8}}}{\sqrt{\lfloor 2\rfloor_3}}$	$\frac{1}{\lfloor 2\rfloor_3}$
(1, −1)	$\frac{-q^{-\frac{1}{8}}}{\sqrt{\lfloor 2\rfloor_3}}$		$\frac{q^{\frac{1}{4}}}{\lfloor 2\rfloor_3}$	$\frac{q^{\frac{1}{8}}}{\sqrt{\lfloor 2\rfloor_3}}$
(−1, 1)	$\frac{-q^{-\frac{1}{8}}}{\sqrt{\lfloor 2\rfloor_3}}$	$\frac{-q^{-\frac{1}{4}}}{\lfloor 2\rfloor_3}$		$\frac{q^{\frac{1}{8}}}{\sqrt{\lfloor 2\rfloor_3}}$
(0, −1)	$\frac{-1}{\lfloor 2\rfloor_3}$	$\frac{-q^{-\frac{1}{8}}}{\sqrt{\lfloor 2\rfloor_3}}$	$\frac{-q^{-\frac{1}{8}}}{\sqrt{\lfloor 2\rfloor_3}}$	

$\sqrt{5}$

The q-CG coefficients for the **5** in **4** × **4**.

	(0, 1)	(1, −1)	(−1, 1)	(0, −1)
(0, 1)				$\frac{q^{\frac{1}{2}}}{\sqrt{\lfloor 5\rfloor_2-1}}$
(1, −1)			$\frac{-q^{\frac{1}{4}}}{\sqrt{\lfloor 5\rfloor_2-1}}$	
(−1, 1)		$\frac{q^{-\frac{1}{4}}}{\sqrt{\lfloor 5\rfloor_2-1}}$		
(0, −1)	$\frac{-q^{-\frac{1}{2}}}{\sqrt{\lfloor 5\rfloor_2-1}}$			

$\sqrt{5}$

The q-CG coefficients for the **1** in **4** × **4**.

C.1.2. $\mathbf{4} \times 5 = \mathbf{4}$.

	(1, 0)	(−1, 2)	(0, 0)	(1, 2)	(−1, 0)
(0, 1)			$\frac{q^{\frac{1}{2}}}{\sqrt{\lfloor 5\rfloor_2}}$	$\frac{q^{\frac{3}{8}}\sqrt{\lfloor 2\rfloor_2}}{\sqrt{\lfloor 5\rfloor_2}}$	$\frac{q^{\frac{3}{8}}\sqrt{\lfloor 2\rfloor_2}}{\sqrt{\lfloor 5\rfloor_2}}$
(1, −1)		$\frac{-q^{\frac{1}{8}}\sqrt{\lfloor 2\rfloor_2}}{\sqrt{\lfloor 5\rfloor_2}}$	$\frac{-1}{\sqrt{\lfloor 5\rfloor_2}}$		$\frac{q^{\frac{3}{8}}\sqrt{\lfloor 2\rfloor_2}}{\sqrt{\lfloor 5\rfloor_2}}$
(−1, 1)	$\frac{q^{\frac{-3}{8}}\sqrt{\lfloor 2\rfloor_2}}{\sqrt{\lfloor 5\rfloor_2}}$		$\frac{-1}{\sqrt{\lfloor 5\rfloor_2}}$	$\frac{-q^{\frac{-1}{8}}\sqrt{\lfloor 2\rfloor_2}}{\sqrt{\lfloor 5\rfloor_2}}$	
(0, −1)	$\frac{q^{\frac{-3}{8}}\sqrt{\lfloor 2\rfloor_2}}{\sqrt{\lfloor 5\rfloor_2}}$	$\frac{q^{\frac{-3}{8}}\sqrt{\lfloor 2\rfloor_2}}{\sqrt{\lfloor 5\rfloor_2}}$	$\frac{q^{\frac{-1}{2}}}{\sqrt{\lfloor 5\rfloor_2}}$		

The q-CG coefficients for the **4** in $\mathbf{4} \times \mathbf{5}$.

We note that the coefficients for $\mathbf{5} \times \mathbf{4} = \mathbf{4}$ are obtained by using the relation

$$\begin{pmatrix} \mathbf{4} & \mathbf{5} & \mathbf{4} \\ m_1 & m_2 & m_1+m_2 \end{pmatrix}_q = \begin{pmatrix} \mathbf{5} & \mathbf{4} & \mathbf{4} \\ m_2 & m_1 & m_1+m_2 \end{pmatrix}_{\frac{1}{q}}$$

from section 5.4.

C.1.3. $\mathbf{5} \times 5 = \mathbf{1}$.

	(1, 0)	(−1, 2)	(0, 0)	(1, −2)	(−1, 0)
(1, 0)					$\frac{q^{\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor+1}}$
(−1, 2)				$\frac{-q^{\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor+1}}$	
(0, 0)			$\frac{1}{\sqrt{\lfloor 4\rfloor+1}}$		
(1, −2)		$\frac{-q^{-\frac{1}{4}}}{\sqrt{\lfloor 4\rfloor+1}}$			
(−1, 0)	$\frac{q^{-\frac{3}{4}}}{\sqrt{\lfloor 4\rfloor+1}}$				

The q-CG coefficients for the **1** in $\mathbf{5} \times \mathbf{5}$.

C.2. *The F- and R-symbols for $so(5)_1$*

The F-symbols which are not unity are give by

$$F_{\mathbf{5}}^{\mathbf{4},\mathbf{4},\mathbf{5}} = F_{\mathbf{4}}^{\mathbf{4},\mathbf{5},\mathbf{5}} = F_{\mathbf{5}}^{\mathbf{5},\mathbf{4},\mathbf{4}} = F_{\mathbf{4}}^{\mathbf{5},\mathbf{5},\mathbf{4}} = \frac{1}{\sqrt{\lfloor 4\rfloor + 1}} \to 1 \tag{C.3}$$

$$F_{\mathbf{5}}^{\mathbf{4},\mathbf{5},\mathbf{4}} = F_{\mathbf{4}}^{\mathbf{5},\mathbf{4},\mathbf{5}} = -\frac{\lfloor 3\rfloor_2}{\lfloor 5\rfloor_2} \to -1 \tag{C.4}$$

$$F_{\mathbf{5}}^{\mathbf{5},\mathbf{5},\mathbf{5}} = \frac{1}{\lfloor 4\rfloor + 1} \to 1. \tag{C.5}$$

In addition, we have

$$F_{\mathbf{4}}^{\mathbf{4},\mathbf{4},\mathbf{4}} = \begin{pmatrix} \frac{-1}{\lfloor 5\rfloor_2 - 1} & -\frac{\sqrt{\lfloor 5\rfloor_2}}{\lfloor 2\rfloor_2^2\sqrt{\lfloor 4\rfloor+1}} \\ -\frac{\sqrt{\lfloor 5\rfloor_2}}{\lfloor 2\rfloor_2^2\sqrt{\lfloor 4\rfloor+1}} & \frac{\lfloor 3\rfloor_2}{(\lfloor 2\rfloor_2)^2} \end{pmatrix} \to \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \tag{C.6}$$

The values of the R-matrix are as follows:

$$R_a^{\mathbf{1},a} = R_a^{a,\mathbf{1}} = 1 \tag{C.7}$$

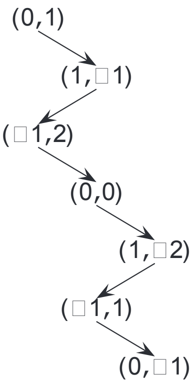

Figure D1. The weights of the G_2 representation **7**.

$$R^{\mathbf{4},\mathbf{4}}_{\mathbf{5}} = -q^{-\frac{1}{4}} \qquad R^{\mathbf{4},\mathbf{4}}_{\mathbf{1}} = -q^{-\frac{5}{4}} \tag{C.8}$$

$$R^{\mathbf{4},\mathbf{5}}_{\mathbf{4}} = R^{\mathbf{5},\mathbf{4}}_{\mathbf{4}} = q^{-1} \tag{C.9}$$

$$R^{\mathbf{5},\mathbf{5}}_{\mathbf{1}} = q^{-2}. \tag{C.10}$$

Appendix D. The case G_2, $k = 1$

The Cartan matrix, its inverse and the quadratic form matrix for G_2 read

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \qquad F_{\mathrm{qf}} = \frac{1}{3}\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}; \tag{D.1}$$

thus, the second root is short, and we have $t_1 = 1$ and $t_2 = 3$, where $t_i = \frac{2}{(\alpha_i,\alpha_i)}$. For G_2, one has that $\overline{\Lambda} = \Lambda$. The weight spaces of the representation $\mathbf{7} = (0, 1)$ is given in figure D1.

The only non-trivial fusion rule is $\mathbf{7} \times \mathbf{7} = \mathbf{1} \times \mathbf{7}$, i.e. the Fibonacci fusion rule. Furthermore, $d_{\mathbf{7}} = \lfloor 11\rfloor_3 - \lfloor 7\rfloor_3 + \lfloor 3\rfloor_3 = \lfloor 7\rfloor_3(\lfloor 5\rfloor_3 - \lfloor 3\rfloor_3 - 1) \to \phi$, where we used $q = \mathrm{e}^{2\pi\mathrm{i}/5}$. The twist factor is $\theta_{\mathbf{7}} = q^2$. In addition, we find $\mathrm{fb}_{\mathbf{1}} = \mathrm{fb}_{\mathbf{7}} = 1$, while the central charge is $\frac{14}{5}$.

D.1. The q-CG coefficients for G_2, $k = 1$

	(0, 1)	(1, −1)	(−1, 2)	(0, 0)	(1, −2)	(−1, 1)	(0, −1)
(0, 1)				$q^{\frac{1}{2}}$	$q^{\frac{5}{12}}\sqrt{\lfloor 2\rfloor_3}$	$q^{\frac{5}{12}}\sqrt{\lfloor 2\rfloor_3}$	$q^{\frac{1}{3}}$
(1, −1)			$-q^{\frac{1}{4}}\sqrt{\lfloor 2\rfloor_3}$	$-q^{\frac{1}{6}}$		$q^{\frac{1}{2}}$	$q^{\frac{5}{12}}\sqrt{\lfloor 2\rfloor_3}$
(−1, 2)		$q^{-\frac{1}{4}}\sqrt{\lfloor 2\rfloor_3}$		$-q^{\frac{1}{6}}$	-1		$q^{\frac{5}{12}}\sqrt{\lfloor 2\rfloor_3}$
(0, 0)	$-q^{-\frac{1}{2}}$	$q^{-\frac{1}{6}}$	$q^{-\frac{1}{6}}$	$q^{-\frac{1}{6}} - q^{\frac{1}{6}}$	$-q^{\frac{1}{6}}$	$-q^{\frac{1}{6}}$	$q^{\frac{1}{2}}$
(1, −2)	$-q^{-\frac{5}{12}}\sqrt{\lfloor 2\rfloor_3}$		1	$q^{-\frac{1}{6}}$		$-q^{\frac{1}{4}}\sqrt{\lfloor 2\rfloor_3}$	
(−1, 1)	$-q^{-\frac{5}{12}}\sqrt{\lfloor 2\rfloor_3}$	$-q^{-\frac{1}{2}}$		$q^{-\frac{1}{6}}$	$q^{-\frac{1}{4}}\sqrt{\lfloor 2\rfloor_3}$		
(0, −1)	$-q^{-\frac{1}{3}}$	$-q^{-\frac{5}{12}}\sqrt{\lfloor 2\rfloor_3}$	$-q^{-\frac{5}{12}}\sqrt{\lfloor 2\rfloor_3}$	$-q^{-\frac{1}{2}}$			

The q-CG coefficients for the **7** in $\mathbf{7} \times \mathbf{7}$. Each coefficient is to be multiplied by $\frac{1}{\sqrt{\lfloor 7\rfloor_3 - 1}}$.

	(0, 1)	(1, −1)	(−1, 2)	(0, 0)	(1, −2)	(−1, 1)	(0, −1)
(0, 1)							$q^{\frac{5}{6}}$
(1, −1)						$-q^{\frac{2}{3}}$	
(−1, 2)					$q^{\frac{1}{6}}$		
(0, 0)				-1			
(1, −2)			$q^{-\frac{1}{6}}$				
(−1, 1)		$-q^{-\frac{2}{3}}$					
(0, −1)	$q^{-\frac{5}{6}}$						

The q-CG coefficients for the **1** in **7** × **7**. Factor: $\frac{1}{\sqrt{\lfloor 11\rfloor_3-\lfloor 7\rfloor_3+\lfloor 3\rfloor_3}}$.

D.2. The F- and R-symbols

The only F-symbols which are not equal to 1 are the following (the numerical values are for $q = e^{2\pi i/5}$):

$$F_{\mathbf{7}}^{\mathbf{7},\mathbf{7},\mathbf{7}} = \begin{pmatrix} \frac{1}{\lfloor 11\rfloor_3-\lfloor 7\rfloor_3+\lfloor 3\rfloor_3} & -\frac{1}{\sqrt{\lfloor 11\rfloor_3-\lfloor 7\rfloor_3+\lfloor 3\rfloor_3}} \\ -\frac{1}{\sqrt{\lfloor 11\rfloor_3-\lfloor 7\rfloor_3+\lfloor 3\rfloor_3}} & -\frac{\lfloor 3\rfloor_3-2}{\lfloor 5\rfloor_3-\lfloor 3\rfloor_3} \end{pmatrix} \to \begin{pmatrix} 1/\phi & -1/\sqrt{\phi} \\ -1/\sqrt{\phi} & -1/\phi \end{pmatrix}. \tag{D.2}$$

In addition, we obtain the following R-symbols:

$$R_a^{\mathbf{1},a} = R_a^{a,\mathbf{1}} = 1 \qquad R_{\mathbf{1}}^{\mathbf{7},\mathbf{7}} = q^{-2} \qquad R_{\mathbf{7}}^{\mathbf{7},\mathbf{7}} = -q^{-1}. \tag{D.3}$$

Appendix E. The case $su(2)_k$

For completeness, we give an explicit expression for the q-Clebsch–Gordan coefficients, as well as the F- and R-symbols in the case of $su(2)_k$ (see, for instance, [31]), using the same basis conventions as we used throughout this paper. In particular, this formula yields 1 for any F-symbol with an identity on any of the outer lines. We use the Dynkin notation to denote the particles; thus, the labels of the particles take the values $a = 0, 1, 2, \ldots, k$, in the case of $su(2)_k$.

The fusion rules can be written as

$$a \times b = \sum_{c=|a-b|}^{\min(a+b,2k-a-b)} c, \tag{E.1}$$

where c increases in steps of two. The quantum dimensions simply read $d_a = \lfloor a+1\rfloor$; the twist factors are $\theta_a = q^{a(a+2)/4}$, while the Frobenius–Schur indicators are $\mathrm{fb}_a = (-1)^a$.

To write the q-Clebsch–Gordan coefficients, we define $\lfloor n\rfloor! = \lfloor n\rfloor\lfloor n-1\rfloor\cdots\lfloor 1\rfloor$ and $\lfloor 0\rfloor! = 1$. Furthermore, for $a \leqslant b+c$, $b \leqslant a+c$, $c \leqslant a+b$ and $a+b+c = 0$ mod 2, we define

$$\Delta(a,b,c) = \sqrt{\frac{\lfloor(a+b-c)/2\rfloor!\lfloor(a-b+c)/2\rfloor!\lfloor(-a+b+c)/2\rfloor!}{\lfloor(a+b+c+2)/2\rfloor!}}. \tag{E.2}$$

With these conventions, the q-Clebsch–Gordan coefficients can, for instance, be written in the following way (see [32] for this particular form, and various others, as well as [33]):

$$\begin{pmatrix} a & b & c \\ k & l & m \end{pmatrix}_q = q^{((a+b-c)(a+b+c+2)+2(al-bk))/16}\Delta(a,b,c)$$
$$\times \sqrt{\frac{}{\lfloor(a-k)/2\rfloor!\lfloor(a+k)/2\rfloor!\lfloor(b-l)/2\rfloor!\lfloor(b+l)/2\rfloor!\lfloor(c-m)/2\rfloor!\lfloor(c+m)/2\rfloor!\lfloor c+1\rfloor}}$$

$$\times \sum_n{}' \frac{(-1)^{n/2} q^{-n(a+b+c+2)/8}}{\lfloor n/2\rfloor!\lfloor (a-k-n)/2\rfloor!\lfloor (b+l-n)/2\rfloor!}$$
$$\times \frac{1}{\lfloor (a+b-c-n)/2\rfloor!\lfloor (c-b+k+n)/2\rfloor!\lfloor (c-a-l+n)/2\rfloor!}, \tag{E.3}$$

where the sum over n is over (non-negative) even integers, such that $\max(0, -(c-b+k), -(c-a-l)) \leqslant n \leqslant \min(a+b-c, a-k, b+l)$, in order that the arguments of the q-factorials are non-negative integers.

One way of writing the F-symbols is as follows [31]:

$$\left[F_d^{a,b,c}\right]_{e,f} = (-1)^{(a+b+c+d)/2}\Delta(a,b,e)\Delta(c,d,e)\Delta(b,c,f)\Delta(a,d,f)\sqrt{\lfloor e+1\rfloor}\sqrt{\lfloor f+1\rfloor}$$
$$\times \sum_n{}' \frac{(-1)^{n/2}\lfloor (n+2)/2\rfloor!}{\lfloor (a+b+c+d-n)/2\rfloor!\lfloor (a+c+e+f-n)/2\rfloor!\lfloor (b+d+e+f-n\rfloor!}$$
$$\times \frac{1}{\lfloor (n-a-b-e)/2\rfloor!\lfloor (n-c-d-e)/2\rfloor!\lfloor (n-b-c-f)/2\rfloor!\lfloor (n-a-d-f)/2\rfloor!}, \tag{E.4}$$

where the sum over n is over (non-negative) even integers, such that $\max(a+b+e, c+d+e, b+c+f, a+d+f) \leqslant n \leqslant \min(a+b+c+d, a+c+e+f, b+d+e+f)$.

In addition, the R-symbols read as follows:

$$R_c^{a,b} = (-1)^{(a+b-c)/2} q^{\frac{1}{8}(c(c+2)-a(a+2)-b(b+2))}. \tag{E.5}$$

Appendix F. The pentagon and hexagon equations

We note that in the presence of fusion multiplicities, the vertices carry an additional label. For completeness, we give the appropriate form of the pentagon and hexagon equations here. The greek index α, associated with a vertex with labels (a, b, c), runs over the values $1, 2, \ldots, N_{a,b}^c$, where $n_{a,b}^c$ is the number of times c appears in the fusion product of $a \times b$. With this notation, we have the following condition, for all possible $j_1, j_2, j_3, j_4, j, j_{12}, j_{123}, j_{34}, j_{234}$ and $\alpha, \beta, \gamma, \eta, \iota, \kappa$:

$$\sum_{j_{23},\delta,\epsilon,\zeta} \left[F_{j_{123}}^{j_1,j_2,j_3}\right]_{(j_{12},\alpha,\beta;j_{23},\delta,\epsilon)} \left[F_j^{j_1,j_{23},j_4}\right]_{(j_{123},\epsilon,\gamma;j_{234},\zeta,\eta)} \left[F_{j_{234}}^{j_2,j_3,j_4}\right]_{(j_{23},\delta,\zeta;j_{34},\iota,\kappa)}$$
$$= \sum_\lambda \left[F_j^{j_{12},j_3,j_4}\right]_{(j_{123},\beta,\gamma;j_{34},\iota,\lambda)} \left[F_j^{j_1,j_2,j_{34}}\right]_{(j_{12},\alpha,\lambda;j_{234},\kappa,\eta)}. \tag{F.1}$$

To write down the hexagon equations, we assume that twisting a vertex does not change the internal label, or in other words, the fusion channel. With that restriction, the hexagon equations read, for all $j_1, j_2, j_3, j, j_{12}, j_{13}$ and $\alpha, \beta, \gamma, \delta$,

$$R_{j_{12}}^{j_1,j_2}\left[F_j^{j_2,j_1,j_3}\right]_{(j_{12},\alpha,\beta;j_{13},\gamma,\delta)} R_{j_{13}}^{j_1,j_3} = \sum_{j_{23},\epsilon,\zeta} \left[F_j^{j_2,j_1,j_3}\right]_{(j_{12},\alpha,\beta;j_{23},\epsilon,\zeta)} R_j^{j_1,j_{23}} \left[F_j^{j_2,j_3,j_1}\right]_{(j_{23},\epsilon,\zeta;j_{13},\gamma,\delta)}, \tag{F.2}$$

and a similar equation for R^{-1}.

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