MATRIX MODELS ON THE FUZZY SPHERE

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Abstract.

Field theory on a fuzzy noncommutative sphere can be considered as a particular matrix approximation of field theory on the standard commutative sphere. We investigate from this point of view the scalar ϕ^4 theory. We demonstrate that the UV/IR mixing problems of this theory are localized to the tadpole diagrams and can be removed by an appropiate (fuzzy) normal ordering of the ϕ^4 vertex. The perturbative expansion of this theory reduces in the commutative limit to that on the commutative sphere.

1. Introduction

The field-theoretical models possessing a finite number of field modes can be presented as matrix models. To this class of models belong, e.g., the models on a finite cubic lattice subjected to some periodic boundary conditions. Another class of such models represent models on a fuzzy sphere, [1]-[3]. The basic idea behind fuzzy sphere is quite simple:

(i) The standard sphere S^2 with radius R is a co-adjoint orbit possessing a Poisson bracket, and it can be considered as a particular "phase-space" with finite volume $4\pi R^2$;

(ii) After "quantization" (deformation), this "phase-space" is effectively divided to cells of the size $\sim \rho^2$ (ρ is a deformation parameter), each representing one field mode. Thus, the total number of field modes is $N \sim R/\rho$. The resulting "quantum" space is known as the fuzzy sphere S^2_F, for details see [4].

In the framework of lattice models it is of great interest to investigate their properties in the continuum limit when, in a proper sense, the lattice spacing a approaches zero: $a \to 0$. For models on a fuzzy sphere the commutative limit $\rho \to 0$ (or equivalently, $N \to \infty$) plays a similar role. A simple scalar field model on a fuzzy sphere with ϕ^4 interaction was proposed in [1].

A potential problem for this program has emerged due to the phenomenon of UV/IR mixing, [5]. The problem was discussed on a noncommutative Moyal space, where fields possess an infinite number of modes, and consequently a regularization procedure is needed. The phenomenon appears to be generic for noncommutative spaces.

It was pointed out in [6] that the UV/IR mixing is present in the ϕ^4 model with naive action introduced in [1]. This work was followed by [7], where the one-loop contribution to the two-point vertex function was calculated explicitely, and it was shown that in the commutative limit there is a finite correction to the expected commutative contribution. Moreover, in the planar limit it incorporates the UV/IR mixing singularity of the Moyal space.

The implications of this result are very serious for the program of using matrix model approximation to continuum field theories. Recently, in [8] we found that there is a quite natural solution and that the problem disappears when the interaction term is properly (fuzzy) normal ordered. The resulting action is therefore the correct starting point for fuzzy lattice approximations.

In the next section we briefly describe the Euclidean scalar field with ϕ^4 interaction on a commutative sphere. Then we present its fuzzy version and formulate all essential results found in [8]. The last section contains a brief summary.

2. Model on a standard commutative sphere.

We consider a real $\Phi(\vec{n})$ defined on a usual sphere S^2 with radius R: $\vec{n} = (n_1, n_2, n_3)$, $\vec{n}^2 = R^2$. The Euclidean field action for $\Phi(\vec{n})$ with $\Phi^4(\vec{n})$ interaction is given as

$$S[\Phi] \; = \; \int dn \, [\frac{1}{2}(\mathcal{L}_i\Phi)^2 + \frac{r}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4] \; , \tag{1}$$

where $dn = \sin\theta d\theta d\varphi$ is the standard measure on S^2 normalized to 4π, and $\mathcal{L}_i = i\varepsilon_{ijk}n_j\partial_{n_k}$, $i,j,k = 1,2,3$, are the usual generators of rotations.

The Euclidean QFT can be formulated, e.g., in terms of functional integrals. If $F[\Phi]$ is some functional depending on field $\Phi = \Phi(\vec{n})$, then its quantum average is defined by

$$\langle F[\Phi]\rangle \;=\; \int D\Phi\, e^{-S[\Phi]}F[\Phi]\ . \tag{2}$$

Here the functional measure $D\Phi$ is normalized by $\langle 1\rangle = 1$. It can be shown that the model specified by (1) and (2) can be formulated rigorously, and moreover, that it possesses the perturbative expansion in λ which is Borel summable.

Therefore, we can restrict ourselves to the standard perturbation expansion of the model based on on (1). It is well known that it contains only finite Feynman diagrams except the tadpole diagram which contribution

$$g_L(r) \;=\; \frac{1}{4\pi}\sum_{l=0}^{L}\frac{2l+1}{l(l+1)+r} \tag{3}$$

is logharithmically divergent in the cut-off parameter L. In interaction picture the interaction term in the action is normal ordered

$$S_{int}[\Phi] \;=\; \int dn\,\frac{\lambda}{4!}:\Phi^4:\ , \tag{4}$$

where the normal ordering is specified by

$$:\Phi^4: = \;\Phi^4 \,-\, 12g(L,t)\Phi^2\ . \tag{5}$$

The parameters r and t are related by $r = t - (\lambda/2)g_L(t)$. This procedure eliminates all tadpole diagrams, and the resulting perturbative series contains only finite Feynman diagrams.

3. Model on the fuzzy sphere

The standard sphere S^2 with radius R is fully characterized by its Cartesian coordinates n_i, $i = 1,2,3$, satisfying relations

$$[n_i,n_j] = n_in_j - n_jn_i = 0\ ,\ \sum_i n_i^2 = R^2\ . \tag{6}$$

In the noncommutative case one replaces these commuting coordinates by operators $\hat n_i$, $i = 1,2,3$, satisfying relations

$$[\hat n_i,\hat n_j] = \hat n_i\hat n_j - \hat n_j\hat n_i = i\rho\varepsilon_{ijk}\hat n_k\ ,\ \sum_i \hat n_i^2 = R^2\ . \tag{7}$$

If $R^2/\rho^2 = L(L+1)$ with L positive integer, then $\hat n_i$, $i = 1,2,3$, can be realized as $(L+1)\times(L+1)$ matrices (in fact, conditions (7) determine the spin $s = L/2$ representation of $SU(2)$ group). The scalar field is now a polynomial in $\hat{n}_i$, i.e. a general $(L+1)\times(L+1)$ matrix $\hat{\Phi} = \Phi(\hat{n})$.

The fuzzy analogs $\hat{\mathcal{L}}_i$, $i = 1,2,3$, of the differential operators $\mathcal{L}_i$, act on an arbitrary matrix $\hat{f} = f(\hat{n})$ as follows:

$$\hat{\mathcal{L}}_i \hat{f} \;=\; [\hat{L}_i, \hat{f}] \ , \ \hat{L}_i = \rho^{-1}\hat{n}_i \ , \ i = 1,2,3 \ . \tag{8}$$

In the fuzzy case, the integral over S^2 is replaced by the normalized trace

$$\int dn\, f(\vec{n}) \;\rightarrow\; \frac{4\pi}{L+1}\,\mathrm{Tr} f(\hat{n}) \ . \tag{9}$$

In the commutative limit $\rho \rightarrow 0$, i.e. $L \rightarrow \infty$, eqs. (8) and (9) reduce, respectively, to the standard definitions of generators of rotations and the usual integral over S^2.

The modifications indicated in (8) and (9) lead to the naïve fuzzy field action proposed in [1]:

$$S_L[\hat{\Phi}] \;=\; \frac{4\pi}{L+1}\,\mathrm{Tr}[\frac{1}{2}(\hat{\mathcal{L}}_i\hat{\Phi})^2 + \frac{r}{2}\hat{\Phi}^2 + \frac{\lambda}{4!}\hat{\Phi}^4] \ . \tag{10}$$

Of course, this gives a tadpole contributions to the pertubation series, which indeed, are finite since the number of field modes is finite.

However, in the noncommutative case the permutation symmetry of the vertex contributions, originated from $\int dn\Phi^4$, is reduced the cyclic symmetry of the fuzzy vertex $4\pi(L+1)^{-1}\,\mathrm{Tr}\hat{\Phi}^4$. This yealds two independent tadpole contributions: instead of 12 tadpole subtractions in (5), we have now 8 subtractions of the planar tadpole diagram and 4 subtractions of the non-planar one. The planar diagram gives the same contribution as the tadpole in the commutative case, whereas the nonplanar possesses a correction. Therefore, the correct fuzzy interaction is, [8]:

$$S_{int}[\hat{\Phi}] \;=\; \frac{4\pi}{L+1}\,\mathrm{Tr} : \hat{\Phi}^4 :] \ , \tag{11}$$

with the normal ordering defined as follows:

$$\mathrm{Tr} : \hat{\Phi}^4 : = \ \mathrm{Tr}\left[\hat{\Phi}^4 \ - \ 12\sum_{lm}\frac{\hat{\Phi}\hat{Y}_{lm}\hat{Y}_{lm}\hat{\Phi}}{l(l+1)+t} \ + \ 2\sum_{lm}\frac{[\hat{\Phi},\hat{Y}_{lm}]^\dagger[\hat{\Phi},\hat{Y}_{lm}]}{l(l+1)+t}\right] \ . \tag{12}$$

Here, $\hat{Y}_{lm}$ are fuzzy analogs of spherical functions determined by the conditions:

$$\hat{\mathcal{L}}_i^2\hat{Y}_{lm} = l(l+1)\hat{Y}_{lm} \ , \ \hat{\mathcal{L}}_3\hat{Y}_{lm} = m\hat{Y}_{lm} \ , \ \frac{4\pi}{L+1}\mathrm{Tr}\hat{Y}^\dagger_{lm}\hat{Y}_{l'm'} = \delta_{ll'}\delta_{mm'} \ . \tag{13}$$

The middle term in (12) is the usual tadpole subtraction and corresponds to a normal ordering in the commutative vertex. The last term is the additional noncommutative subtraction that is necessary to obtain the correct commutative limit.

4. Conclusions

The correct matrix model that represents a lattice regularization of the commutative theory is given by the field action with the interaction term given by (11) and (12). It can be put into the form

$$\tilde{S}_L[\hat{\Phi}] \; = \; \frac{4\pi}{L+1}\,\mathrm{Tr}[\frac{1}{2}\hat{\Phi}(\hat{\mathcal{L}}^2+\frac{\lambda}{2}Q_L(\hat{\mathcal{L}}^2)+t-g_L(t))\hat{\Phi}+\frac{\lambda}{4!}\hat{\Phi}^4]\;. \qquad (14)$$

The term $(\lambda/2)g_L(t)$, given by the middle term in (12), corresponds to the usual tadpole subtraction (the parameter r is replaced by $t-(\lambda/2)g_L(t)$, exactly, like in the commutative theory). The term $(\lambda/2)Q_L(\hat{\mathcal{L}}^2)$ is given by the manifestly positive last term in (12). It interpreted as the momentum dependent wave-function renormalization since $Q_L(\hat{\mathcal{L}}^2)$ is a power series in $\hat{\mathcal{L}}^2$ which starts at $\hat{\mathcal{L}}^2$, and therefore we could have written (see, [7], [8]):

$$\hat{\mathcal{L}}^2+\frac{\lambda}{2}Q_L(\hat{\mathcal{L}}^2) \; = \; \hat{\mathcal{L}}^2 Z_L(\hat{\mathcal{L}}^2)\;.$$

This term exactly cancels the unwanted momentum dependent quadratic terms in the effective action arising from non-planar diagrams in the fuzzy theory. This cancellation then guarantees that the continuum limit of this theory is the standard scalar ϕ^4 on the sphere.

The modified action (14) can serve for the non-perturbative definition of the fuzzy quantum averages

$$\langle F[\hat{\Phi}]\rangle_L \; = \; \int D_L\hat{\Phi}\, e^{-\tilde{S}_L[\hat{\Phi}]}F[\hat{\Phi}]\;. \qquad (15)$$

Here, $D_L\hat{\Phi} \; = \; N_L\Pi_{lm}dc_{lm}$ is a finite-dimensional measure associated to arbitrary field configurations

$$\hat{\Phi} \; = \; \sum_{l=0}^{L}\sum_{m=-L}^{+L} c_{lm}\hat{Y}_{lm}\;,\; c_{l,-m}=c^*_{l,m} \;-\; \text{complex}\;, \qquad (16)$$

with the normalization constant N_L fixed by $\langle 1\rangle_L=1$. The action $\tilde{S}_L[\hat{\Phi}]$ in (15) guarantees the correct commutative limit $L\to\infty$ of quantum averages $\langle F[\hat{\Phi}]\rangle_L$ for any fixed field functional $F[\hat{\Phi}]$ not depending explicitely on L.

It would be desirable to investigate in the same spirit the ϕ^4 theory in four-dimensional fuzzy spaces, e.g. as a suitable canditate could serve the fuzzy space $S_F^2 \times S_F^2$. However, in this case, the problem will be more severe since there will be additional residual non-local differences for two- and four-point functions. It will be therefore more difficult to establish the model which will reproduce the commutative limit.

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