Extremal norms for positive linear inclusions.

Oliver Mason†,*   Fabian Wirth†

June 18, 2013

Abstract

For finite-dimensional linear semigroups which leave a proper cone invariant it is shown that irreducibility with respect to the cone implies the existence of an extremal norm. In case the cone is simplicial a similar statement applies to absolute norms. The semigroups under consideration may be generated by discrete-time systems, continuous-time systems or continuous-time systems with jumps. The existence of extremal norms is used to extend results on the Lipschitz continuity of the joint spectral radius beyond the known case of semigroups that are irreducible in the representation theory interpretation of the word.

Keywords: Joint spectral radius, extremal norm, linear switched systems, linear semigroups

AMS Classification: 15B48, 15A60, 34D08, 52A21

1 Introduction

In this paper we investigate the exponential growth rate of linear semigroups which leave a proper cone K invariant. In the case of matrix products this exponential growth rate is commonly known as the joint spectral radius of a set of matrices and we will use this name for matrix semigroups in general. For matrix semigroups generated by discrete-time systems, continuous time systems or switched systems with jumps it is shown that an irreducibility condition of the semigroup with respect to the cone K guarantees the existence of an extremal norm. In addition this norm can be chosen to be monotone with respect to the cone K. This has been obtained previously for positive semigroups with respect to the cone $\mathbb{R}^n_+$ in [1] and for general cones in the discrete time case in [15]. Here we follow the basic idea of [1], but have to adjust several arguments to be able to deal with general proper cones. The approach is distinct from the ideas presented in [15].

The results are used to extend regularity results for the joint spectral radius, in that the joint spectral radius is irreducible in a neighbourhood of K-irreducible semigroups.

The joint spectral radius of sets of matrices for discrete or continuous linear inclusions and associated extremal norms have been studied in [17, 26, 42, 43, 16, 28, 31, 9], see also the survey [36]. The joint spectral radius plays a key role in characterising growth rates of solutions of inclusions and hence in their stability analysis [20]. Motivated by the practical importance of systems whose state variables are constrained to remain non-negative (given non-negative initial conditions), there has been significant interest recently in studying positive inclusions and positive switched systems [10, 18, 21, 32]. Also in several recent proposals for efficient computation of the joint spectral radius, positivity has played an important role, see [6, 15] and the lifting techniques discussed for instance in [22]. In [33] positivity with respect to arbitrary positive cones has been considered. Also it has been noted e.g. in [31] that positivity properties yield criteria for the uniqueness (up to scaling) of Barabanov norms, a concept we describe below. Note that in general it is hard to decide, whether a given set of matrices share an invariant cone, [34]. In some cases,

*Hamilton Institute, National University of Ireland, Maynooth, Ireland, oliver.mason@nuim.ie

†Institut für Mathematik, Universität Würzburg, Würzburg, Germany, wirth@mathematik.uni-wuerzburg.de
F. Wirth is supported by the DFG under grant Wi1458/10.

however, this is clear in the way that the matrices are defined. In particular, for the positive orthant $\mathbb{R}^n_+$ or the cone or positive semidefinite matrices.

The defining property of positive inclusions, the positivity constraint on their dynamics, has motivated a number of particular questions in their stability analysis. One such example is in the study of *copositive Lyapunov functions* and linear copositive Lyapunov functions in particular. Necessary and sufficient conditions for the existence of a common linear copositive Lyapunov function in the case of a switched positive system with 2 modes were given in [30]; these were then extended to the general case in [23]. Copositivity has subsequently been studied for more general cones in [8], while its application to the stability and stabilisation of switched positive systems has been thoroughly investigated in [12], [11] and [13].

In [29], the question of D-stability for positive inclusions was considered. Separate sufficient and necessary conditions for a positive linear inclusion to be D-stable were described. These conditions are intimately connected to the existence of common linear copositive Lyapunov functions for a related inclusion. This theme was developed in [7] where a single necessary and sufficient condition was described for a system whose constituent systems are described by irreducible matrices. The matrix theoretic notion of irreducibility shall play a key role again in the current paper.

It was shown in [18] that the stability of a 2-dimensional positive inclusion is equivalent to the stability of the convex hull of its associated matrices. Unfortunately this result fails to be true in general and a specific counterexample for 3-dimensional systems has been described in [10]. In studying general linear inclusions, the concepts of extremal and Barabanov norms play an important role as such norms can be used to explicitly characterise the growth rate of an inclusion. Barabanov norms for positive linear inclusions have been recently considered in [39]; an explicit closed-form expression for a Barabanov norm for 2-dimensional discrete positive inclusions is derived.

In this paper, we consider extremal norms for positive inclusions in both the discrete and continuous case. We show that for positive inclusions, a matrix-theoretic notion of irreducibility is sufficient for the existence of an extremal norm. This is novel as the concept of irreducibility we consider is weaker than that used to establish the existence of extremal norms in the work of Barabanov, [2] and others. Our result also relates the existence of an extremal norm to a property of the convex hull of the matrices associated with the inclusion.

We use the above results to extend regularity results for the joint spectral radius [25, 42]. In the case of positive systems irreducibility in the sense of nonnegative matrices is sufficient for local Lipschitz properties of the joint spectral radius. We emphasise that this notion of irreducibility is distinct from that utilised for general inclusions. Our results show that an interesting observation of [24] for the case of nonnegative 2×2 matrices is a consequence of a general property of sets of nonnegative matrices and monotone norms.

The paper is organised as follows: In the ensuing Section 1.1 we recall several known results about proper cones and the properties of matrices that leave such cones invariant. In Section 2 we describe three principles by which linear semigroups can be generated and recall a common definition of the joint spectral radius. In Section 3 we recall the definition of extremal norms. As the first main result it is shown that a K-irreducible positive semigroup admits a monotone extremal norm. The results of Section 3 are used in Section 4 to obtain Lipschitz continuity results for the joint spectral radius. These results extend previous results in [42] which used a notion of irreducibility from representation theory to obtain Lipschitz continuity properties.

1.1 Preliminaries

Throughout the paper, $\mathbb{R}$ and $\mathbb{R}^n$ denote the field of real numbers and the vector space of all n-tuples of real numbers, respectively. For $x \in \mathbb{R}^n$ and $i = 1, \ldots, n$, x_i denotes the ith coordinate of x. Similarly, $\mathbb{R}^{n\times n}$ denotes the space of $n \times n$ matrices with real entries and for $A \in \mathbb{R}^{n\times n}$, A_{ij} denotes the (i, j)th entry of A. The convex hull of a set $C \subset \mathbb{R}^n$ is denoted by $\operatorname{conv} C$.

We now recall standard concepts regarding proper cones, as discussed in [41, 35, 38]. As usual, a cone $K \subset \mathbb{R}^n$ is a nonempty set satisfying $rK \subset K$ for all real $r > 0$. We will always consider a proper cone K, that is a cone, which is (i) convex, so that $x + y \in K$ for all $x, y \in K$, (ii) pointed, i.e. $K \cap -K = \{0\}$, (iii) closed and (iv) full, i.e. the interior of K is nonempty. The interior of a set $C \subset \mathbb{R}^n$ is denoted by $\operatorname{int} C$, the closure by $\operatorname{cl} C$ and its boundary by ∂C. The ball of radius $\varepsilon > 0$ around $0 \in \mathbb{R}^n$ is denoted by $B(0, \varepsilon)$.

Given a proper cone K we consider the partial order induced by K on $\mathbb{R}^n$. For vectors $x, y \in \mathbb{R}^n$, we write: $x \geq_K y$ if $x - y \in K$; $x >_K y$ if $x \geq_K y$ and $x \neq y$; $x \gg_K y$ if $x - y \in \operatorname{int} K$. A base B of a cone K is a set with the properties that

$$0 \notin \operatorname{cl} B \quad \text{and} \quad K = \mathbb{R}_+ B = \{rx \mid x \in B, r \geq 0\}.$$

We will always consider compact bases, that are given as the intersection of a hyperspace X in $\mathbb{R}^n$ with K. It is known that a cone has such a base, if and only if it is pointed.

It will be useful to study norms, which are adapted to the nonnegative setting. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is called *monotone* with respect to the ordering induced by K if

$$x \geq_K y \geq_K 0 \quad \Rightarrow \quad \|x\| \geq \|y\|. \tag{1}$$

For any proper cone and the corresponding order, a monotone norm on $\mathbb{R}^n$ exists, see [27, p. 38].

A matrix $A \in \mathbb{R}^{n \times n}$ is called nonnegative, or cone preserving, with respect to a proper cone K if $AK \subset K$. The set of cone-preserving linear maps is denoted by $\pi(K)$. We say that A is K-positive if $A\,(K \setminus \{0\}) \subset \operatorname{int} K$.

Following [37] we say that A is exponentially K-nonnegative, if $e^{At} \in \pi(K)$, i.e., $e^{At}K \subset K$, for all $t \geq 0$. This property is also known as *cross-positivity* of A with respect to the cone K, see [35, 14]. We denote the set of exponentially K-nonnegative matrices by $\pi_{\exp}(K)$. Also A is exponentially K-positive, if e^{At} is K-positive, for all $t > 0$.

For the concept of irreducibility of a cone-preserving map $A \in \pi(K)$, recall that a face F of a proper cone K is a cone contained in K, which also has the property, that

$$x \in F \text{ and } x \geq_K y \geq_K 0 \quad \Longrightarrow \quad y \in F. \tag{2}$$

The faces $\{0\}$ and K are the trivial cases of faces of K. Now $A \in \pi(K)$ is called K-reducible, if $AF \subset F$ for some nontrivial face F of K and K-irreducible, if it is not K-reducible. We will need the following characterisation of K-irreducibility, [41, Theorem 4.1 and Lemma 4.2]: an $A \in \pi(K)$ is K-irreducible, if and only if one of the following equivalent conditions is satisfied:

(IR1) A does not have an eigenvector on ∂K,

(IR2) $(I + A)^{n-1}(K \setminus \{0\}) \subset \operatorname{int} K$.

(IR3) For every compact base B of K we have $(I + A)^{n-1} B \subset \operatorname{int} K$.

(IR4) For every compact base B of K and all choices of positive coefficients $\alpha_i > 0, i = 1, \ldots, n-1$ we have for corresponding linear combination of the powers of A that

$$\left(\sum_{i=1}^{n-1} \alpha_i A^i \right) B \subset \operatorname{int} K.$$

The equivalence of K-irreducibility and (IR1) is proved in [41], as well as the fact that K-irreducibility implies (IR2). The remaining implications are trivial.

Remark 1.1 Note that from condition (IR2) or (IR3) we see that K-irreducibility is an open property in $\pi(K)$ in the topology induced by the norm topology on $\mathbb{R}^{n \times n}$. Also the set of K-positive matrices is open in $\mathbb{R}^{n \times n}$.

An exponentially K-nonnegative matrix has the property that e^{At} is K-irreducible for all t with the possible exception of a discrete set if and only if A has no eigenvector in ∂K, see [35, Lemma 8]. We require a slightly different view of this statement as follows. Again let K be a proper cone. For any face F of K we denote by H_F the span of F, that is the smallest linear space containing F. Recall, that for any vector $x \in \partial K$, $x \neq 0$, there exists a face F of K, such that $x \in \operatorname{int}_{H_F} F$, where $\operatorname{int}_{H_F} F$ denotes the interior of F relative to the subspace H_F, [41, Lemma 2.1].

Lemma 1.2 *Let $A \in \mathbb{R}^{n\times n}$ be exponentially K-nonnegative. The following are equivalent:*

(i) A has no eigenvector in ∂K,

(ii) e^{At} is K-irreducible for all $t \geq 0$ with the possible exception of a discrete set,

(iii) there exists a $t > 0$ such that e^{At} is K-irreducible.

(iv) there does not exist a nontrivial face F of K such that H_F is A-invariant.

Proof: The equivalence of (i) and (ii) is proved in [35, Lemma 8]. The implication (ii) $\Rightarrow$ (iii) is obvious. If (iii) holds, then e^{At} does not have an eigenvector in ∂K by (IR1). As the eigenvectors of A are also eigenvectors of e^{At} we obtain (i). If (iv) does not hold, then there exists a nontrivial face F of K such that $A_{|H_F}$ is exponentially F-nonnegative. By [35, Theorem 6] this implies that $A_{|H_F}$ has an eigenvector in F. This is also an eigenvector of A so that (i) is false.

Finally, let $Ax = \lambda x$ for $x \in \partial K \setminus \{0\}$. Now let F with span H_F such that $x \in \operatorname{int}_{H_F} F$. For any $t > 0$ we have that $e^{At}x = e^{\lambda t}x$ and for $0 \neq y \in F$ we may choose an $\alpha > 0$ such that $\alpha y \ll_F x$, as $x \in \operatorname{int}_{H_F} F$. This implies $\alpha y \leq_K x$ and so $0 \leq_K e^{At}y \leq_K e^{At}x = \alpha e^{\lambda t}x$. By the defining property of a face this implies that $e^{At}y \in F$. As $t > 0$ was arbitrary we see that $e^{At}F \subset F$ for all $t > 0$ from which it follows that $e^{At}H_F \subset H_F$ for all $t > 0$ and so $AH_F \subset H_F$. This concludes the proof. □

We note that in the proof of $\neg$ (i) $\Rightarrow$ $\neg$ (iv), we have followed ideas already used in [41]. With a slight but common abuse of terminology, we call an exponentially K-nonnegative matrix A irreducible, if A satisfies one of the equivalent conditions of Lemma 1.2.

Remark 1.3 Concerning Lemma 1.2 (ii), note that the corresponding statement in [35] always speaks of an at most countable exceptional set of times t at which e^{At} is not irreducible. As the proof in [35] uses analyticity arguments the formulation we use here is actually the statement proved in the original paper. □

We will also need the following two observations about cones and spectra of matrices.

Lemma 1.4 *Let $K \subset \mathbb{R}^n$ be a proper cone and let B be a compact base of K. For every compact $C \subset \operatorname{int} K$ there is a $\delta > 0$ such that*

$$\delta x \ll_K y\,, \quad \forall\, x \in B, y \in C\,. \tag{3}$$

Proof: Let $y \in C$. By assumption $y \gg_K 0$, so $0 \in \operatorname{int}(y - K)$. Let $\varepsilon > 0$ be small enough so that $B(0,\varepsilon) \subset \operatorname{int}(y-K)$. As B is compact there exists a $\delta_y > 0$ such that $\delta_y B \subset B(0,\varepsilon/2)$. Then if $\|y - z\| < \varepsilon/2$ we have

$$(y - z) + \delta_y B \subset B(0,\varepsilon) \subset \operatorname{int}(y - K)$$

and so $\delta_y B \subset \operatorname{int}(z-K)$. Thus for every $y \in C$ there exists a neighbourhood U_y and a $\delta_y > 0$ such that $\delta_y x \ll_K z$ for all $x \in B, z \in U_y$. Choose a finite subcover $U_{y_1}, \ldots, U_{y_\ell}$ of the cover $\{U_y \mid y \in C\}$. Then $\delta := \min\{\delta_{y_1}, \ldots, \delta_{y_\ell}\} > 0$ satisfies (3). □

Lemma 1.5 *Let $K \subset \mathbb{R}^n$ be a proper cone and $A \in \pi(K)$. If there exist $x \in K, c > 0$ such that*

$$Ax \gg_K cx\,,$$

then the spectral radius $r(A) > c$.

Proof: This follows readily from Corollary 1.3.34 of [4]. $\square$

A cone in $\mathbb{R}^n$ is called *simplicial* if it is given as the conical hull of exactly n linearly independent vectors. A particular example is the positive orthant $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$. Simplicial cones are precisely the cones which induce a lattice structure, i.e. an ordering which admits the definition of maximum and minimum. We define $\max_K\{x,y\}$ as the unique $z \in \mathbb{R}^n$ with the property that $z \geq_K x$ and $z \geq_K y$ and

$$w \geq_K x \text{ and } w \geq_K y \quad \Longrightarrow \quad w \geq_K z\,.$$

The absolute value $|x|_K$ of a vector $x \in \mathbb{R}^n$ with respect to a simplicial cone K is then given by

$$|x|_K := \max{}_K\{x, -x\}\,.$$

Again in the case that $K = \mathbb{R}^n_+$ we have that $|x| = |x|_{\mathbb{R}^n_+}$ is defined by $|x|_i := |x_i|, i = 1, \dots, n$.

If the cone is simplicial it is usual to sharpen the definition of a monotone norm to the requirement that

$$|x|_K \geq_K |y|_K \quad \Rightarrow \quad \|x\| \geq \|y\|\,.$$

This is equivalent to the requirement that $\|x\| = \|\,|x|_K\,\|$ for all $x \in \mathbb{R}^n$, see [3, Theorem 2], [19, Theorem 5.5.10]. Norms with the latter property are called *absolute*. Since there is the potential of misinterpretation of the term monotone, we will use monotone to denote the property defined in (1) and we will always speak of absolute norms, when we have a simplicial cone.

2 Linear Inclusions and the Joint Spectral Radius

We now discuss semigroups of matrices $\mathcal{S} \subset \mathbb{R}^{n\times n}$ that have an associated concept of time and thus a growth rate. There are several ways of introducing such semigroups and we discuss three of them. The first corresponds to discrete time switched systems, the second to continuous time switched systems and the third one encompasses switched differential algebraic systems, [36, 40].

A. The Discrete Time Case

For a compact set $\mathcal{M} \subset \mathbb{R}^{n\times n}$ we consider the linear inclusion

$$x(t+1) \in \{Mx(t) \mid M \in \mathcal{M}\}\,, t \in \mathbb{N}\,. \tag{4}$$

Solutions of (4) corresponding to the initial value x_0 are given by sequences $\{x(t)\}_{t\in\mathbb{N}}$ where for each $t \in \mathbb{N}$ there exists an $A(t) \in \mathcal{M}$ such that $x(t+1) = A(t)x(t)$. The evolution operators generated by $\mathcal{M}$ are therefore the sets

$$\mathcal{S}_t := \{A(t-1)\dots A(0) \mid A(s) \in \mathcal{M}, s = 0, \dots, t-1\}\,,$$

and the associated matrix semigroup is $\mathcal{S} := \bigcup_{t\in\mathbb{N}} \mathcal{S}_t$, where we set $\mathcal{S}_0 := \{I\}$. If we want to emphasise that $\mathcal{S}$ is generated by $\mathcal{M}$ in the discrete time setting, we write $\mathcal{S} = \mathcal{S}(\mathcal{M}, \mathbb{N})$.

B. The Continuous Time Case

In the continuous time setting we define linear inclusions as follows. Given a compact set of matrices $\mathcal{M}$ we consider a linear inclusion of the form

$$\dot{x} \in \{Mx \mid M \in \mathcal{M}\}\,. \tag{5}$$

The evolution operators defined by (5) are given by solutions to the differential equation

$$\dot{\Phi}_\sigma(t) = A(t)\Phi_\sigma(t)\,, \quad \Phi_\sigma(0) = I\,,$$

where $\sigma := A : \mathbb{R}_+ \to \mathcal{M}$ is measurable. It is possible to consider only piecewise continuous functions A with locally finitely many discontinuities; this neither changes the notions of positivity nor of stability discussed below. The map σ is called the switching signal defining the differential equation and the notation Φ_σ is a reminder that it is defined via a particular switching signal. In this case the set of time t evolution operators is given by

$$\mathcal{S}_t := \{\Phi_\sigma(t) \mid \sigma : [0,t] \to \mathcal{M} \text{ measurable}\}$$

and again $\mathcal{S} := \bigcup_{t\in\mathbb{R}_+} \mathcal{S}_t$, where we set $\mathcal{S}_0 := \{I\}$. If we want to emphasise that $\mathcal{S}$ is generated by $\mathcal{M}$ in the continuous time setting, we write $\mathcal{S} = \mathcal{S}(\mathcal{M}, \mathbb{R}_+)$.

C. Switched Linear Systems with Jumps

We now discuss a class of impulsive systems which encompasses the case of switched DAEs, as shown in [40]. Consider a compact set $\mathcal{M} \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, where each pair $(A, \Pi) \in \mathcal{M}$ has the property that

$$\Pi^2 = \Pi\,, \quad A\Pi = \Pi A\,.$$

For a piecewise constant and right-continuous switching signal $\sigma : \mathbb{R}_+ \to \mathcal{M}$, $t \mapsto (A_{\sigma(t)}, \Pi_{\sigma(t)})$, with discontinuities $0 = t_0 < t_1 < t_2 < \ldots < t_k \to \infty$ we define the corresponding evolution operator for $t \in [t_k, t_k + 1)$ by

$$\Phi_\sigma(t,0) = e^{A_{\sigma(t)}(t-t_{k-1})}\Pi_{\sigma(t_{k-1})} \ldots e^{A_{\sigma(t_1)}(t_2-t_1)}\Pi_{\sigma(t_1)}e^{A_{\sigma(t_0)}(t_1-t_0)}\Pi_{\sigma(t_0)}\,. \tag{6}$$

Again, for every $t \geq 0$, $\mathcal{S}_t$ is defined as the set of all possible evolution operators that can be defined by (6) and $\mathcal{S} := \bigcup_{t\in\mathbb{R}_+} \mathcal{S}_t$. It is shown in [40, Lemma 6] that this defines a semigroup. Also it is shown in Proposition 10 of that reference that the sets $\mathcal{S}_t$ are bounded as subsets of $\mathbb{R}^{n\times n}$ if and only if the set of projections

$$\mathcal{M}_\Pi := \{\Pi \mid \exists\ A \in \mathbb{R}^{n\times n} \text{ such that } (A, \Pi) \in \mathcal{M}\}$$

is product bounded. The set of projections is product bounded if the discrete semigroup $\mathcal{S}(\mathcal{M}_\Pi, \mathbb{N})$ is bounded.

Growth Rates

Given a semigroup defined in one of the ways we have discussed so far, we define the joint spectral radius of $\mathcal{S}$ by setting

$$\rho(\mathcal{S}) := \lim_{t\to\infty} \sup\{\|S\| \mid S \in \mathcal{S}_t\}^{1/t}\,, \tag{7}$$

where we will suppress the fact that depending on the situation at hand $t \in \mathbb{N}$ or $t \in \mathbb{R}_+$.

It is well known that the limit exists, is independent of the norm considered, and characterises the maximal and uniform exponential growth of solutions to (4), resp. (5) or (6) [2, 36, 20, 40]. We will need the following property of the joint spectral radius, which is independent of the particular way in which the semigroup is defined:

$$\rho(\mathcal{S}) := \limsup_{t\to\infty} \sup\{r(S) \mid S \in \mathcal{S}_t\}^{1/t}\,. \tag{8}$$

Also in the discrete and the continuous time case we have by [2] the convexity relation

$$\rho(\mathcal{S}(\mathcal{M})) = \rho(\mathcal{S}(\operatorname{conv}\mathcal{M}))\,. \tag{9}$$

An immediate consequence of (8) and (9) is the property that

$$r(A) \leq \rho(\mathcal{S}(\mathcal{M}))\,, \quad \forall A \in \operatorname{conv}\mathcal{M}\,. \tag{10}$$

This property is proved using extremal norms in [2, Part I] and an alternative argument for this relation is provided in [5, Lemma 1].

In the following we concentrate on K-positive switched systems for a regular cone K.

It is clear that in the discrete time case $\mathcal{S}(\mathcal{M}, \mathbb{N}) \subset \pi(K)$, if and only if $\mathcal{M} \subset \pi(K)$, whereas in the continuous time case $\mathcal{S}(\mathcal{M}, \mathbb{R}_+) \subset \pi(K)$ if and only if $\mathcal{M} \subset \pi_{\exp}(K)$. From these two relations it is easy to see, that in the case of switched systems with jumps, we have that K-positivity is equivalent to the to requirements (i) $\mathcal{M}_\Pi \subset \pi(K)$ and (ii) $A\Pi \in \pi_{\exp}(K)$ for all $(A, \Pi) \in \mathcal{M}$.

For positive systems irreducibility of matrices plays a decisive role.

Definition 2.1 (Irreducibility of Inclusions) *Let $K \subset \mathbb{R}^n$ be a proper cone. A semigroup $\mathcal{S} \subset \pi(K)$ is called K-irreducible, if there exists a $t > 0$ such that $\operatorname{conv} \mathcal{S}_t$ contains a K-irreducible element.*

While the previous definition has the advantage of being independent of the particular definition of the semigroup, it is instructive to point out what the definition amounts to in the different cases. To this end the following observation is of interest. In the continuous-time case we provide an argument for matrices of the form $A + \lambda I, A \in \pi(K), \lambda \in \mathbb{R}$. Denoting $\Lambda := \{\lambda I \mid \lambda \in \mathbb{R}\}$, it is known that $\pi(K) + \Lambda \subset \pi_{\exp}(K)$. For general cones the two sets are distinct, but for polyhedral cones equality holds, [35, 14].

Proposition 2.2 *Let $K \subset \mathbb{R}^n$ be a proper cone.*

(i) If $\emptyset \neq \mathcal{M} \subset \pi(K)$, then $\operatorname{conv} \mathcal{M}$ contains a K-irreducible element if and only if for every nontrivial face F of K there exists an $A \in \mathcal{M}$ such that $AF \not\subset F$.

(ii) If $\emptyset \neq \mathcal{M} \subset \pi(K) + \Lambda$ is bounded, then $\operatorname{conv} \mathcal{M}$ contains a K-irreducible element if and only if for every nontrivial face F of K there exists an $A \in \mathcal{M}$ such that $A \operatorname{span} F \not\subset \operatorname{span} F$.

Proof: (i) As $A \in \pi(K)$ is K-irreducible if and only if rA is for every $r > 0$, we may replace every nonzero $A \in \mathcal{M}$ by $A/\|A\|$ and obtain a bounded set $\mathcal{M}$. It is thus sufficient to prove the claim for bounded sets $\mathcal{M} \subset \pi(K)$.

If there exists a nontrivial face F of K such that $AF \subset F$ for all $A \in \mathcal{M}$, then the same is clearly true for all $A \in \operatorname{conv} \mathcal{M}$ and so $\operatorname{conv} \mathcal{M}$ does not contain a K-irreducible element.

Conversely, assume that for every nontrivial face F of K there exists an $A \in \mathcal{M}$ such that $AF \not\subset F$. By Remark 1.1 it is sufficient to show that $\operatorname{cl} \operatorname{conv} \mathcal{M}$ contains a K-irreducible element. Let $\mathcal{Q} = \{A_1, A_2, \ldots\}$ be a dense sequence lying in $\mathcal{M}$ and choose a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ such that

$$\sum_{k=1}^{\infty} \varepsilon_k = 1 \,, \qquad \text{and} \qquad \varepsilon_k > 0 \,, \;\; \forall k \in \mathbb{N} \,.$$

By construction $\overline{A} := \sum_{k=1}^{\infty} \varepsilon_k A_k \in \operatorname{cl} \operatorname{conv} \mathcal{M}$. For every nontrivial face F of K we may choose an $A \in \mathcal{M}$ such that $AF \not\subset F$ and as this is an open property there exists an index j such that $A_j F \not\subset F$. Now for any $B_1, B_2 \in \pi(K)$ we have as a consequence of (2) that $(B_1 + B_2) F \subset F$ implies $B_i F \subset F, i = 1, 2$. With this argument it follows that $\overline{A} F \not\subset F$ and as F was arbitrary this shows that $\overline{A}$ is K-irreducible.

(ii) This follows from (i), by considering $\mathcal{M} + rI \subset \pi(K)$ for some $r > 0$ large enough. □

Proposition 2.3 *Let $K \subset \mathbb{R}^n$ be a proper cone.*

(i) Let $\mathcal{S} = \mathcal{S}(\mathcal{M}, \mathbb{N}) \subset \pi(K)$ be generated via the discrete inclusion (4). Then $\mathcal{S}$ is K-irreducible if and only if $\operatorname{conv} \mathcal{M}$ contains a K-irreducible element.

(ii) Let $\mathcal{S} = \mathcal{S}(\mathcal{M}, \mathbb{R}_+) \subset \pi(K)$ be generated via the continuous inclusion (5). Then $\mathcal{S}$ is K-irreducible if and only if for every nontrivial face F of K there exists an $A \in \mathcal{M}$ such that $A \operatorname{span} F \not\subset \operatorname{span} F$.

In case that $\mathcal{M} \subset \pi(K) + \Lambda$ the latter statement is equivalent to the existence of an irreducible, exponentially nonnegative $A \in \operatorname{conv} \mathcal{M}$.

Proof: (i) As $\mathcal{S}_1 = \mathcal{M}$, it is clear that if $\operatorname{conv} \mathcal{M}$ contains a K-irreducible element, then $\mathcal{S}$ is irreducible. Conversely, if $\operatorname{conv} \mathcal{M}$ does not contain a K-irreducible element, then by Proposition 2.2 (i) there exists a nontrivial face F of K such that $AF \subset F$ for all $A \in \mathcal{M}$. It follows that for any $t \in \mathbb{N}$ we have $SF \subset F$ for all $S \in \mathcal{S}_t$. Then $SF \subset F$ for all $S \in \operatorname{conv} \mathcal{S}_t$ and as t is arbitrary, $\mathcal{S}$ is K-reducible.

(ii) If there exists a nontrivial face F of K such that for all $A \in \mathcal{M}$ we have $AH_F \subset H_F$, then for all $A \in \mathcal{M}$ and all $t \geq 0$ we have $e^{At}F \subset F$. By classical relaxation arguments, we have that products of the form

$$e^{A_k t_k} e^{A_{k-1} t_{k-1}} \dots e^{A_1 t_1}, \quad A_j \in \mathcal{M}, t_j > 0, j = 1, \dots, k, \sum_{j=1}^{k} t_j = t$$

lie dense in $\mathcal{S}_t(\mathcal{M})$. This shows that $SF \subset F$ for all $S \in \operatorname{conv} \mathcal{S}_t$ and as $t > 0$ is arbitrary it follows that $\mathcal{S}$ is reducible.

Conversely, assume that for every nontrivial face F of K there exists an $A \in \mathcal{M}$ such that $AH_F \not\subset H_F$. Choosing a dense sequence $\mathcal{Q} = \{A_1, A_2, \dots\}$ lying in $\mathcal{M}$ we have as before that for every nontrivial face F of K there exists an $A_j \in \mathcal{M}$ such that $A_j H_F \not\subset H_F$. For every index j consider the set $T_j \subset \mathbb{R}_+$ of times t for which the invariant subspaces of $e^{A_j t}$ coincide with those of A_j. It is well known that T_j is the complement of a set of Lebesgue measure 0. It follows that there exists a $\bar{t} \in \bigcap_{j \in \mathbb{N}} T_j$. Thus for all nontrivial faces F of K there exists an index j such that $e^{A_j \bar{t}} F \not\subset F$. It then follows from Proposition 2.2 (i) that $\operatorname{conv} \mathcal{S}_{\bar{t}}$ contains an irreducible element.

The final statement follows from Proposition 2.2 (ii).

□

3 Extremal Norms

In the analysis of linear inclusions extremal and Barabanov norms play an interesting role. In this paper we restrict our attention to extremal norms. Indeed, in the situations we consider we cannot guarantee that a Barabanov norm exists.

Definition 3.1 *Let $\mathcal{S} = \bigcup_{t \geq 0} \mathcal{S}_t \subset \mathbb{R}^{n \times n}$ be a semigroup. A norm v on $\mathbb{R}^n$ is called extremal for $\mathcal{S}$, if for all $x \in \mathbb{R}^n$ and all $t \geq 0$ we have*

$$v(Sx) \leq \rho(\mathcal{S})^t v(x), \quad \forall S \in \mathcal{S}_t. \tag{11}$$

In particular, this means that if the semigroup is exponentially stable, which is equivalent to $\rho(\mathcal{S}) < 1$, then an extremal norm is a Lyapunov function that not only characterises exponential stability but also the precise growth rate of the system.

The interesting fact is that existence of a K-irreducible element in the convex hull of some $\mathcal{S}_t$ guarantees the existence of an extremal norm. Note that in general extremal norms need not exist. It is known that existence is equivalent to the boundedness of the semigroup $\{\rho^{-t}(\mathcal{S})S \mid t \geq 0, S \in \mathcal{S}_t\}$, [26], but this is a criterion that is hard to check in general. An additional benefit of the norm constructed here is that it can be chosen to be monotone.

Theorem 3.2 *Let $K \subset \mathbb{R}^n$ be a proper cone. Let $\mathcal{S} \subset \pi(K)$ be a K-irreducible semigroup. If $\rho(\mathcal{S}) = 1$, then $\mathcal{S}$ is bounded.*

Proof: Let B be a compact base of K. Let $t > 0$ be such that $\operatorname{conv} \mathcal{S}_t$ contains a K-irreducible element, which we denote by A. Assume to the contrary that $\mathcal{S}$ is unbounded. Thus there exists a sequence $\{S_k\}_{k \in \mathbb{N}} \subset \mathcal{S}$, a sequence $c_k \to \infty$ and vectors $x_k \in B$ such that

$$S_k x_k \in c_k B. \tag{12}$$

As A is irreducible, we have using (IR2) that

$$(I + A)^{n-1} B \subset \operatorname{int} K. \tag{13}$$

As B is compact, so is the continuous image $(I+A)^{n-1}B$ and by Lemma 1.4 there exists a constant $\delta > 0$ such that for all $x \in B$ we have

$$(I + A)^{n-1} B \gg_K \delta x. \tag{14}$$

Combining (12) and (14) we obtain that

$$(I + A)^{n-1} S_k x_k \gg_K \delta c_k x_k \,. \tag{15}$$

By Lemma 1.5 we obtain $r((I + A)^{n-1} S_k) > \delta c_k$. Now by the binomial theorem

$$(I + A)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} A^k$$

and so by defining

$$\eta_n := \left(\sum_{k=0}^{n-1} \binom{n-1}{k} \right)^{-1} \tag{16}$$

we see that $\eta_n (I + A)^{n-1} S_k$ is a convex combination of elements of the set

$$\overline{M} = \{S_k, AS_k, \ldots, A^{n-1} S_k\} \,.$$

On the other hand A is a convex combination of elements in $\mathcal{S}_t$ and inductively we see, that A^p is a convex combination of elements in $\mathcal{S}_{tp}$. Therefore there exists a set $\tilde{M}_k \subset \mathcal{S}$ such that

$$\eta_n (I + A)^{n-1} S_k \in \operatorname{conv} \tilde{M}_k \,.$$

Also, for k large enough $r(\eta_n (I + A)^{n-1} S_k) > \eta_n \delta c_k > 1$. Using (10) this implies

$$\rho(\tilde{M}_k, \mathbb{N}) > 1 \,, \tag{17}$$

and by (8) some product of the matrices in $\tilde{M}_k$ has a spectral radius larger than 1. Now $\mathcal{S}(\tilde{M}_k, \mathbb{N}) \subset \mathcal{S}$ and so an element in $\mathcal{S}$ has a spectral radius larger than 1. Again using (8) this contradicts $1 = \rho(\mathcal{S})$. This contradiction completes the proof. □

The previous result now allows us to define monotone extremal norms, or even absolute, extremal norms in the case that K is simplicial.

Theorem 3.3 *Let K be a proper cone in $\mathbb{R}^n$. If $\mathcal{S} \subset \pi(K)$ is a K-irreducible semigroup, then there exists a monotone extremal norm v for $\mathcal{S}$. If K is simplicial, then the norm v may be chosen to be absolute.*

Proof: As $\mathcal{S}$ is irreducible there exist $t > 0$ and a K-irreducible $A \in \mathcal{S}_t$. Now $r(A)$ is a simple eigenvalue of A, [27], and so necessarily $r(A) > 0$. Using (10) this implies $\rho(\mathcal{S}) > 0$. Thus considering the semigroup $\{\rho^{-t}(\mathcal{S}) S \mid t \geq 0, S \in \mathcal{S}_t\}$, we may assume without loss of generality that $\rho(\mathcal{S}) = 1$. Using Theorem 3.2 it follows that $\mathcal{S}$ is bounded.

An extremal norm may then be defined in the following way, see also [26]. Let $\| \cdot \|$ be a K-monotone norm on $\mathbb{R}^n$. Then define $v : \mathbb{R}^n \to \mathbb{R}$ by setting, for $x \in \mathbb{R}^n$,

$$v(x) := \sup\{\|Sx\| \mid S \in \mathcal{S}\} \,. \tag{18}$$

It is clear that v is positively homogeneous and positive definite, as $I \in \mathcal{S}$, and well-defined by boundedness of $\mathcal{S}$. The triangle inequality for v follows from

$$v(x + y) = \sup\{\|S(x + y)\| \mid S \in \mathcal{S}\} \leq \sup\{\|Sx\| + \|Sy\| \mid S \in \mathcal{S}\} \leq v(x) + v(y) \,.$$

Using the assumption $\rho(\mathcal{S}) = 1$ and the semigroup property of $\mathcal{S}$ we obtain extremality of v from

$$v(Sx) = \sup\{\|TSx\| \mid T \in \mathcal{S}\} \leq v(x) \,. \tag{19}$$

Finally the monotonicity of v is inherited from the monotonicity of $\|\cdot\|$ as follows. For $0 \leq_K x \leq_K y$ we have $0 \leq_K Sx \leq_K Sy$ for $S \in \mathcal{S}$ and so $\|Sx\| \leq \|Sy\|$ for all $S \in \mathcal{S}$. Hence

$$v(x) = \sup\{\|Sx\| \mid S \in \mathcal{S}\} \leq \sup\{\|Sy\| \mid S \in \mathcal{S}\} = v(y) \,.$$

If K is simplicial, then we choose an absolute norm $\|\cdot\|$ to perform a variant of the construction described above. In this case we define

$$v(x) := \sup\{\|S|x|_K\| \mid S \in \mathcal{S}\}\,.$$

By definition v satisfies $v(x) = v(|x|_K)$ for all $x \in \mathbb{R}^n$. Also if $0 \leq_K |x|_K \leq_K |y|_K$, then

$$v(x) = \sup\{\|S|x|_K\| \mid S \in \mathcal{S}\} \leq \sup\{\|S|y|_K\| \mid S \in \mathcal{S}\} = v(y)\,. \tag{20}$$

The triangle inequality for v then follows from

$$\begin{aligned} v(x+y) = v(|x+y|_K) &\overset{(20)}{\leq} v(|x|_K + |y|_K) = \sup\{\|S(|x|_K + |y|_K)\| \mid S \in \mathcal{S}\} \\ &\leq \sup\{\|S|x|_K\| + \|S|y|_K)\| \mid S \in \mathcal{S}\} \leq v(|x|_K) + v(|y|_K) = v(x) + v(y)\,. \end{aligned}$$

Positive definiteness and homogeneity are again clear and the extremality property follows as in (19). This concludes the proof. □

Combining the characterisation of K-irreducibility from Proposition 2.3 we immediately obtain the following corollary.

Corollary 3.4 *Let $K \subset \mathbb{R}^n$ be a proper cone.*

(i) If $\mathcal{M} \subset \pi(K)$ is compact and if $\operatorname{conv}\mathcal{M}$ *contains a K-irreducible element, then there exists a K-monotone, extremal norm v for $\mathcal{S}$ generated by* (4).

(ii) If $\mathcal{M} \subset \pi_{\exp}(K)$ is a compact and if for every nontrivial face F of K there exists an $A \in \mathcal{M}$ such that $A \operatorname{span} F \not\subset \operatorname{span} F$, then there exists a K-monotone, extremal norm v for $\mathcal{S}$ generated by (5).

4 Regularity of the Joint Spectral Radius

For K-irreducible matrices $A \in \mathbb{R}^{n\times n}_+$ it is well known that the spectral radius $\rho(A)$ is a simple eigenvalue of A and that all eigenvalues λ of A of modulus equal to the spectral radius are simple. It is a consequence of standard perturbation theory, that under these conditions the spectral radius as a function of the entries of a matrix is Lipschitz continuous on a neighbourhood of A. In this section we show that by the previous results the same is true for positive linear inclusions that are K-irreducible. This result complements the result of [42], where it was shown that the joint spectral radius is Lipschitz continuous on the set of compact matrix sets that are *irreducible* in the sense of representation theory. In this context this name is a bit misleading, because in the nomenclature of [42] a set of matrices is irreducible, if no subspace other than the trivial ones, $\{0\}$ and $\mathbb{R}^n$, is invariant under all matrices in $\mathcal{M}$. This property is not implied by the assumptions in Theorem 3.3. To see this, consider a pair of positive matrices with respect to $K = \mathbb{R}^{n\times n}_+$ with a common eigenvector (take a set of row stochastic matrices for example). Such a set will automatically satisfy our assumptions but will clearly have a common invariant subspace spanned by the common eigenvector. Hence the set will not be irreducible in the sense used in the work of Barabanov and others. On the other hand, if a set of nonnegative or Metzler matrices has no nontrivial common invariant subspace, it will be K-irreducible in our sense. Hence our assumption is strictly weaker than the usual one.

Let $A \in \mathbb{R}^{n\times n}$ and $\mathcal{M} \subset \mathbb{R}^{n\times n}$ be closed. Then we define the distance from A to $\mathcal{M}$ by

$$\operatorname{dist}(A, \mathcal{M})\} := \min\{\|A - M\| \mid M \in \mathcal{M}\}\,.$$

Note that as $\mathcal{M}$ is closed there exists a $B \in \mathcal{M}$ such that $\|A - B\| = \operatorname{dist}(A, \mathcal{M})\}$. The Hausdorff distance between compact sets of matrices $\mathcal{M}, \mathcal{N}$ is then defined by

$$H(\mathcal{M}, \mathcal{N}) := \max\{\max_{A\in\mathcal{M}}\{\operatorname{dist}(A, \mathcal{N})\}, \max_{B\in\mathcal{N}}\{\operatorname{dist}(B, \mathcal{M})\}\}\,.$$

The particular value of the Hausdorff distance depends on the norm we have chosen on $\mathbb{R}^{n\times n}$, if we want to emphasise this we write $H_{\|\cdot\|}(\mathcal{M}, \mathcal{N})$.

Theorem 4.1 *Let $K \subset \mathbb{R}^n$ be a proper cone.*

(i) In the discrete time case the joint spectral radius is locally Lipschitz continuous on the set

$$\mathcal{P}_{\mathbb{N}} := \{\mathcal{M} \subset \pi(K) \mid \mathcal{M} \text{ is compact and } K\text{-irreducible}\}$$

endowed with the Hausdorff metric.

(ii) In the continuous time case the joint spectral radius is locally Lipschitz continuous on the set

$$\mathcal{P}_{\mathbb{R}_+} := \{\mathcal{M} \subset \pi_{\exp}(K) \mid \mathcal{M} \text{ is compact and } K\text{-irreducible}\}$$

endowed with the Hausdorff metric.

In the proof we follow the idea of [42]. There the proof is based on the consideration of the eccentricity of extremal norms corresponding to different sets. In general, the eccentricity of a norm v with respect to a norm $\|\cdot\|$ is defined by

$$\mathrm{ecc}_{\|\cdot\|}(v) := \frac{\max\{v(x) \mid \|x\| = 1\}}{\min\{v(x) \mid \|x\| = 1\}}\,. \tag{21}$$

Note that for any $A \in \mathbb{R}^{n\times n}$ we have for the induced operator norm that

$$\frac{1}{\mathrm{ecc}_{\|\cdot\|}(v)}\|A\| \le v(A) \le \mathrm{ecc}_{\|\cdot\|}(v)\|A\|\,. \tag{22}$$

The decisive property is now that the eccentricity of absolute extremal norms is bounded on compact subsets of $\mathcal{P}_{\mathbb{N}}$. We note that an analogous statement to [43, Lemma 4.1] is false here, because we cannot exclude the possibility of positively homogeneous functions that have an extremality property and vanish on a subspace. An example to this effect is given by the pair of matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

one of which is clearly irreducible and for which the following function is extremal, but of course not a norm:

$$w(x) := \left\| \begin{bmatrix} 1 & 1 \end{bmatrix} x \right\|_2\,.$$

Proposition 4.2 *Let $K \subset \mathbb{R}^n$ be a proper cone. Let $\mathcal{X} \subset \mathcal{P}_{\mathbb{N}}$ be compact (as a subset of the metric space $(\mathcal{P}_{\mathbb{N}}, H)$) and let $\|\cdot\|$ be K-monotone, then there exists a bound $0 < C < \infty$ such that for all sets $\mathcal{M} \in \mathcal{X}$ there exists a K-monotone norm v, which is extremal for $\mathcal{S}(\mathcal{M}, \mathbb{N})$ and satisfies*

$$\mathrm{ecc}_{\|\cdot\|}(v) < C\,. \tag{23}$$

Proof: We show the property locally in a neighbourhood of $\mathcal{M} \in \mathcal{P}_{\mathbb{N}}$, then the assertion follows by a standard compactness argument.

So let $\mathcal{M} \in \mathcal{P}_{\mathbb{N}}$ and apply Theorem 3.3 to choose a K-monotone, extremal norm for $\mathcal{M}$. We claim that there is a neighbourhood of $\mathcal{M}$ in $\mathcal{P}_{\mathbb{N}}$ for which (23) holds. If this is false then we may pick sequences $\mathcal{M}_k \to \mathcal{M}$ and $C_k \to \infty$ such that every K-monotone, extremal norm of $\mathcal{M}_k$ has eccentricity exceeding C_k.

As norms are convex functions, a norm is extremal for $\mathcal{M}$ if and only if it is extremal for $\mathrm{conv}\,\mathcal{M}$. Thus we may assume that all $\mathcal{M}_k$ and $\mathcal{M}$ are convex. In particular, by assumption there is a K-irreducible matrix $\overline{M} \in \mathcal{M}$. By Remark 1.1, for k large enough there are K-irreducible matrices $\overline{M}_k \in \mathcal{M}_k$ with $\overline{M}_k \to \overline{M}$.

As the joint spectral radius is continuous, [42], we know that $\rho(\mathcal{M}_k) \to \rho(\mathcal{M}) > 0$. This shows that $\rho^{-1}(\mathcal{M}_k)\mathcal{M}_k \to \rho(\mathcal{M})^{-1}\mathcal{M}$. As this rescaling does not change extremal norms, we may assume that all joint spectral radii involved are equal to 1.

Let $\|\cdot\|$ be a K-monotone, extremal norm for $\mathcal{M}$. We construct K-monotone extremal norms v_k for $\mathcal{M}_k$ using (18). This is possible by the construction in Theorem 3.3. Note in particular, that this implies $v_k(x) \ge \|x\|$ for all $x \in \mathbb{R}^n$.

By (IR2) $\overline{M}$ is K-irreducible if and only if $(I+\overline{M})^{n-1}$ is K-positive. Let B be a compact base of K. As $\overline{M}_k \to \overline{M}$ there exists by Lemma 1.4 a constant $\delta > 0$ and an index k_0, such that for all $k \geq k_0$ and all $x \in B$ we have

$$(I+\overline{M}_k)^{n-1} B \gg_K \delta x \,. \tag{24}$$

If $\mathrm{ecc}_{\|\cdot\|}(v_k) > C_k \to \infty$, then as $v_k(\cdot) \geq \|\cdot\|$ it follows from the definition of v_k that for all k sufficiently large there are $S_k \in \mathcal{S}(\mathcal{M}_k)$ and $x_k \in B$ such that (with η_n defined by (16))

$$S_k x_k \in \frac{1}{\eta_n \delta} B \,, \tag{25}$$

and so

$$\eta_n (I+M_k)^{n-1} S_k x_k \gg_K x_k \,. \tag{26}$$

As in the final step of the proof of Theorem 3.2 the combination of (26) and (25) leads to a contradiction to the assumption that $\rho(\mathcal{M}_k) = 1$. This contradiction to the assumption of unbounded eccentricity concludes the proof. □

Now the proof of Theorem 4.1 can be completed following the steps outlined in [42, 43].

Proof: (of Theorem 4.1) (i) In the discrete-time case, let $\mathcal{X} \subset \mathcal{P}$ be compact, let C be as in Proposition 4.2. Pick $\mathcal{M}, \mathcal{N} \in \mathcal{X}$ and an absolute extremal norm v for $\mathcal{M}$. Recall that by definition this implies for the induced matrix norm, also denoted by v, that $v(A) \leq \rho(\mathcal{M})$ for all $A \in \mathcal{M}$. Then for any $B \in \mathcal{N}$, we may choose $A \in \mathcal{M}$ such that $v(A-B) \leq \mathrm{dist}_v(B, \mathcal{M})$ and we obtain

$$v(B) \leq v(A) + v(B-A) \leq \rho(\mathcal{M}) + H_v(\mathcal{M}, \mathcal{N}) \,,$$

where H_v is the Hausdorff distance defined using v. This yields $\rho(\mathcal{N}) \leq \rho(\mathcal{M}) + H_v(\mathcal{M}, \mathcal{N})$. Using (22), we see that $H_v(\mathcal{M}, \mathcal{N}) \leq C H(\mathcal{M}, \mathcal{N})$ and by symmetry the assertion follows.

(ii) The continuous time follows as in [42] by noting that the map $\mathcal{M} \to \mathcal{S}_t(\mathcal{M})$ defines a Lipschitz continuous set-valued map. If $\mathcal{M}$ consists of exponentially K-nonnegative matrices, then $\mathcal{S}_t(\mathcal{M}) \subset \pi(K)$ and the irreducibility property is preserved for almost all t. In this way the continuous-time case is a direct consequence of the discrete-time case. □

Note that the result of Theorem 4.1 does not yield the full force of the statement for single K-irreducible matrices. There we may obtain Lipschitz continuity of the spectral radius on a neighbourhood which may also include matrices not in $\pi(K)$. So far our result is restricted to neighbourhoods of K-nonnegative matrix sets, but we expect it can be extended to larger neighbourhoods of irreducible sets of nonnegative matrices. We note however that K-positivity is an open property. We thus obtain immediately

Corollary 4.3 *Let $K \subset \mathbb{R}^n$ be a proper cone.*

(i) The (discrete-time) joint spectral radius is locally Lipschitz continuous on the set of compact subsets of

$$\{A \in \pi(K) \mid \ A \text{ is } K\text{-positive}\} \,,$$

endowed with the Hausdorff metric.

(ii) The (continuous-time) joint spectral radius is locally Lipschitz continuous on the set of compact subsets of

$$\{A \in \pi_{\exp}(K) \mid \ A \text{ is exponentially } K\text{-positive}\} \,,$$

endowed with the Hausdorff metric.

5 Conclusions

In this paper we have considered linear inclusions defining positive systems. We show that under a generalised irreducibility assumption absolute extremal norms exist. As an application local Lipschitz continuity of the joint spectral radius on certain positive linear inclusions is proved. The characterisation of irreducibility in terms of the data of a semigroup of continuous-time systems with jumps similar to the results of Propositions 2.2 and 2.3 remains an open question.

References

[1] M. Ait Rami, V. S. Bokharaie, O. Mason, and F. Wirth. Extremal norms for positive linear inclusions. In *Proc. 20th Int. Symposium on Mathematical Theory of Networks and Systems, MTNS 2012*, Melbourne, Australia, 2012.

[2] N. E. Barabanov. Lyapunov indicator of discrete inclusions. I–III. *Autom. Remote Control*, 49:152–157, 283–287, 558–565, 1988.

[3] F. Bauer, J. Stoer, and C. Witzgall. Absolute and monotonic norms. *Numerische Mathematik*, 3(1):257–264, 1961.

[4] A. Berman and R. J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Classics in Applied Mathematics. SIAM, Philadelphia, PA, USA, 1987.

[5] V. D. Blondel and Y. Nesterov. Computationally efficient approximations of the joint spectral radius. *SIAM Journal on Matrix Analysis and Applications*, 27(1):256–272, 2005.

[6] V. D. Blondel, Y. Nesterov, and J. Theys. On the accuracy of the ellipsoid norm approximation of the joint spectral radius. *Lin. Alg. Appl.*, 394:91–107, 2005.

[7] V. S. Bokharaie, O. Mason, and F. Wirth. On the d-stability of linear and nonlinear positive switched systems. In *The 19th International Symposium on the Mathematical Theory of Networks and Systems (MTNS)*, 2010.

[8] S. Bundfuss and M. Dür. Copositive Lyapunov functions for switched systems over cones. *Systems and Control Letters*, 58(5):342–345, 2009.

[9] X. Dai. Extremal and Barabanov semi-norms of a semigroup generated by a bounded family of matrices. *Journal of Mathematical Analysis and Applications*, 2011.

[10] L. Fainshil, M. Margaliot, and P. Chigansky. On the stability of positive linear switched systems under arbitrary switching laws. *IEEE Transactions on Automatic Control*, 54(4):897–899, 2009.

[11] E. Fornasini and M. E. Valcher. Linear copositive Lyapunov functions for continuous-time positive switched systems. *IEEE Transactions on Automatic Control*, 55(8):1933–1937, 2010.

[12] E. Fornasini and M. E. Valcher. Stabilizability of discrete-time positive switched systems. In *2010 49th IEEE Conference on Decision and Control (CDC)*, pages 432–437. IEEE, 2010.

[13] E. Fornasini and M. E. Valcher. Stability and stabilizability criteria for discrete-time positive switched systems. *IEEE Transactions on Automatic Control*, 57(5):1208–1221, 2012.

[14] P. Gritzmann, V. Klee, and B.-S. Tam. Cross-positive matrices revisited. *Lin. Alg. Appl.*, 223:285–305, 1995.

[15] N. Guglielmi and V. Protasov. Exact computation of joint spectral characteristics of linear operators. *Foundations of Computational Mathematics*, 13(1):37–97, 2013.

[16] N. Guglielmi and M. Zennaro. An algorithm for finding extremal polytope norms of matrix families. *Lin. Alg. Appl.*, 428(10):2265–2282, 2008.

[17] L. Gurvits. Stability of discrete linear inclusions. *Lin. Alg. Appl.*, 231:47–85, 1995.

[18] L. Gurvits, R. Shorten, and O. Mason. On the stability of switched positive linear systems. *IEEE Transactions on Automatic Control*, 52(6):1099–1103, 2007.

[19] R. A. Horn and C. R. Johnson. *Matrix Analysis.* Cambridge University Press, New York, NY, USA, 1985.

[20] R. Jungers. *The Joint Spectral Radius: Theory and Applications*. Springer-Verlag, 2009.

[21] R. M. Jungers. On asymptotic properties of matrix semigroups with an invariant cone. *Lin. Alg. Appl.*, 437(5):1205–1214, 2012.

[22] R. M. Jungers, N. Guglielmi, and A. Cicone. Lifted polytope methods for stability analysis of switching systems. *arXiv preprint arXiv:1207.5123*, 2012.

[23] F. Knorn, O. Mason, and R. Shorten. On linear co-positive Lyapunov functions for sets of linear positive systems. *Automatica*, 45(8):1943–1947, 2009.

[24] V. Kozyakin. Structure of extremal trajectories of discrete linear systems and the finiteness conjecture. *Autom. Rem. Control*, 68(1):174–209, 2007.

[25] V. Kozyakin. An explicit Lipschitz constant for the joint spectral radius. *Lin. Alg. Appl.*, 433(1):12–18, 2010.

[26] V. S. Kozyakin. Algebraic unsolvability of problem of absolute stability of desynchronized systems. *Autom. Rem. Control*, 51(6):754–759, 1990.

[27] M. Krasnosel'skii, E. Lifshits, and A. Sobolev. *Positive Linear Systems: The Method of Positive Operators*. Heldermann-Verlag, Berlin, 1989. Transl. from the Russian by Jürgen Appell.

[28] M. Maesumi. Optimal norms and the computation of joint spectral radius of matrices. *Lin. Alg. Appl.*, 428(10):2324–2338, 2008.

[29] O. Mason, V. S. Bokharaie, and R. Shorten. Stability and d-stability for switched positive systems. *Positive Systems*, pages 101–109, 2009.

[30] O. Mason and R. Shorten. On linear copositive Lyapunov functions and the stability of switched positive linear systems. *Automatic Control, IEEE Transactions on*, 52(7):1346–1349, 2007.

[31] I. Morris. Criteria for the stability of the finiteness property and for the uniqueness of Barabanov norms. *Lin. Alg. Appl.*, 433(7):1301–1311, 2010.

[32] A. Peperko. Bounds on the generalized and the joint spectral radius of Hadamard products of bounded sets of positive operators on sequence spaces. *Linear Algebra and its Applications*, 437(1):189 – 201, 2012.

[33] V. Protasov, R. Jungers, and V. Blondel. Joint spectral characteristics of matrices: A conic programming approach. *SIAM Journal on Matrix Analysis and Applications*, 31(4):2146–2162, 2010.

[34] V. Y. Protasov. When do several linear operators share an invariant cone? *Lin. Alg. Appl.*, 433(4):781–789, 2010.

[35] H. Schneider and M. Vidyasagar. Cross-positive matrices. *SIAM J. on Numerical Analysis*, 7(4):508–519, 1970.

[36] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability theory for switched and hybrid systems. *SIAM Review*, 49(4):545–592, 2007.

[37] R. J. Stern and H. Wolkowicz. Exponential nonnegativity on the ice cream cone. *SIAM Journal on Matrix Analysis and Applications*, 12(1):160–165, 1991.

[38] B.-S. Tam. A cone-theoretic approach to the spectral theory of positive linear operators: The finite-dimensional case. *Taiwanese Journal of Mathematics*, 5(2):pp–207, 2001.

[39] R. Teichner and M. Margaliot. Explicit construction of a Barabanov norm for a class of positive planar discrete-time linear switched systems. *Automatica*, 48(1):95–101, 2012.

[40] S. Trenn and F. Wirth. Linear switched DAEs: Lyapunov exponents, a converse Lyapunov theorem, and Barabanov norms. In *51st IEEE Conference on Decision and Control (CDC2012)*, pages 2666–2671. IEEE, 2012.

[41] J. S. Vandergraft. Spectral properties of matrices which have invariant cones. *SIAM Journal on Applied Mathematics*, 16(6):1208–1222, 1968.

[42] F. Wirth. The generalized spectral radius and extremal norms. *Lin. Alg. Appl.*, 342:17–40, 2002.

[43] F. Wirth. On the structure of the set of extremal norms of a linear inclusion. In *CDC-ECC'05. 44th IEEE Conference on Decision and Control, and 2005 European Control Conference*, pages 3019–3024, 2005.