On global convergence of consensus with nonlinear feedback, the Lure problem, and some applications

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***Abstract*—We give a rigorous proof of convergence of a recently proposed consensus algorithm with output constraint. Examples are presented to illustrate the efficacy and utility of the algorithm.**

I. Introduction

We consider nonlinear systems described by

$$x(k+1) = P(k)x(k) + \mu\big(r - g(x(k))\big)e \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $P(k)$ is a $n \times n$ row stochastic matrix, $e = [1\ 1\ \dots\ 1]^T$, μ and r are scalars while g is a scalar valued function. Equation (1) describes a consensus problem subject to an output constraint. It basically says that if consensus is achieved, it must be achieved subject to the equilibrium constraint $g(x^*) = r$. That is, at equilibrium

$$g(x^*) \quad = \quad r, \tag{2}$$

with $x_i^* = x_j^*$ for all $i, j \in \{1, 2, ..., n\}$. Equation (1) is of interest as it arises in many situations in the study of the *internet of things* (IOT). For example, in some situations a group of agents are asked to achieve a fair allocation of a constrained resource; TCP is an algorithm that strives to achieve this objective in internet congestion control. Recently, similar ideas have been applied in the context of the charging of electric vehicles, smart grid applications, and in the regulation of pollution in an urban context [1]–[3]. A second application arises when one wishes to optimise an objective function subject to certain privacy constraints. For example, collaborative cruise control systems are emerging in which a group of vehicles on a stretch of road share information to determine an advised speed limit that minimises fuel consumption of the swarm subject to some constraint (traffic flow, pollution constraints) [4]. Since each car is individually optimised for a potentially different speed, the technical challenge is for the group of cars to agree on a common speed without an individual revealing any of its inner workings to other vehicles. Another example in this direction arises in the context of deploying services from parked cars as part of an IBM Research project. Here, privacy preserving algorithms of the form of (1) have been deployed and demonstrated to show great promise in the context of load balancing across batteries from a fleet of parked vehicles [5]. Other examples of this nature abound. For example, in many applications a number of sensing devices are asked to agree on a common value (a consensus problem). Each device is subject to some sensing error. The objective is then to find the common value that is most likely; namely, minimises some group-wide uncertainty without the individual uncertainty functions being revealed to other agents. The proliferation of such applications is a direct consequence of large scale connectivity of both devices and people. This connectivity has given rise to a new wave of research focussed on addressing societal inefficiencies in a manner that has hitherto been impossible [6], [7]. At the heart of these engineering applications is the idea that individual things (agents) orchestrate their behaviour to achieve a common goal. Typically, these problems have a common property in that one tries either implicitly or explicitly to solve a consensus problem with an input. As we have mentioned, for reasons of privacy, usually one does not attempt to solve such problems in a fully distributed manner. Neither, for reasons of robustness, scale, and communication overhead, does one attempt to solve them in a centralised manner. Rather, one uses a mix of local communication, and limited broadcast information, to solve these problem in a manner that conceals the private information of each of the individual agents. Implicit and explicit consensus algorithms that exploit local and global communication strategies are proposed and studied in [8]. Equation (1) is perhaps the simplest algorithm of the explicit consensus algorithm with inputs, admitting a very simple intuitive understanding, which can be explained as follows. It is well known that a row stochastic matrix P operates on a vector $x \in \mathbb{R}^n$ such that $\max(x) - \min(x) \geq \max(Px) - \min(Px)$ where $\max(x)$ and $\min(x)$ are defined as the maximum and minimum component in vector x, respectively. Since the addition of $\big(r - g(x(k))\big)e$ does not affect this contraction, intuition suggests that $x_i(k) - x_j(k) \to 0$ as k increases and eventually, the dynamics of (1) will be governed by the following scalar **Lure system**:

$$y(k+1) = y(k) + \mu(r - g(y(k)e)), \tag{3}$$

with $x_i(k) \approx y(k)$ asymptotically for all i. Intuition further suggests, as long as (3) is stable, then so is (1). A plausibility argument along these lines, in support of (1), is given in [8]. However, no formal stability proof is given in that paper. Our objective in this brief note is to address this and to establish conditions on the function g for which global uniform asymptotic stability is assured, thereby giving a rigorous proof of convergence in the process. The general setup we study can be formulated as a special case of the systems studied in

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[9]. In this reference the authors prove local synchronization results for a general class of nonlinear time-varying systems. In contrast to the assumptions of that paper we require fewer differentiability assumptions and state conditions which ensure global convergence. Another contribution of the present paper is to illustrate several generic situations from practical IOT based optimisation problems where this algorithm applies.

II. Notation, Conventions and Preliminary Results

1. Notation. We denote the standard basis in $\mathbb{R}^n$ by the vectors $e_1, \dots, e_n$. Note that $e = \sum_{i=1}^n e_i$. A matrix $P \in \mathbb{R}^{n\times n}$ is called row stochastic, if all its entries are nonnegative and if all its row sums equal one. The row sum condition is equivalent to $Pe = e$, that is, e is an eigenvector of P corresponding to the eigenvalue 1. Hence there is a single transformation which achieves upper block triangularisation of all row stochastic matrices. Let $\{v_2, \dots, v_n\}$ be a basis for the $n-1$ dimensional subspace $e^\perp := \{x \in \mathbb{R}^n \,:\, e^T x = 0\}$. Then $\{e, v_2, \dots, v_n\}$ is a basis of $\mathbb{R}^n$. Consider now the transformation matrix $T := \begin{bmatrix} e & v_2 & \dots & v_n \end{bmatrix}$ which represents a change of basis from the standard basis to the new basis. Under this transformation, a row stochastic matrix P is transformed as follows:

$$T^{-1}PT = \begin{bmatrix} 1 & c \\ 0 & Q \end{bmatrix}. \tag{4}$$

2. Facts about consensus. Given a sequence of row stochastic matrices $\{P(k)\}_{k\in\mathbb{N}}$, consider the time-varying linear system

$$x(k+1) = P(k)x(k). \tag{5}$$

A solution of (5) is represented by the left products of the matrix sequence in the following sense: a sequence $\{x(k)\}_{k\in\mathbb{N}}$ is a solution of (5) corresponding to the initial condition $x(0) = x_0$ if and only if for all $k \in \mathbb{N}$,

$$x(k) = \Phi(k)x_0 \tag{6}$$

where

$$\Phi(k) := P(k-1)\cdots P(0), \quad \forall k \in \mathbb{N}. \tag{7}$$

The sequence $\{P(k)\}_{k\in\mathbb{N}}$ is called weakly ergodic if the difference between each pair or rows converges to zero, i.e. if for all i, j we have

$$\lim_{k\to\infty} \left(e_j^T - e_i^T\right)\Phi(k) = 0. \tag{8}$$

This is equivalent to system (5) being a consensus system, that is, every solution $\{x(k)\}_{k\in\mathbb{R}^n}$ of (5) satisfies

$$\lim_{k\to\infty} x_j(k) - x_i(k) = 0 \tag{9}$$

for all i, j. The sequence $\{P(k)\}_{k\in\mathbb{N}}$ is strongly ergodic if it is weakly ergodic and, in addition, the limit $\lim_{k\to\infty}\Phi(k)$ exists. By a result of Chatterjee and Seneta [10] weak and strong ergodicity are equivalent for left products of row stochastic matrices. This is equivalent to every solution of (5) satisfying

$$\lim_{k\to\infty} x(k) \in E. \tag{10}$$

or, equivalently,

$$\lim_{k\to\infty} \Phi(k)x_0 \in E \tag{11}$$

for all $x_0 \in \mathbb{R}^n$ where $E := \operatorname{span}\{e\}$ is the space of consensus vectors.

We call the sequence $\{P(k)\}_{k\in\mathbb{N}}$ uniformly strongly ergodic, if all tail sequences $\{P(k)\}_{k=k_0}^\infty$ are strongly ergodic for all $k_0 \in \mathbb{N}$. Note that a sequence can be strongly ergodic and not uniformly strongly ergodic. For instance, if one of the matrices in the sequence has rank 1 and all the subsequent matrices are the identity matrix.

Using the transformation (4) a system equivalent to (5) is given by

$$\begin{aligned} z(k+1) &= T^{-1}P(k)Tz(k) \\ T^{-1}P(k)T &:= \begin{bmatrix} 1 & c(k) \\ 0 & Q(k) \end{bmatrix}. \end{aligned} \tag{12}$$

It is then clear that $\{P(k)\}$ is strongly ergodic if and only if

$$\lim_{k\to\infty} Q(k)\dots Q(0) = 0. \tag{13}$$

A useful property in the study of products of row stochastic matrices is the observation that for any row stochastic matrix P we have

$$\min(x) \le \min(Px) \le \max(Px) \le \max(x) \tag{14}$$

for all $x \in \mathbb{R}^n$, where for any vector $y \in \mathbb{R}^n$,

$$\min(y) := \min\{y_1 \dots, y_n\}, \qquad \max(y) := \max\{y_1 \dots, y_n\}.$$

As the associated difference of maximum and minimum plays the role of a Lyapunov function we introduce the notation

$$V(x) := \max(x) - \min(x). \tag{15}$$

Clearly, (14) implies that $V(Px) \le V(x)$. Also, the sequence $\{P(k)\}_{k\in\mathbb{N}}$ is strongly ergodic, if and only if

$$\lim_{k\to\infty} V\big(\Phi(k)x_0\big) = 0 \tag{16}$$

for all $x_0 \in \mathbb{R}^n$ where $\Phi(k)$ is given by (7).

In this note the standard norm used is the Euclidean norm $\|x\| = \sqrt{x^T x}$. Note that any vector $x \in \mathbb{R}^n$ can be uniquely decomposed as

$$x = \overline{x}e + x_\perp$$

where

$$\overline{x} := (1/n)e^T x \tag{17}$$

is the mean of the components of x and

$$x_\perp := x - \overline{x}e \;\in e^\perp, \tag{18}$$

i.e. $e^T x_\perp = 0$. Hence

$$\operatorname{dist}(x, E) = \|x_\perp\| \tag{19}$$

where $\operatorname{dist}(x, E) := \inf\{\|x - z\| \,:\, z \in E\}$ is the distance of a vector $x \in \mathbb{R}^n$ to the consensus set E. Note also that $V(x) = V(x_\perp)$ and $\|x_\perp\|_\infty \le V(x_\perp) \le 2\|x_\perp\|_\infty$ where, for any vector $z \in \mathbb{R}^n$, $\|z\|_\infty = \max_i |z_i|$.

III. Consensus under Feedback

Consider a sequence of row stochastic matrices $\{P(k)\}_{k\in\mathbb{N}}$ and a continuous function $G:\mathbb{R}^n\to\mathbb{R}$. Then, the system,

$$\begin{aligned} x(k+1) &= F(k,x(k)) \\ F(k,x) &:= P(k)x + G(x)e \end{aligned} \tag{20}$$

can be regarded as consensus system under feedback. In later statements, further differentiability assumptions will be imposed on G as required. Associated with (20) we consider the one-dimensional system

$$\begin{aligned} y(k+1) &= h(y(k)) \\ h(y) &:= y + G(ye)\,. \end{aligned} \tag{21}$$

This is the aforementioned Lure system and, as we shall see, the dynamics of the consensus system (20) is strongly related to the dynamics of (21). Unless stated otherwise we consider the systems (20) and (21) with initial time $k_0=0$. A few comments on results that hold uniformly with respect to all initial times are made where appropriate.

A. Local Stability Results

We begin with the following elementary observations.

Lemma 1 *Let $\{P(k)\}_{k\in\mathbb{N}}$ be a sequence of row stochastic matrices and $G:\mathbb{R}^n\to\mathbb{R}$. If $\{y(k)\}_{k\in\mathbb{N}}$ is a solution of (21) then $\{y(k)e\}_{k\in\mathbb{N}}$ is a solution of (20).*

Proof: This follows from $P(k)e=e$. □

The next result tells us that the consensus system under feedback (20) is also a consensus system.

Lemma 2 *Let $\{P(k)\}_{k\in\mathbb{N}}$ be a sequence of row stochastic matrices which is strongly ergodic. Then for every solution $\{x(k)\}_{k\in\mathbb{N}}$ of (20) we have*

$$\lim_{k\to\infty} \operatorname{dist}(x(k),E) = 0\,. \tag{22}$$

Proof: Consider any solution $\{x(k)\}_{k\in\mathbb{N}}$ of (20) and let $x_0=x(0)$. Since $\{P(k)\}_{k\in\mathbb{N}}$ is strongly ergodic we have $\lim_{k\to\infty} V(\Phi(k)x_0)=0$, where $\Phi(k)$ is given by (7). On the other hand,

$$\begin{aligned} V(x(k+1)) &= V\left(P(k)x(k)+G(x(k))e\right) \\ &= V\left(P(k)x(k)\right)\,. \end{aligned}$$

This shows by induction that for all $k\in\mathbb{N}$ we have $V(x(k)) = V(\Phi(k)x_0)$. Hence, $\lim_{k\to\infty} V(x(k))=0$, which is equivalent to $\lim_{k\to\infty} x_j(k)-x_i(k)=0$ for all i,j. This is the same as the desired result (22). □

We now consider the local stability of (20) and see that it is determined by the stability of the induced system (21) on the consensus space. As we have no global concerns no Lipschitz property of G is required. Initially, it is sufficient that G be continuous.

Theorem 3 *Let $\{P(k)\}_{k\in\mathbb{N}}$ be a strongly ergodic sequence of row stochastic matrices and $G:\mathbb{R}^n\to\mathbb{R}$ be continuous. Suppose that y^* is a locally asymptotically stable fixed point of the one dimensional system (21). Then y^*e is a locally asymptotically stable fixed point at time $k_0=0$ for (20).*

If the sequence $\{P(k)\}_{k\in\mathbb{N}}$ is uniformly strongly ergodic, then y^*e is asymptotically stable for all initial times $k_0\in\mathbb{N}$.

Proof: Suppose that y^* is a locally asymptotically stable fixed point for system (21). Let W be a local Lyapunov function which guarantees this stability property. That is, $W(y^*)=0$ and there is an open neighborhood U of y^* such that $W(y)>0$ and $W(h(y))<W(y)$ for all $y\in U\setminus\{y^*\}$. Without loss of generality we may assume U to be a forward invariant set of (21). Define the arithmetic mean of the entries of $x\in\mathbb{R}^n$ by

$$\bar{x} := (1/n)e^T x\,. \tag{23}$$

For $\varepsilon>0$ such that $W^{-1}([0,\varepsilon])\subset U$ is a compact set we may choose $\delta>0$ sufficiently small, so that

$$W(h(y)+d)<\varepsilon \qquad \text{for} \qquad W(y)<\epsilon \quad \text{and} \quad |d|\le\delta\,.$$

This is possible by continuity of all the functions involved and by the decay property of the Lyapunov function W.

Now note that for any $x\in\mathbb{R}^n$, $Px = P(\overline{x}e + x_\perp) = \overline{x}e + Px_\perp$. Hence

$$\overline{Px}-\overline{x} = \overline{Px_\perp} = (1/n)e^T P x_\perp\,. \tag{24}$$

Given a sufficiently small $\varepsilon>0$ and an appropriate δ as above, choose $\eta>0$ such that $V(x)\le\eta$ and $W(\bar{x})\le\epsilon$ implies for any row stochastic matrix P that

$$|\overline{Px}-\overline{x}| + |G(x)-G(\overline{x}e)| < \delta\,.$$

This is possible by uniform continuity of G on a bounded neighborhood of y^*e. Consider now the neighborhood of y^*e given by

$$N_\varepsilon := \{x\in\mathbb{R}^n \,:\, \overline{x}\in U,\, W(\overline{x})<\varepsilon, V(x)<\eta\}\,.$$

We claim that N_ε is forward invariant at all times $k\in\mathbb{N}$. Indeed, if $x(k)\in N_\varepsilon$, then we obtain

$$\begin{aligned} \overline{x}(k+1) &= \overline{P(k)x(k)} + G(x(k)) \\ &= \overline{x}(k) + G(\overline{x}(k)e) + d \\ &= h(\overline{x}(k)) + d \end{aligned}$$

where $d=\overline{P(k)x(k)}-\overline{x}(k)+G(x(k))-G(\overline{x}(k)e)$. Hence $|d|<\delta$ from which it follows that $W(\overline{x}(k{+}1))<\varepsilon$. Referring to the argument in the proof of Lemma 2

$$V(x(k+1)) = V(P(k)x(k)) \le V(x(k)) < \eta.$$

As ε,η were arbitrary, this shows stability of y^*e. To show local attractivity, let $x_0\in N_\varepsilon$ for $\varepsilon>0$ sufficiently small so that stability holds. Note that by Lemma 2 and by stability we have that $\omega(x_0)\subset Ue\subset E$ where $\omega(x_0)$ is the ω-limit set of the solution corresponding to x_0. Suppose that $ye\in\omega(x_0)$ and $y\neq y^*$. Then as the trajectory starting in ye converges to y^*e it follows that $y^*e\in\omega(x_0)$. However, the assumption that y^*e and ye are in the ω-limit set contradicts the stability of y^*e. Hence $\{x(k)\}_{k\in\mathbb{N}}$ converges to y^*e. □

We now extend the previous result to local exponential stability. To this end we call a sequence of row stochastic matrices $\{P(k)\}_{k\in\mathbb{N}}$ exponentially ergodic if it is strongly ergodic and there exist scalars $M \geq 1$, $0 < r < 1$ such that for all $k \in \mathbb{N}$

$$\|\Phi(k) - \Phi_\infty\| \leq Mr^k .$$

The sequence is called uniformly exponentially ergodic, if it is uniformly strongly ergodic and the constants M, r can be chosen such that for all $k, k_0 \in \mathbb{N}$ with $k \geq k_0$ there exists a matrix Φ_∞ such that $\|\tilde{\Phi}(k, k_0) - \Phi_\infty\| \leq Mr^{(k-k_0)}$; where

$$\tilde{\Phi}(k, k_0) \quad := \quad P(k-1)\cdots P(k_0) .$$

Theorem 4 *Let $\{P(k)\}_{k\in\mathbb{N}}$ be an exponentially ergodic. sequence of row stochastic matrices and $G : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Suppose that y^* is a locally exponentially stable fixed point of the one dimensional system* (21). *Then, $y^* e$ is a locally exponentially stable fixed point at time $k_0 = 0$ for* (20). *If the sequence $\{P(k)\}_{k\in\mathbb{N}}$ is uniformly exponentially ergodic, then $y^* e$ is locally uniformly exponentially stable.*

Proof: Consider the linearisation of the one-dimensional map defining (21). By the assumption of exponential stability its modulus must satisfy

$$|h'(y^*)| < 1 \tag{25}$$

where $h'(y^*) = 1 + DG(y^* e)e$ and DG is the derivative of G, which we interpret as a row vector. We now compute the derivative of F with respect to x at $x = y^* e$ and time k to obtain

$$\frac{\partial F}{\partial x}(k, y^* e) = P(k) + eDG(y^* e). \tag{26}$$

If we now consider the transformation T which results in in (12) and using $T^{-1}e = e_1$ we see that

$$T^{-1}\frac{\partial F}{\partial x}(k, y^* e)T = \begin{bmatrix} 1 & c(k) \\ 0 & Q(k) \end{bmatrix} + e_1 DG(y^* e)T . \tag{27}$$

Two things are noticeable when considering this equation. First the resulting transformed matrix is of the form

$$\begin{bmatrix} \lambda & \tilde{c}(k) \\ 0 & Q(k) \end{bmatrix} , \tag{28}$$

where only the first row is affected by G and λ is independent of k. Secondly,

$$\lambda = 1 + DG(y^* e)e = h'(y^*) . \tag{29}$$

Hence $|\lambda| < 1$. By assumption $\|Q(k)Q(k-1)\ldots Q(0)\| \leq Mr^k$ for suitable constants $M \geq 1, r \in (0, 1)$. It now follows that the linearised system of (20) at the fixed point $y^* e$ is exponentially stable. It follows by standard linearisation theory, that the nonlinear system is locally exponentially stable at $y^* e$. If the sequence $Q(k)Q(k-1)\ldots Q(0)$ converges to zero uniformly exponentially, this shows local uniform exponential stability of $y^* e$ for the nonlinear system. □

B. Global Stability Results

To obtain global stability results we first need the following boundedness result.

Lemma 5 *Let $\{P(k)\}_{k\in\mathbb{N}}$ be a strongly ergodic sequence of row stochastic matrices and suppose that $G : \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies the following conditions.*

(i) There exists an $\varepsilon > 0$ such that G satisfies a Lipschitz condition with constant $L > 0$ on the set

$$B_\varepsilon(E) := \{x \in \mathbb{R}^n \,:\, \mathrm{dist}(x, E) \leq \varepsilon\} .$$

(ii) There exists constants $\beta, \gamma > 0$ such that

$$|h(y)| \leq |y| - \gamma \qquad \textit{when} \quad |y| \geq \beta$$

where $h(y) = y + G(ye)$.

Then every trajectory of (20) *is bounded.*

Proof: Consider any solution $\{x(k)\}_{k\in\mathbb{N}}$ of (20) with $x(0) = x_0$. By Lemma 2 there exists a $k_0 \in \mathbb{N}$ such that $x(k) \in B_\varepsilon(E)$ for all $k \geq k_0$. We can express $x(k)$ as $x(k) = \overline{x}(k)e + x_\perp(k)$ where $\overline{x}(k) = (1/n)e^T x(k)$ and $x_\perp(k) := x(k) - \bar{x}(k)e$. It follows from (22) that $\lim_{k\to\infty} \|x_\perp(k)\| = 0$. Hence boundedness of the sequence $\{\bar{x}(k)\}_{k\in\mathbb{N}}$ implies boundedness of $\{x(k)\}_{k\in\mathbb{N}}$. Considering the evolution of $\overline{x}(k)$ we obtain that, for $k \geq k_0$,

$$\begin{aligned} |\overline{x}(k+1)| &= |\overline{P(k)x(k)} + G(\overline{x}(k)e + x_\perp(k))| \\ &\leq |\overline{x}(k) + G(\overline{x}(k)e)| \\ &+ |\overline{P(k)x(k)} - \overline{x}(k)| \\ &+ |G(\overline{x}(k)e + x_\perp(k)) - G(\overline{x}(k)e)| \\ &\leq |h(\overline{x}(k))| + |(1/n)e^T P(k)x_\perp(k)| + L\|x_\perp(k)\| \\ &\leq |h(\overline{x}(k))| + \tilde{L}\|x_\perp(k)\| \end{aligned}$$

where $l := \sup_{k\in\mathbb{N}}(1/n)\|e^T P(k)\|$ and $\tilde{L} := l + L$. Hence

$$|\overline{x}(k+1)| \leq |h(\overline{x}(k))| + \tilde{L}\|x_\perp(k)\| .$$

It now follows from hypothesis (ii) that whenever $|\bar{x}(k)| \geq \beta$, we must have

$$|\bar{x}(k+1)| \leq |\bar{x}(k)| - \gamma + \tilde{L}\|x_\perp(k)\|.$$

Since $\lim_{k\to\infty} \|x_\perp(k)\| = 0$, there exists a $k_* \geq k_0$ such that $\tilde{L}\|x_\perp(k)\| \leq \gamma$ for all $k > k_*$. Thus,

$$|\bar{x}(k+1)| \leq |\bar{x}(k)| \qquad \text{when} \quad k \geq k_* \text{ and } |\bar{x}(k)| \geq \beta.$$

This implies boundedness of $\{\bar{x}(k)\}_{k\in\mathbb{N}}$ and completes the proof. □

Comment: As an example of a general class of functions which satisfy hypothesis (ii) of Lemma 5, consider any strict contraction mapping h on $\mathbb{R}$, i.e., for a suitable constant $c \in (0, 1)$,

$$|h(x) - h(y)| \leq c|x - y| , \quad \forall x, y \in \mathbb{R}.$$

By the Banach contraction theorem, there is a unique fixed point y^* such that $h(y^*) = y^*$. Hence,

$$\begin{aligned} |h(y)| \leq |h(y) - y^*| + |y^*| &\leq c|y - y^*| + |y^*| \\ \leq c|y| + (1+c)|y^*| &= |y| - (1-c)|y| + (1+c)|y^*|. \end{aligned}$$

and hypothesis (ii) is assured with $\beta = \frac{1+c}{1-c}|y^*|$.

Finally, we state a global result on asymptotic or exponential stability. In spirit, the following two results are closely related to [9, Theorem 1]. Note that we obtain a global result and are only concerned with fixed points, not general attractors. Also no assumption on the invertibility of the Jacobian is required.

Theorem 6 *Let $\{P(k)\}_{k\in\mathbb{N}} \subset \mathbb{R}^{n\times n}$ be a strongly ergodic sequence of row stochastic matrices and and suppose that G satisfies all conditions of Lemma 5. If y^* is a globally asymptotically stable fixed point of* (21) *then, y^*e is a globally asymptotically stable fixed point for system* (20).

Proof: The assumptions of Theorem 3 are met and so it only remains to show global attractivity. Note that, by Lemma 5 all solutions of (20) are bounded. By Lemma 2 the ω-limit sets corresponding to all initial conditions lie in E. So consider an ω-limit set $\omega(x_0)$ and assume that $ye \in \omega(x_0)$ but $y \neq y^*$. Let U be a neighborhood of y^*e on which local stability holds according to Theorem 3. We may assume $\mathrm{dist}(ye, U) > 0$. As $ye \in E$ it follows from Lemma 1 that all solutions $x(\cdot; k_0, ye)$ with the initial condition $x(k_0) = ye$ satisfy $\lim_{k\to\infty} x(k; k_0, ye) = y^*e$. Note that on E the system is time-invariant, so that there exists a time K, such that for all k_0 we have $x(k_0 + K; k_0, ye) \in U$. By assumption (i) the maps $x \mapsto P(k)x + G(x)e$ are equicontinuous on $B_\varepsilon(E)$. Choose $\eta > 0$ such that

$$B_{\eta,\infty}(E) := \{x \in \mathbb{R}^n \; ; \; \mathrm{dist}_\infty(x, E) = \min_{r\in\mathbb{R}} \|x - re\|_\infty < \eta\}$$

is contained in $B_\varepsilon(E)$. The set $B_{\eta,\infty}(E)$ is forward invariant under all $F(k, \cdot)$, because if $\mathrm{dist}_\infty(x, E) = \|x - r_x e\|_\infty < \eta$, then as $\|P\|_\infty = 1$ for all row stochastic matrices

$$\mathrm{dist}_\infty(P(k)x + G(x)e, E) \leq \|P(k)(x - r_x e)\|_\infty < \eta \, .$$

Thus there exists a sufficiently small neighborhood U_2 of ye such that for all $k_0 \in \mathbb{N}$ the solution corresponding to the initial condition $x(k_0) \in U_2$ satisfies $x(k_0 + K; k_0, x(k_0)) \in U$. But then by local stability, it follows that $x(k; k_0, x(k_0)) \in U$ for all $k \geq k_0 + K$. We thus arrive at a contradiction, if $ye \in \omega(x_0)$, then there exists a sequence $k_\ell \to \infty$ so that $\lim x(k_\ell; 0, x_0) = ye$. But then $x(k_\ell; 0, x_0) \in U_2$ for a sufficiently large ℓ and hence $x(k; 0, x_0) \in U$ for all $k \geq k_\ell + K$. Hence no subsequence of $\{x(k)\}$ converges to ye. This contradiction completes the proof. □

The previous result can be sharpened, if we assume exponential stability of the fixed point of (21). We omit the proof, which uses the same arguments as the proof of Theorem 6.

Theorem 7 *Let $\{P(k)\}_{k\in\mathbb{N}} \subset \mathbb{R}^{n\times n}$ be a uniformly exponentially ergodic sequence of row stochastic matrices and suppose that $G : \mathbb{R}^n \to \mathbb{R}$ is differentiable and satisfies conditions (i) and (ii) of Lemma 5. Then y^*e is globally uniformly exponentially stable for system* (20).

C. Switched Systems

Given a compact set of row stochastic matrices $\mathcal{P} \subset \mathbb{R}^{n\times n}$, we may consider the switched system

$$x(k+1) = P(k)x(k) + G(x(k))e \, , \tag{30}$$

where $P(k) \in \mathcal{P}$. The results obtained so far have some immediate consequences for consensus under feedback with arbitrary switching. It is well-known that all sequences $\{P_k\}_{k\in\mathbb{N}} \in \mathcal{P}^{\mathbb{N}}$ are strongly ergodic if and only if all sequences in $\mathcal{P}^{\mathbb{N}}$ are uniformly exponentially ergodic [9]. In this case we call $\mathcal{P}$ uniformly ergodic. The rate of convergence towards E is in fact given by the projected joint spectral radius [9].

With this in mind the results obtained so far have immediate consequences for switched systems of the form (30). We note one of these consequences.

Corollary 8 *Let $\mathcal{P}$ be a compact set of row stochastic matrices that is uniformly ergodic and suppose that $G : \mathbb{R}^n \to \mathbb{R}$ is differentiable and satisfies conditions (i) and (ii) of Lemma 5. Then y^*e is globally uniformly exponentially stable for the switched system* (30) *under arbitrary switching.*

IV. Applications

In this section, we show how to apply the results of the previous sections to solve the optimisation problems. The following set-up resembles that in [8], [11]. The significant extension here is the application of (20) to solve optimised consensus problems, rather than the regulation problems with fairness constraints that were considered in those papers. Specifically, we consider a scenario in which there are N agents connected in a network through communication links. Let $\underline{\mathrm{N}}$ denote the set $\{1, 2, ..., N\}$ for indexing the agents. We assume that each communication link between the agents is directional and time-varying. Associated with each agent is a utility function f that has a different meaning depending on the application at hand. Finally, we assume that each agent has access to a simple broadcast signal; and so there is no significant constraint on the communication topology. This setting is depicted in Fig.1 [11].

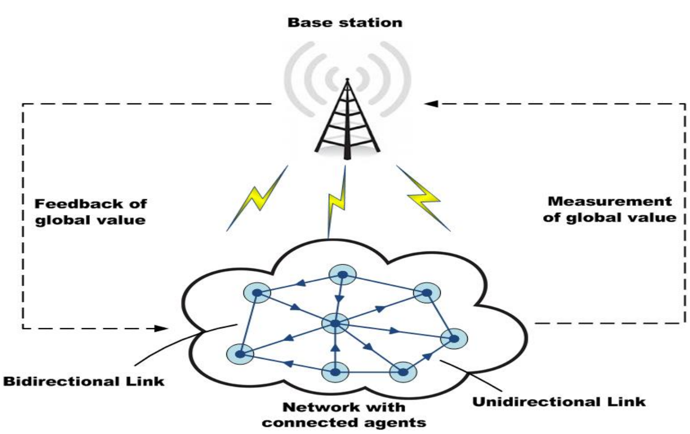

Fig. 1. Schematic diagram of the optimisation framework [11]

The problem we wish to solve is to find an optimal consensus point satisfying $x^* = y^*e$ such that the following optimisation problem is solved:

$$\begin{aligned} &\underset{x\in\mathbb{R}^{\mathrm{N}}}{\text{minimise}} \quad \sum_{i=1}^{N} f_i(x_i) \\ &\text{subject to:} \quad x_i = x_j, \; \forall i, j \in \underline{\mathrm{N}} \, . \end{aligned} \tag{31}$$

We wish to use an iterative feedback scheme of the form (20) to solve the optimisation problem. We will require that (31) has a unique solution and derive the specific form for G in (20) from first order optimality conditions. To this end, it follows from elementary optimisation theory that when the f_i's are strictly convex, the optimisation problem will be solved if and only if there exists a unique $y^* \in \mathbb{R}$ satisfying

$$\sum_{i=1}^{N} f_i'(y^*) = 0, \tag{32}$$

where f' denotes the first derivative of the utility function f. With this in mind we apply a feedback signal $G(x) = -\mu \sum_{i=1}^{N} f_i'(x_i)$ where $\mu \in \mathbb{R}$ is a parameter to be determined. This gives rise to the following dynamical system

$$x(k+1) = P(k)x(k) - \mu \sum_{i=1}^{N} f_i'(x_i(k))e \tag{33}$$

where we assume that the sequence $\{P(k)\}_{k\in\mathbb{N}}$ satisfies the conditions of uniform strong ergodicity specified in Section II. As we assume that the f_i are strictly convex, their derivatives are strictly increasing. We assume that each f_i' has a strictly positive and bounded growth, i.e., there exist constants $d_{\min}^{(i)}$ and $d_{\max}^{(i)}$; such that for any $a \neq b$

$$0 < d_{\min}^{(i)} \leq \frac{f_i'(a) - f_i'(b)}{a-b} \leq d_{\max}^{(i)} \qquad \forall i \in \underline{\mathrm{N}}. \tag{34}$$

We claim that provided μ is chosen according to

$$0 < \mu < 2\left(\sum_{i=1}^{N} d_{\max}^{(i)}\right)^{-1} \tag{35}$$

then (33) is uniformly globally asymptotically stable at the unique optimal point x^*e of the optimisation problem (31). First, we consider the scalar system of (33) which is given by

$$y(k+1) = y(k) - \mu \sum_{i=1}^{N} f_i'(y(k)). \tag{36}$$

Note first that the fixed point condition for (36) is $\sum_{i=1}^{N} f_i'(y^*) = 0$. So that a fixed point y^* of (36), gives rise, by Lemma 1 to a fixed point of (33), which satisfies the necessary and sufficient conditions for optimality (32). Now, we wish to use Theorem 6 to show global asymptotic stability. To this end, we need to verify the system (33) satisfies all the conditions required in Theorem 6. The condition (35) ensures in fact that the right hand side of (36) is in fact a strict contraction on $\mathbb{R}$. It follows from our comments after Lemma 5 that the assumption (ii) of Lemma 5 is satisfied. To show the Lipschitz condition (i) note that by (34) each f_i' is globally Lipschitz. As the coordinate functions are globally Lipschitz and sums of globally Lipschitz functions retain that property we obtain condition (ii).

To illustrate this application we consider an application with 20 agents with $f_i(x_i) = a_i \cdot x_i^2 + b_i \cdot x_i + c_i$ where a_i, b_i and c_i are constant parameters chosen in the range (0,1). In addition, the time-varying topology of the network is designed such that the uniform strongly connectivity of agents is guaranteed. μ is chosen to be 0.01. This is accordance with (33). The simulation results are presented in Figure 2.

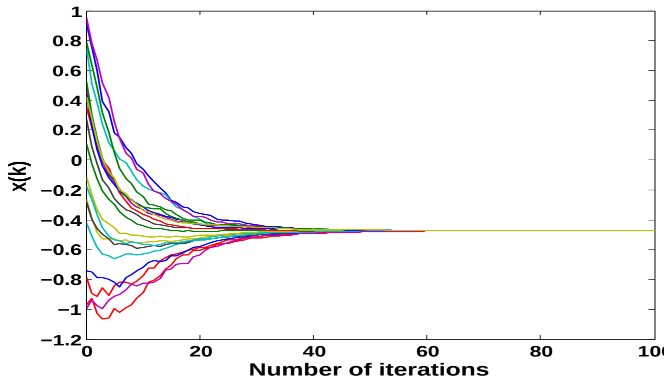

Fig. 2. Dynamics of the state variables $x(k)$ with $\eta = 0.01$ and $\mu = 0.01$

V. Conclusion

In this note we present a rigorous proof of stability and convergence of a recently proposed consensus system with feedback. Examples are given to illustrate the usefulness of the algorithm. For other smart grid applications see [3].

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