Z-POLYNOMIALS AND RING COMMUTATIVITY

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Abstract. We characterise polynomials f with integer coefficients such that a ring with unity R is necessarily commutative if $f(x)$ is central for all $x \in R$. We also solve the corresponding problem without the assumption that the ring has a unity.

1. Introduction

In [4] and [1], characterisations were given for the polynomials f with integer coefficients such that a ring R is necessarily commutative whenever $f(x) = 0$ for all $x \in R$. Here we characterise those polynomials f such that a ring R is necessarily commutative whenever R satisfies the weaker condition that $f(x)$ is central for all $x \in R$. The fact that these two classes of polynomials are different follows from the observation that a ring satisfying the identity $x^2 - 2x = 0$ is necessarily commutative, while there are easy examples to show that there are non-commutative rings where $x^2 - 2x$ is central for all x.

Throughout this paper, $f(X) = \sum_{i=1}^{n} a_i X^i \in X\mathbb{Z}[X]$. Given a ring R, we write $f(R) = 0$ if $f(x) = 0$ for all $x \in R$, and we write $f(R) \subset Z(R)$ if $f(x) \in Z(R)$ for all $x \in R$; here $Z(R)$ is the centre of R. For us, a ring does not necessarily have a unity, unless this is assumed.

Given a class $\mathcal{F}$ of rings, we denote by $C_0(\mathcal{F})$ and $C_Z(\mathcal{F})$ the sets of polynomials $f \in X\mathbb{Z}[X]$ that force a ring $R \in \mathcal{F}$ to be commutative whenever $f(x)$ always lies in $\{0\}$ or $Z(R)$, i.e.,

$$C_0(\mathcal{F}) = \{f(X) \in X\mathbb{Z}[X] \ : \ (R \in \mathcal{F} \text{ and } f(R) = 0) \Longrightarrow R \text{ commutative}\},$$
$$C_Z(\mathcal{F}) = \{f(X) \in X\mathbb{Z}[X] \ : \ (R \in \mathcal{F} \text{ and } f(R) \subset Z(R)) \Longrightarrow R \text{ commutative}\}.$$

We are mainly interested in two classes $\mathcal{F}$: the class of all rings $\mathcal{R}$, and the class of all rings with unity $\widetilde{\mathcal{R}}$. For each prime p, we also define the class $\mathcal{R}_p$ of rings such that $p^k R \subset Z(R)$ for some $k \in \mathbb{N}$, and the class $\widetilde{\mathcal{R}}_p := \widetilde{\mathcal{R}} \cap \mathcal{R}_p$. We refer to polynomials in $C_Z(\mathcal{R})$ as *Z-polynomials*, and polynomials in $C_0(\mathcal{R})$ as *C-polynomials*.

A well-known result of Jacobson [3, Theorem 11] shows that for $n > 1$, $X^n - X$ is a C-polynomial. More generally, Herstein [2] showed that if $a_1 = \pm 1$, then f is not just a C-polynomial, but also a Z-polynomial. In view of that result, we call f a *Herstein polynomial* if $a_1 = \pm 1$.

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Using Herstein's result, the second author and Laffey [4] showed that f is a C-polynomial if and only if f is either a Herstein polynomial, or f satisfies the following set of three conditions: $a_1 = \pm 2$, a_2 is odd, and $\sum_{i=2}^{n} a_i$ is odd. We will see that Z-polynomials form a more restrictive class than C-polynomials. In fact Z-polynomials coincide with Herstein polynomials; see Proposition 4.

Our characterisation of $C_Z(\widetilde{\mathcal{R}})$ is not as simple to state as that of $C_Z(\mathcal{R})$. It involves the following family of conditions indexed by a prime p:

There is at least one non-multiple of p among the numbers

$$T_p := \{a_1\} \cup \{b_j \mid 0 \le j < p-1\},$$

where

$$b_j = \sum_{\substack{1 \le i \le n \\ i \equiv j \pmod{p-1}}} i a_i, \qquad 0 \le j < p-1.$$

Whenever the above condition holds, we say that f *satisfies the T_p condition.*

Theorem 1. *Suppose $f(X) = \sum_{i=1}^{n} a_i X^i \in \mathbb{Z}[X]$. Then $f \in C_Z(\widetilde{\mathcal{R}})$ if and only if the greatest common divisor of the numbers $\{a_i\}_{i=1}^{n}$ is 1, and f satisfies the T_p condition for all primes $p \le n$ that divide a_1.*

By comparison, we note that the main result in [1] states that a polynomial $f(X) \in X\mathbb{Z}[X]$ lies in $C_0(\widetilde{\mathcal{R}})$ if and only if the greatest common divisor of the numbers $\{a_i\}_{i=1}^{n}$ is 1, and f satisfies the S_p condition for all primes $p \le n/2$ that divide a_1, where the S_p condition involves a set S_p is defined by:

$$S_p := T_p \cup \{c_j \mid 0 \le j < p-1\},$$

where

$$c_j = \sum_{\substack{1 \le i \le n \\ i \equiv j \pmod{p-1}}} a_i, \qquad 0 \le j < p-1.$$

After reducing the problem to understanding $C_Z(\widetilde{\mathcal{R}}_p)$ for all primes p in Section 2, we prove the main results in Section 3.

2. Reduction to prime powers

There is one rather obvious necessary condition for $f \in C_Z(\widetilde{\mathcal{R}}_p)$: given any prime p, the ring $GL_2(\mathbb{F}_p)$ is non-commutative and of characteristic p, so if every coefficient of f is divisible by p then $f \notin C_0(\widetilde{\mathcal{R}}_p) \supset C_Z(\widetilde{\mathcal{R}}_p)$. Thus every a polynomial in $C_Z(\widetilde{\mathcal{R}})$ (or in $\bigcap_{p \text{ prime}} C_Z(\widetilde{\mathcal{R}}_p)$) is *primitive*, i.e. the greatest common divisor of its coefficients is 1.

The rest of this section is dedicated to proving the following lemma which reduces the task of characterizing $C_Z(\widetilde{\mathcal{R}})$ to that of characterizing $C_Z(\widetilde{\mathcal{R}}_p)$ for all primes p.

Lemma 2. $C_Z(\widetilde{\mathcal{R}}) = \bigcap_{p \text{ prime}} C_Z(\widetilde{\mathcal{R}}_p)$.

As a first step, the following simple lemma shows that commutativity of a ring R such that $mR \subset Z(R)$ follows from commutativity of its subrings R_p satisfying $p^k R_p \subset Z(R)$ for some $k \in \mathbb{N}$ and prime factor p of m.

Lemma 3. *Suppose $mR \subset Z(R)$, where $m \in \mathbb{N}$ has prime factorisation $m = \prod_{p|m} p^{k_p}$. For each prime factor p of m, let $m_p := m/p^{k_p}$ and $R_p := m_p R$. Then*

(a) *R_p is an ideal in R, and $p^{k_p} R_p \subset Z(R)$.*
(b) *Every $x \in R$ can be written in the form*

$$x = z + \sum_{p|m} x_p\,, \qquad z \in Z(R),\ x_p \in R_p\,.$$

(c) *$xy = yx$ whenever $x \in R_p$, $y \in R_q$, and p, q are distinct prime factors of m.*
(d) *R is commutative if and only if each R_p is commutative.*

Proof. Part (a) is immediate. As for (b), since the greatest common divisor of the numbers $\{m_p : p \mid n\}$ is 1, we can choose $n_p \in \mathbb{Z}$ such that $\sum_{p|m} n_p m_p$ equals 1 mod m, and then $x - \sum_p n_p(m_p x) \in Z(R)$.

We next prove (c). Let $x = m_p x'$, $y = m_q y'$. Since m divides $m_p m_q$, we can use distributivity repeatedly to get

$$xy = ((m_p m_q)x')y' = y'((m_p m_q)x') = yx\,.$$

Finally for (d), the "only if" part is trivial. Conversely, suppose that each of the rings R_p is commutative. Given $x, y \in R$, we write

$$x = z + \sum_{p|m} x_p\,, \qquad y = w + \sum_{p|m} y_p\,,$$

where $z, w \in Z(R)$, and $x_p, y_p \in R_p$ for $p \mid m$. Using distributivity we expand xy into a sum of products of pairs of elements from the set $\{z, w\} \cup \Big(\bigcup_{p|m} \{x_p, y_p\}\Big)$. Bearing in mind (c), we see that the factors in each of these products commute, and so $xy = yx$. □

The *degree* $\deg(f)$ and *codegree* $\operatorname{codeg}(f)$ of a nonzero polynomial $f(X) = \sum_{i=1}^n a_i X^i$ are the largest and smallest $i \in \mathbb{N}$, respectively, such that $a_i \neq 0$.

Proof of Lemma 2. Clearly $C_Z(\widetilde{\mathcal{R}}) \subset \bigcap_{p \text{ prime}} C_Z(\widetilde{\mathcal{R}}_p)$, so we need only prove the reverse implication. Suppose therefore that $f \in \bigcap_{p \text{ prime}} C_Z(\widetilde{\mathcal{R}}_p)$, so f is necessarily primitive. Suppose also that $f(R) \subset Z(R)$ for some given unital ring R. f must be of degree at least 1. We write $f(X) = \sum_{i=1}^n a_i X^i \in \mathbb{Z}[X]$, where $a_n \neq 0$ and $n \in \mathbb{N}$, so $1 \leq \operatorname{codeg}(f) \leq \deg(f) = n$.

If $\operatorname{codeg}(f) < \deg(f)$ then $g(X) := 2^n f(X) - f(2X)$ defines another nonzero polynomial such that $\operatorname{codeg}(g) = \operatorname{codeg}(f)$ and $\deg(g) \leq \deg(f) - 1$. In fact

$$g(X) = \sum_{i=1}^{n-1} (2^n - 2^i) a_i X^i \,.$$

Also note that $g(R) \subset Z(R)$. Iterating this reduction procedure we eventually get a nonzero monomial such that $h(R) \subset Z(R)$. If $\deg(h) > 1$, then simply replace h by $H(X) := h(X+1) - h(1)$. Then $\deg(H) = \deg(h)$ and $\operatorname{codeg}(H) = 1$, so if we again repeat the reduction procedure we eventually get a polynomial $F(X) = mX$, $m \in \mathbb{N}$, such that $F(R) \subset Z(R)$. Thus $mR \subset Z(R)$.

Define m_p and R_p as in Lemma 3, and let

$$R'_p = \{m_p x + b \cdot 1 \mid x \in R \,, n \in \mathbb{Z}\} \,.$$

Then for each prime factor p of m, R'_p is a subring of R, $1 \in R'_p$, and $p^k R'_p \subset Z(R)$, so $R'_p \in \widetilde{\mathcal{R}}_p$. Since also $f(R'_p) \subset Z(R) \cap R'_p = Z(R'_p)$ for all p, and $f \in C_Z(\widetilde{\mathcal{R}}_p)$, each R'_p is commutative. Thus also each R_p is commutative, and so R is commutative by Lemma 3. But R is an arbitrary ring satisfying $f(R) \subset Z(R)$, so we deduce that $f \in C_Z(\widetilde{\mathcal{R}})$, as required. □

3. Proofs of results

We first state and prove our characterisation of $C_Z(\mathcal{R})$.

Proposition 4. *The classes of Z-polynomials and Herstein polynomials coincide.*

Proof. The fact that Herstein polynomials are Z-polynomials is Herstein's main result in [2]. Conversely, as mentioned in the Introduction, it is shown in [4] that if $f(X) = \sum_{i=1}^{n} a_i X^i \in \mathbb{Z}[X]$ is a C-polynomial, then either it is a Herstein polynomial or $a_1 = \pm 2$. Thus to establish our result, it suffices to exhibit a non-commutative ring R such that $f(R) \subset Z(R)$ whenever a_1 is even.

This is rather easy to do: we simply take $(R, +, \cdot)$ to be the ring of 3×3 matrices over $\mathbb{Z}_2$ of the form

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

This ring is not commutative since, for instance,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \neq \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

However $2x = x^3 = 0$ for all $x \in R$, Moreover since $xyz = 0$ for all $x, y, z \in R$, it follows that $x^2 \in Z(R)$ for all x. Thus if a_1 is even, then $f(x) = a_2 x^2 \in Z(R)$ for all $x \in Z(R)$. □

We now turn to the proof of Theorem 1. The main step is the following characterisation of $C_Z(\widetilde{\mathcal{R}}_p)$.

Theorem 5. *Suppose $f(X) = \sum_{i=1}^n a_i X^i \in \mathbb{Z}[X]$, and let p be a prime. Then $f \in C_Z(\widetilde{\mathcal{R}}_p)$ if and only if f satisfies the T_p condition.*

Proof. We prove sufficiency of the T_p condition. We may assume that $R \in \widetilde{\mathcal{R}}$ is such that $p^k R \subset Z(R)$ for some $k \in \mathbb{N}$. When considering $f(R) \subset Z(R)$ for such rings, we may treat the coefficients of f as being either elements of $\mathbb{Z}_{p^k}$, or elements of $\mathbb{Z}$, as suits us.

If $p \nmid a_1$, then a_1 is a unit mod p^k, so $g(X) := a_1^{-1} f(X) \in \mathbb{Z}_{p^k}[X]$ has the form $X + \sum_{i=2}^n d_i X^i$, and so it is a Herstein polynomial when we view its coefficients as being integers. In particular the condition $g(R) \subset Z(R)$ forces characteristic p^k rings $R \in \widetilde{\mathcal{R}}$ to be commutative. We may therefore assume that $p \mid a_1$.

Suppose that there exists i, $0 \leq i < p-1$, such that $p \nmid b_i$. We treat $f(X)$ as a polynomial in $\mathbb{Z}_{p^k}[X]$, but let us also write $f_p(X)$ for $f(X)$ when instead viewed as an element of $\mathbb{Z}_p[X]$. Expanding $f_p(X+t)$ for $t \in \mathbb{Z}_p$, we see that the coefficient of X is $s_p(t) := \sum_{i=1}^n i a_i t^{i-1}$. Let $S_p(X) := \sum_{i=0}^{p-1} b_i X^i \in \mathbb{Z}_p[X]$. By Fermat's Little Theorem, $s_p(t) = S_p(t)$ for all $t \in \mathbb{Z}_p$. The fact that $p \nmid b_i$ for some i means that S_p is not the zero polynomial, and so it has at most $p-1$ roots. Thus there exists $t \in \mathbb{Z}_p$ such that $s_p(t) \neq 0$. It follows that the coefficient of X in the expansion of $f(X + t \cdot 1)$ is coprime to p for some $t \in \mathbb{Z}_{p^k}$. Fixing this value of t and picking $k \in \mathbb{Z}_{p^k}$ which is equivalent to t mod p, we get a polynomial $g(X) := f(X+k) - f(k) \in \mathbb{Z}_{p^k}[X]$ such that $g(R) \subset Z(R)$ and such that the coefficient of X in g is a unit mod p^k. This implies the commutativity of R as before.

We now prove the converse. Suppose therefore that the T_p condition fails for a given function f. Let R be the ring of matrices

$$x = \begin{pmatrix} \alpha & \beta & \delta \\ 0 & \alpha & \gamma \\ 0 & 0 & \alpha \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_p$. For brevity, let us call $\alpha, \beta, \gamma, \delta$, the *first*, *second*, *third*, and *fourth coordinates* of x, respectively.

Given such a matrix x, it can be verified inductively that for all $i > 1$,

$$x^i = \begin{pmatrix} \alpha^i & i\alpha^{i-1}\beta & * \\ 0 & \alpha^i & i\alpha^{i-1}\gamma \\ 0 & 0 & \alpha^i \end{pmatrix}, \tag{1}$$

where $*$ equals $i\alpha^{i-1}\delta + \binom{i}{2}\alpha^{i-2}\beta\gamma$ (and α^0 is defined to be 1, even for $\alpha = 0$), but the actual value does not affect subsequent calculations.

Consider now $f(x)$. Because $t^p = t$ for all $t \in \mathbb{Z}_p$, it follows from (1) that the second coordinate of $f(x)$ equals $a_1\beta + \sum_{i=0}^{p-2} d_i \alpha^{p+i-2}\beta$, where $d_1 = b_1 - a_1$ and

$d_i = b_i$ for every other index in this sum, and the numbers b_i are as in the T_p condition. Now T_p fails to hold, so a_i and all the b_is are divisible by p, and so $p|d_i$ for $0 \leq i < p-1$. It follows that the second coordinate of $f(x)$ equals zero, and similarly we see that the fourth coordinate of $f(x)$ is 0. Thus $f(x)$ has the form

$$\begin{pmatrix} \varepsilon & 0 & \zeta \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix},$$

for some $\varepsilon, \zeta \in \mathbb{Z}_p$. But it is readily verified that all such matrices lie in the centre of R, so we have shown that $f(x) \in Z(R)$ for all $x \in R$ whenever T_p fails. Now $R \in \widetilde{\mathcal{R}}_p$, and it is non-commutative regardless of p, since for instance

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $f \notin C_Z(\widetilde{\mathcal{R}}_p)$ if the T_p condition fails. □

Proof of Theorem 1. Since

$$C_Z(\widetilde{\mathcal{R}}) = \bigcap_{p \text{ prime}} C_Z(\widetilde{\mathcal{R}}_p),$$

it follows that the polynomials in $C_Z(\widetilde{\mathcal{R}})$ are precisely those for which the T_p condition holds for all primes p. If the gcd of the coefficients is not 1, then all coefficients a_i are divisible by some prime p, and certainly f does not satisfy the T_p condition. Thus by Theorem 5, $f \notin C_Z(\widetilde{\mathcal{R}})$.

For the converse direction, since T_p trivially holds when p does not divide a_1, it suffices to show that the T_p condition holds for all primes $p > n$ as long as the gcd of the coefficients is 1. Because $p > n$, all the sums in the T_p condition involve at most one term. Thus, since the gcd of the coefficients is 1, there exists $i \leq n < p$ such that $p \nmid ia_i = b_i$. □

The characterisation for quadratic polynomials is particularly simple, and follows immediately from Theorem 1.

Corollary 6. *Suppose $f(X) = a_1X + a_2X^2 \in \mathbb{Z}[X]$. Then $f \in C_Z(\widetilde{\mathcal{R}})$ if and only if a_1 is odd.*

According to [4], a polynomial f lies in $C_Z(\mathcal{R})$ if and only if it is a Herstein polynomial. Comparing this with Corollary 6 or Theorem 1, it is easy to give examples of polynomials in $C_Z(\widetilde{\mathcal{R}}) \setminus C_Z(\mathcal{R})$, for instance $3X + X^2$ or $5X + 2X^3$. Comparing Theorem 1 with the characterisation of $C_0(\widetilde{\mathcal{R}})$ in [1], it is easy to give examples of polynomials in $C_0(\widetilde{\mathcal{R}}) \setminus C_Z(\widetilde{\mathcal{R}})$, for instance $3X^2 + 2X^3$ or X^2.

Lastly we note that the examples proving necessity in Theorem 5 (and so also in Theorem 1) involve only finite rings of prime characteristic. Thus if $\mathcal{F}$ is the

set of all finite rings with unity, then $C_Z(\mathcal{F}) = C_Z(\widetilde{\mathcal{R}})$, while if $\mathcal{F}$ consists of all finite rings with unity and characteristic p, then $C_Z(\mathcal{F}) = C_Z(\widetilde{\mathcal{R}}_p)$. This is analogous to the fact that if $\mathcal{F}$ is the set of all finite rings (without the assumption of unity), then $C_Z(\mathcal{F}) = C_Z(\mathcal{R})$ because the proof in [4] uses only finite rings to prove necessity.

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