On Algebraic Minimal Surfaces in $\mathbb{R}^3$ Deriving from Charge 2 Monopole Spectral Curves

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Dedicated to James Eells

Abstract.

We give formulae for minimal surfaces in $\mathbb{R}^3$ deriving, via classical osculation duality, from elliptic curves in a line bundle over $\mathbb{P}_1$. Specialising to the case of charge 2 monopole spectral curves we find that the distribution of Gaussian curvature on the auxiliary minimal surface reflects the monopole's structure. This is elucidated by the behaviour of the surface's Gauss map.

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§1. Introduction

Let $\pi : \mathbb{T} \longrightarrow \mathbb{P}_1$ be the projection map from the total space of the holomorphic tangent bundle of the complex projective line to its base and suppose that S is a holomorphic curve in $\mathbb{T}$; the collection of global sections osculating S determines an auxiliary null holomorphic curve in $\mathrm{H}^0(\mathbb{P}_1, \mathcal{O}(\mathbb{T}))$. This is best understood, after embedding $\mathbb{T}$ in $\mathbb{P}_3$, in terms of classical osculation duality; the geometry of the null curve is of course encoded in S. See [3], [4], [9] for further details.

If S is described by a pair of meromorphic functions (g, f) on a Riemann surface M, then, with respect to a certain choice of basis for $\mathrm{H}^0(\mathbb{P}_1, \mathcal{O}(\mathbb{T}))$, the coordinate functions of the auxiliary null curve $\psi : M^* \longrightarrow \mathbb{C}^3$ are given by the classical Weierstrass formulae:

$$\psi_1 = -((1-g^2)f^{(2)}/2 + gf^{(1)} - f)/2 \tag{1}$$

$$\psi_2 = -i((1+g^2)f^{(2)}/2 - gf^{(1)} + f)/2 \tag{2}$$

$$\psi_3 = (gf^{(2)} - f^{(1)})/2 \tag{3}$$

where M^* is obtained from M by deleting a finite number of points, and $f^{(1)} = f'/g'$, $f^{(2)} = (f^{(1)})'/g'$, etc.

ϕ =Re(ψ) describes an algebraic minimal surface in $\mathbb{R}^3$; g may be identified with the Euclidean Gauss map of ϕ. (The choice of basis agrees with [4], cf. [3].)

Now, any elliptic curve may be realised as a double cover of $\mathbb{P}_1$, lying in $\mathbb{T}$, branched at four distinct points on the zero section. It turns out that for any such realisation, there exists a Weierstrass $\wp$-function such that the coordinate functions of the auxiliary null curve may be written, after rescaling and application of a transformation in SO(3, $\mathbb{C}$), in the form:

$$\omega_1(u) = \sqrt{e_2 - e_3}\,\xi_{10}^3(2u) \tag{4}$$

$$\omega_2(u) = \sqrt{e_3 - e_1}\,\xi_{20}^3(2u) \tag{5}$$

$$\omega_3(u) = \sqrt{e_1 - e_2}\,\xi_{30}^3(2u) \tag{6}$$

where $\xi_{j\,0}(u)$ denotes the square root of $\wp(u) - e_j$, whose residue at the origin is 1. (The notation follows [10].)

The auxiliary algebraic minimal surfaces are complete, finitely branched and of total curvature equal to -8π. Their geometry is 'localised' in the sense that 'from infinity they look like a finite number of planes passing through the origin', cf. [6], [7], [8].

Much of their structure may be deduced from properties of classical osculation duality [9], together with (4)-(6), but in general there appears to be little point in pursuing explicitness any further. However, there is a family of elliptic curves for which further calculation may be justified; these are the spectral curves of SU(2) monopoles of charge 2, first described by Hurtubise [5]. The fact that there is a minimal surface canonically associated to an SU(2) monopole on $\mathbb{R}^3$ was first observed in [1] and [4]; however, the significance of its geometry for the monopole has never been elucidated.

In §2 we derive the explicit form of (4)-(6) for charge 2 monopole spectral curves. In §3 we describe the most salient features of the auxiliary minimal surfaces and their Gauss maps. In §4 we describe the dependence of certain key aspects of the geometry of the minimal surfaces on the elliptic modulus $k \in [0, 1)$, used to parameterise the orbits of the SO(3, $\mathbb{R}$) action on the Atiyah-Hitchin manifold $\mathcal{M}_2^0$, [2]; this facilitates the study of the auxiliary 'solitonic family' of minimal surfaces determined by a geodesic on $\mathcal{M}_2^0$.

§2. Formulae for charge 2 monopole null curves

Let ζ be an affine coordinate on $\mathbb{P}_1$ and (ζ, η) be the coordinates given by $(\zeta, \eta) \to \eta d/d\zeta$. The spectral curve of a non-axially symmetric centred charge 2 monopole may by rotation of $\mathbb{R}^3$ be brought to the reduced form:

$$\eta^2 = r_1\zeta^3 - r_2\zeta^2 - r_1\zeta, \quad r_1,\ r_2 \in \mathbb{R},\ r_1 > 0, \tag{7}$$

see [5]. The lattice is rectangular; let the real and imaginary periods be ω_1 and ω_2 respectively. Monopole non-singularity constrains the periods, so that $\omega_1 = 2\sqrt{r_1}$.

The substitutions: $\zeta = \tilde{\zeta} + k_2$ and $\eta = k_1\tilde{\eta}$, where $k_1 = \frac{1}{2}\sqrt{r_1}$ and $k_2 = r_2/3r_1$ reduce (4) to

$$\tilde{\eta}^2 = 4\tilde{\zeta}^3 - g_2\tilde{\zeta} - g_3, \tag{8}$$

where $g_2 = 12k_2^2 + 4$ and $g_3 = 8k_2^3 + 4k_2$, [5]. So, if $\wp(u)$ is the Weierstrass $\wp$-function determined by g_2 and g_3, then the spectral curve is uniformised by $\zeta = g(u) = \wp(u) + k_2$ and $\eta = f(u) = \frac{\omega_1}{4}\wp'(u)$.

Observe that $z^3 - (1 + 3k_2^2)z - k_2(1 + 2k_2^2) = (z + k_2)(z^2 - k_2 z - (1 + 2k_2))$; since the lattice is rectangular, $e_1 > e_3 > e_2$, and hence $k_2 = -e_3$.

In [2], (4) is rewritten, in terms of $0 < k < 1$:

$$\eta^2 = K(k)^2\zeta(kk'(\zeta^2 - 1) + (k^2 - (k')^2)\zeta), \tag{9}$$

where $k^2 + k'^2 = 1$, and K is Legendre's complete elliptic integral. Note that $g_2 = 4(1 - k^2 + k^4)/3k^2k'^2$, and $g_3 = 4(k^2 - 2)(k^2 + 1)(2k^2 - 1)/27k^3k'^3$. Let S_k denote the spectral curve above.

Lemma 2.1 *For $\wp(u)$, with g_2, g_3 determined by k as above:*

$$\begin{aligned} 1 + 2\frac{k'}{k}(\wp(u) - e_3) - (\wp(u) - e_3)^2 &= \wp'(u)\xi_{10}(2u) & (10) \\ -1 + 2\frac{k}{k'}(\wp(u) - e_3) + (\wp(u) - e_3)^2 &= -\wp'(u)\xi_{20}(2u) & (11) \\ 1 + (\wp(u) - e_3)^2 &= -\wp'(u)\xi_{30}(2u) & (12) \end{aligned}$$

Proof. (12) is given (modulo a typo) in §333 of [10]; both (10) and (11) may be derived from it.

With respect to k-monopole coordinates, $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$, cf. [2], the null curve given by osculation of S_k is represented by

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} -k & 0 & k' \\ k' & 0 & k \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{13}$$

Theorem 2.2 *Suppose that $k \in (0,1)$, determines g_2, g_3, as above, and $\wp(u)$ is the associated Weierstrass function. The null curve that is generated by osculation of the spectral curve S_k, uniformised by $g(u) = \wp(u) - e_3$, $f(u) = \omega_1\wp'(u)/4$, has components with respect to k-monopole coordinates given by:*

$$\Psi_1(u) = -k\frac{\omega_1}{4}\xi_{10}^3(2u) \tag{14}$$

$$\Psi_2(u) = k'\frac{\omega_1}{4}\xi_{20}^3(2u) \tag{15}$$

$$\Psi_3(u) = -i\frac{\omega_1}{4}\xi_{30}^3(2u) \tag{16}$$

Proof. (16) follows from a direct calculation (substituting $g(u) = \wp(u) - e_3$, $f(u) = \omega_1\wp'(u)/4$, into (2)), together with (12). Observe that

$$\Psi_1'(u) \quad = \quad \frac{k}{4}g'(u)f^{(3)}(u)(1 - g(u)^2 + 2\frac{k'}{k}g(u)).$$

Now, differentiating (2) and (16), and comparing using (12), yields $f^{(3)}(u) = -3\omega_1\wp'(2u)/\wp'(u)^2$; hence it follows from (10) that

$$\Psi_1(u) = -k\frac{\omega_1}{4}\xi_{10}^3(2u) + \Psi_1(\frac{\omega_1}{4}).$$

A quarter-period formula gives $\Psi_1(\omega_1/4) = 0$.

The formula for Ψ_2 follows in a similiar way from (11), together with $\Psi_2(\omega_2/4) = 0$.

Corollary 2.3 *The branched metric induced on the spectral curve by* $\phi = \mathrm{Re}(\Psi)$ *has the form:*

$$ds^2(u) = \frac{9}{32}|\wp'(2u)|^2\{k^2|\wp(2u) - e_1| + k'^2|\wp(2u) - e_2| + |\wp(2u) - e_3|\}|du|^2.$$

§3. The geometry of charge 2 monopole minimal surfaces and their Gauss maps

Let $\tau : \mathbb{T} \longrightarrow \mathbb{T}$, be the real structure given in local coordinates by $\tau(\zeta, \eta) = (-\overline{\zeta}^{-1}, -\overline{\eta}\overline{\zeta}^{-2})$. Recall that $\mathbb{T}$ may be used to parameterise the set of oriented lines in $\mathbb{R}^3$, τ reverses orientation along lines and SO(3, $\mathbb{R}$) acts on $\mathbb{T}$ via bundle maps, cf. [2], [4]. Let D be the subgroup of SO(3, $\mathbb{R}$) comprising the identity element together with rotations through 180^o about the axes $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ in $\mathbb{R}^3$. S_k parameterises the monopole's *spectral lines*; it is invariant under τ and D, [5]. These observations, together with basic properties of osculation duality yield the following, except (iv), which requires (14)-(16).

Proposition 3.1 *Osculation of the spectral curve* S_k, $k \neq 0$, *gives a complete branched minimal immersion* $\phi : S_k \setminus \{0, \omega_1/2, \omega_2/2, \omega_3/2\} \longrightarrow \mathbb{R}^3$, *with the following properties.*

(i) ϕ *factors through the twice punctured Klein bottle* $S_k/\tau \setminus \{[0], [\omega_1/2]\}$.

(ii) The total Gaussian curvature of the induced branched metric on $S_k \setminus \{0, \omega_1/2, \omega_2/2, \omega_3/2\}$ *equals* -8π.

(iii) The minimal surface has two ends; they are perpendicular to the two spectral lines through the monopole's centre.

(iv) There are six branch points, of order 1, on the minimal surface in $\mathbb{R}^3$*; they are at:*

$$\pm\beta_1 = \pm\frac{K(k)}{2k}(0, 1, 0), \;\; \pm\beta_2 = \pm\frac{K(k)}{2k'}(0, 0, k^2), \;\; \pm\beta_3 = \pm\frac{K(k)}{2}(0, k^2, 0).$$

(v) The image of ϕ is invariant under D.

Since they lie on the Higgs axis we call $\pm\beta_1, \pm\beta_3$ the *Higgs branch points.*

Remark. The positions of the branch points are tied to the moduli space orbit parameter k, through the monopole non-singularity constraint of [5].

S_k is hyperosculated at the sixteen quarter-period points; the points of order 2 give the ends, while the remaining points give the zeros in the metric. These pass, via ϕ, to the two ends and the six branch points of the minimal surface respectively. Locally, near a zero in the induced metric, S_k may be described by $\eta = \sigma(\zeta) + a_4\zeta^4 + \mathcal{O}(\zeta^5)$, where $\sigma(\zeta)$ is quadratic, and $a_4 \in \mathbb{C}^*$. So at each zero in the metric the minimal surface is locally a translated perturbation of a rescaled associate surface of the minimal surface determined by osculation of $\eta = \zeta^4$. The latter incorporates three rays emanating from the origin at 120^o, cf. Figure 6 in [7]. This *triple curve intersection structure* is present in twisted form at each of the six branch points since it is preserved by higher order perturbations and multiplication by $a_4 \neq 0$.

With (14)-(16) in hand it is easy to deduce that the vertical line through $\omega_1/4$ is mapped by ϕ onto the Higgs axis between β_1 and β_3, while the vertical line through $3\omega_1/4$ is mapped by ϕ onto the Higgs axis between $-\beta_1$ and $-\beta_3$. We denote the union of these two line segments on the Higgs axis by $\Gamma_{\text{Higgs}}(k)$.

Again, it is easy to see that the horizontal line through $\omega_2/4$ is mapped (monotonically) to a D-invariant curve in the $(\underline{e}_2, \underline{e}_3)$-plane, with vertices at the four branch points $\beta_2, \beta_3, -\beta_3$ and $-\beta_2$, which we denote $\Gamma_{\text{Star}}(k)$. That part of $\Gamma_{\text{Star}}(k)$ lying in the first quadrant is concave up.

Now let $\mathcal{K}$ denote the Gaussian curvature of the branched metric ds^2 on $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ induced by ϕ, and recall that

$$\int_{\mathbb{C}/(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)} \mathcal{K} dA = -\int_{\mathbb{C}/(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2)} \frac{4|g'|^2}{(1+|g|^2)^2} dxdy, \tag{17}$$

where dA is the area form of ds^2, cf. [7], [8]. The integral on the right is just the area induced by the Gauss map g. Let

$$G = \frac{4|g'|^2}{(1+|g|^2)^2}.$$

The Gauss map $\gamma_\Psi : \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) \longrightarrow Q_1$, with respect to k-monopole coordinates, is given by differentiating (14)-(16):

$$\gamma_\Psi(u) = [-k\xi_{10}(2u),\ k'\xi_{20}(2u),\ -i\xi_{30}(2u)].$$

It follows that $g_\Psi : \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) \longrightarrow \mathbb{C} \cup \{\infty\}$, is given by:

$$g_\Psi(u) = h^{-1} \circ \gamma_\Psi(u) = \frac{-i\xi_{30}(2u)}{k\xi_{10}(2u) + ik'\xi_{20}(2u)},$$

where $h : \mathbb{C} \cup \{\infty\} \longrightarrow Q_1$, $h(\zeta) = [1 - \zeta^2, i(1 + \zeta^2), -2\zeta]$.

Remark. Let $\gamma_\phi : \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) \longrightarrow S^2$, be the Euclidean Gauss map of the minimal surface $\phi = \mathrm{Re}(\Psi)$. Observe that g_Ψ agrees with γ_ϕ, after the latter is composed with stereographic projection from $-\underline{e}_3$ to the $(\underline{e}_1, \underline{e}_2)$-plane in $\mathbb{R}^3$.

Since

$$g_\Psi(u) = -(k'\xi_{20}(2u) + ik\xi_{10}(2u))/\xi_{30}(2u) \quad \text{and} \quad g'_\Psi(u) = -2ig_\Psi(u)/\xi_{30}(2u),$$

we have

$$G(u) = \frac{8}{k^2|\wp(2u) - e_1| + k'^2|\wp(2u) - e_2| + |\wp(2u) - e_3|}. \tag{18}$$

It is easy to see that G is symmetric about the quarter-period lines.

The following properties of the Gauss map elucidate the behaviour of G in the limits $k \to 0, 1$; we discuss this in §4.

Theorem 3.3 *(i)As u passes along the quarter-period line from $\omega_1/4$ to $\omega_1/4 + \omega_2$, the Gauss map $\gamma_\phi(u)$ winds once around the great circle on S^2 that projects to the $\underline{e}_1$-axis under stereographic projection (from $-\underline{e}_3$). The analogue holds on the quarter-period line through $3\omega_1/4$.*

(ii) As u passes from $\omega_2/4$ to $\omega_2/4 + \omega_1$, the Gauss map $\gamma_\phi(u)$ winds once around the great circle on S^2 that projects to the $\underline{e}_2$-axis under stereographic projection (from $-\underline{e}_3$).

Proof. This follows easily from the behaviour of g_Ψ, which is real on the vertical quarter-period lines used, while purely imaginary on the horizontal.

§4. Remarks on k dependence and curvature concentration

¿From $g(u) = \wp(u) - e_3$, it is easy to see that as k approaches 0 or 1, the two ends line up. As k passes from values close to 1 to values close to 0, the angle between the direction of the ends changes by nearly 180^o.

As $k \to 1$, the monopole solution separates into two particles centred at the *star centres*; these are the points along the Higgs axis at $\pm(0,\ K(k)/2,\ 0)$ respectively. It follows from (9) that in this limit the configuration of spectral lines approximates the two 'stars' comprising all the oriented lines through the star centres, cf. [2]. It follows from 2.2 that as $k \to 1$, $\Gamma_{\mathrm{Higgs}}(k)$ shrinks to the star centres.

The branch points on the null curve in $\mathbb{C}^3$ that project to the Higgs branch points approach 'exponentially close' to $\mathbb{R}^3$, relative to the separation distance $K(k) \sim -\log k'$, as $k \to 1$. In fact it follows from 2.2 that the same is true along the points on the null curve that project to $\Gamma_{\mathrm{Higgs}}(k)$. Now, it is easy to see that if $S \subset \mathbb{T}$ is the spectral curve of a monopole then the affine null planes in $\mathbb{C}^3$ osculating the auxiliary null curve intersect $\mathbb{R}^3$ in the spectral lines of the monopole. These facts, together with some elementary calculations yield:

Theorem 4.1 *As $k \to 1$, every normal line to the minimal surface, along $\Gamma_{\text{Higgs}}(k)$, becomes exponentially close, relative to separation distance, to a spectral line of the monopole.*

For a monopole of charge $\ell \geq 1$, the total energy is $4\pi\ell$, while the total Gaussian curvature on the auxiliary minimal surface equals $-4\pi\ell$. Now we interpret this coincidence 'locally' for $\ell = 2$, using (17).

It follows from (18) that as $k \to 1$, G localises on the vertical quarter-period lines mapping to $\Gamma_{\text{Higgs}}(k)$. This is elucidated by the behaviour of the Gauss map along $\Gamma_{\text{Higgs}}(k)$: the normal vector along either segment of $\Gamma_{\text{Higgs}}(k)$ turns through 360^o for all $k \in (0, 1)$; but the segments shrink as $k \to 1$, thus the conformality of the Gauss map means that as $k \to 1$, narrower and narrower strips around the vertical quarter period lines mapping to $\Gamma_{\text{Higgs}}(k)$ are stretched by the Gauss map to cover almost the entire sphere. Thus the Gaussian curvature *concentrates* on $\Gamma_{\text{Higgs}}(k)$, as $k \to 1$, into two 'lumps' at the star centres.

As $k \to 0$, the monopole solution approaches the axially symmetric case and particle structure is lost, [2]. The segments comprising $\Gamma_{\text{Higgs}}(k)$ stretch out and the curvature there 'unwinds'; instead it concentrates on $\Gamma_{\text{Star}}(k)$.

There is a remarkable dance performed by the six branch points as k varies, which may be followed through 3.1(iv). It is noteworthy that the branch points 'escaping to infinity' in the limits $k \to 0, 1$ are highly attenuated and do not carry any 'bare' curvature away to infinity. More precisely, it can be shown that:

(i) There exist constants $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, such that for any fixed $k \in (0, 1)$, sufficiently close to 0*, and* $0 < |h| < \sqrt{k}$*,*

$$|\mathcal{K}(\frac{\omega_1}{4} + \frac{h}{2})| \leq \alpha_1 k^5 |h|^{-2} + \alpha_2 k^4 + \alpha_3 k^3 |h|^2 + O(|h|^4).$$

(ii) There exist constants $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$, such that for any fixed $k \in (0, 1)$, sufficiently close to 1, and $0 < |h| < \sqrt{k'}$*,*

$$|\mathcal{K}(\frac{\omega_2}{4} + \frac{h}{2})| \leq \delta_1 k'^5 |h|^{-2} + \delta_2 k'^4 + \delta_3 k'^3 |h|^2 + O(|h|^4).$$

This may be deduced from:

$$|\mathcal{K}(u)| \leq \frac{4}{9kk'K^2(k)|(\wp(2u) - e_1)(\wp(2u) - e_2)(\wp(2u) - e_3)^3|}.$$

Remarks. (1) It is easy to argue, at least heuristically, that a monopole's spectral lines must enter regions of space where the fields are changing rapidly. This explains the structure of the configurations of spectral lines as $k \to 1$. The behaviour of the monopoles' 'Gauss maps' $g = \pi|_{S_k} = g_\Psi$, in this limit, elucidates the connection between the distribution of a monopole's energy and the twisting of its spectral lines. The measure induced on a spectral curve by $\pi|_S$ would appear to merit further study in general. In the charge 3 case one might investigate its relationship with the Weierstrass points of S.

(2) The above results facilitate the study of the families of minimal surfaces generated by geodesics on $\mathcal{M}_2^0$. It is easy to see that many of the main features of the 90^o scattering of monopoles in a direct collision, cf. [2], are closely reflected by the behaviour of the Gaussian curvature on the auxiliary minimal surfaces. In particular, the interchange of the roles of the Higgs axis and the third axis at collision results in a 90^o scattering of the lumps of Gaussian curvature. The loss of particle structure close to collision is also reflected in the behaviour of the Gaussian curvature.

(3) An earlier version of this paper, which includes more details and some pictures, may be found at arXiv:math.DG/0203223.

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