# Wavelength-controlled variable-order optical fractional Fourier transform

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The relationship between optical fractional Fourier transforms (OFRTs) obtained at different wavelengths is derived by use of the *ABCD* matrix formalism. It is shown that varying the wavelength while retaining the same optical system can be used to control the order of the OFRT. The advantage of this method of varying OFRT order is that no variation in the characteristics of the bulk optics is required. A general experimental verification of the theory is provided by showing the exact equivalence of two OFRT systems of different order when they are replayed using the same input function at different wavelengths. 

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The fractional Fourier transform (FRT) was introduced in optics to describe wave propagation in gradient-index media in 1993,[1,2] and its relationship with the Wigner distribution function[3] and its bulk optical implementations were described soon after. The FRT order indicates the domain into which it transforms and an order $a = 1$ is simply the Fourier transform. The FRT is linear and separable in the $x$ and $y$ directions and can be optically implemented with different continuously variable orders.[4] It has the properties $\Im_a[f_0(x_0)](x_a) = \Im_{a+4n}[f_0(x_0)](x_{a+4n})$ for all integers $n$, and $\Im_a[f_0(x_0)](x_a) = \Im_{a+2m}[f_0(x_0)](-x_{a+2m})$ for all odd integers $m$.

The *ABCD* matrix formalism is an effective tool for the synthesis, analysis, and characterization of lossless optical systems[5–7] and has received much attention in the literature.[3,8–12] It allows us to model an optical system by multiplying component matrices to determine the total system matrix. Although the wavelength of the coherent light used is often included as a parameter in the transformation integral, it frequently does not appear in the *ABCD* matrix. For symmetrical homogeneous systems we can easily incorporate the wavelength into the matrix while maintaining the correspondence between the matrix and the integral transformations.

Typical the order and scale of the transform are directly related to the free-space propagation distances and the lens focal lengths in the optical system. For every implementation of the optical fractional Fourier transform (OFRT) the overall wave integral of the system is generally given by the equation

$$u_a(x_a) = \Im_{a,f,\lambda}[u_0(x_0)](x_a) = \frac{1}{j(2\pi\lambda f \sin\phi)^{1/2}} \int u_0(x_0) \times \exp\left[\frac{j\pi}{\lambda f}\left(\frac{{x_0}^2 + {x_a}^2}{\tan\phi} - \frac{2x_0 x_a}{\sin\phi}\right)\right]\mathrm{d}x_0 . \quad (1)$$

$\Im_{a,f,\lambda}$ is the operator for the transform, $\phi = a\pi/2$, $\lambda$ is the wavelength of the light, and $f$ represents a focal length that, together with the wavelength, decides the scale factor of the coordinates. The corresponding *ABCD* matrix including the scaling factor is

$$\begin{bmatrix} \cos\phi & \lambda f \sin\phi \\ -\sin\phi/\lambda f & \cos\phi \end{bmatrix} . \quad (2)$$

The general mathematical scale relationship between FRTs of different orders has been discussed.[13] However, to our knowledge, no experimental validation has been carried out and no applications have been proposed.

Using the matrix formalism, we first show that using a FRT optical system at one wavelength is completely equivalent to carrying out a FRT operation of a different order at a different wavelength. If we change the wavelength used in the OFRT system defined by Eq. (1) and matrix (2), we obtain the *ABCD* matrix given by matrix (2), where we substitute $\lambda_1$ for $\lambda$, which we can rewrite as

$$\begin{bmatrix} \sqrt{\lambda_1 f} & 0 \\ 0 & 1/\sqrt{\lambda_1 f} \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \times \begin{bmatrix} 1/\sqrt{\lambda_1 f} & 0 \\ 0 & \sqrt{\lambda_1 f} \end{bmatrix} . \quad (3)$$

The inner matrix represents a normalized FRT transform, and the corresponding integral is given by Eq. (1) with $f\lambda = 1$. We note that the OFRT can be described as a scaling up by $\lambda_1/f$, then the carrying out of a normalized FRT, followed by a scaling down by the same amount as we scaled up by initially. We can further rewrite expression (3) as follows:

$$\begin{bmatrix} \sqrt{\lambda_1 f} & 0 \\ 0 & 1/\sqrt{\lambda_1 f} \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \times \begin{bmatrix} \sqrt{\lambda/\lambda_1} & 0 \\ 0 & 1/\sqrt{\lambda_1/\lambda} \end{bmatrix}\begin{bmatrix} 1/\sqrt{\lambda f} & 0 \\ 0 & \sqrt{\lambda f} \end{bmatrix} . \quad (4)$$

Focusing on the two inner matrices, we now discuss the scaling property of the normalized FRT. The FRT of some order of a scaled function is equivalent to the

 

FRT of a different order of the unscaled function with additional chirping and scaling. We show this equivalence in the following operator and matrix equations:

$$\Im_a\left[\frac{1}{\sqrt{M}}u_0\left(\frac{x_0}{M}\right)\right](x_a) = \frac{1}{\sqrt{M_1}} \times \exp(j\pi {x_{a_1}}^2 q)\Im_{a_1}[u_0(x_0)]\left(\frac{x_{a_1}}{M_1}\right), \quad (5)$$

$$\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} M & 0 \\ 0 & 1/M \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ q & 1 \end{bmatrix} \times \begin{bmatrix} M_1 & 0 \\ 0 & 1/M_1 \end{bmatrix}\begin{bmatrix} \cos\phi_1 & \sin\phi_1 \\ -\sin\phi_1 & \cos\phi_1 \end{bmatrix}, \quad (6)$$

where $M$ is the scale that we apply to our input function before we apply a FRT of order $a$, $\phi = a\pi/2$, and $\phi_1 = a_1\pi/2$, where $a_1$, $M_1$, and $q$ are defined as

$$a_1 = \frac{2\phi_1}{\pi} = \frac{2}{\pi}\tan^{-1}\left(\frac{\tan\phi}{M^2}\right),$$

$$M_1 = \frac{\sin\phi}{M\sin\phi_1},$$

$$q = \frac{\sin^2\phi_1 - \sin^2\phi}{\sin\phi\cos\phi}. \quad (7)$$

The left-hand side of expression (6) is equivalent to the inner two matrices in expression (4), where $M = \sqrt{\lambda/\lambda_1}$. We are now in a position to rewrite expression (4) as follows:

$$\begin{bmatrix} \sqrt{\lambda_1 f} & 0 \\ 0 & 1/\sqrt{\lambda_1 f} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ q & 1 \end{bmatrix}\begin{bmatrix} M_1/\sqrt{\lambda f} & 0 \\ 0 & \sqrt{\lambda f}/M_1 \end{bmatrix} \times \begin{bmatrix} \cos\phi_1 & \lambda f\sin\phi_1 \\ -\sin\phi_1/\lambda f & \cos\phi_1 \end{bmatrix}. \quad (8)$$

The equivalence of expressions (3) and (8) allows us to draw the following conclusions: For a given input, the output obtained from a FRT optical system of order $a$ operating at some wavelength $\lambda_1$ is equivalent to the output from a second FRT optical system of order $a_1$ operating at a different wavelength $\lambda$, which has been scaled by some factor $M_2$ and multiplied by a chirp function. We clarify this equivalence using the operator notation introduced in Eq. (1) and define $M_2$ and the chirp function below:

$$\Im_{a,f,\lambda_1}[u_0(x_0)](x_a) = \frac{1}{\sqrt{M_2}} \times \exp(j\pi q_1 {x_a}^2)\Im_{a_1,f,\lambda}[u_0(x_0)]\left(\frac{x_{a_1}}{M_2}\right), \quad (9)$$

where

$$a_1 = \frac{2}{\pi}\tan^{-1}\left(\frac{\lambda_1\tan\phi}{\lambda}\right) = \frac{2}{\pi}\tan^{-1}(k\tan a\pi/2),$$

$$M_2 = \sqrt{\lambda_1 f}\,\frac{M_1}{\sqrt{\lambda f}} = \frac{k\sin\phi}{\sin\phi_1},$$

$$q_1 = \frac{q}{(\sqrt{\lambda_1 f})^2} = \frac{\sin^2\phi_1 - \sin^2\phi}{\lambda_1 f\sin\phi\cos\phi}, \quad (10)$$

where $k = \lambda_1/\lambda$. In many cases[14–16] we are interested only in finding the intensity of the FRTs. These results are consistent with those presented previously.[13] Clearly, as we change the wavelength for any OFRT system we are changing the order of the FRT. In this way any existing OFRT system, which can be described by an equation similar to Eq. (1), can be used as a variable OFRT optical system provided that we have suitable light sources. We note the dependence of our new order $a_1$ on the ratio of the two wavelengths and on the physical system and that the model presented here can be applied to systems in which the scale (focal-length value $f$) of the OFRT system changes when the order of the system changes, for example, Lohmann[3] type I and type II systems. We can therefore relate the outputs of such a system, for different orders, using the same optical scale factor. If $a = 1$, implying a standard Fourier transform, then $a_1$ is also equal to 1, the additional chirp factor disappears and the magnification factor $M_2$ reduces to the ratio of the wavelengths as expected. The change in fractional order depends on the value of $k$ and also on the value of the original order $a$. Figure 1 shows the relationship between $a$ and $a_1$ for five different ratios of $k$: $k = 0.5$ (the case when the new wavelength is half of the original wavelength), $k = 0.77092$ (the experimental case presented), and $k = 1.0, 2.0, 4.0$. For known values of $\lambda$ and $\lambda_1$ (i.e., for known values of $k$) we can determine the relationship between $a_1$ and $a$ and plot this relationship as has been done for five values of $k$ in Fig. 1. To determine the new order $a_1$ we simply plot a line vertically from the value of $a$ until we intersect the relationship curve and then plot a line horizontally from this intersection point.

Equations (9) and (10) were tested numerically for a large number of wavelengths, fractional orders, and input functions. To validate this theory experimentally we used two monochromatic light sources of wavelengths 633 and 488 nm. As simple an experimental setup as possible, requiring standard equipment available in all optical laboratories, was used. The lenses were of focal length 10 cm. The input field was produced by illuminating a commercial spatial frequency filter pinhole of specified diameter 15 ± 5 $\mu$m with a plane wave.

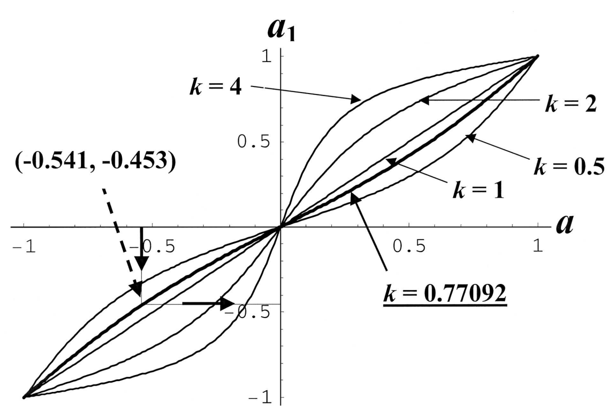


Fig. 1. Relationship between $a$ and $a_1$ for different values of the parameter $k = \lambda_1/\lambda$. The pair of values used in the experiment is shown $(-0.541, -0.453)$.

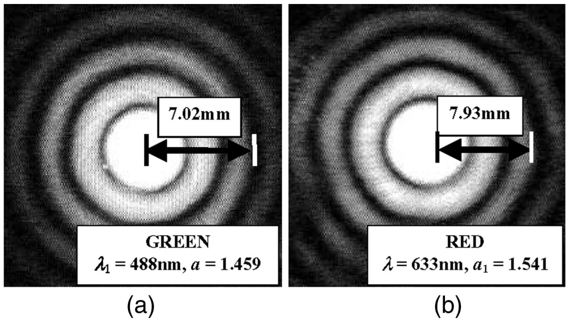


Fig. 2. Output function from the Cai and Wang OFRT system with (a) $a = 1.459$ and $\lambda_1 = 488$ nm, (b) $a_1 = 1.541$ and $\lambda = 633$ nm.

From Eq. (10) we can clearly see that the larger the difference in wavelength, the greater the change in order produced. The amount of change also depends on the initial order $a$. In this case we had at our disposal red ($\lambda = 633$ nm) and green ($\lambda_1 = 488$ nm) laser outputs. By use of these wavelengths and a fixed bulk optical system, it was determined that there could not be a very large change in order no matter what initial order was used. Therefore to validate the theory, the values of $a$ that would provide the largest deviation for a particular $a_1$ were determined. In the case of $\lambda = 633$ nm and $\lambda_1 = 488$ nm (for which $k = 0.77092$; see Fig. 1), these values of $a$ were found to be $-1.459$, $-0.541$, 0.541, and 1.459, giving corresponding values of $a_1$ of 0.459, $-0.459$, 0.459, and $-0.459$. Examining these values, we find that scaling factor $M_2$ is negative in the first and last cases. However, using the property that, for all odd integers $m$, $\Im_a[f_0(x_0)](x_a) = \Im_{a+2m}[f_0(x_0)](-x_{a+2m})$, we equivalently obtain $a_1$-order values $-1.541$, $-0.459$, 0.459, and 1.541, for which $M_2$ is always positive. The difference in FRT order for all these pairs is 0.082.

To validate Eq. (9) we use the Cai and Wang OFRT system,[10] as it allows us to specify different fractional orders for the same standard focal length $f$. Our implementation is identical to that shown in Fig. 1 in Ref. 10, and we use the notation defined there. The diameter of the pinhole was estimated experimentally to be 15.9 $\mu$m. For ease of implementation and practical reasons concerning the distances between the lenses, the 10-cm focal lengths, and the finite aperture sizes, the order $a = 1.459$ was chosen, giving $a_1 = 1.541$. This is illustrated in Fig. 1, where we have subtracted 2 from both $a$ and $a_1$, giving inverted but otherwise equivalent values. The distances for $a = 1.459$ were calculated to be $d_1 = d_3 = 54.7$ mm and $d_2 = 145.3$ mm (see Fig. 1 in Ref. 10). The input function was illuminated with the green laser ($\lambda_1 = 488$ nm), and the resulting intensity of the FRT of order $a = 1.459$, shown in Fig. 2(a), was recorded with a Sony CCD VCL 1437 camera. The required distances to implement the optical FRT system of order $a_1 = 1.541$ were found to be $d_1 = d_3 = 62.6$ mm and $d_2 = 134.4$ mm. When the input function is illuminated with the red laser ($\lambda = 633$ nm), the resulting output intensity is shown in Fig. 2(b).

For the input function used, the general form of the output (sombrero or broad-brimmed hat type intensity distribution) is predicted to be the same and is measured to be unchanged in the different fractional domains. The effect of varying the order is to change the size of the output intensity pattern. Equation (10) predicts that $1/M_2 = 0.878$. For the case of the green laser, the radius of the second ring was measured to be 7.02 mm, and for the case of the red laser it was measured to be 7.93 mm. Multiplying $1/M_2$ by 7.93 mm gives 6.96 mm, which is within $100 \times (0.06/7.02) = 0.85\%$ of the measured radius value for the green case. In conclusion, the variation predicted by the theory is in good agreement with our experimental results.

Why is this result of practical significance? Recently[15] it was shown that by capturing a sequence of OFRTs of the field reflected from a rough surface for two different FRT orders one can simultaneously determine independent displacements and rotations of the surface by use of speckle photographic techniques. In this Letter we have theoretically shown how order and wavelength are intimately related and have validated this prediction experimentally. Since varying the illumination wavelength will cause the order of a fixed OFRT to vary, this technique will allow simple implementation of a speckle photographic variable-order OFRT-based system.

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