Pergamon

Control Eng. Practice, Vol. 3, No. 4, pp. 495-503, 1995

0967-0661/95 $9.50 + 0.00

0967-0661(95)00021-6

MULTIVARIABLE CONTROL USING THE SINGULAR VALUE DECOMPOSITION IN STEEL ROLLING WITH QUANTITATIVE ROBUSTNESS ASSESSMENT

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(Received April 1993; in final form November 1994)

Abstract: The shape control problem for a Sendzimir cold rolling steel mill has been well documented. However, application of the *Singular Value Decomposition* (SVD) allows valuable insight to be gained into the control problem and produces a superior control strategy. In addition, singular values provide a natural basis for robustness analysis which is important in the mill context, due to the multi-pass, multi-schedule operation with resulting frequent changes in the plant parameters.

Keywords: Multivariable control systems; Rolling mills; Shape control; Singular value decomposition; Robustness;

1. THE Z-MILL AND SHAPE CONTROL

1.1 Shape of Steel Strip

Shape (in the current context, a misnomer) refers to the *stress* distribution in steel strip. A strip with perfect shape has a uniform internal stress distribution, so that if cut into narrow strips, it will lie flat on a flat surface. Bad shape can cause strip to buckle or tear (in the extreme). Shape measurement is performed by measuring a differential *tension* profile across the strip (see Fig.1) at 8 (modelled) equally-spaced points. The output of the system is therefore a profile, represented in vector form.

Strip shape may be controlled by bending the rolls of the mill, causing selective elongation of the strip at points where the rolls are closest. 'Long' or loose sections of the strip have associated compressive stress, while 'short' or tight sections suffer from tensile stress.

1.2 The Sendzimir Mill Model

The Z-mill has an ASEA 'Stressometer' for measuring the differential tension (or stress) profile across the strip. This device is mounted 2.91 m downstream of the roll gap and produces 8 (modelled) output measurements.

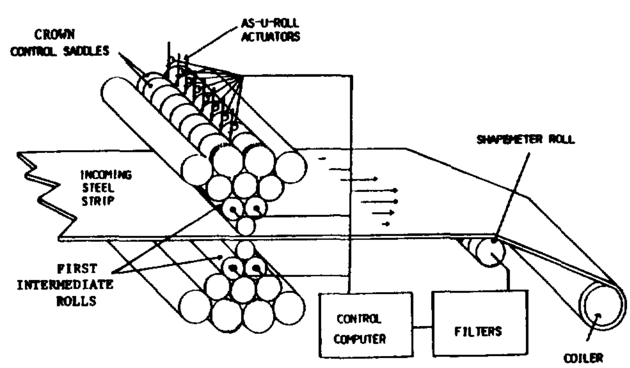

Fig. 1: Sendzimir 20-roll cold rolling steel mill

Two separate types of actuation for roll bending are provided on this mill (Fig.1). The 'As-U-Rolls' provide the equivalent of 8 independent (but equally spaced) point loads, while the first intermediate rolls are tapered, with lateral movement in or out of the mill creating roll bending. The upper and lower sets of the first intermediate rolls have opposite tapers, allowing both sides of the strip to be influenced equally, if necessary.

The Z-mill model, therefore, has 8 outputs and 10 outputs (8 AUR and 2 FIR). The rolling cluster is the most complex part of the system and accounts for all of the interaction between the 8 (modelled) paths in the system. Linearized gain matrices (G_a for the AUR's and G_i for the FIR's) relate changes in the roll-gap shape profile to changes in the positions of the AUR's and FIR's respectively. Diagonal dynamic blocks account for the actuators, strip dynamics (between roll-gap and shapemeter) and the shapemeter. The mill model is therefore of the form:

$$y = h(s)\,[G_a\, f_a(u_a) \quad G_i\, f_i u_i)\,] \qquad G_a \;\varepsilon\; \Re^{8x8} \;,\quad G_i \;\varepsilon\; \Re^{8x2} \tag{1}$$

where h(s) includes dynamics due to the strip and shapemeter, and the nonlinear functions $f_a(.)$ and $f_i(.)$ represent the AUR and FIR actuators respectively. Note that:

$$h(s) = \frac{e^{-0.582s}}{(1+1.064s)(1+0.74s)} \tag{2}$$

for a medium strip speed (5 $\rightarrow$ 10 m/s). A disturbance, d(s), is included in the mill model to account for the shape of the incoming steel strip.

1.3 Actuator Linearization

The control problem may be eased somewhat if the different nonlinear functions, $f_a(.)$ and $f_i(.)$, can be equalized and linearized to some nominal linear (scalar) transfer function p(s). Such a representation may be obtained using a simple describing function approach (Ringwood, 1994). Both actuator sets have similar block diagrams (see Fig. 2) but different parameter values. Identifying the dominant elements of the sub-system as the relay (with dead zone), proportional gain and the integrator (motor), the describing function (DF) is evaluated for the relay with dead zone as:

$$DF = (4/\pi x)\,(1 - \delta^2/x^2)^{1/2}\, s \tag{3}$$

where x is the input to the relay (and is measurable). If the above expression is used as an effective gain representation, a closed-loop transfer function for the actuator subsystem may be obtained as:

$$g_{act}(s) = \frac{1}{1 + (\pi x / 4k_c k_i)(1 - \delta^2 / x^2)^{1/2}\, s}\,. \tag{4}$$

This has the form of a linear, first-order transfer function, with time constant:

$$\tau = (\pi x / 4k_c k_i)\,(1 - \delta^2/x^2)^{1/2}. \tag{5}$$

If now a simple first-order compensator of the form:

$$T(s) = (1 + \tau s)/(1 + \tau_e s) \tag{6}$$

is placed in cascade with each of the actuator sets, with τ evaluated (on-line) from eq.(5) as

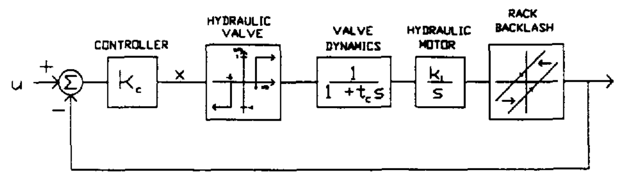

Fig. 2: Actuator block diagram

appropriate for the AUR's and FIR's, then the combination should yield a linear first-order system with constant parameters of the form:

$$p(s) = 1/(1 + \tau_e\, s) \tag{7}$$

where τ_e may be chosen by the designer. Note that τ_e may not be chosen arbitrarily fast, due to the limit on the rate of change of the hydraulic motor position (dependent on the motor gain). A value of $\tau_e = 2.0$ is appropriate for the application.

The transfer function matrix for the complete system (with the linearizing precompensators in place) now becomes:

$$G(s) = g(s)\;[G_a \;\; G_i] \tag{8}$$

where:

$$g(s) = h(s)\,p(s) \quad \varepsilon \quad \Re(s).$$

2. PREVIOUS APPROACHES TO THE CONTROL PROBLEM

Classical approaches to the multivariable design problem would suggest diagonalisation (or an attempt to make the system diagonally dominant) at a number of selected frequencies or in a general sense over all frequencies (MacFarlane, 1970; MacFarlane and Kouvaritakis, 1977). Indeed, from equation (8), it would seem that it is possible to decouple the system exactly over all frequencies, since all the interaction occurs in the *constant gain* matrices G_a and G_i. However, two factors complete the issue. Firstly, it may be noted that the matrix:

$$G_m = [G_a \;\; G_i\,] \quad \varepsilon \quad \Re^{8x10} \tag{9}$$

is non-square and secondly, G_a is not of full rank resulting in a rank less than 8 for G_m.

In (Grimble and Fotakis, 1982; Ringwood *et al*, 1990; Ringwood and Grimble, 1983), attention is focused on the AUR system only, resulting in a square G_m matrix. An s-domain optimal control formulation (Ringwood and Grimble, 1983) suggests precompensating the forward path with a G_m^{-1} block with resulting single-loop optimisation.

The problem of singularity is addressed using two approaches.

Approach 1 (Grimble and Fotakis, 1982)

The system output shape profile is parameterised in terms of a set of coefficients which reflect the components of low-order ($1 \rightarrow 4$) polynomial profiles present in the output. A 'parameterisation' matrix $P \in \Re^{4x8}$ multiples the output shape profile to give 4 effective outputs. In addition, a P^T term is applied at the system input, allowing the AUR actuator demand to be specified in parameterised form. This limits the allowable bending of the rolls to 4^{th} order - good from mechanical considerations. The resulting 4x4 system is full rank and may now be inverted.

Approach 2 (Ringwood et al, 1990)

In this approach an effective 'pseudo-inverse' of G_m is obtained by decomposing the system into its eigencomponents and neglecting the 'small' eigenvalues (and associated eigenvectors). This approach has much similarity with approach 1 but the effective 'parameterisation' matrix is formed by the eigenvectors corresponding to the four largest eigenvalues. Also, a measure of the degree of singularity of G_m is available from the eigenvalue spectrum, suggesting the reduced dimension size.

$$G_m = T \Lambda T^T \quad \text{(orthonormal eigenvectors)} \qquad (10)$$

$$\text{Pseudo-inverse} \quad G_m^+ = T_1 \Lambda_1^{-1} T_1^T \qquad (11)$$

where: $T_1 \in \Re^{8x4}$, $\Lambda_1 \in \Re^{4x4}$

This approach has the appeal that the parameterisation used (the eigenvectors, which roughly represent $1^{st} \rightarrow 4^{th}$-order polynomial profiles) represent the *natural* bending modes present in the mill, with the result that the precompensator, Λ_1^{-1}, is diagonal, reducing the required computations, since system diagonalisation is effectively performed by the eigenvector matrices.

Approach 3 (Ringwood and Grimble, 1990)

The analysis in (Ringwood and Grimble, 1990) considers the complete system (including the first intermediate rolls) and adopts the same parameterisation as that of Grimble and Fotakis (1982). Again, an input parameterisation is applied to the AUR's resulting in six effective inputs (4 AUR and 2 FIR). The resulting TFM is:

$$G_{red}(s) = g(s)\ P\ [G_a P^T \quad G_i] \in \Re(s)^{4x6}. \qquad (12)$$

The system (now of full rank, 4) may be diagonalised using a (non-unique) right inverse, such that:

$$P\ [G_a P^T \quad G_i]\ A^+ = I_4 \qquad (13)$$

In particular, if the Moore-Penrose inverse (Ben-Israel and Greville, 1974) is chosen, then

$$A^+ = A^T(AA^T)^{-1} \quad , \ A = P\,[G_a P^T \quad G_i] \qquad (14)$$

and the norm of the control input vector to the system is minimised.

3. THE SINGULAR VALUE DECOMPOSITION

3.1 Definition of SVD (Klema and Laub, 1980)

Let $M \in \Re_r^{m \times n}$, where the subscript denotes a matrix of rank r. Then M may be decomposed as:

$$M = U \Sigma V^T \qquad (15)$$

where

$$\Sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad S = diag(\sigma_1, \ldots\ldots, \sigma_r)$$

and

$$\sigma_1 \geq \sigma_2 \geq \ldots\ldots\ \sigma_r \geq 0$$

The numbers $\sigma_1,\ldots,\sigma_r$ are the *singular values* of M and are the positive square roots of the eigenvalues (which are non-negative) of M^TM (or equivalently MM^T). The columns of U are the *left singular vectors* of M which are the orthonormal eigenvectors of MM^T. The columns of V are the *right singular vectors* of M which are the orthonormal eigenvectors of M^TM. Consequently, both U and V are unitary matrices.

3.2 Calculation of the SVD

From the definition above, it would seem that the obvious method for the calculation of the singular value decomposition is to use eigensystem routines. It has been shown (Klema and Laub, 1980), however, that such an approach can result in a badly distorted answer, resulting from roundoff error in finite precision. Instead, a method based on the implicitly shifted QR algorithm is recommended (Golub and Kahan, 1965). To further complicate the issue, the precise coding of such an algorithm is important. MATLAB provides a reliable SVD algorithm based on LINPACK routines.

3.3 Some Useful Properties of the SVD

Property 1

The SVD provides a decomposition (diagonalization) method for non-square systems. This is clear from the above definition.

Property 2

The singular value spectrum is a good indicator of the singularity (rank) of a matrix, whereas the eigenspectrum is not. If λ is an eigenvalue of M, it can be shown (Doyle, 1979) that:

$$\underline{\sigma} \leq |\lambda| \leq \bar{\sigma} \quad (16)$$

where $\underline{\sigma}$ and $\bar{\sigma}$ denote the smallest and largest singular values of M, respectively. Note that it is possible for the smallest eigenvalue to be much larger than $\underline{\sigma}$. The quantity $\underline{\sigma}/\bar{\sigma}$ is known as the condition number with respect to inversion (Wilkinson, 1965).

Property 3

The SVD may be used to solve linear least-squares problems. Given a set of overdetermined equations:

$$Mx = b \quad , \quad M \in \Re^{m \times n}$$

the unique solution x of the smallest 2-norm which minimises $|b - Mx|_2^2$ is given by:

$$x = M^+ b \quad (17)$$

where

$$M^+ = U \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^T$$

This is equivalent to the Moore-Penrose inverse given in equation (14) above.

Property 4

The singular values of a matrix give norm measures of that matrix. In particular:

$$|M|_2 = \bar{\sigma} \; and \; |M|_F^2 = \sigma_1^2 + \sigma_2^2 + \; ...... , \; + \sigma_r^2 . \quad (18)$$

Thus the spectral norm and the Frobenius norm are known if the singular values are known. Norm calculations play an important part in robustness calculations.

These properties will now be utilised in the analysis and control of the Z-mill system.

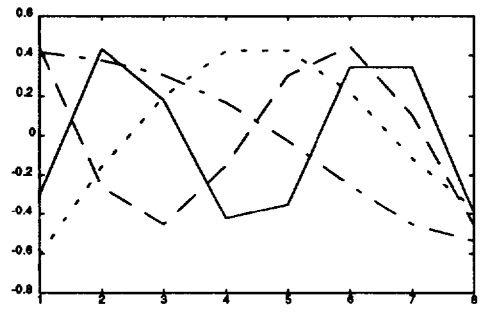

Fig. 3: Left singular vectors of mill matrix, G_m

4. SVD APPLICATION TO THE SHAPE CONTROL PROBLEM

From Property 3 above (equation (17)) it is clear that the SVD may be used to evaluate a non-square (generalised) inverse for a system. Thus, the SVD may be used to diagonalise the multivariable component in the Z-mill model, G_m, given in equation (9). However, upon examination of a typical singular value spectrum for G_m, evaluated as:

9.96 7.60 4.19 1.48 0.33 0.25 0.091 0.025

it would appear that a separation condition exists such that:

$$\mu_1 = \min_{1 \leq i \leq 4}\{\sigma_i\} >> \max_{5 \leq i \leq 8}\{\sigma_i\} = \mu_2 \quad (19)$$

In view of this separation condition, it would seem appropriate to concentrate the control design on the larger singular values. Some factors which support this decision are:

- An inverse which relies on the full singular value spectrum is likely to be sensitive to small variations in G_m, due to the relatively poor condition number. G_m is known to contain modelling inaccuracies - from physical considerations (symmetry of the mill rolling cluster) the matrix G_a should be 'singular' and two different modelling exercises (Gunawardene, 1982; Dutton, 1983) have resulted in poor agreement on the values of the matrix gains.

- Under normal rolling practice on the Z-mill, no attempt is made to control shape components representing polynomial profiles greater than fourth order. Figure 3 displays the first four left singular vectors which are seen to be roughly linear, quadratic, cubic and quartic in profile. These indicate the natural bending modes present in the mill. Therefore, concentration of the design on the first four singular values accords with rolling practice.

- The size of the four smallest singular values (effective gains associated with polynomial orders 4 → 8) indicate the excessive amount of control effort associated with setting up these high-order bending modes.

Note that a similar separation condition to (19) exists for the eigenvalues of G_a. However, not only does the current analysis cater for the non-square system including the FIR's, but it also concentrates on the *singular values*, which are better indicators of singularity (or relative singularity).

4.1 Feedback Control Design

With reference to the separation condition in (19) above, partition the matrices containing the left and right singular vectors of G_m as:

$$G(s) = g(s)\begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \qquad (20)$$

where:

$$U_1, U_2 \in \Re^{8\times 4} \quad , \quad V_1 \in \Re^{10\times 4}$$

$$V_2 \in \Re^{10\times 6} \quad , \quad \Sigma_1, \Sigma_2 \in \Re^{4\times 4}$$

Equation (20) can alternatively be expressed as:

$$G(s) = g(s)\left[U_1\Sigma_1V_1^T + U_2\Sigma_2V_2^T\right] \qquad (21)$$

Now choose a forward path compensator:

$$K(s) = k(s)K = k(s)V_1\Sigma_1^{-1}U_1^T \qquad (22)$$

so that the system is diagonalised with respect to the Σ_1 singular values and the high-order shape profiles present in the output are ignored, via the U_1^T parameterisation. k(s) is a scalar transfer function chosen to give suitable closed-loop dynamics. Figure 4 shows the compensated system, where the reference and output are specified as (8-point) shape *profiles*. With a minor rearrangement, Figure 5 shows the system represented in parametric form, where the reference and output

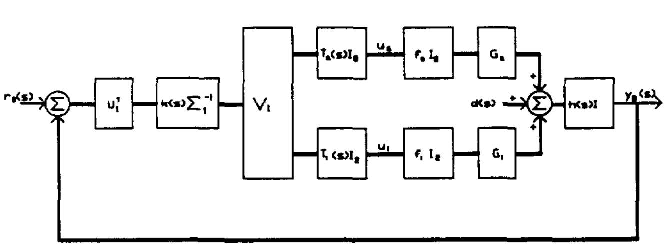

Fig. 4: 8 x 8 compensated system

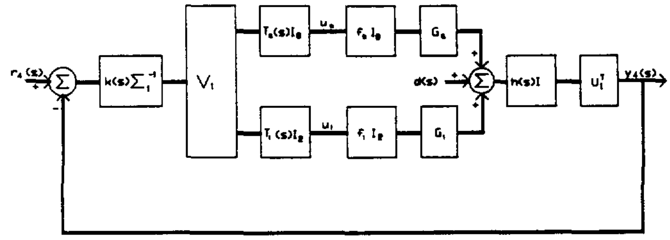

Fig. 5: 4 x 4 compensated system

are specified in terms of 1st → 4th-order polynomial coefficients. That the system of Fig.5 is decoupled may be observed from Fig.6, which shows (as an example) the open-loop response due to a step in parameter 4 only.

The dynamic system design was performed using classical (scalar) frequency response techniques. For a medium strip speed:

$$k(s) = \frac{200(2s+1)}{(1000s+1)(0.9s+1)}. \qquad (23)$$

4.2 Simulation Results

A full nonlinear simulation was used to assess the performance of the SVD-based controller. A uniformly flat shape profile was demanded (desired shape parameters 1 → 4 set = 0) with a constant (but non-zero) disturbance profile being introduced to simulate poor incoming strip shape.

The output shape profile variations with time are shown in Fig.7. The initial profile (time = 0 to 3 secs) represents the shape disturbance profile appearing at the system output. After 3 seconds, control is applied and only a high (> 4th) order residual profile remains at the end of the simulation run. Figure 8 shows the variations in the 4 shape parameters. Note that the 1st → 4th-order coefficients converge to zero (as required) but the high-order coefficients (corresponding to the U_2 vectors) are unaffected.

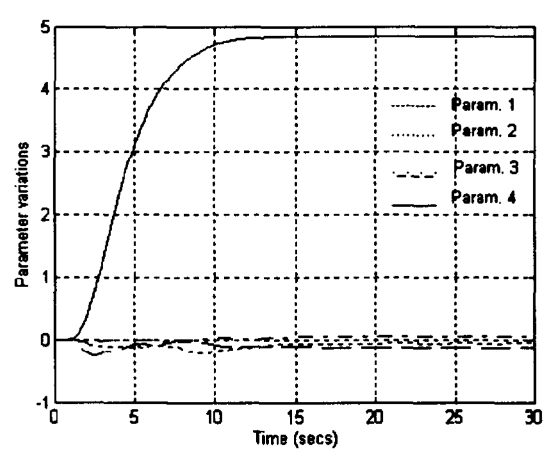

Fig. 6: Response to setpoint step in parameter 4

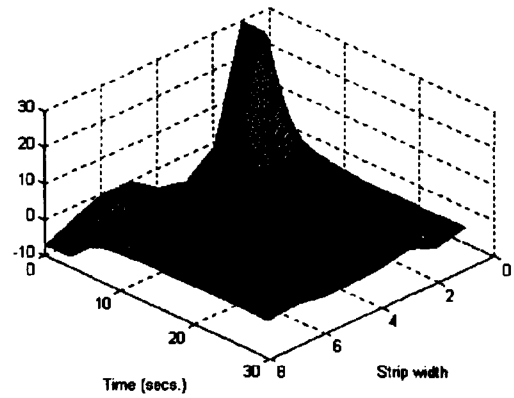

Fig. 7: Shape profile variations

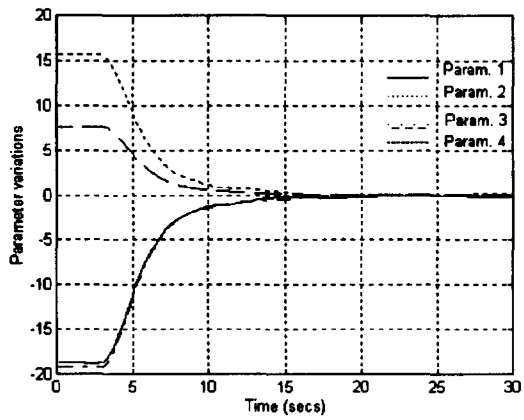

Fig. 8: Shape parameter variations

5. A QUANTITATIVE ROBUSTNESS MEASURE

The Z-mill processes more than 3500 different material sizes and types (with associated changes in roll diameters, roll taper gradients, etc.). Since it is desirable to use one controller with a number of different mill schedules, it is important to have a measure of the allowable variations in G_m which retain stability (note that g(s) is constant for a given strip speed).

Singular value analysis provides an excellent framework for robustness analysis due to:

(a) the analogy of singular values with *gain* for scalar systems, and

(b) the relationship of singular values with matrix norms.

In addition, since the Z-mill control design is SVD-based, particularly simple robustness measures result. Robustness measures based on singular values are widely reported in the literature, see (Doyle, 1979 ; Cruz *et al.*, 1981; Postlethwaite *et al.*, 1981). The analysis shown here follows the general spirit of these approaches. Two robustness result variants are developed. The first is based on the 8x8 system, while the second concentrates on the 4x4 system. The second approach is necessary for systems where it is required to incorporate integral action into the dynamic controller.

5.1 Problem Formulation

It is required to determine allowable variations in G_m, such that stability of the closed-loop system is retained. An additive perturbation, Δ_m, is considered, resulting in the perturbed mill matrix of $(G_m+\Delta_m)$.

The stability of the perturbed system is described by the return difference (with reference to Fig. 4) as:

$$\underline{\sigma}(I_8+(G_m+\Lambda_m)K\,gk(s)) \;>\; 0 \quad \text{for} \quad \mathrm{Re}(s)\geq 0. \quad (24)$$

The condition in (24) specifies a requirement that the return difference matrix (or characteristic polynomial matrix) must not have any roots (i.e. its determinant must be non-zero) in the right-half s-plane (Re(s) ≥ 0). Since $[I_8+(G_m+\Lambda_m)K\,gk(s)]$ is strictly proper and analytic and bounded in the interior of D, the suprema are achieved on the imaginary axis, so (24) may be replaced by:

$$\underline{\sigma}(I_8+(G_m+\Lambda_m)K\,gk(j\omega)) \;>\; 0 \;,\;\; \omega\geq 0 \quad (25)$$

Equation (25) provides the basis for the following robustness developments.

5.2 Calculation for 8x8 System

Equation (25) may be replaced by:

$$\underline{\sigma}(I_8+G_mK\,gk(j\omega)) \;>\; \overline{\sigma}(\Lambda_mK\,gk(j\omega)) \qquad \omega\geq 0 \quad (26)$$

using the relation:

$$\underline{\sigma}(A+B) \;\geq\; \underline{\sigma}(A) \;-\; \overline{\sigma}(B) \quad (27)$$

Using a second relation:

$$\overline{\sigma}(AB) \;\leq\; \overline{\sigma}(A)\,\overline{\sigma}(B) \quad (28)$$

equation (26) may be further modified to:

$$\underline{\sigma}(I_8+G_mK\,gk(j\omega)) > \overline{\sigma}(\Lambda_m)\,\overline{\sigma}(K\,gk(j\omega)) \quad \omega\geq 0\,. \quad (29)$$

Equation (29) describes a condition for the stability of the perturbed system in terms of an upper bound on the largest singular value of the perturbation matrix. This condition may be further simplified to:

$$\sigma_4 > \bar{\sigma}(\Lambda_m) \sup_{\omega\geq 0}(|gk(j\omega)| / \min(1, |1+gk(j\omega)|)) \quad (30)$$

using the simplifications indicated in Appendices 1 and 2 and taking the supremum over frequency. Equation (30) describes a condition for stability of the perturbed system based on the open-loop frequency response, the maximum singular value of the perturbation matrix and the smallest singular value of the reduced-dimension nominal system. Two comments about this result are noteworthy:

- Since the maximum gain of the open-loop frequency response and not the closed-loop frequency response is used, equation (30) cannot be used with systems containing integral action.
- Some conservatism is built into the calculations via equations (27) and (28).

5.3 Calculation for 4x4 System

With reference to equation (25) and Fig. 5, the stability of the perturbed system is described by the return difference:

$$\underline{\sigma}(I_4 + U_1^T(G_m + \Lambda_m)V_1\Sigma_1^{-1} gk(j\omega)) \; > \; 0 \quad , \quad \omega \geq 0. \quad (31)$$

Equation (31) may be recast as:

$$\underline{\sigma}(I_4 + U_1^T G_m V_1 \Sigma_1^{-1} gk(j\omega)).$$
$$(I_4 + (1+gk(j\omega))^{-1} gk(j\omega) U_1^T \Lambda_m V_1 \Sigma_1^{-1}) > 0 , \omega \geq 0 \quad (32)$$

Now, since

$$U_1^T G_m V_1 \Sigma_1^{-1} \; = \; I_4$$

and the nominal system has been designed to be closed-loop stable, i.e. $(1+gk)^{-1}gk \neq 0$, $\omega \geq 0$, the stability of the perturbed system is completely described by the inequality:

$$\underline{\sigma}(I_4 + (1+gk(j\omega))^{-1} gk(j\omega) U_1^T \Lambda_m V_1 \Sigma_1^{-1}) \; > \; 0$$

$$\omega \geq 0. \quad (33)$$

Now, repeatedly using relation (27), equation (33) may be reduced to:

$$\gamma\, \bar{\sigma}(\Sigma_1^{-1})\bar{\sigma}(\Lambda_m) \; < \; 1$$

or

$$\bar{\sigma}(\Lambda_m) \; < \; 1/[\gamma\, \bar{\sigma}(\Sigma_1^{-1})] \quad (34)$$

where:

$$\gamma \; = \; \sup_{\omega\geq 0}\left(\frac{gk(j\omega)}{(1+gk(j\omega))}\right) \quad (35)$$

The result in equation (34) is intuitively appealing, requiring a tredeoff between the maximum value of the closed-loop frequency response, the largest singular value of the perturbation matrix and the largest singular value of the controller precompensator matrix, which is directly related to the smallest singular value of the reduced-dimension, unperturbed mill matrix. Again, some conservatism is built into the result in the progression from equation (33) to equation (34).

6. SOME PROPERTIES OF THE SVD CONTROLLER

Two comments on the SVD-based controller are appropriate. The first concerns the use of the SVD as a tool to solve overdetermined or underdetermined equations. In both of these equations, a number of possible solutions exist, and a common approach is to minimise the norm (spectral norm) of the error (overdetermined equations) or minimise the norm of solution (undetermined equations) in order to obtain a unique answer.

The Z-mill shape control problem may be represented as a set of underdetermined equations - it is required to determine 10 control inputs from 8 error signals. The problem may be stated (omitting the dynamics, and assuming the reference signal to be zero, for convenience) as:

$$G_m \, u \; = \; y \quad (36)$$

where

$$u \in \Re^{10} \quad , \quad y \in \Re^{8}$$

and it is desired to find the control input, **u**, which diagonalises the system. Property 3 in Section 3 implies that when the SVD is used to determine a non-square inverse, the Moore-Penrose inverse is evaluated. The Moore-Penrose inverse minimises the norm of the solution, which in the case of equation (36), results in the minimisation of $\mathbf{u}^T\mathbf{u}$. Recall that **u** are the actuator inputs (see Fig. 4).

In the other solution to the non-square Z-mill problem (Ringwood and Grimble, 1990) (which will be referred to as the R&G controller), a Moore-Penrose inverse is also evaluated. However, the control signal vector ($\in R^6$) which is minimised in that case passes through a deparameterisation stage (to obtain 10 signals) before reaching the actuators.

It would seem, then, that the SVD solution is more appropriate in its signal minimisation.

The second point concerns the computational burden of the compensator. Both the SVD scheme and the R&G scheme have 4x8 output parameterisations. Both schemes have input deparameterisation (to the actuators) also, 8x4 for the R&G controller and 10x4 for the SVD case. However, the compensating matrix in the R&G controller is a full 6x4, whereas the SVD has a diagonal matrix of dimension 4 (Σ_1^{-1}). Table 1 summarises the computations:

Table 1 : Computational comparison of schemes

operation	R&G scheme	SVD scheme
o/p param.	32(x) & 28(+)	32(x) & 28(+)
i/p deparam.	32(x) & 24(+)	40(x) & 30(+)
comp.	24(x) & 18(+)	4(x)
Total	88(x) & 70(+)	76(x) & 58(+)

Since the steel strip can have speeds of up to 15 m/s, a sampling period in the 10's of milliseconds would not be inappropriate, so the saving of 12(x) and 12(+) in computational load may be helpful. The dynamic compensation for both schemes is identical.

The reason for the slightly lower computational effort for the SVD scheme is that the parameterisation itself diagonalises the system, the compensator merely equalising the gains in each of the resultant 4 separate loops. It would seem that this scheme concentrates on the 'natural' bending modes in the system, without forcing an alien parameterisation as in the R&G scheme, which results in the extra decoupling effort required.

7. CONCLUSIONS

The Z-mill shape control problem has been recast in an SVD framework. This would seem to be the natural setting for the problem, considering such features as control signal minimisation, ease of decoupling (and associated lighter computational burden) and basis for robustness calculations. An important feature is an analytical robustness measure, which gives an indication of the number of controller precompensators which must be stored in order to cover all operational cases. Since the precompensator is diagonal, storage requirements are less than for previous schemes.

APPENDIX 1

Simplification of $\bar{\sigma}(K\,gk(j\omega))$.

From (22), $$K = V_1\Sigma_1^{-1}U_1^T$$

Now let $$S = \Sigma_1^{-1}\,gk(j\omega),$$

so that $$S = \mathrm{diag}\,(gk(j\omega)/\sigma_i) \qquad 1 \le i \le 4$$

$$\Rightarrow \qquad K\,gk(j\omega) = V_1\,S\,U_1^T. \qquad \text{(A1.1)}$$

The SVD of the matrix $(K\,gk(j\omega))$ may be found (from Section 2) by computing the eigensystem for $(K\,gk(j\omega))^H(K\,gk(j\omega))$, where $(\ )^H$ denotes the complex conjugate transposed. Using equation (A.1.1) and noting that V_1 is unitary,

$$(K\,gk(j\omega))^H(K\,gk(j\omega)) = U_1\,S^*S\,U_1^T \qquad \text{(A1.2)}$$

where $(\ )^*$ denotes complex conjugation. The eigenvalues of $(K\,gk(j\omega))$ are therefore given as:

$$(gk^*(j\omega)gk(j\omega)/\sigma_i^2) \quad , \quad 1 \le i \le 4$$

and

$$\bar{\sigma}(K\,gk(j\omega)) = |gk(j\omega)|/\sigma_4. \qquad \text{(A1.3)}$$

APPENDIX 2

Simplification of $\underline{\sigma}(I_8 + K\,gk(j\omega))$.

Initially, consider

$$G_mK = U\Sigma V^T V_1\Sigma_1^{-1}U_1^T$$

or

$$G_mK = \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} V_1\Sigma_1^{-1}U_1^T \qquad \text{(A2.1)}$$

Noting that: $V_1^TV_1 = I_4$ and $V_2^TV_1 = 0$

equation (A2.1) may be simplified to:

$$G_mK = \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \end{bmatrix}\begin{bmatrix} \Sigma_1^{-1} \\ 0 \end{bmatrix} U_1^T$$

or

$$G_mK = U\,U_1^T. \qquad \text{(A2.2)}$$

Now, since $UU^T = I_8$,

$$(I_8 + K\,gk(j\omega)) = UU^T\left[I_8 + U_1U_1^T gk(j\omega)\right]UU^T$$

$$(I_8 + K\,gk(j\omega)) = U\left[U^T U + (U^T U_1)(U_1^T U) gk(j\omega)\right]U^T$$

or

$$(I_8 + K\,gk(j\omega)) = U\left[I_8 + \begin{bmatrix} I_4 & 0 \\ 0 & 0 \end{bmatrix} gk(j\omega)\right]U^T.$$

Finally,

$$(I_8 + K\,gk(j\omega)) =$$

$$U\begin{bmatrix}\begin{bmatrix}1+gk(j\omega) & & & \\ & 1+gk(j\omega) & & \\ & & 1+gk(j\omega) & \\ & & & 1+gk(j\omega)\end{bmatrix} & 0 \\ 0 & I_4\end{bmatrix}U^T$$

By examining the eigenspectrum of $(I_8 + K\,gk(j\omega))^H (I_8 + K\,gk(j\omega))$ in a manner similar to that in Appendix 1, the singular values of $(I_8 + K\,gk(j\omega))$ may be found to be:

$$(|1+gk(j\omega)|, |1+gk(j\omega)|, |1+gk(j\omega)|, |1+gk(j\omega)|, 1, 1, 1, 1)$$

so that:

$$\underline{\sigma}(I_8 + K\,gk(j\omega)) = \min\left[1, |1+gk(j\omega)|\right]. \qquad \text{(A2.3)}$$

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