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Bundle stabilisation and positive Ricci curvature

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Abstract

If E is the total space of a vector bundle over a compact Ricci non-negative manifold, it is known that $E \times \mathbb{R}^p$ admits a complete metric of positive Ricci curvature for all sufficiently large p. In this paper we establish a small, explicit lower bound for the dimension p.

MSC: 53C20

Keywords: Positive Ricci curvature; Non-negative Ricci curvature; Bundles

1. Introduction

It is easy to see from [1, §9G] that the total space of any fibre bundle with base and fibres compact and Ricci positive will admit a Ricci positive metric—at least if the structural group is a Lie group and acts by isometries on the fibre. (All metrics under consideration will be complete.) This result was first pointed out by Nash [5]. In [5] and [2], the existence of a Ricci positive metric for any vector bundle with fibre dimension at least two and compact Ricci positive base was established. In [3] it was shown that if M is a complete manifold (either compact or non-compact) with non-negative Ricci curvature, then $M \times \mathbb{R}^n$ admits a Ricci positive metric, provided $n \geqslant 3$. This raises the question of which vector bundles over Ricci non-negative manifolds admit Ricci positive metrics.

The existence of Ricci positive metrics on bundles where base and fibre are compact and Ricci non-negative, and on vector bundles over compact Ricci non-negative manifolds was investigated in [4]. In both cases it was shown that the manifold admits a positive Ricci curvature metric after 'stabilising', that is, after forming the Cartesian product with some Euclidean space $\mathbb{R}^p$ for p sufficiently large. The required minimum value of p depends on the base and fibre dimensions, as well as how twisted the bundle is. It would seem difficult to compute a suitable value for p in most cases, and it is likely that suitable values arising from the method in the paper are very large. Belegradek and Wei ask whether it is possible to obtain a realistic lower bound for p. The aim of this paper is to address this question. Specifically, we prove:

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Theorem A. *Let E be the total space of a bundle with compact Ricci non-negative base, either compact Ricci non-negative fibre or a vector space as fibre, with structural group a Lie group acting by isometries on the fibre in the compact fibre case, and an orthogonal group in the case of a vector bundle. Suppose the fibre has dimension n. Then $E \times \mathbb{R}^p$ admits a complete metric of positive Ricci curvature provided $p > 24n + 1$.*

Thus in our case, the lower bound for p is independent of both the base dimension and the twisting of the bundle E. It is not clear if this lower bound is optimal.

It follows easily from Theorem A that if N denotes the three-dimensional Heisenberg manifold, then $N \times \mathbb{R}^p$ admits a Ricci positive metric whenever $p > 25$. It is interesting to compare this with [8, p. 17] where it is shown that $p > 26$ is sufficient.

This paper is laid out as follows. In Section 2 we describe the form our metric on $E \times \mathbb{R}^p$ will take. Our approach to computing curvatures is to use the O'Neill formulas for Riemannian submersions. This necessitates computation of A-tensors, and this is carried out in Section 3. We present the Ricci curvature formulas for our metric in Section 4, and finally we prove Theorem A in Section 5.

2. Metric overview

Consider a fibre bundle with total space E, base B^m, fibre F^n and Lie structural group G. We will assume that both B and F are equipped with Ricci non-negative metrics (G-invariant in the case of F), and that B is compact. Given a principal connection Θ on the associated principal G-bundle, it follows from [7] that there is a unique submersion metric g on E with totally geodesic fibres, which agrees with the given metrics on B and F, and whose horizontal distribution is associated to Θ. From the O'Neill formulas for the Ricci curvature of a Riemannian submersion [1, §9D] it is clear that this submersion metric will not necessarily have positive Ricci curvature.

We will study the curvature of such bundles after 'stabilising'—that is we will study the curvature of $E \times \mathbb{R}^p$ for some p. There are various ways in which we could view such a product. For example, we could view this as a bundle with base B and fibre $F \times \mathbb{R}^p$, where the structural group G acts trivially on $\mathbb{R}^p$. We will adopt the viewpoint, however, where we regard $E \times \mathbb{R}^p$ as a bundle with base $B \times \mathbb{R}^p$, fibre F and structure group G. If $\pi : E \to B$ denotes the bundle projection for E, then clearly the bundle projection for $E \times \mathbb{R}^p$ is given by $\pi' : E \times \mathbb{R}^p \to B \times \mathbb{R}^p$, where $\pi' = \pi \times \mathbb{I}$ with $\mathbb{I}$ the identity on $\mathbb{R}^p$.

Let us consider metrics on such an object. We will equip $E \times \mathbb{R}^p$ with a submersion metric as described above. To do this we need to specify a G-invariant fibre metric, a base metric and a G-invariant horizontal distribution on the associated principal bundle. We assume that there exists a G-invariant Ricci non-negative metric on F: label such a metric $\hat{g}$. On the base we will assume a warped product metric of the form

$$dr^2 + f^2(r)\, ds^2_{p-1} + h^2(r)\check{g}$$

where $\check{g}$ is a given Ricci non-negative metric on B, r denotes the radial parameter of $\mathbb{R}^p$, $f(r)$ is a smooth function which is odd at $r = 0$ satisfying $f'(0) = 1$, $f(r) > 0$ for $r > 0$, and $h(r)$ is a smooth positive function which is even at $r = 0$.

Let P denote the principal G-bundle associated to E. Fix a principal connection on P. This is equivalent to giving a (right) G-invariant distribution of horizontal subspaces $\mathcal{H}$ for P. There is a natural way in which this can be 'extended' to a horizontal distribution $\tilde{\mathcal{H}}$ for $P \times \mathbb{R}^p$, the principal bundle associated to $E \times \mathbb{R}^p$. At any point $x \in P$, we have a subspace $\mathcal{H}_x \subset T_x P$. For any $y \in \mathbb{R}^p$, the subspace $\tilde{\mathcal{H}}_{(x,y)}$ is defined to be the space

$$\mathcal{H}_x \oplus \mathbb{R}^p \subset T_x P \oplus T_y\mathbb{R}^p = T_{(x,y)}(P \times \mathbb{R}^p).$$

Clearly $\{\tilde{\mathcal{H}}_{(x,y)} \mid (x, y) \in P \times \mathbb{R}^p\}$ is a horizontal distribution, and moreover is G-invariant since $\{\mathcal{H}_x \mid x \in E\}$ is G-invariant. Thus $\tilde{\mathcal{H}}$ corresponds to a principal connection on $P \times \mathbb{R}^p$. Associated to $\mathcal{H}$ respectively $\tilde{\mathcal{H}}$ we obtain horizontal distributions on E respectively $E \times \mathbb{R}^p$ which we will denote $\mathcal{D}$ respectively $\tilde{\mathcal{D}}$. It is easy to check that $\tilde{\mathcal{D}}_{(e,y)} = \mathcal{D}_e \oplus T_y\mathbb{R}^p$ at any point $(e, y) \in E \times \mathbb{R}^p$.

Given this data, a result of [7] guarantees us a complete submersion metric on $E \times \mathbb{R}^p$ with totally geodesic fibres, agreeing with the given data. Let us denote this new metric $\tilde{g}$. From now on, all quantities labelled with a tilde will relate to this metric.

3. A-tensor calculations

We now compute the A-tensor, $\tilde{A}$, of $\tilde{g}$. See [1, §9C] for definitions and details. In particular we will compare $\tilde{A}$ with the A-tensor of the submersion metric g on E determined by $\check{g}$, $\hat{g}$ and $\mathcal{D}$. We will denote this latter tensor simply A. First we select local vector fields on $E \times \mathbb{R}^p$. Let X and X' be horizontal fields tangent to E, U a (vertical) vector field tangent to the fibres of E, W and W' vector fields tangent to $S^{p-1} \subset \mathbb{R}^p$, and d/dr the radial vector field on $\mathbb{R}^p$. Note that there is a canonical way of extending the vector fields X and X' to $\tilde{\mathcal{D}}$-horizontal vector fields on $E \times \mathbb{R}^p$, and of extending U to a vertical vector field on $E \times \mathbb{R}^p$. *In the following discussion we will use the same symbols to denote both the vector fields on E and the 'equivalent' fields on* $E \times \mathbb{R}^p$, *the context making clear which is to be understood*. Moreover, we will also be considering the Lie brackets of these vector fields for both bundles. However, it is clear that $[X, X']$ and $[X, U]$ on $E \times \mathbb{R}^p$ are just the canonical extensions of the equivalent Lie brackets on E. We will therefore use the same notation for these Lie brackets in both contexts.

Lemma 3.1. $\tilde{A}_X X' = A_X X'$.

Proof. By [1, 9.24] we have

$$\tilde{A}_X X' = \frac{1}{2}\tilde{\mathcal{V}}[X, X'] = \frac{1}{2}\mathcal{V}[X, X'] = A_X X'.$$

Here, $\tilde{\mathcal{V}}$ (respectively $\mathcal{V}$) denotes vertical projection onto the tangent space to the fibres in $E \times \mathbb{R}^p$ (respectively E). □

Lemma 3.2. *The following expressions all vanish identically*: $\tilde{A}_{d/dr} d/dr$, $\tilde{A}_{d/dr} W$, $\tilde{A}_{d/dr} X$, $\tilde{A}_W X$, $\tilde{A}_W W'$.

Remark. For horizontal vector fields in $E \times \mathbb{R}^p$ (that is, vector fields tangent to the distribution $\tilde{\mathcal{D}}$), the A-tensor is anti-symmetric. Thus the above two lemmas essentially cover all possibilities for $\tilde{A}$ evaluated on a pair of $\tilde{\mathcal{D}}$-horizontal vector fields.

Proof of lemma. We show that the $\tilde{\mathcal{V}}$-components of the Lie brackets $[d/dr, d/dr]$, $[d/dr, W]$, $[d/dr, X]$ and $[W, X]$ all vanish.

By antisymmetry of the Lie bracket it follows that $[d/dr, d/dr]$, and hence $\tilde{\mathcal{V}}[d/dr, d/dr]$ must vanish.

An elementary calculation shows that $[d/dr, W]$ is a vector field tangent to S^{p-1}, so its vertical component must be zero. The same is of course true for $[W, W']$.

Since $\tilde{A}$ is a tensor, the quantities $\tilde{\mathcal{V}}[W, X]$ and $\tilde{\mathcal{V}}[d/dr, X]$ only depend on X at any given point. Hence we are free to extend a given $\mathcal{D}$-horizontal vector to any $\tilde{\mathcal{D}}$-horizontal vector field X without altering $\tilde{\mathcal{V}}[\bullet, X]$. Let us extend a given vector to a $\mathcal{D}$-horizontal vector field on E in any way, and then extend to $E \times \mathbb{R}^p$ in the canonical way. In particular, the resulting vector field X is independent of the $\mathbb{R}^p$ coordinate. With these assumptions it is easy to show that $[d/dr, X]$ vanishes, and $[W, X]$ is a vector field tangent to S^{p-1} and so has no vertical component. □

We now turn our attention to expressions taking the form $\tilde{A}_\bullet U$, where the dot represents a horizontal vector field.

Lemma 3.3. *We have* $\tilde{A}_{d/dr} U \equiv 0$, $\tilde{A}_X U = \frac{1}{h^2} A_X U$ *and* $\tilde{A}_W U \equiv 0$.

Proof. First let us consider $\tilde{A}_{d/dr} U$. By definition of the A-tensor, we have

$$\tilde{A}_{d/dr} U = \tilde{\mathcal{H}} \tilde{\nabla}_{d/dr} U$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the metric $\tilde{g}$ and $\tilde{\mathcal{H}}$ is the operator taking the $\tilde{\mathcal{D}}$-horizontal component. We construct a local coordinate system for $E \times \mathbb{R}^p$ as follows. First choose a local coordinate system $x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}$ for E, such that $x_1, \dots, x_n$ are coordinates in the fibres. Next, extend this to a local coordinate system $y_1, \dots, y_{n+m+p}$ on $E \times \mathbb{R}^p$, where $y_i = x_i$ for $1 \leqslant i \leqslant n+m$, $y_{n+m+1} = r$ and $y_{n+m+2}, \dots, y_{n+m+p}$ are local coordinates on S^{p-1}. It is an elementary computation to show that the Christoffel symbols $\tilde{\Gamma}^k_{n+m+1\,i}$ vanish identically for all $1 \leqslant i \leqslant n$ and for all k, and hence $\tilde{\nabla}_{d/dr}\partial/\partial y_i$ vanishes for all $1 \leqslant i \leqslant n$. The vertical vector field U can be expressed as

$$U = \sum_{i=1}^{n} f_i \partial/\partial y_i$$

where the f_i are functions on $B \times \mathbb{R}^p$. We therefore have

$$\tilde{\nabla}_{d/dr} U = \sum_{i=1}^{n} f_i \nabla_{d/dr}\partial/\partial y_i + \sum_{i=1}^{m} \partial f_i/\partial r \partial/\partial y_i = \sum_{i=1}^{m} \partial f_i/\partial r \partial/\partial y_i.$$

Clearly, this is a vertical vector, so

$$\tilde{A}_{d/dr} U = \tilde{\mathcal{H}}\tilde{\nabla}_{d/dr} U = 0.$$

Next we consider the term $\tilde{A}_X U$. To evaluate this, we must consider the following expressions: $\tilde{g}(\tilde{A}_X U, d/dr)$, $\tilde{g}(\tilde{A}_X U, W)$ and $\tilde{g}(\tilde{A}_X U, X')$. In all cases, these expressions are equal to $-\tilde{g}(\tilde{A}_X \bullet, U)$ (see [1, 9.21(d)]). Since $\tilde{A}_X d/dr$ and $\tilde{A}_X W$ both vanish (as discussed above), we conclude that $\tilde{A}_X U$ has no component in any $\mathbb{R}^p$ direction.

For $\tilde{g}(\tilde{A}_X U, X')$ we have:

$$\begin{aligned}\tilde{g}(\tilde{A}_X U, X') &= -\tilde{g}(\tilde{A}_X X', U) = -\tilde{g}(A_X X', U) \\ &= -g(A_X X', U) = +g(A_X U, X') = \frac{1}{h^2(r)}\tilde{g}(A_X U, X').\end{aligned}$$

Since $\tilde{A}_X U$ is tangent to E, we deduce that

$$\tilde{A}_X U = \frac{1}{h^2} A_X U.$$

It remains to compute $\tilde{A}_W U$. Considering the expressions $\tilde{g}(\tilde{A}_W U, d/dr)$, $\tilde{g}(\tilde{A}_W U, W)$ and $\tilde{g}(\tilde{A}_W U, X)$ in similar fashion to the above shows that $\tilde{A}_W U$ vanishes identically. □

Our curvature computations (Section 4) will be based on the O'Neill formulas for Riemannian submersions [1, §9D]. These formulas involve A-tensor terms, including a certain A-tensor derivative. We now turn our attention to computing this for $\tilde{A}$.

Consider a local orthonormal family of vector fields for $(B, \check{g})$ which form a basis for the tangent spaces. Lift these to horizontal vector fields in E, and extend and scale these fields in the obvious way to horizontal orthonormal fields in (some local region of) $E \times \mathbb{R}^p$. Call these fields $X_1, \ldots, X_m$. Pick orthonormal fields $U_1, \ldots, U_n$ spanning fibre directions in E, and again extend in the obvious way to fields defined locally on $E \times \mathbb{R}^p$. Similarly, choosing local orthonormal vector fields spanning the tangent spaces of $(S^{p-1}; ds^2_{p-1})$ gives rise to local orthonormal vector fields $W_1, \ldots, W_{p-1}$ tangent to S^{p-1} in $E \times \mathbb{R}^p$.

We consider the following derivative of $\tilde{A}$ (see [1, 9.33(e)]):

$$\check{\delta}\tilde{A} := -\sum_{i=1}^{m} (\tilde{\nabla}_{X_i}\tilde{A})_{X_i} - \sum_{i=1}^{p-1} (\tilde{\nabla}_{W_k}\tilde{A})_{W_k} - (\tilde{\nabla}_{d/dr}\tilde{A})_{d/dr}.$$

From a Ricci curvature point of view $\check{\delta}\tilde{A}$ arises in the O'Neill formulas in the form $\tilde{g}((\check{\delta}\tilde{A})\bullet, U)$, where $\bullet$ represents a $\tilde{\mathcal{D}}$-horizontal vector field. We will evaluate this expression in the various possible cases.

Lemma 3.4. *Denoting the Levi-Civita connection of g by ∇, we have*:

$$\tilde{\nabla}_{X_i} X_j = \nabla_{X_i} X_j;$$

$$\tilde{\nabla}_{X_i} \frac{d}{dr} = \frac{h'}{h} X_i;$$

$$\tilde{\nabla}_{X_i} W_k = 0;$$
$$\tilde{\mathcal{V}}\tilde{\nabla}_{X_i} U = \mathcal{V}\nabla_{X_i} U.$$

Proof. These formulas follow from the Koszul formula [6, p. 22] by a straightforward computation. □

Lemma 3.5.

$$\tilde{g}\big((\tilde{\nabla}_{W_k}\tilde{A})_{W_k}\bullet, U\big) = 0;$$
$$\tilde{g}\big((\tilde{\nabla}_{d/dr}\tilde{A})_{d/dr}\bullet, U\big) = 0;$$

where the dot represents either a horizontal or vertical vector field.

Proof. In the first case we have

$$\tilde{g}\big((\tilde{\nabla}_{W_k}\tilde{A})_{W_k}\bullet, U\big) = \tilde{g}\big(\tilde{\nabla}_{W_k}(\tilde{A}_{W_k}\bullet), U\big) - \tilde{g}(\tilde{A}_{\tilde{\nabla}_{W_k} W_k}\bullet, U) - \tilde{g}\big(\tilde{A}_{W_k}(\tilde{\nabla}_{W_k}\bullet), U\big).$$

As discussed above, $A_{W_k}\bullet$ vanishes identically, so we have

$$\tilde{g}\big((\tilde{\nabla}_{W_k}\tilde{A})_{W_k}\bullet, U\big) = -\tilde{g}(\tilde{A}_{\tilde{\nabla}_{W_k} W_k}\bullet, U).$$

By the Koszul formula

$$\tilde{g}(\tilde{\nabla}_{W_k} W_k, \bullet) = W_k\tilde{g}(W_k, \bullet) + \tilde{g}\big([\bullet, W_k], W_k\big).$$

From this it is clear that $\tilde{g}(\tilde{\nabla}_{W_k} W_k, \bullet)$ can only be non-zero if $\bullet$ has a component tangent to S^{p-1}. Therefore $\tilde{\nabla}_{W_k} W_k$ is a vector tangent to S^{p-1}. It follows that

$$\tilde{g}\big((\tilde{\nabla}_{W_k}\tilde{A})_{W_k}\bullet, U\big) = 0.$$

Similar arguments show that

$$\tilde{g}\big((\tilde{\nabla}_{d/dr}\tilde{A})_{d/dr}\bullet, U\big) = 0. \qquad \square$$

Combining the results of Lemma 3.5 with the definition of $\check{\delta}\tilde{A}$ we get:

Corollary 3.6. *If $\bullet$ represents a $\tilde{\mathcal{D}}$-horizontal vector field, then*

$$\tilde{g}\big((\check{\delta}\tilde{A})\bullet, U\big) = -\sum_{i=1}^{m}\tilde{g}\big((\tilde{\nabla}_{X_i}\tilde{A})_{X_i}\bullet, U\big).$$

Consider the expression

$$\tilde{g}\big((\tilde{\nabla}_{X_i}\tilde{A})_{X_i}\bullet, U\big) = \tilde{g}\big(\tilde{\nabla}_{X_i}(\tilde{A}_{X_i}\bullet), U\big) - \tilde{g}(\tilde{A}_{\tilde{\nabla}_{X_i} X_i}\bullet, U) - \tilde{g}\big(\tilde{A}_{X_i}(\tilde{\nabla}_{X_i}\bullet), U\big).$$

By Lemmas 3.2 and 3.4 we see that this expression vanishes if the dot represents d/dr or a W_k. It remains to study the case $\tilde{g}((\tilde{\nabla}_{X_i}\tilde{A})_{X_i} X, U)$.

Lemma 3.7. $\tilde{g}((\tilde{\nabla}_{X_i}\tilde{A})_{X_i} X, U) = g((\nabla_{X_i} A)_{X_i} X, U)$.

Proof. We have

$$\tilde{g}\big((\tilde{\nabla}_{X_i}\tilde{A})_{X_i} X, U\big) = \tilde{g}\big(\tilde{\nabla}_{X_i}(\tilde{A}_{X_i} X), U\big) - \tilde{g}(\tilde{A}_{\tilde{\nabla}_{X_i} X_i} X, U) - \tilde{g}\big(\tilde{A}_{X_i}(\tilde{\nabla}_{X_i} X), U\big).$$

Consider $\tilde{g}(\tilde{\nabla}_{X_i}(\tilde{A}_{X_i} X), U)$. Since $\tilde{A}_{X_i} X = A_{X_i} X$ (Lemma 3.1) and $\tilde{\mathcal{V}}\tilde{\nabla}_{X_i} U = \mathcal{V}\nabla_{X_i} U$ from Lemma 3.4, we deduce that

$$\tilde{g}\big(\tilde{\nabla}_{X_i}(\tilde{A}_{X_i} X), U\big) = g\big(\nabla_{X_i}(A_{X_i} X), U\big).$$

Next consider $\tilde{g}(\tilde{A}_{\tilde{\nabla}_{X_i} X_i} X, U)$. It follows from Lemmas 3.1, 3.4 and the fact that $\tilde{A}_U \bullet$ vanishes for any vertical vector field U that

$$\tilde{g}(\tilde{A}_{\tilde{\nabla}_{X_i} X_i} X, U) = g(A_{\nabla_{X_i} X_i} X, U).$$

Finally, consider $\tilde{g}(\tilde{A}_{X_i}(\tilde{\nabla}_{X_i} X), U)$. Again, we deduce easily that

$$\tilde{g}\big(\tilde{A}_{X_i}(\tilde{\nabla}_{X_i} X), U\big) = g\big(A_{X_i}(\nabla_{X_i} X), U\big)$$

and the lemma is proved. □

Putting these results together we can establish

Proposition 3.8. *If X is a vector field tangent to B, W a vector field tangent to S^{p-1} and U a vertical vector field we have*

$$\tilde{g}\big((\check{\delta}\tilde{A})X, U\big) = \frac{1}{h^2} g\big((\check{\delta}A)X, U\big);$$
$$\tilde{g}\big((\check{\delta}\tilde{A})W, U\big) = 0;$$
$$\tilde{g}\left((\check{\delta}\tilde{A})\frac{d}{dr}, U\right) = 0.$$

Proof. The second and third equations follow immediately from the discussion above. For the remaining case, note that $g((\check{\delta}A)X, U)$ is defined in terms of g-orthonormal vectors for the horizontal space of the bundle E. The vectors hX_i form such a family. Therefore

$$\sum_{i=1}^{m} g\big((\nabla_{X_i} A)_{X_i} X, U\big) = \frac{1}{h^2}\sum_{i=1}^{m} g\big((\nabla_{hX_i} A)_{hX_i} X, U\big) = \frac{1}{h^2} g\big((\check{\delta}A)X, U\big). \qquad \square$$

4. Curvature formulas

The Ricci curvatures of the submersion metric $\tilde{g}$ are given by the O'Neill formulas [1, §9D]. These express the Ricci curvatures of $\tilde{g}$ in terms of the Ricci curvatures of the fibre $(F, \hat{g})$, the base manifold $(B, \check{g})$ of E, the sphere (S^{p-1}, ds^2_{p-1}), the warping functions $f(r)$, $h(r)$, and tensors A and T. In the case of $\tilde{g}$, the fibres are all totally geodesic (guaranteed by Vilms [7]), which means that $T \equiv 0$.

We consider the effect of rescaling the metric $\tilde{g}$ in fibre directions. Specifically, we introduce a function $\theta(r)$ and replace the metric $\hat{g}$ in fibre directions by the metric $\theta^2(r)\hat{g}$. The curvature effect of such a scaling was computed in [9] for the case of a general Riemannian submersion with totally geodesic fibres. A term by term analysis using the Koszul formula yielded the following expressions.

Lemma 4.1. *Consider a Riemannian submersion with totally geodesic fibres $F \hookrightarrow M \to B$. Denote the submersion metric by $\langle\,,\,\rangle$. Let X and Y denote horizontal vector fields, and U and V denote vertical vector fields. Rescale the submersion metric in fibre directions by θ^2, where $\theta : B \to \mathbb{R}^+$. Then the Ricci curvature,* Ric*, of this new metric is given by*

$$\mathrm{Ric}(U, V) = \mathrm{Ric}_{\hat{g}}(U, V) - (\dim F - 1)\langle U, V\rangle\|\nabla\theta\|^2 + \theta^4(AU, AV) + \langle U, V\rangle\theta\Delta\theta;$$
$$\mathrm{Ric}(U, X) = \theta\dim F\langle A_X U, \nabla\theta\rangle - \theta^2\big\langle(\check{\delta}A)X, U\big\rangle;$$
$$\mathrm{Ric}(X, Y) = \mathrm{Ric}_{\check{g}}(X, Y) - 2\theta^2(A_X, A_Y) - \frac{\dim F}{\theta}\mathrm{Hess}_\theta(X, Y).$$

Using these equations, a long but elementary calculation yields the curvature formulas below. We have used Y_i and V_j to denote the orthonormal vector fields on (E, g) spanning $\mathcal{H}$ respectively $\mathcal{V}$ which generate the vector fields X_i and U_j on $E \times \mathbb{R}^p$. We have done this so that the A-tensor terms of the right-hand side of these formulas is independent of the parameter r. In particular note that we have used Lemmas 3.1, 3.3 and Proposition 3.8 to write the terms involving the tensor $\tilde{A}$ for $\tilde{g}$ in terms of the tensor A on (E, g).

Proposition 4.2. *The Ricci curvatures of the metric obtained from scaling $\tilde{g}$ in fibre directions by the function $\theta^2(r)$ are as follows*:

$$\mathrm{Ric}(d/dr, d/dr) = -(p-1)\frac{f''}{f} - m\frac{h''}{h} - n\frac{\theta''}{\theta};$$
$$\mathrm{Ric}(X_i, X_i) = \frac{1}{h^2}\mathrm{Ric}_{\check{g}}(Y_i, Y_i) - \frac{h''}{h} - (m-1)\frac{h'^2}{h^2} - (p-1)\frac{f'h'}{fh} - n\frac{\theta' h'}{\theta h} - 2\frac{\theta^2}{h^4}(A_{Y_i}, A_{Y_i});$$
$$\mathrm{Ric}(W_k, W_k) = -\frac{f''}{f} + (p-2)\frac{1-f'^2}{f^2} - m\frac{h'f'}{hf} - n\frac{\theta' f'}{\theta f};$$
$$\mathrm{Ric}(U_j, U_j) = \frac{1}{\theta^2}\mathrm{Ric}_{\hat{g}}(V_j, V_j) - \frac{\theta''}{\theta} - (n-1)\frac{\theta'^2}{\theta^2} - m\frac{\theta' h'}{\theta h} - (p-1)\frac{\theta' f'}{\theta f} + \frac{\theta^2}{h^4}(AV_j, AV_j);$$
$$\mathrm{Ric}(X_i, U_j) = -\frac{\theta}{h^3}g\big((\check{\delta}A)Y_i, V_j\big);$$
$$\mathrm{Ric}(d/dr, X_i) = \mathrm{Ric}(d/dr, W_k) = \mathrm{Ric}(X_i, W_k) = \mathrm{Ric}(d/dr, U_j) = \mathrm{Ric}(W_k, U_j) = 0.$$

5. Proof of Theorem A

We are now in a position to prove Theorem A. We simply need to choose appropriate scaling functions $f(r)$, $h(r)$ and $\theta(r)$ which make the Ricci curvature formulas in Proposition 4.2 positive. We will need the quantities (A_{Y_i}, A_{Y_i}) and $|g((\check{\delta}A)Y_i, V_j)|$ to be bounded above on (E, g). This is automatic in the case of a compact fibre. In the case where E is a vector bundle, the discussion in [4, Example 4.1] shows that a suitable choice of (Ricci non-negative) fibre metric will give boundedness.

Proof. Let us set

$$f(r) = L(1+r)^{\frac{1}{2}};$$
$$h(r) = (1+r)^{-\frac{1}{q}};$$
$$\theta(r) = k(1+r)^{-(2+\alpha)};$$

where $q \in \mathbb{N}$, $k \in \mathbb{R}^+$ and α, $L \in (0, 1)$. Note that the resulting metric does not satisfy the necessary boundary conditions at $r = 0$ to give a smooth metric on $E \times \mathbb{R}^p$. Leaving this point aside for the moment, let us check that these functions give us Ricci positivity for $r > 0$. A simple calculation shows that

$$\mathrm{Ric}(d/dr, d/dr) = \frac{1}{(1+r)^2}\left[\frac{p-1}{4} - \frac{m}{q}\left(1 + \frac{1}{q}\right) - n(2+\alpha)(3+\alpha)\right].$$

This is strictly positive for all r provided that

$$\frac{p-1}{4} > \frac{m}{q}\left(1 + \frac{1}{q}\right) + n(2+\alpha)(3+\alpha).$$

Since we are free to choose q as large as we like and α as small as we like, we can always fulfil this requirement provided

$$p > 24n + 1.$$

For the X_i directions, suppose that $(A_{Y_i}, A_{Y_i}) < K_i < \infty$. We then have

$$\mathrm{Ric}(X_i, X_i) \geqslant \frac{1}{(1+r)^2}\left[-\frac{1}{q}\left(1 + \frac{1}{q}\right) + \frac{p-1}{2q} - \frac{m-1}{q^2} - \frac{(2+\alpha)n}{q} - \frac{2k^2 K_i}{(1+r)^{2+2\alpha-\frac{4}{q}}}\right].$$

This will be strictly positive for all r if, for q sufficiently large and α sufficiently small, we have

$$\frac{p-1}{2} > 1 + 2n - 2k^2 K_i q.$$

Choosing k sufficiently small compared to K_i and q (for example setting $k = \frac{1}{q \max_i\{\sqrt{K_i}\}}$) we just need to ensure that

$$p > 4n + 3.$$

For W_k we have

$$\mathrm{Ric}(W_k, W_k) = \frac{1}{(1+r)^2}\left[\frac{1}{4} + \frac{p-2}{4}\left(\frac{4+4r-L^2}{4L^2(1+r)^2}\right) + \frac{m}{2q} + \frac{(2+\alpha)n}{2}\right].$$

Clearly this is always strictly positive.

For U_j we have

$$\mathrm{Ric}(U_j, U_j) \geqslant \frac{1}{(1+r)^2}\left[-(n-1)(2+\alpha)^2 - (2+\alpha)(3+\alpha) - \frac{(2+\alpha)m}{q} + \frac{(p-1)(2+\alpha)}{2}\right].$$

For q sufficiently large and α sufficiently small it suffices to keep $p - 1 > 4(n-1) + 6$, or equivalently

$$p > 4n + 3.$$

Finally, we consider the influence of the (non-zero) 'mixed' curvature term $\mathrm{Ric}(X_i, U_j)$. In order for this term not to upset Ricci positivity, we need to ensure that

$$\mathrm{Ric}(X_i, X_i)\,\mathrm{Ric}(U_j, U_j) > \left[\mathrm{Ric}(X_i, U_j)\right]^2. \tag{$\star$}$$

Notice from the above lower bounds that by choosing q sufficiently large and α sufficiently small we can make the lower bounds for $\mathrm{Ric}(X_i, X_i)$ respectively $\mathrm{Ric}(U_j, U_j)$ arbitrarily close to $\lambda/(1+r)^2$ respectively $\mu/(1+r)^2$ where

$$\lambda = \frac{1}{q}\left(\frac{p-1}{2} - 2n - 1\right),$$
$$\mu = p - 1 - 4(n-1) - 6.$$

As noted at the start of Section 5, on (E, g) we can assume that $|g((\check{\delta} A)Y_i, V_j)| < \Lambda_{ij}$ for some constants Λ_{ij}. Thus for a suitably large choice of q and suitably small choice of α the inequality ($\star$) will be true provided

$$\frac{\lambda\mu}{(1+r)^4} > \Lambda_{ij}^2 k^2 (1+r)^{-4-2\alpha+\frac{6}{q}},$$

or more simply

$$\lambda\mu > \Lambda_{ij}^2 k^2 (1+r)^{-2\alpha+\frac{6}{q}}.$$

By choosing k sufficiently small we can guarantee this, provided $-2\alpha + 6/q < 0$. For any choice of α we can arrange for this simply by choosing q larger, if necessary.

So far we have shown that for $r > 0$ the Ricci curvatures of the metric created by setting $f(r)$, $h(r)$ and $\theta(r)$ as above are all strictly positive. We now turn our attention to the problem of adjusting the metric near $r = 0$ to create a well-defined smooth metric on $E \times \mathbb{R}^p$, without introducing any negative Ricci curvatures.

Consider a constant $r_0 \in (0, 1)$. We will adjust all our scaling functions over the interval $r \in [0, r_0]$.

First of all we alter θ and h near $r = 0$. We require both of these functions to be smooth, even, and positive at $r = 0$. We will consider θ: the arguments for h are analogous.

Suppose we change $\theta(r)$ for $r \in [0, r_0]$ by instead setting $\theta(r) = d - cr^2$ at these values of r, where c and d are positive constants. This ensures that θ is even and positive at $r = 0$, and by choosing c and d appropriately we can ensure the resulting function is C^1 at $r = r_0$. Specifically, suppose that

$$\lim_{r \to r_0^+} \theta'(r) = -s \quad (\approx -(2+\alpha)k \text{ since } r_0 \text{ is small}).$$

With the above form for θ on $[0, r_0]$ we have

$$\lim_{r \to r_0^-} \theta'(r) = -2cr_0.$$

Therefore for first derivatives to agree at $r = r_0$ we need

$$c = \frac{s}{2r_0}.$$

We can then select a value for d to achieve a C^1 join at $r = r_0$, namely

$$d = sr_0/2 + k(1 + r_0)^{-(2+\alpha)}.$$

It is easy to see that over $[0, r_0]$ the quantity $|\theta'/\theta|$ can be bounded above independently of r_0. Moreover, $-\theta''/\theta \approx 2c/d$ for r_0 sufficiently small, and since d can be bounded above and below by positive constants independent of r_0 we see that $-\theta''/\theta$ can be made arbitrarily large, simply by choosing r_0 sufficiently small. Thus we can ensure that the $-\theta''/\theta$ term dominates in the expression for $\mathrm{Ric}(U_j, U_j)$ for $r \in [0, r_0]$.

By a similar argument we can ensure that the term $-h''/h$ dominates the expression for $\mathrm{Ric}(X_i, X_i)$. In addition the expressions for $\mathrm{Ric}(d/dr, d/dr)$ and $\mathrm{Ric}(W_k, W_k)$ are clearly still positive for $r \in [0, r_0]$.

We must also consider the mixed curvature term. By choosing r_0 sufficiently small, it is clear that we can keep the variation in $\mathrm{Ric}(X_i, U_j)$ arbitrarily small over $[0, r_0]$. On the other hand, as noted above we can choose the lower bounds on $\mathrm{Ric}(X_i, X_i)$ and $\mathrm{Ric}(U_j, U_j)$ to be as large as we like by choosing r_0 sufficiently small. Thus the mixed curvature term cannot upset positive Ricci curvature for $r \in [0, r_0]$, at least if r_0 is sufficiently small.

Let us fix once and for all a suitable value for r_0, and turn our attention to the function $f(r)$. For $r \in [0, r_0]$ set $f(r) = \sin r$. This will then satisfy the required boundary conditions at $r = 0$, namely f is odd with $f'(0) = 1$. In order that this curve intersects the original function $f(r) = L(1 + r)^{1/2}$ at $r = r_0$ we need to select a sufficiently small value for $L \in (0, 1)$. We fix such a value of L. Note that 'sufficiently small' here depends on the chosen value for r_0. Clearly, for $r \in [0, r_0]$ the derivative of our modified f is greater than or equal to the derivative of the original function, and the values taken by the modified f are less than or equal to those of the original function. Thus our modification results in the quantity f'/f being at least as large after the modification as it was before. As a result it is clear that this modification cannot affect the positivity of the Ricci curvature.

The resulting function $f(r)$ is, however, only C^0 at $r = r_0$. However we can smooth the function over an arbitrarily small interval $[r_0 - \epsilon, r_0]$ in a way that does not diminish f', does not increase f and is everywhere concave down. Hence Ricci positivity can be retained over this deformation.

We now have a well-defined metric on $E \times \mathbb{R}^p$, which is smooth except at $r = r_0$ because of the lack of smoothness of $\theta(r)$ and $h(r)$. We observe that the dependence of the Ricci curvature on θ'' and h'' is linear. Since we can smooth both functions keeping θ'' and h'' between their values on either side of the deformation interval, and since we can keep this interval arbitrarily small, we see that the smoothing can be performed keeping the Ricci curvature positive.

This concludes the proof of Theorem A. □

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