An analysis of the opportunities for creative reasoning in undergraduate calculus courses

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We report here on a study of the opportunities for creative reasoning afforded to first year undergraduate students. This work uses the framework developed by Lithner (2008) which distinguishes between imitative reasoning (which is related to rote learning and mimicry of algorithms) and creative reasoning (which involves plausible mathematically-founded arguments). The analysis involves the examination of notes, assignments and examinations used in first year calculus courses in DCU and NUI Maynooth with the view to classifying the types of reasoning expected of students. As well as describing our use of Lithner's framework, we discuss its suitability as a tool for classifying reasoning opportunities in undergraduate mathematics courses.

INTRODUCTION

In this project, we aim to study the opportunities for creative reasoning afforded to first year undergraduate students using the framework developed by Lithner (2008) to characterise different types of reasoning. He defines reasoning as 'the line of thought adopted to produce assertions and reach conclusions in task-solving' (Lithner 2008, p 257). His definition includes both high and low quality arguments and is not restricted to formal proofs. For this reason, the framework is useful in studying the thinking processes required to solve problems in calculus courses, where often proofs are not given or required but students are expected to make plausible arguments and conclusions. Lithner distinguishes between *imitative reasoning* (which is related to rote learning and mimicry of algorithms) and *creative reasoning* (which involves plausible mathematically-founded arguments). In this project, we use this framework to classify the reasoning opportunities available in a range of first year calculus modules offered in DCU and NUI Maynooth. We are considering both courses for specialist and non-specialist students, as well as compulsory and non-compulsory modules. (Note that by specialist students we mean students who intend to take a degree in mathematics, while courses for non-specialists are often called service courses.)

Studies have shown (for example Boesen et al. 2010) that the types of tasks assigned to students can affect their learning and that the use of tasks with lower levels of cognitive demand leads to rote-learning by students and a consequent inability to solve unfamiliar problems or to transfer mathematical knowledge to other areas competently and appropriately. It is therefore important to investigate whether first year students in our universities are given sufficient opportunities to develop their reasoning and thinking skills. This research is particularly timely given the current focus on how best to foster critical thinking skills in undergraduate students (HEA & NCCA 2011). The development of mathematical reasoning and thinking skills is also crucial for prospective mathematics teachers, whose work demands much more than rote-learning of mathematical procedures (Ball, Thames and Phelps 2008).

In this paper, we will outline the framework used in our analysis and give some examples of the classification of tasks from the courses under review.

LITERATURE REVIEW

Transition to university is widely acknowledged as a difficult process and students often find that the transition in mathematics is especially problematic (Clarke and Lovric 2009). Students' difficulties in first year seem to stem from the new thinking skills and levels of understanding expected of them (Gueudet 2008). Students grapple with notions such as function, limit, the role of definitions, and rigorous proof. These topics are encountered by millions of students worldwide including engineers, scientists, future teachers, as well as mathematics specialists. It is often said that the study of mathematics promotes the development of thinking skills, indeed Dudley (2010) states that the purpose of mathematics education is to teach reasoning. However, there is a sense of unease amongst some commentators that students 'can pass courses via mimicry and symbol manipulation' (Fukawa-Connelly 2005, p. 33) and that most students learn a large number of standardised procedures in their mathematics courses but not the 'working methodology of the mathematician' (Dreyfus 1991, p. 28) and thus may not develop conceptual understanding or problem-solving skills. Some studies have been carried out, notably in the UK and in Sweden, to investigate if there is evidence for these comments. Pointon and Sangwin (2003) developed a question taxonomy to classify a total of 486 course-work and examination questions used on two first year undergraduate mathematics courses. They concluded that:

> (i) the vast majority of current work may be successfully completed by routine procedures or minor adaption of results learned verbatim and (ii) the vast majority of questions asked may be successfully completed without the use of higher skills (p.8).

In Sweden, Bergqvist (2007) used Lithner's framework to analyse 16 examinations from introductory calculus courses in four universities. She found that 70% of the examination questions could be solved using imitative reasoning alone and that 15 of the 16 examinations could be passed without using creative reasoning.

Recent studies in Ireland (Lyons, Lynch, Close, Sheerin, and Boland 2003, Hourigan and O'Donoghue 2007) have found that procedural skills are emphasized in second level classrooms and that technical fluency is prized over mathematical understanding. This can lead to problems when students progress to third level (Hourigan and O'Donoghue 2007). In this study, we aim to investigate whether assessment in first year undergraduate courses in Ireland resembles that of Sweden and the UK and if the emphasis on procedures and algorithms at second level persist in university modules.

CONCEPTUAL FRAMEWORK

In this project a *task* will be any piece of student work including homework assignments, tests, presentations, group work etc. Lithner (2008) distinguishes between imitative and creative reasoning. Imitative reasoning (IR) has two main types: memorised (MR) and algorithmic (AR). In order to be classified as MR a reasoning sequence should have the following features:

> 1. The strategy choice is founded on recalling a complete answer.
>
> 2. The strategy implementation consists only of writing it down. (Lithner 2008, p. 258)

This type of reasoning is seen most often at the undergraduate level when students are asked to recall a definition or to state and prove a specific theorem. Algorithmic reasoning is characterised by

> 1. The strategy choice is to recall a solution algorithm. […]
>
> 2. The remaining reasoning parts of the strategy implementation are trivial for the reasoner, only a careless mistake can prevent an answer from being reached. (Lithner 2008, p. 259)

Lithner calls a reasoning sequence creative if it has the following three properties:

> 1. Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created.
>
> 2. Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible.
>
> 3. Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning. (Lithner 2008, p. 266).

The creative reasoning (CR) classification can be further divided into two subcategories: Local creative reasoning; and Global creative reasoning. A task is said to require local creative reasoning (LCR) if it is solvable using an algorithm but the student needs to modify the algorithm locally. A task is classified in the global creative reasoning (GCR) category if it does not have a solution that is based on an algorithm and requires creative reasoning throughout (Bergqvist 2007). We note that some minor adjustments to the framework were found to be necessary. These are discussed below.

METHODOLOGY

In this study we classify tasks from four first year calculus courses; two at DCU and two at NUI Maynooth. The courses include a business mathematics module, two modules for science students, as well as a module for pure mathematics students. These four modules span the range of first year calculus courses offered to students in Ireland.

The data in this project consist of the following types: lecture notes, textbooks, assignments, examination questions. We collected all the relevant information with the cooperation of the module lecturers. The data analysis of each module is currently being carried out by two independent researchers from the research team who do not work in the home university of the module. This inter-rating approach will ensure reliability of the analysis of the course material from the different modules (see e.g. Chapter 5 of Cohen, Manion and Morrison (2000)).

We began the analysis by classifying exercises from a calculus textbook, in order to gain some experience and to discuss and agree on our classification methods. All four of the authors classified these sample tasks independently and then met to finalize our procedures. These procedures are in line with those presented by Lithner (2008) and Bergqvist (2007). The researchers first construct a solution to the task and this is then compared to the course notes and textbook examples. Using Lithner’s framework, the researchers decide whether the task could be solved using imitative reasoning or whether creative reasoning is needed. We found that the most difficult decisions concerned the classification of tasks into the LCR or GCR categories, and so we adapted the framework in the following way: In order to be consistent we decided that we would classify a task as LCR if the solution was based on an algorithm but students had to modify one sub-procedure. We decided to classify a task as GCR if two or more sub-procedures were new, if a proof aspect was the novel element, or if mathematical modeling was the novel element.

EXAMPLES

In this section we will present some examples of tasks classified using the Lithner reasoning framework. We will concentrate on one topic in order to be coherent and to be better able to compare categories. We will consider the topic of quadratic equations, which is important in many calculus and pre-calculus courses.

In the course in question, the lecture notes and the textbook (Jacques 2009) discuss solutions of quadratic equations using the quadratic formula as well as factoring, and give examples which illustrate both methods. The questions below are taken from the exercises in Section 2.1 of the text and were assigned as tutorial problems by the lecturer.

Task 1*: Solve the following quadratic equations, rounding your answers to 2 decimal places, if necessary:*

(a) $x^2 - 15x + 56 = 0$*; (b)* $2x^2 - 5x + 1 = 0$*; (c)* $4x^2 - 36 = 0$*;*
(d) $x^2 - 14x + 49 = 0$*; (e)* $3x^2 + 4x + 7 = 0$*; (f)* $x^2 - 13x + 200 = 16x + 10$.

Task Analysis:

Solution method: Students could use the quadratic formula or factorization here. The solutions are:

a) $x^2 - 15x + 56 = (x-7)(x-8)$, so the solutions are $x = 7{,}8$;
b) using the quadratic formula we have $x = \frac{5 \pm \sqrt{17}}{4}$, so to 2 decimal places $x = 2.28, 0.22$;
c) $4x^2 - 36 = 4(x-3)(x+3)$, so the solutions are $x = -3, 3$;
d) $x^2 - 14x + 49 = (x-7)^2$, so there is just one solution at $x = 7$;
e) using the quadratic formula we have $x = \frac{-4 \pm \sqrt{-68}}{6}$, so there are no real solutions;
f) subtracting $16x + 10$ from both sides gives $x^2 - 29x + 190 = 0$ and since $x^2 - 29x + 190 = (x-10)(x-19)$, the solutions are $x = 10, 19$.

Text Analysis:

- *Occurrences in the notes*: The quadratic formula is given on page 14 of section 2.1 and it is used in examples on pages 16, 17 and 18 of that section. The factor method and an example can be found on page 19. Examples of rearrangements similar to (f) occur on pages 18 and 29.
- *Occurrences in the text*: The quadratic formula can be found on page 132 of the textbook and it is used in examples on pages 132, 133 and 134. The factor method is explained on pages 134 and 135 of the book and used in examples on page 135. An example on page 141 includes a rearrangement similar to part (f).

Argument and conclusion:

This is an Imitative Reasoning (IR) task, specifically it is an Algorithmic Reasoning (AR) task. The students just need to use the algorithms from the notes and the textbook.

Task 2*: Write down the solutions to the following equation:*

$$(x-2)(x+1)(4-x) = 0.$$

Task Analysis: Solution Method: Since $(x-2)(x+1)(4-x) = 0$, we conclude that $x = 2, -1{,}4$.

Text Analysis:

- *Occurrences in the notes*: The factor method and an example can be found on page 19, but there is no example with three factors.
- *Occurrences in the text*: The factor method is given on pages 134 and 135 of the book and used in examples on page 135; however the examples do not cover the case of three factors.

Argument and conclusion:

This is a Creative Reasoning (CR) task, specifically it is a Local Creative Reasoning (LCR) task. The students can use the factor method algorithm from the notes and the textbook however they need to modify it to handle the three factors.

Task 3*: One solution of the quadratic equation*

$$x^2 - 8x + c = 0$$

is known to be $x = 2$. *Find the second solution.*

Task Analysis:

Solution Method: Since $x = 2$ is a solution, we can see that $2^2 - 8(2) + c = 0$, i.e. $c = 12$. Using this, we can solve $x^2 - 8x + 12 = 0$ using either the factor method or the quadratic formula to get that the second solution is $x = 6$.

Text Analysis:

- *Occurrences in the notes*: The factor method and the use of the quadratic formula can be found in the notes; however there is no example of this type there.
- *Occurrences in the text*: There are examples using the factor method and the quadratic formula in the text but there is nothing similar to this question.

Argument and conclusion:

This is a Creative Reasoning (CR) task, specifically it is a Global Creative Reasoning (GCR) task. The notes and textbook do not contain an algorithm that the students can follow; they need to create a new mathematically plausible strategy to find the value of *c*.

DISCUSSION

We note first that the analysis of all tasks for the different courses has not yet been completed. Thus we cannot yet discuss the proportions of tasks in each category or compare modules; this will be reported on at a later date.

Of the tasks classified to date, we have not found any that lie in the MR (Memorised Reasoning) category. It will be of interest to see if this category appears in exams.

As noted above, the classification is not always straightforward, especially when deciding between LCR and GCR. Similar difficulties arise in distinguishing between AR and LCR. For example, it can be difficult to decide whether a reasoning element should be regarded as novel or not: this can be subjective. In order to counteract this, the inter-rating approach was used, with clear guidelines agreed on categorization and the use of discussions to resolve borderline cases. It was also found necessary to amend Lithner's framework slightly in order to fit our purposes.

A further difficulty is that we do not know what other learning experiences the student has had – for example in secondary school, in tutorials, in Mathematics Learning Support Centres, etc. We can only classify tasks using the information we have from the notes and textbook. This is a possible weakness in the study. However, it should be noted that this difficulty mirrors the situation in which the lecturer finds him or herself: they must make decisions on teaching and assessment in the absence of detailed knowledge of their students' prior learning experiences.

Classifications like this can help us as lecturers to make sure we balance our assignments and examinations to ensure that students are presented with an appropriate variety of reasoning tasks, and to avoid an over-emphasis on rote-learning tasks. The results of the full analysis will provide us with a detailed picture of the reasoning opportunities available to first year calculus students in our courses. By highlighting this process, we hope to provide a useful tool for other mathematics lecturers involved in curriculum design.

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