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# Students' views of example generation tasks

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**We report here on students' views of example generation tasks assigned to them in two first year undergraduate Calculus courses. The design and use of such tasks was undertaken as part of a project which aimed to afford students opportunities to develop their thinking skills and their conceptual understanding. In interviews with 10 students, we found that on the whole they viewed the example generation tasks as unfamiliar and sometimes difficult, but also as beneficial for promoting conceptual understanding and independent thinking. In addition, some students characterized these tasks as 'the backwards ones'.**

## 1. Introduction

Examples are used by mathematicians in a number of ways: to help them understand a statement or definition, to help them generate an argument, to help decide whether a statement is in fact false (Alcock, 2004) and to increase their confidence in a particular result (Weber & Mejia-Ramos, 2011). Feynman famously said

> I can't understand anything in general unless I'm carrying along in my mind a specific example and watching it go. (Feynmann *et al.*, 1997, p. 244)

This article is concerned with tasks that require students to construct their own examples of mathematical objects; in that regard, we are following the definition given by Edwards (2011) that an example is 'a mathematical object that satisfies certain criteria' (p. 17). We will not consider the use of examples in other contexts; in particular we will not deal with worked examples or model solutions at all. The example generation tasks that will be discussed here were designed as elements of a set of unfamiliar tasks assigned to students in an effort to provide them with greater opportunities to develop the types of mathematical thinking skills used by experts. This was motivated by concerns reported in the research literature regarding predominant approaches to the teaching and learning of mathematics at second-level in Ireland. (This is discussed in some detail in the following section 'The Task Design Project'.)

Mathematicians construct their own examples of concepts and counter-examples to conjectures (Mason & Watson, 2008). In order to allow learners to experience the practices of mathematicians, it would seem sensible to encourage students not only to use the examples provided for them but to generate their own examples to help them understand definitions and results, and to obtain insight into the underlying structure of the mathematical concepts involved. Mason & Watson (2008) explain that when learners are asked to construct several examples they 'look for the scope and breadth of generality possible' (p. 197), thereby seeking to make sense of the underlying relationships, properties and structure of the concept or theorem under discussion. Indeed, Watson & Shipman (2008) believe that the idea of using learner-generated examples to motivate conceptual understanding is implicit in the writings of mathematicians who have urged others to create their own examples when interpreting mathematical text (e.g. Halmos (1983)). Selden & Selden (1998) explain how coming up with examples requires students to develop and use different cognitive skills than those used in carrying out an algorithm: students must look at a mathematical object in terms of its properties and usually have no pre-learned algorithm to show them the 'correct way' to do this.

Many of the research studies examining experts' and students' responses to example generation tasks focus on the strategies they use to generate examples and the processes participants engage in to transition between different strategies (e.g. Hazzan & Zazkis (1999), Antonini (2006), Edwards (2011), Iannone et al. (2011), Saglam & Dost (2015)). However, there is little literature documenting the inclusion of example generation tasks in undergraduate mathematics courses in practice. Pointon & Sangwin (2003) examined 486 questions included in coursework and examinations for two core (compulsory) mathematics modules for first year undergraduate students at a university in the UK. Their classification scheme had eight classes of questions: namely, factual recall; carry out a routine calculation or algorithm; classify some mathematical object; interpret situation or answer; prove, show, justify (general argument); extend a concept; construct example/instance; criticize a fallacy. Only 2.4% of the 486 questions inspected required students to construct an example. However, Sangwin (2003) reports on a focus group held with six undergraduate students who had been invited to test some example generation (or 'create an instance') tasks and discuss their experiences. His students demonstrated a mature understanding of the purposes of the tasks, appreciating the higher level skills required for completion, with one student commenting that the tasks were to 'test your understanding of the subject, rather than your ability to turn a handle' (p. 825). Meehan (2002) reports on example generation exercises, which were incorporated into an introductory analysis course at an Irish university, and discusses the contribution examples make in enabling students to develop a better-informed concept image.

In Pennyslvania, Dahlberg & Housman (1997) interviewed 11 third and fourth year undergraduate mathematics students in an attempt to study their evoked concept images having been presented with a formal definition of a new concept. They inferred that the students in their study who employed an example generation learning strategy were more effective in attaining an initial understanding of the new concept introduced to them than those who primarily employed other learning strategies. The students who spontaneously generated their own examples learned a significant amount despite having only been presented with the concept definition, and were better able to identify the correctness of conjectures involving the concept and provide explanations than the other students. Dahlberg & Housman (1997) also found that some of their students were reluctant to engage in example generation: these students lacked confidence in their answers and repeatedly sought assurance from the interviewer.

Hazzan & Zazkis (1999) presented pre-service elementary teachers with three example generation tasks and collected data on the strategies they used to generate the examples sought. They inferred that barriers to example generation were often emotional rather than mathematical. They believe the freedom given in an example generation task and the fact that the solution is not unique can be troubling for students and result in a desire to 'quit and avoid making choices when there is no one definite way to proceed' (p. 11). Despite this, they maintain that the decision making opportunities provided by example generation tasks are important, and that learning occurs when students practice making judgements and come to terms with the freedom inherent in such tasks. Moreover, they refer to example generation tasks as 'inverses' of standard, more familiar tasks because the usual roles of what is given and what is to be found are reversed.

While many mathematics education researchers seem to suggest that asking learners to generate examples of mathematical concepts is an effective way of learning about novel concepts, Iannone *et al.* (2011) argue that this suggestion has limited empirical support. The results of a study they carried out with 53 undergraduate students did not support the hypothesis that generating examples of a particular type of function led to better proof production than reading worked examples when the function was first introduced. They conclude that the 'teaching strategy of example generation is not yet understood well enough to be a viable pedagogical recommendation' (p. 1). Furinghetti *et al.* (2011), while acknowledging that the role of examples is often considered to be crucial in problem solving, caution that a focus on examples may also 'make students stick to the explorative stage and inhibit the need for generalization' (p. 219), with students often considering that checking examples constitutes a means of proof. Furthermore, while Edwards (2011) believes that students with a rich and varied example space are more likely to have concept images that are aligned with formal theory, he advises that the use of certain types of example generation tasks as a pedagogical tool may not benefit students who are struggling with the *role* of mathematical definitions as well as their content.

However, from the perspective of a mathematician, mathematics is 'a domain of creativity and discovery in its articulation, proof, and application' (Mason & Watson, 2008, p. 191). Antonini (2006) describes the generation of an example satisfying certain properties as 'an open-ended task, a problem with many answers that students can solve by various approaches' (p. 57). Example generation tasks promote active engagement in mathematics and provide an opportunity for learners to be assertive and creative. Watson & Mason (2005) describe how 'making choices for yourself is energizing; being trusted to make choices is empowering' (p. ix). Bills *et al.* (2006) have also spoken about example generation exercises as transferring the initiative to the learner. In being constructive in this way, students can experience freedom and constraint, and can identify and explore the dimensions of variation possible in the choice of examples available, as well as come to an appreciation of aspects of invariance in the midst of change (Mason & Johnston-Wilder, 2004).

In this article, we will discuss self-reported views of students on the example generation tasks assigned to them. We do not aim to present a measure of the effectiveness of such tasks here.

## 2. Task design project

The tasks of interest here were assigned to first year undergraduate students taking Calculus modules taught by the first two authors. They were designed by the authors as part of a project on developing mathematical tasks to promote conceptual understanding. In the USA, the National Research Council (NRC) (2001) described 'conceptual understanding' as the 'comprehension of mathematical concepts, operations and relations' (p. 116) and go on to say that the term 'refers to an integrated and functional grasp of mathematical ideas' (p. 118). The NRC emphasized the importance of being able to organize knowledge and make connections between concepts, as well as an appreciation as to why an idea is important, and an ability to explain why a method works. We will use the NRC's description of conceptual understanding here. We should note that conceptual understanding was just one of five interdependent strands of mathematical proficiency identified by the NRC; the others were procedural fluency, strategic competence, adaptive reasoning, and productive disposition. For instance, procedural fluency was defined as the 'ability to carry out procedures flexibly, accurately, efficiently and appropriately' (NRC, 2001, p. 116).

There has been very little research on mathematics education at university level in Ireland, but research on mathematics education at the senior cycle of post-primary school has identified a predominantly procedural or instrumental approach to mathematics teaching and learning. The State Examinations Commission's report on students' performance on the state Mathematics examination at the end of post-primary school showed they had inadequate understanding of concepts (SEC, 2005). Others have found the state examinations were predictable and rewarded the learning of rules and their application in familiar contexts (Elwood & Carlisle, 2003). Because of the backwash effect of assessment on teaching and learning, 'shaping both what is taught and how it is taught' (Conway & Sloane, 2005, p. 28), it was found that Irish classrooms tended to be focussed on the use of algorithmic procedures, with very little emphasis on conceptual understanding, and that students appeared unable to apply techniques learnt in unfamiliar contexts (Lyons *et al.*, 2003). However, it is useful to keep in mind the NRC's (2001) caution that pitting skill, or procedural fluency, against conceptual understanding can create a false dichotomy as the two are interwoven: understanding makes learning skills easier and less prone to be forgotten, but a certain level of skill is required to successfully learn many mathematical concepts with understanding (NRC, 2001, p. 122). More recently in the Irish context, O'Sullivan (2014) has undertaken an analysis of mathematical tasks in three textbook series used at senior cycle of post-primary schools in Ireland. He describes a task as 'novel' when either the skills required for its completion or the concept it involves (or both) are not familiar to the solver from preceding tasks or expository sections in the textbook chapter in which it occurs. On the one hand, for a task to be classified as novel, its successful completion requires the solver to significantly adapt the method outlined in the worked examples provided or used for previous tasks. On the other hand, if a particular type of task is repeated a number of times in an exercise set, incorporating only a superficial change in the expression, numerical values or context involved, the 'novelty' level of the repetitive tasks is likely to be lower than that of the original task according to O'Sullivan's framework. Of some 1838 tasks on the topic of patterns, sequences and series, O'Sullivan found only 5% to be novel overall, with the percentage of novel tasks in a single textbook ranging from 2% to 9%. These results are supported by O'Keeffe & O'Donoghue (2012) who found that less than 18% of problems on statistics, probability and geometry in a selection of textbooks used at senior cycle of post-primary schools in Ireland are 'non-routine problems'. For O'Keeffe & O'Donoghue, 'non-routine problems' are 'problems which cannot be answered by a routine procedure or problems in which it is not immediately obvious what one must do' (2012, p. 23).

Given this context, the authors endeavoured to design a series of unfamiliar tasks for first year undergraduate students in an effort to give them opportunities to develop their thinking skills and to address the perceived imbalance in the opportunities provided for students to develop conceptual understanding as well as procedural fluency. Here an 'unfamiliar task' is taken to be one for which students have no algorithm, well-rehearsed procedure or previously demonstrated process to follow. Following Lithner's (2000) observation that students often rely heavily on past experience when solving problems, the authors hoped, by presenting the students with unfamiliar tasks, to discourage such reliance and help them to develop the flexibility in their thinking and reasoning characteristic of mathematicians. Rittle-Johnson *et al.* (2015) acknowledged the importance of novelty in conceptual tasks in order to compel students to use their knowledge of concepts to solve the problem: 'a critical feature of conceptual tasks is that they be relatively unfamiliar to participants, so that participants have to derive an answer from their conceptual knowledge, rather than implement a known procedure for solving the task' (p. 3). Selden & Selden (1998) have also suggested that the difficulties students experience on encountering a new concept may point to their excessive dependence on explicit instruction. Moreover, Arslan (2010) found that traditional instruction in a Differential Equations course (dominated by a procedural approach and the algebraic solution of equations) for pre-service teachers was not sufficient for conceptual learning, and, consequently, he highlighted the need for contemporary approaches to be adopted.

In order to identify a set of task types that would be appropriate for use in an undergraduate Calculus course in an Irish context, we drew on the work of Mason & Johnston-Wilder (2004a), Swan (2008), and the recommendations of Cuoco *et al.* (1996). Cuoco *et al.* (1996) remarked that mathematics curricula are usually given in terms of mathematical content and often fail to mention the 'mathematical habits of mind' involved in developing the mathematical results studied. They proposed that students need to conjecture, experiment, visualize, describe, invent, generalize and be able to use mathematical language precisely. Mason & Johnston-Wilder (2004a) suggested that questions posed to students should involve the practices employed by research mathematicians and hence, they proposed that the following words be used when designing tasks: 'exemplifying, specialising, completing, deleting, correcting, comparing, sorting, organising, changing, varying, reversing, altering, generalising, conjecturing, explaining, justifying, verifying, convincing, refuting' (p. 109). Swan (2008) described five task types he selected in order to promote conceptual understanding among secondary school students: classifying mathematical objects, interpreting multiple representations, evaluating mathematical statements, creating problems, and analysing reasoning and solutions. He claimed that these types of tasks collectively help students to gain an appreciation for the importance of definitions and the properties of objects, to develop new mental images, to appreciate the possibility of more than one solution to a problem, as well as enhancing students' reasoning skills and encouraging their use of examples and counter-examples.

Thus, the task types we identified as appropriate for the students enrolled in our Calculus courses were tasks requiring students to generate examples, evaluate statements, analyse reasoning, conjecture, generalize, visualize, and/or use definitions. The tasks designed were assigned to students and subsequently evaluated through the collection of data using a variety of means, one of which was conducting interviews with a small sample of students. We will focus on the example generation tasks in this article. A selection of other tasks designed along with a more detailed discussion of the rationale for the task framework used can be found in Breen & O'Shea (2011).

## 3. Sample tasks

Two example generation tasks designed in this study, together with the rationale behind their design, are shown below.

Question A: Find examples of the following:

(a) *Polynomials $P(x)$ and $Q(x)$ such that $P(4) = 0 = Q(4)$ and* $\lim_{x\to 4} \frac{P(x)}{Q(x)} = 0$.

(b) *Polynomials $P(x)$ and $Q(x)$ such that $P(4) = 0 = Q(4)$ and* $\lim_{x\to 4} \frac{P(x)}{Q(x)} = 1$.

(c) *Polynomials $P(x)$ and $Q(x)$ such that $P(4) = 0 = Q(4)$ and* $\lim_{x\to 4} \frac{P(x)}{Q(x)} = -2$.

(d) *Polynomials $P(x)$ and $Q(x)$ such that $P(4) = 0 = Q(4)$ and* $\lim_{x\to 4} \frac{P(x)}{Q(x)} = \infty$.

(e) *Polynomials $P(x)$ and $Q(x)$ such that $P(4) = 0 = Q(4)$ and* $\lim_{x\to 4} \frac{P(x)}{Q(x)}$ *does not exist.*

Students had studied limits of rational functions in class and were assigned several problems involving the calculation of such limits for given functions. Question A aimed to develop students' *understanding* of the limits of rational functions. In the situations presented in the questions (that is, where the limits of both the numerator and the denominator are zero), students often instinctively feel that the limit of the rational function must then be 0/0 and so does not exist. The task designer hoped that by attempting this task, students would gain an understanding of the different outcomes that can arise from this situation and an appreciation of the reasons for these different outcomes. In addition, it was envisaged that students' facility with constructing polynomials with certain properties would be enhanced and that students' skills for dealing with unfamiliar tasks would be developed.

Question B:

(a) *Give three examples of functions that are continuous everywhere.*
(b) *Give an example of a function, f, that is not continuous at 0 because f(0) does not exist.*
(c) *Give an example of a function, f, that is not continuous at 0 because* $lim_{x\to 0}f(x)$ *does not exist.*
(d) *Give an example of a function, f, which is not continuous at 0 because* $lim_{x\to 0}f(x)$*does not exist, although f(0) exists.*
(e) *Give an example of a function, f, which is not continuous at 0 because* $lim_{x\to 0}f(x)\neq f(0)$*, although both exist.*

The definition of continuity and examples of functions which were continuous or not continuous at a particular point had been discussed in class. Question B was designed to reinforce the idea that different types of discontinuity exist and it was hoped that the students' subsequent reflection on the range of examples they had constructed would contribute to their understanding in this regard. Being asked for a sequence of examples provided students with an opportunity to explore which features they could change and in what way, that is, to explore the dimensions of variation possible. Efforts were made to incorporate some of Watson & Mason's (2005) strategies for learner-generated examples to some extent: (a) uses 'make up another or more like this'; while constraints are added sequentially in (c), (d) (p. 153).

Each problem set assigned to students (and the final examination) contained unfamiliar non-procedural tasks as well as some more procedural tasks. For example, the following procedural task (taken from Larson *et al.* (2008)) appeared on the same problem set as Question A above:

Question C: *Find the limit (if it exists):* $lim_{x\to 3}\frac{x^2-x-6}{x^2-5x+6}$.

## 4. Methodology

The first two authors are mathematics lecturers in different third level institutions in Ireland and were teaching Differential Calculus modules in their respective institutions in the academic year 2011–2012. The tasks designed in this project were presented to students as homework assignments in these modules. All the students registered for these modules had chosen to study Mathematics: 130 students at National University of Ireland, Maynooth (NUIM) undertaking a BA or Finance degree and 35 students at St Patrick's College, Drumcondra (SPD) studying for a BEd (Primary) or BA degree. A number of tutorial sessions were observed by a research assistant as part of the overall project; however, we will not report on that data here. The NUIM module took place in the first semester of 2011–2012. At the end of the NUIM module, all 130 registered students were emailed and invited to be interviewed by the researcher observer (the third author) on their experiences in relation to the tasks assigned; eight students responded immediately and because of timetable constraints five students were interviewed. In December 2011, three of these were interviewed individually (students labelled F, I, J), and the remaining two were interviewed together (G and H—we will refer to this as the joint interview). The SPD module took place over both semesters of the academic year. The interviews for this cohort of students were held in May 2012. We decided to seek volunteers from the observed tutorial groups with a view to collating data from the observations and the interviews. The fourteen SPD students who had previously attended the observed tutorials were invited to volunteer to participate in interviews. Six students volunteered but because of timetable constraints only five were interviewed individually (students with pseudonyms A–E). The interviews were semi-structured and of 16 minutes duration on average, ranging from 13 to 22 minutes. They were audio-recorded and were fully transcribed by the research assistant. The interviewees' identities were not revealed to the first and second authors (module lecturers) unless the interviewees chose to do so themselves.

The interviewer had a selection of problem sheets available for the students to consult during the interview. For the NUIM students, three of seven problem sheets assigned during the course were available, while for SPD students there were four of ten. The NUIM problem sheets provided included three types of unfamiliar tasks: example generation, evaluating statements, using definitions. All seven types of unfamiliar tasks appeared on the four SPD problem sheets. The Supplementary Appendix contains two of the problem sheets available in the interviews, one from each institution.

The interview schedule produced in advance of the semi-structured interviews outlined the following questions: *What do you think of your mathematics course (at university)? How is it different from school mathematics? Different types of questions were used on the problem sheets in the Calculus course, were you aware of the difference? Can you give examples? What did the tasks help you learn? Which tasks aided your development of conceptual understanding?*

The transcripts were read and coded independently by all three authors using a general inductive approach (Thomas, 2006). They met on various occasions to discuss and compare codes. An important concept to emerge from the analysis was that of student views of the example generation tasks; these tasks were mentioned more frequently than any other type of task. The first two authors independently recoded the transcripts with this concept in mind. After discussion, they combined their codes into categories or aspects of the students' views. The categories that emerged were as follows: example generation tasks as unfamiliar tasks; example generation tasks and understanding; benefits of example generation tasks; characterization of example generation tasks as 'the backward ones'.

## 5. Results

We report here on the observations of students on the example generation tasks during the interviews.

### 5.1 *Example generation tasks as unfamiliar tasks*

During each interview, the interviewer asked students if they were aware that their lecturer had been using some unfamiliar task types on assignments and if so, she followed up by asking for an example of an unfamiliar task. Student J spoke about task content rather than task type and felt that everything was unfamiliar. Student H did not get an opportunity to answer the question in the focus group. All of the other students had noticed task types that were not familiar to them from school, and all of them chose examples from the tasks designed by the authors. Six of these eight students chose example generation tasks (Students A, C, E, F, G, I). In fact when asked about unfamiliar tasks, Student G immediately said:

*G: The backwards ones?*
*Interviewer: Ya, kind of. Which ones do you mean by the backwards ones?*

*G: Well, she usually gives us the answers and asks us to find the question. [. . .] I think it's a clever idea because it means that you have to understand every detail of how to get it. [refers to Question A above as an example – note that in each part of this task the value of the limit is given to the students and they are asked to generate examples of rational functions with the specified limits].*

Student C had a similar view:

*C: [Question B] as well, that's different to last year, because we had to give examples, we were given the examples, maybe work backwards kind of, find out if the limit does exist but you were trying to – it gets you thinking more, kind of have to find one where the limit doesn't exist, and stuff like that.*

The other students mentioned that these tasks were different from the ones they had seen at school, and some said they were different even from those in their university textbook. Note that although the students were assigned more than one example generation task over the course of their module, the tasks were always on different topics and did not have predictable solutions. This seems to have ensured that students still viewed these tasks as unfamiliar.

Many of the students reported having difficulty when attempting the example generation tasks (Students A, B, C, E, F, G, H, J) and described them as hard, tricky and even overwhelming. Their difficulties seem to stem mainly from the lack of a familiar solution procedure; Students F, G and H all remarked on the difficulty of knowing where to start, or deciding on the first step. When asked how she felt when she saw Question B initially, Student B said

*B: You kind of feel disheartened sometimes when you don't know how to do things.*

Both Student B and Student E said that it was important to have a mixture of familiar and unfamiliar tasks. Student B followed the quote above by saying:

*B: Like every now and then it's nice to have the ones that are familiar and then it makes you feel a bit more confident in approaching the rest of them.*

### 5.2 Example generation tasks and conceptual understanding

The students were asked which types of tasks helped them to develop conceptual understanding. Five of the students spoke about example generation questions, while two other students chose an example generation question when asked to choose between two questions from a problem set. Student A was slow to give an example for a question which helped her develop conceptual understanding so the interviewer asked her to compare Question B and the preceding question on the same problem sheet (questions 1 and 2 in Supplementary Appendix B). She said:

*A: I think the second one, finding the examples of functions that are continuous somewhere [Question B]. Because you're kind of applying what you know. You have to use your basic theorem or definition of continuity there. [. . .] So you're kind of bringing together what you know from things whereas in [the preceding question] you kind of – you're told what to do. So you're literally just kind of following a procedure.*

Student F was asked to compare two questions on rational functions (Questions A & C from above). He said that Question C was useful in learning some techniques but did not help explain the concepts involved, while the other task 'is more focussed on like why you're using certain things instead of just like how'.

In explaining their choices of tasks which helped them to develop conceptual understanding, the students who mentioned example generation tasks spoke about the need to use definitions and also how the lack of an obvious procedure or algorithm forced them to come up with a solution strategy themselves. Student B echoed Student A's remarks above when she said that Question B required a student to 'go back on what continuity means and then try and apply it to the different ones', thus recognizing that reviewing and applying the definition was necessary in this question. Student E chose the same question and when comparing it to a more procedural question on the same topic said

> *E: [Question B] is more about like why it works. You have to come up with the examples yourself.*

In the joint interview, Students G and H both commented on the benefits of example generation tasks. Student H chose an example generation task (Question A) and said:

> *H: At first I found it difficult to understand how to go about approaching that, ahm, but after working out how to do it, it certainly cemented in my mind as to what way to approach that question. It gave me a good understanding of how to come up with these functions, you know, rather than just find the limit of a function, you know, which is fairly standard, fairly easy to do.*

Student G added that the example generation tasks 'are really good for learning but they are also very tricky and take a long time'. She observed that these types of tasks might not be suitable examination questions but when the interviewer asked her if they were useful in developing understanding she agreed and said that 'they make you think more'. Student I chose Question A and said that he learned more from the 'conceptual type rather than the how-something-works type questions'. He expanded on the reasons for this by saying:

> *I: Because you had to actually work not only backward but forwards in answering the questions and you can see clearly why it works.*

### *5.3 Students' views on the benefits of example generation tasks*

Students were not asked explicitly to speak about the benefits of example generation tasks but many of them did so anyway during the interviews. Apart from developing conceptual understanding, the major benefit referred to by the interviewees was that this type of task made them think more and think for themselves (Students A, C, E, F, G, H, I). For example, Students E and H made the following remarks

> *E: I think the 'give the example' one, because you have to think for yourself and like come up with it, it's not like filling into the formula, it kind of proves you understand it more.*

> *H: You really have to think more about these and understand the concepts and the different – em-possible solutions that may be and why one solution isn't going to work.*

Some of the students (A, B, E, I, F) appreciated the fact that since there was no familiar procedure or algorithm on hand to solve these problems, then they had to think for themselves. Student A described how with procedural questions she

> *A: just launched straight into that because I know what I'm doing, whereas [for Question B] I'd have to take more time*

She later added that even if she was not able to complete such a task or got it wrong that she still learned from the experience. Students also felt that 'in these ones [example generation questions] you have to like completely understand it to get the answer' (Student B), and we saw earlier that Student H expressed a similar sentiment. We saw previously that Students A and B mentioned another benefit when they spoke about drawing on previous knowledge when working on example generation tasks.

### *5.4 Characterisation of example generation tasks as 'the backwards ones'*

We have already seen from the quotes above that some of the interviewees spontaneously used the analogy of 'backwards' when speaking about the example generation tasks. In fact four of the ten interviewees (C, G, I, J) did this. Student J reported having difficulties with many of the tasks, and when asked how she felt when she first saw Question A she said:

> *J: I wanted to scream. [laughs] [. . .] I think it was very hard because it's kind of working backwards from what I'd used to do. Normally I'm given the function, I have to work from the function, not the other way around.*

These students seemed to draw a distinction between questions that asked them to do something like find a limit or show that a function is continuous and those that asked for examples of functions with a specified limit or with certain properties. Student G was the most vocal on this issue. We have already seen that when asked if there were any unfamiliar task types on the assignment sets, she immediately said 'the backward ones', ones she describes as questions where 'she [the lecturer] usually gives us the answers and asks us to find the question'. Shortly afterwards, she explained the difference between these questions and the ones she had seen in school. She is asked if she is used to a certain type of question in school and replies:

> *G: Yes, certainly: this is the question, what's the answer. Whereas here [at university] they may give you the domain of something and you are asked to get the function of it. [. . .] In this one [Question A] she gives you the answer to the limit, and asks you to find the function of the limit. It's the opposite way.*

Her companion in the joint interview, Student H, then said that he had not thought of the example generation tasks in that way, but that he now understood what Student G meant. It seems that this characterization of example generation tasks as 'backwards' has resonance with these students.

## 6. Discussion

A number of the interviewees independently characterized the example generation tasks assigned as 'the backwards ones' or spoke of the need to 'work backwards' to tackle them. Perhaps the association of example generating problems with a backwards motion may be seen as providing support to claims that mathematics learning at post-primary level in Ireland focusses on procedural step-by-step learning, in which the order of steps plays a significant role. The students' initial reaction here was to relate a new task with those familiar to them and it would seem that they were familiar with working only in a particular direction. Hazzan & Zazkis (1999) also describe example generation tasks as 'inverses' of standard, more familiar tasks. The act of reversing a familiar procedure, or 'undoing', can itself provide a valuable learning opportunity leading to a creative range of possibilities which students can explore (Mason & Johnston-Wilder, 2004). In fact, Watson & Mason (2005) state that 'task reversal is a technique that many teachers have used for some time' (p. 15). It may be that this reversal is a useful way of disrupting an algorithmic approach.

The tactic of inverting the usual order of things, and asking students to produce mathematical objects that are usually given in advance, provides an opportunity to introduce students to the ways of thinking and practicing of mathematicians and finds resonance with teaching strategies in the constructivist paradigm. Comments from the students in relation to having to think for themselves in order to construct their own examples or indeed complete other unfamiliar tasks, lends support to the idea of such tasks transferring the initiative to the learner and giving them scope to be both assertive and creative (Watson & Mason, 2005; Bills *et al.*, 2006).

As the construction of examples requires different cognitive skills than following or applying algorithms, it can, as Selden & Selden (1998) describe, 'be disconcerting' for students. Hazzan & Zazkis (1999) surmised that barriers to example generation can be emotional rather than mathematical, perhaps because of the freedom inherent in an example generation task. Dahlberg & Housman (1997) also found that some of their students were reluctant to engage in example generation, lacking confidence in their answers and seeking assurance from the interviewer. However, in the interviews reported on here, while students acknowledged the difficulties they experienced with the example generation tasks, a number of them agreed, when asked, that they felt a sense of satisfaction having completed the tasks for themselves (Students A, F, I).

The interviewees also commented on the benefits of more familiar or procedural tasks, particularly for building student confidence. As mentioned previously, the National Research Council (2001, p. 116) asserts that mathematical proficiency has five strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition. Furthermore, it sees these strands as interwoven and posits that all are necessary in order to learn mathematics. With this in mind, the assignments for the courses discussed here consisted of a variety of tasks of different types, levels and focus (see Supplementary Appendices). Indeed, since no single strategy or task type is likely to be universally successful in developing mathematical thinking or proficiency, Mason & Johnston-Wilder (2004) advocate a 'mixed economy' (p. 6) of tasks.

The students' reflections on their own learning through the unfamiliar tasks were generally positive: most of them recognized the purposes of the tasks and acknowledged the effectiveness of example generation problems as learning tools, in particular, for promoting conceptual understanding. This lends support to the claim that the creation of examples can provide an arena for conceptual learning (Watson & Mason, 2005, p. 13). In fact, Watson & Mason (2005) go further in stating that 'until you can construct your own examples, both generic and extreme, you do not fully appreciate a concept' (p. 32). Dahlberg & Housman's (1997) findings endorse this as their data revealed that example generation was crucial for understanding a new concept and that the initial concept image evoked for a new concept was most sophisticated in those students who employed example generation. This is not surprising given mathematicians' views of the role of examples. As Halmos (1983) explained:

> *A good stock of examples, as large as possible, is indispensable for a thorough understanding of a concept, and when I want to learn something new, I make it my first job to build one.* (p. 63, 1983)

In the study reported here, it is the students themselves who remark on the usefulness of example generation tasks in building conceptual understanding. In reflecting on their experiences of the different types of tasks they encountered during the Calculus module (both familiar or routine and unfamiliar), many of the students interviewed identified example generation tasks as helping them gain conceptual understanding of the material. Indeed, some of them spoke about the tasks forcing them to make connections between topics (Student A) and others commented on the importance of 'why' rather than 'how' in these questions (Student F for example), which fits well with the NRC's (2001) definition of conceptual understanding. The views expressed by the students in our study would seem to provide evidence for the assertion of Mason & Watson (2008) that when learners are asked to construct several examples they seek to make sense for themselves of the underlying relationships, properties and structure of the concept or theorem involved. Similar to the students working with Sangwin (2003), the interviewees here demonstrated a mature understanding of the purposes of these tasks and seemed to appreciate the higher level skills required for their completion.

As mentioned above, some comments made by the interviewees could be interpreted as indicating a predominantly procedural approach to mathematics learning in the past. Findings of Lithner (2000) and Selden & Selden (1998) in relation to students' dependence on past experience and explicit instruction, together with the students' views reported here, endorse the decision to present students with unfamiliar tasks, such as example generation tasks, to discourage an over-reliance on past experience, move away from an over-emphasis on procedures and algorithms and help them to develop the flexibility in their thinking and reasoning characteristic of mathematicians. While the students in this study were assigned a number of example generation tasks over the course of the module, each task required them to generate examples of different types of mathematical objects (e.g. give an example of a function with a particular domain, give an example of a particular with a particular limit at a certain point, give an example of a function which is not differentiable at a certain point). Thus, while students may have developed their own technique or algorithm in response to a particular task and used it successfully to generate the examples desired, it is unlikely that the same approach would work for other example generation tasks assigned. At no point during the lectures or tutorials for the course was a procedure for generating particular types of examples demonstrated. Thus, in the context of this study, it would seem reasonable to consider each new example generation task as an unfamiliar one for students.

However, it would be wise to keep in mind the experience of Iannone *et al.* (2011). They were surprised, given the strong arguments made in the research literature in support of the benefits of example generation for students' understanding of mathematical concepts, to find that generating examples did not lead to better proof production than reading worked examples. They suggest several ways of accounting for their findings: example generation may facilitate understanding of a new concept but not facilitate proof production; their tasks may have been poorly designed or were not sufficiently discriminatory to detect an effect; or, while some strategies for generating examples may prove useful in learning and understanding a concept, such strategies were not the ones most frequently observed in their sample. In any case, further empirical research should be carried out to ascertain the nature of the link between example generation tasks and the developmental of conceptual understanding. While the findings in the study reported here are positive, they relate to self-reported views of students rather than objective measures of conceptual understanding. Nevertheless, there is evidence here that the unfamiliar tasks that resonated most with students were the example generation tasks. Based on our experience, we would encourage other university mathematics lecturers to include these types of tasks on assignments.

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## Supplementary material

Appendices can be found in the Supplementary material available at *Teaching Mathematics and its Applications* online.

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